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RESEARCH ARTICLE

Decomposition-integral: unifying Choquet and the concave integrals

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Abstract This paper introduces a novel approach to integrals with respect to capacities. Any random variable is decomposed as a combination of indicators. A prespecified set of collections of events indicates which decompositions are allowed and which are not. Each allowable decomposition has a value determined by the capacity. The decomposition-integral of a random variable is defined as the highest of these values. Thus, different sets of collections induce different decomposition-integrals. It turns out that this decomposition approach unifies well-known integrals, such as Choquet, the concave and Riemann integral. Decomposition-integrals are investigated with respect to a few essential properties that emerge in economic contexts, such as concavity (uncertainty-aversion), monotonicity with respect to stochastic dominance and translation-covariance. The paper characterizes the sets of collections that induce decomposition-integrals, which respect each of these properties.

Keywords Capacity · Non-additive probability · Decision making · Decomposition-integral · Concave integral · Choquet integral

JEL Classification C71 · D80 · D81 · D84

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1 Introduction

In economics, and particularly in the decision theory under uncertainty, a rational decision maker is often described as an expected utility maximizer. The expected utility is calculated with respect to (w.r.t.) some prior probability over the state space. Although expected utility theory is useful and convenient to work with, different experiments, among which the Ellsberg's paradox (1961), show that decision makers often violate this theory.

1.1 Non-additive integral

Schmeidler (1986) proposed a theory of decision making, where the belief of the decision maker is represented by a non-additive probability (henceforth referred to as *capacity*). The representation of the belief by a capacity might reflect an incomplete or imprecise information the decision maker has about the uncertain aspects of the decision problem under consideration. Schmeidler (1986) proposed a model where the expected value of a random variable is calculated according to Choquet integral (1955): among all alternatives (in this literature, they are called *acts*), the decision maker chooses the one that maximizes Choquet expected utility.

As an integration scheme, Choquet integral poses two essential properties and lacks one. On one hand, it is monotonic w.r.t. first-order stochastic dominance and it is translation-covariant. That is, Choquet expected value of a portfolio with an added constant is equal to the expected value of the original portfolio plus the constant. On the other hand, a diversification, according to Choquet integral, does not necessarily have an advantage. In formal words, according to Choquet integral, the expected value of two portfolios mixed together is not necessarily greater than, or equal to, the mixture of the expected values of the two portfolios calculated separately.

Lehrer (2009) introduced the concave integral with respect to capacities, which differs from Choquet integral. It hinges on the idea underlying the Lebesgue integral and thus respects uncertainty-aversion. The concave integral is based on decomposition of random variables to simple ingredients. A *decomposition* is a representation of a random variable as a positive linear combination of indicators.[1] A capacity assigns to each decomposition a value: replacing each indicator by the value of its corresponding event, transforms the decomposition to a linear combination of numbers. This value enables the decision maker to evaluate any portfolio, even when the information available is incomplete or imprecise. The expected value of a random variable, according to the concave integral, is defined as the maximum value obtained among all its decompositions.

Not only the concave integral can be expressed in terms of decompositions, but Choquet integral can also be described in these terms. While the concave integral does not impose any restriction on the decompositions allowed, Choquet integral does. A chain of events is a sequence of decreasing events w.r.t. inclusion. A Choquet decomposition is a decomposition that uses only chains. Like the con-

[1] An indicator of event A, denoted $\mathbb{I}_A$, is the random variable that attains the value 1 on A and the value 0, otherwise.

cave integral, Choquet integral of a random variable is defined as the maximum value obtained among its decompositions, but in this case, only among its Choquet decompositions.

Based on the decomposition method, this paper develops a new notion of integral w.r.t. capacities: the *decomposition-integral*. This integral scheme is determined by a set of collections that dictates which decompositions are allowed and which are not. For instance, when all possible decompositions are allowed, the decomposition-integral coincides with the concave integral, and when only Choquet decompositions are allowed, the decomposition-integral coincides with Choquet integral. It turns out that the decomposition approach to integration unifies many other integral schemes, including Riemann and Shilkret (see Shilkret 1971).

A decision maker who holds a non-additive belief would like to use it in order to choose the best act. However, different integration methods might result in different evaluations and ultimately in different decisions. One of the advantages of the decomposition method is that it clarifies the trade-off between different integration methods w.r.t. essential properties. Once this trade-off is well formulated, the decision maker can compare between the various available integration schemes and choose the one that owns the properties she value most.

Few essential properties are maintained by all decomposition-integrals, regardless of the particular set of collections used. It is said that one random variable is greater than another if the former obtains a higher value than the latter in every possible state. It turns out that when one random variable is higher than another, its decomposition-integral is greater than that of the other. A similar property remains valid when comparing two capacities. A capacity is greater than another if it assigns every event a higher value than the other. Regardless of the set of collections used, the decomposition-integral of the same random variable w.r.t. two capacities maintains the order among the capacities. Furthermore, decomposition-integral is homogeneous[2] and is independent of irrelevant events.[3] However, there are essential properties that are respected by some decomposition-integrals but not by other, depending on the sets of collections used.

We study in depth three properties of this type: concavity (uncertainty-aversion), monotonicity w.r.t. first-order stochastic dominance and translation-covariance. It turns out, for instance, that uncertainty-aversion and monotonicity w.r.t. first-order stochastic dominance cannot live together. Roughly speaking, the concave integral is the only plausible scheme that respects uncertainty-aversion, while Choquet integral is the only plausible scheme that respects monotonicity w.r.t. first-order stochastic dominance, as well as translation-covariance. This kind of a trade-off is essential for a decision maker to understand before using an integration scheme in order to compare, for instance, between two portfolios, or two working groups (as in the motivating example given in Sect. 2).

[2] The integral is homogeneous if for every random variable X, and for every positive number c, $\int cX\mathrm{d}v = c\int X\mathrm{d}v$.

[3] The integral is independent of irrelevant events if for every $A \subseteq N$, $\int \mathbb{I}_A \mathrm{d}v = \int \mathbb{I}_A \mathrm{d}v_A$, where v_A is defined over A, $v_A(T) = v(T)$ for every $T \subseteq A$.

1.2 Other integral schemes and unifying approaches

There are well-known integral schemes that can be expressed in terms of decompositions. A decomposition of a random variable is *partitional* if any two of its indicators are disjoint (i.e., they obtain the value 1 on disjoint events). Riemann integral (or Pan-integral, see Wang and Klir 1992) coincides with the decomposition-integral when the set of collections allows only partitional decompositions. Another example of decomposition-integral is Shilkret integral (see Shilkret 1971). Suppose that the collections allowed to be used for decompositions consist of only one event. In this case, the linear combination consists of merely one indicator. Obviously, in this case, there is no way to obtain any random variable as one indicator multiplied by a positive scalar. This is why the integral scheme allows also sub-decompositions. A sub-decomposition of a random variable is a linear combination of indicators, but unlike a decomposition, it does not necessarily coincide with the random variable—it may be smaller. Using the language of decomposition-integrals, Shilkret integral of a random variable is the maximum among all its sub-decompositions that employs only one indicator.

Another well-known concept for integration w.r.t. capacities is Sugeno (1974), also known as the Fuzzy integral. When the capacity takes only the values zero and one (a simple game, in the terminology of cooperative games), Sugeno integral coincides with Choquet integral (see Murofushi and Sugeno 1993), but it does not coincide with the expected value when the capacity is additive. Sugeno integral is not generalized by the decomposition approach. That is, there is no set of collections that induces a decomposition-integral, which coincides with Sugeno integral.

Other unifying approaches were proposed in the literature. One approach (see de Campos et al. 1991) unifies Choquet and Sugeno integrals through four essential properties. Another approach (see Klement et al. 2010), which builds on Choquet, Sugeno and Shilkret integrals, defines a universal integral. Both methods use different binary operations instead of the regular addition and multiplication, and both do not generalize the concave integral. It is worth noting also that these unifying approaches do not necessarily coincide with the Lebesgue integral (i.e., the expectation) when the underlying capacity is a probability distribution.

1.3 Organization

Section 2 provides a motivating example. Section 3 introduces the notion of decompositions and the way they are used to define the decomposition-integral. It is shown that the decomposition-integral generalizes the concave, Choquet, Riemann and Shilkret integrals. Section 4 studies a few properties of integral schemes: positive homogeneity, coincidence with the expectation whenever the capacity is a probability distribution, monotonicity and additivity. Section 5 examines three essential properties that Choquet and the concave integrals do not commonly share. Concavity (the main property of Lehrer's concave integral) is discussed first, then monotonicity w.r.t. stochastic dominance and finally, the property of translation-covariance. The sets of collections that induce decomposition-integrals which respect each of these properties are fully characterized. The dual approach to the decomposition-integral is discussed in Sect. 6.

The paper ends with Sect. 7.1, which reviews in a brief and partial way the literature on the Choquet and the concave integrals.

2 A motivating example

Three workers work on a joint project. However, each worker is willing to put a different amount of time on the project, and moreover, the workers' output depends on the team working together. For instance, if workers 1 and 2 are working one month together, they complete 0.9 of the project. We say then that $v(12) = 0.9$. The following figures provide a full information about the teams' productivity rates per month. $v(1) = v(2) = v(3) = 0.2$, $v(23) = 0.5$, $v(13) = 0.8$ and $v(123) = 1$. We denote by X_i the time (in months fractions) that worker i is willing to invest on the project. Let $X_1 = 1, X_2 = 0.4, X_3 = 0.6$. This means, for instance, that worker 1 is willing to invest one month on the project. The question is what is the maximal product that can be obtained, given the workers' willingness to invest (henceforth, time endowment) and the teams' productivity rates.

Suppose that team $\{1, 2\}$ is working 0.4 of a month together and team $\{1, 3\}$ is working 0.6 of a month together. This way all workers exhaust their time endowment, and the total product is $v(1, 2) \cdot 0.4 + v(1, 3) \cdot 0.6 = 0.9 \cdot 0.4 + 0.8 \cdot 0.6 = 0.84$. With this team structure, the output is 84 % of the project. It turns out that this is the maximum that can be produced. In other words, any other team structure would result in a smaller product. This method is akin to what is later referred to as the concave integral (Lehrer 2009).

Suppose, however, that the players are not free to choose the teams they are working with the way they want. Rather, the entire group should start working together, and then, workers gradually leave without returning to work again on the project. Under these constraints, the maximum that the workers could produce is attained when $\{1, 2, 3\}$ work 0.4 of a month together, 2 leaves and let $\{1, 3\}$ work 0.2 of a month together, and finally $\{1\}$ works 0.4 of a month alone. The output is then $1 \cdot 0.4 + 0.8 \cdot 0.2 + 0.2 \cdot 0.4 = 0.64$. That is, due to the constraint on teams formation, the output reduces to 64 %. This method is the one induced by the Choquet integral (Choquet 1955).

While the method related to the concave integral seems to be more suitable to measuring the productivity of a group, the method defined by the Choquet integral is extensively used in the theory of decision making under uncertainty. The question arises as to what makes one method more suitable than the other in one context and less so in another. Furthermore, these two methods suggest that there might exist other methods, possibly more suitable for applications in some other contexts.

In order to address these issues, we define a large family of integration schemes that contains both the concave and the Choquet integrals. We examine the schemes in this family vis-a-vis a few essential properties that are significant in various economic contexts. In particular, the paper characterizes those schemes in the family that satisfy concavity (which is equivalent to ambiguity aversion—see Schmeidler 1989), monotonicity with respect to stochastic dominance (which is used in ordering stochastic prospects—see, for instance, Hadar and Russell 1969; Bawa 1975) and translation-covariance (which is one of the axioms that characterize coherent risk measures—see Artzner et al. 1999). This study enables decision makers to choose an adequate integration scheme, depending on the case under consideration, when the need arises.

3 Capacity, decompositions and integrals

3.1 Capacity and a decomposition of a random variable

Let N be a finite set ($|N| = n$). A *collection* D is a set of subsets of N. That is, $D \subseteq 2^N$. A *capacity* v over N is a function $v : 2^N \to [0, \infty]$ satisfying: (i) $v(\phi) = 0$; and (ii) $S \subseteq T \subseteq N$ implies $v(S) \leq v(T)$.

A *random variable* (r.v. or simply, a variable) X over N is a function $X : N \to \mathbb{R}$. A subset of N will be called an *event*. For any event $A \subseteq N$, $\mathbb{I}_A$ denotes the indicator of A, which is the random variable that takes the value 1 over A and the value 0 otherwise.

The paper deals with non-negative random variables, and therefore, when we say a random variable, we refer to a non-negative one.

Definition 1 Let X be a random variable.

1. A *sub-decomposition* of X is a finite summation $\sum_{i=1}^{k} \alpha_i \mathbb{I}_{A_i}$ such that
   (i) $\sum_{i=1}^{k} \alpha_i \mathbb{I}_{A_i} \leq X$; and
   (ii) $\alpha_i \geq 0$ and $A_i \subseteq N$ for every $i = 1, \ldots, k$.
2. Let D be a collection. $\sum_{i=1}^{k} \alpha_i \mathbb{I}_{A_i}$ is a D-*sub-decomposition* of X if it is a sub-decomposition of X and $A_i \in D$ for every $i = 1, \ldots, k$.

We say that $\sum_{i=1}^{k} \alpha_i \mathbb{I}_{A_i}$ is a *decomposition* of X if equality replaces inequality in (i). That is, $\sum_{i=1}^{k} \alpha_i \mathbb{I}_{A_i}$ is a decomposition of X if it is a sub-decomposition of X, and $\sum_{i=1}^{k} \alpha_i \mathbb{I}_{A_i} = X$. A similar definition applies to D-decomposition of X.

Suppose, for instance, that $D = 2^N$ and $X = \mathbb{I}_N$. Then, $X = \sum_{i=1}^{n} \mathbb{I}_{\{i\}}$, and at the same time, $X = \mathbb{I}_N$. Both decompositions use subsets in D.

3.2 Decompositions and integrals

Using the terminology of D-decompositions, we can reiterate the definition of the concave integral w.r.t. the capacity v (see Lehrer 2009):

$$\int^{\text{cav}} X \mathrm{d}v = \max\left\{\sum_{i=1}^{k} \alpha_i v(A_i);\ \sum_{i=1}^{k} \alpha_i \mathbb{I}_{A_i} \text{is } 2^{\mathrm{N}}\text{-sub-decomposition of } X\right\}. \quad (1)$$

Note that since v is monotonic w.r.t. inclusion, one can replace sub-decomposition in Eq. (1) by decomposition. That is,

$$\int^{\text{cav}} X \mathrm{d}v = \max\left\{\sum_{i=1}^{k} \alpha_i v(A_i);\ \sum_{i=1}^{k} \alpha_i \mathbb{I}_{A_i} \text{is } 2^{\mathrm{N}}\text{-decomposition of } X\right\}. \quad (2)$$

In words, $\int^{\text{cav}} X\mathrm{d}v$ is the maximum of the values $\sum_{i=1}^{k} \alpha_i v(A_i)$ among all possible decompositions of X. The concave integral imposes no restriction over the decompositions being used: all possible decompositions are taken into account when considering the maximum.

We show that Choquet integral can also be expressed in terms of decompositions. However, unlike the concave integral, Choquet integral does impose restrictions. We recall first the traditional definition of the Choquet integral. Let σ be a permutation on N, such that $X_{\sigma(1)} \leq \cdots \leq X_{\sigma(n)}$. The Choquet integral of a r.v. X, denoted $\int^{\text{Ch}} X\mathrm{d}v$, is defined by the following summation: $\sum_{i=1}^{n}(X_{\sigma(i)} - X_{\sigma(i-1)})v(A_i(X))$, where $X_{\sigma(0)} = 0$ and $A_i(X) = \{\sigma(i), \ldots, \sigma(n)\}$, $i = 1, \ldots, n$. We say that two subsets A and B of N are *nested* if either $A \subseteq B$ or $B \subseteq A$. A collection D is called a *chain* if any two events $A, B \in D$ are nested. Denote by $\mathcal{F}^{\text{Ch}}$ the set of all chains.

The following proposition states that the Choquet integral is the maximum of $\sum_{i=1}^{k} \alpha_i v(A_i)$, among all decompositions in which every A_i and A_j are nested.

Proposition 1

$$\int^{\text{Ch}} X\mathrm{d}v = \max\left\{\sum_{i=1}^{k}\alpha_i v(A_i);\ \sum_{i=1}^{k}\alpha_i \mathbb{I}_{A_i} \textit{is}\ \mathcal{F}^{\text{Ch}}\text{-sub-decomposition of } X\right\} \tag{3}$$

$$= \max\left\{\sum_{i=1}^{k}\alpha_i v(A_i);\ \sum_{i=1}^{k}\alpha_i \mathbb{I}_{A_i} \textit{is}\ \mathcal{F}^{\text{Ch}}\text{-decomposition of } X\right\}. \tag{4}$$

Stated differently, Choquet integral is the maximum of $\sum_{i=1}^{k} \alpha_i v(A_i)$, over all decompositions in which every A_i and A_j are nested. The proof is deferred to the "Appendix".

Since any chain is a subset of 2^N, it is evident from Eqs. (2) and (4) that

$$\int^{\text{Ch}} \cdot\,\mathrm{d}v \leq \int^{\text{cav}} \cdot\,\mathrm{d}v.$$

Example 1 Let $N = \{1, 2, 3\}$, $v(N) = 1$, $v(12) = v(13) = 1/2$, $v(23) = 11/12$ and $v(1) = v(2) = v(3) = 1/3$. Define $X = (3, 5, 2)$ to be a variable over N. The decomposition $X = 3\mathbb{I}_{12} + 2\mathbb{I}_{23}$ is the one at which the maximum of the right-hand side of (1) is obtained. Therefore, the concave integral of X is

$$\int^{\text{cav}} X\mathrm{d}v = 3 \cdot (1/2) + 2 \cdot (11/12) = \frac{10}{3}.$$

On the other hand, Choquet integral of X is obtained at the chain $\{(2), (12), (123)\}$, where the decomposition of X is $2\mathbb{I}_2 + 1\mathbb{I}_{12} + 2\mathbb{I}_N$ and

$$\int^{\text{Ch}} X\mathrm{d}v = 2 \cdot (1/3) + 1 \cdot (1/2) + 2 \cdot 1 = \frac{19}{6}.$$

3.3 Allowable decompositions and the decomposition-integral

In this part we show that the method of sub-decomposition enables us to unify many well-known and useful methods of integration under one general method. Suppose that $\mathcal{F}$ is a set of collections. A sub-decomposition of X is *$\mathcal{F}$-allowable* if it is a D-sub-decomposition of X, with the restriction that $D \in \mathcal{F}$. In other words, it has the form $\sum_{A_i \in D} \alpha_i \mathbb{I}_{A_i}$, where $D \in \mathcal{F}$. Thus, in the sub-decomposition of X, only events from the same collection D in $\mathcal{F}$ are allowed to be used. The key concept of this paper is introduced in the following definition.

Definition 2 The *decomposition-integral w.r.t.* the set of collections $\mathcal{F}$ is defined as follows.

$$\int_{\mathcal{F}} X \mathrm{d}v = \max \left\{ \sum_{A \in D} \alpha_A v(A);\ \sum_{A \in D} \alpha_A \mathbb{I}_A \text{ is } \mathcal{F}\text{-allowable sub-decomposition of X} \right\}. \tag{5}$$

The integral $\int_{\mathcal{F}} \cdot \, \mathrm{d}v$ is the maximum over all sub-decompositions that use only A_i's from the same collection $D \in \mathcal{F}$. The sub-decomposition attaining the maximum in (5) is called the *v-optimal* sub-decomposition (or decomposition) of X w.r.t. $\mathcal{F}$. When no ambiguity arises, we just call it an optimal sub-decomposition (or decomposition) of X.

Remark 1 The decomposition-integral is defined as the maximum over the set on the RHS of (5). Considering the maximum rather than the supremum is justified because for any collection $D \in \mathcal{F}$, the set of vectors

$$\left\{ (\alpha_A)_{A \in D};\ \alpha_A \geq 0 \text{ for every } A \in D \text{ and } \sum_{A \in D} \alpha_A \mathbb{I}_A \text{ is } D\text{-sub-decomposition of } X \right\}$$

is compact, and the function $\sum_{A \in D} \alpha_A v(A)$ defined over this set is continuous. Therefore, for any collection $D \in \mathcal{F}$,

$$\max \left\{ \sum_{A \in D} \alpha_A v(A);\ \sum_{A \in D} \alpha_A \mathbb{I}_A \text{ is } D\text{-sub-decomposition of } X \right\}$$

exists. Since there are finitely many collections D in $\mathcal{F}$, writing the maximum in (5) is justified.

The following example illustrates the reason why in Definition 2 we allow for sub-decompositions and do not insist on decompositions.

Example 2 (Example 1 continued) Consider $\mathcal{F}$ defined as follows.

$$\mathcal{F} = \{\{(1), (23)\}, \{(12)\}, \{(2), (13)\}\}.$$

Here, $\mathcal{F}$ consists of three collections. It turns out that a sub-decomposition, rather than a decomposition, attains the maximum in (5). The optimal sub-decomposition of X is $3\mathbb{I}_{(1)} + 2\mathbb{I}_{(23)}$ obtained at the collection $\{(1), (23)\}$, and $\int_{\mathcal{F}} X \mathrm{d}v = 3 \cdot (1/3) + 2 \cdot (11/12) = \frac{34}{12}$.

Denote by $\mathcal{F}^{\mathrm{cav}}$ the set of collections consisting of merely the collection 2^N. Then, $\int_{\mathcal{F}^{\mathrm{cav}}} \cdot \mathrm{d}v = \int^{\mathrm{cav}} \cdot \mathrm{d}v$. Proposition 1 states that[4] $\int_{\mathcal{F}^{\mathrm{Ch}}} \cdot \mathrm{d}v = \int^{\mathrm{Ch}} \cdot \mathrm{d}v$. Hence, the concave and Choquet integral differ from each other in the decompositions that the respective sets of collections allow. While the concave integral allows for all possible decompositions, the Choquet integral allows for chain decompositions (or Choquet decompositions) only. Since the set of collections $\mathcal{F}^{\mathrm{cav}}$ allows for all decompositions, the following statement (given without a proof) is obtained.

Proposition 2 *Suppose that $\mathcal{F}$ is a set of collections. Then,*

$$\int_{\mathcal{F}} \cdot \mathrm{d}v \leq \int^{\mathrm{cav}} \cdot \mathrm{d}v$$

for every v.

In other words, of all the decomposition-integrals, the concave integral is the highest.

3.4 Riemann integral, Shilkret integral and the minimum

It turns out that other integration schemes also conform to the decomposition method. A partition of N is a collection $D = \{A_1, A_2, \ldots, A_k\}$ consisting of pairwise disjoint events whose union is N itself. Denote by $\mathcal{F}^{part}$ the set of all partitions of N. The integral $\int_{\mathcal{F}^{part}} \cdot \mathrm{d}v$ is Riemann integral (or Pan-integral—see Wang and Klir 1992).

Consider now the set $\mathcal{F}^{sing} = \{\{A\};\ A \subseteq N\}$. This set of collections consists of all the singletons whose members are events. The maximum in (5) is obtained at the event that maximizes $\alpha v(A)$, subject to the constraint that $\alpha \mathbb{I}_A \leq X$. Formally,

$$\int_{\mathcal{F}^{sing}} X \mathrm{d}v = \max\left\{\sum_i \alpha_i v(A_i);\ \sum_i \alpha_i \mathbb{I}_{A_i} \text{ is } \mathcal{F}^{sing}\text{-allowable sub-decomposition of } X\right\}$$

$$= \max\left\{\alpha v(A);\ \alpha \mathbb{I}_A \leq X,\ A \subseteq N,\ \alpha \geq 0\right\} = \max\left\{\alpha \cdot v(X \geq \alpha);\ \alpha \geq 0\right\}.$$

The right-hand side is the scheme known as Shilkret integral of X w.r.t. v.

Another natural set of collections is the one consisting of a single member: an algebra of sets. We say that D is an *algebra* of sets if it is closed under unions and complement, that is, if $A, B \in D$ implies that $A \cup B$ and $N \setminus A$ are also in D. It

[4] Coincidentally, the notation $\mathcal{F}^{\mathrm{Ch}}$, derived from the word chain, resonates with the notation $\int^{\mathrm{Ch}}$ that derives from Choquet.

might occur that a decision maker is forced or would like to rely only on events in an algebra D. This might happen, for instance, when the decision maker suspects that the information embedded in the capacity about events out of the algebra is unreliable. In this case, employing the integral $\int_{\{D\}} X \mathrm{d}v$ to evaluate the random variable X seems to be a natural choice. In Zhang (2002), the unambiguous events are represented by a λ-system. The main difference between an algebra and a λ-system is that the latter is not required to be intersection-closed. Zhang (2002) and Nehring (1999) show a connection between λ-systems and the Choquet integral. Nehring (1999) shows that for some preferences of the DM, the set of unambiguous events is in fact an algebra.

Finally, consider the set of collections $\mathcal{F}$ that consists of $\{N\}$ alone. Then,

$$\int_{\mathcal{F}} X \, \mathrm{d}v = \min X \cdot v(N).$$

4 Properties of the decomposition-integral

This section examines the family of decomposition-integrals with respect to four natural properties.

4.1 Positive homogeneity of degree one

The decomposition-integral is *positive homogeneous* for any set of collections $\mathcal{F}$. This means that for every $\lambda > 0$, $\int_{\mathcal{F}} \lambda X \mathrm{d}v = \lambda \int_{\mathcal{F}} X \mathrm{d}v$ for every X, v and $\mathcal{F}$.

4.2 The decomposition-integral and additive capacities

The integral w.r.t. a general capacity is meant to generalize the notion of expectation in case the capacity is probability. Riemann, Choquet and the concave integrals indeed coincide with the expectation whenever v is a probability, while Shilkret integral does not. The objective of this chapter is to find conditions on the set of collections which guarantee that the decomposition-integral coincides with the expectation in case the capacity is a probability distribution. Denote by $\mathbb{E}_P(X)$ the expectation of X w.r.t. probability P.

Proposition 3 *$\mathbb{E}_P(X) = \int_{\mathcal{F}} X dP$ for every r.v. X and every probability P, if and only if every X has a D-decomposition with $D \in \mathcal{F}$.*

Proof Let P be probability with full support (i.e., $P(i) > 0$ for every $i \in N$) and suppose that $\mathbb{E}_P(X) = \int_{\mathcal{F}} X dP$ for every r.v. X. In order to attain the value $\mathbb{E}_P(X)$, $\mathcal{F}$-allowable sub-decomposition of X needs to be a decomposition of X. Thus, every X has D-decomposition with $D \in \mathcal{F}$. As for the inverse direction, suppose that every X has a D-decomposition, which is $\mathcal{F}$-allowable. When P is additive, any decomposition of X induces the same value, $\mathbb{E}_P(X)$. Thus, $\mathbb{E}_P(X) = \int_{\mathcal{F}} X dP$ for every X. □

4.3 Monotonicity

The first observation regarding monotonicity refers to fixed sets of collections and capacity. Fix v and $\mathcal{F}$, and suppose that $X \leq Y$. Then, $\int_{\mathcal{F}} X \mathrm{d}v \leq \int_{\mathcal{F}} Y \mathrm{d}v$.

The second observation refers to comparison between two capacities. Fix a set of collections $\mathcal{F}$. If for every $D \in \mathcal{F}$ and every $A \in D$, $v(A) \geq u(A)$, then for every r.v. X, $\int_{\mathcal{F}} X \mathrm{d}u \leq \int_{\mathcal{F}} X \mathrm{d}v$.

The third observation refers to the comparison between two sets of collections. Any set of collections $\mathcal{F}$ induces a decomposition-integral. The question arises as whether any two different sets of collections induce different integrals. The answer to this question is negative. The following proposition characterizes the circumstances in which the decomposition-integral w.r.t. $\mathcal{F}$ is always smaller than, or equal to, that w.r.t. $\mathcal{F}'$. For this purpose, we need the following definition and lemma.

Definition 3 Fix a collection $C \subseteq 2^N$ of subsets of N. We say that C is an *independent collection* if the variables $\mathbb{I}_A$, $A \in C$ are linearly independent.

In other words, C is an independent collection if for every variable X, there are no two different C-decompositions of X. The $C = \{(12), (1)\}$ is an independent collection, while $C = \{(12), (1), (2)\}$ is not because $\mathbb{I}_{(1)}, \mathbb{I}_{(2)}$, and $\mathbb{I}_{(12)}$ are linearly dependent. This is demonstrated also by the fact that $\mathbb{I}_{(1)} + \mathbb{I}_{(2)}$ and $\mathbb{I}_{(12)}$ are two different decompositions of the same variable, which employ indicators of events from C.

Lemma 1 *Fix v, $\mathcal{F}$ and X. Suppose that an optimal $\mathcal{F}$-allowable sub-decomposition of X is obtained by a D-sub-decomposition of X, where $D \in \mathcal{F}$. Then, there is an independent collection $C \subseteq D$ and a C-sub-decomposition, which is an optimal $\mathcal{F}$-allowable sub-decomposition of X.*

The proof is postponed to the "Appendix".

Proposition 4 *Suppose that $\mathcal{F}$ and $\mathcal{F}'$ are two sets of collections. Then, $\int_{\mathcal{F}} \cdot \, \mathrm{d}v \leq \int_{\mathcal{F}'} \cdot \, \mathrm{d}v$ for every v, if and only if for every $D \in \mathcal{F}$ and every independent collection $C \subseteq D$, there is $D' \in \mathcal{F}'$ such that $C \subseteq D'$.*

Proof Suppose that for every $D \in \mathcal{F}$ and independent collection $C \subseteq D$, there is $D' \in \mathcal{F}'$ such that $C \subseteq D'$. Fix v and X. Let an optimal sub-decomposition of X w.r.t. $\mathcal{F}$ be obtained at D. By Lemma 1, there is an independent collection $C \subseteq D$ and an optimal C-sub-decomposition of X. By assumption, there is $D' \in \mathcal{F}'$ that contains C as a subset. Thus, there is a D'-sub-decomposition of X that achieves at least the level attained by the D-sub-decomposition of X. Thus, $\int_{\mathcal{F}} X \, \mathrm{d}v \leq \int_{\mathcal{F}'} X \, \mathrm{d}v$ and since X is arbitrary, $\int_{\mathcal{F}} \cdot \, \mathrm{d}v \leq \int_{\mathcal{F}'} \cdot \, \mathrm{d}v$.

Now assume that $\int_{\mathcal{F}} \cdot \, \mathrm{d}v \leq \int_{\mathcal{F}'} \cdot \, \mathrm{d}v$ for every v. Suppose, to the contrary of the proposition, that there are collections C and $D \in \mathcal{F}$ (C is not necessarily in $\mathcal{F}$) such that C is an independent collection, $C \subseteq D$ and no $D' \in \mathcal{F}'$ contains C as a subset. We construct v and X such that $\int_{\mathcal{F}'} X \, \mathrm{d}v < \int_{\mathcal{F}} X \, \mathrm{d}v$ and thereby getting a contradiction. Consider the smallest capacity such that $v(A) = \frac{|A|}{|N|}$ for every $A \in C$. That is,

$v(B) = 0$, unless $A \subseteq B$ for some $A \in C$, in which case $v(B) = \frac{|A|}{|N|}$, where A is the largest set in C such that $A \subseteq B$. Define $X = \sum_{A \in C} \mathbb{I}_A$. Thus, $\int_{\mathcal{F}} X \, \mathrm{d}v \geq \sum_{A \in C} \frac{|A|}{|N|}$. Define P to be a uniform distribution—the probability that assigns each point in N a weight of $\frac{1}{|N|}$. Note that

$$\mathbb{E}_P(X) \leq \int_{\mathcal{F}} X \mathrm{d}v. \tag{6}$$

Suppose that the optimal sub-decomposition of X w.r.t. $\mathcal{F}'$ is obtained at D'. Denote this sub-decomposition as $\sum_{B \in D'} \beta_B \mathbb{I}_B$. Thus,

$$\int_{\mathcal{F}'} X \, \mathrm{d}v = \sum_{B \in D'} \beta_B v(B). \tag{7}$$

We can assume that each $B \in D'$ whose β_B is strictly positive contains at least one $A \in C$ as a subset (since otherwise, $v\,(B) = 0$). Denote $A(B)$ the largest event in C that is a subset of $B,\ B \in D'$. We obtain,

$$\sum_{B \in D'} \beta_B \mathbb{I}_{A(B)} \leq \sum_{B \in D'} \beta_B \mathbb{I}_B \leq X, \tag{8}$$

which implies together with the definition of v,

$$\sum_{B \in D'} \beta_B v(B) = \sum_{B \in D'} \beta_B v(A(B)) = \mathbb{E}_P\left(\sum_{B \in D'} \beta_B \mathbb{I}_{A(B)}\right) \leq \mathbb{E}_P(X). \tag{9}$$

Due to Eqs. (6), (7) and (9), we obtain

$$\int_{\mathcal{F}'} X \, \mathrm{d}v \leq \int_{\mathcal{F}} X \, \mathrm{d}v. \tag{10}$$

We show now that this inequality is strict.

There exist two cases. The first is when every $A \in C$ has $B \in D'$ such that $A = A(B)$. Since C is not a subset of D', there is $A \in C$ such that $A \notin D'$, implying that $A \subsetneq B$. This, in turn, implies that $\sum_{B \in D'} \beta_B \mathbb{I}_{A(B)} \neq \sum_{B \in D'} \beta_B \mathbb{I}_B$. Thus,

$$\sum_{B \in D'} \beta_B \mathbb{I}_{A(B)} \neq X. \tag{11}$$

Since P assigns to every point in N a positive probability, Eqs. (8) and (11) imply

$$\mathbb{E}_P\left(\sum_{B \in D'} \beta_B \mathbb{I}_{A(B)}\right) < \mathbb{E}_P(X).$$

Thus, Eqs. (6), (7) and (9) imply that the inequality (10) is strict, which is a contradiction.

The second case is where not every $A \in C$ has $B \in D'$ such that $A = A(B)$. It means that not every $A \in C$ appears in $\sum_{B \in D'} \beta_B \mathbb{I}_{A(B)}$. Since C is an independent collection, there are no two different C-decompositions of X. Recall that X was defined as a decomposition that involves all $A \in C$. It implies that, ignoring zero coefficients, $\sum_{B \in D'} \beta_B \mathbb{I}_{A(B)}$ cannot be a different decomposition of X. Thus, $\sum_{B \in D'} \beta_B \mathbb{I}_{A(B)} \neq X$. As in the previous case, this implies that $\mathbb{E}_P\left(\sum_{B \in D'} \beta_B \mathbb{I}_{A(B)}\right) < \mathbb{E}_P(X)$ which in turn implies that the inequality (10) is strict, which is a contradiction. □

4.4 Additivity

A well-known property of Choquet integral is comonotonic additivity. Two variables X and Y are *comonotone* if for every $i, j \in N$, $(X(i) - X(j))(Y(i) - Y(j)) \geq 0$. It turns out that this property can be expressed in terms of sets of collections and optimal decompositions. Consider the set of collections $\mathcal{F}^{\text{Ch}}$ (recall, it consists of all chains). Then, X and Y are comonotone if and only if the optimal decompositions of X and Y use the same D in $\mathcal{F}^{\text{Ch}}$. Comonotonic additivity means that if X and Y use the same D for their optimal decomposition, then $\int_{\mathcal{F}^{\text{Ch}}} X \mathrm{d}v + \int_{\mathcal{F}^{\text{Ch}}} Y \mathrm{d}v = \int_{\mathcal{F}^{\text{Ch}}} (X + Y) \mathrm{d}v$. A natural question arises as to whether this is a general property of the decomposition-integral. That is, whether for any set of collections $\mathcal{F}$, if X and Y use the same $D \in \mathcal{F}$ for their optimal sub-decomposition, then $\int_{\mathcal{F}} X \mathrm{d}v + \int_{\mathcal{F}} Y \mathrm{d}v = \int_{\mathcal{F}} (X + Y) \mathrm{d}v$.

The answer to this question proves to be negative. Indeed, consider the set of collections $\mathcal{F}^{part}$ (recall, the one consisting of all partitions of N), and a capacity v, defined on $N = \{1, 2\}$ as follows: $v(1) = v(2) = 1/3$ and $v(12) = 1$. Define $X = (\varepsilon, 1)$, $Y = (1, \varepsilon)$, $\varepsilon > 0$. Assume that ε is small enough, so that the optimal decomposition of both X and Y use $D = \{(1), (2)\}$. In this case, $\int_{\mathcal{F}} X \mathrm{d}v = \int_{\mathcal{F}} Y \mathrm{d}v = (1/3)(1 + \varepsilon)$. As for the sum $X+Y$, taking $D' = \{(12)\}$ yields $\int_{\mathcal{F}} (X + Y) \mathrm{d}v = 1+\varepsilon$, which is strictly greater than $\int_{\mathcal{F}} X \mathrm{d}v + \int_{\mathcal{F}} Y \mathrm{d}v = (2/3)(1 + \varepsilon)$.

The following proposition refers to additivity in case two integrands use the same $D \in \mathcal{F}$ for their optimal decomposition w.r.t. to $\mathcal{F}$ and a specific v.

Fix a set of collections $\mathcal{F}$ and a capacity v. We say that the variable Y is *leaner than* the variable X if there exist (i) an optimal decomposition of Y: $\sum_{A \in C'} \beta_A \mathbb{I}_A$ with $\beta_A > 0$, $A \in C'$; and (ii) an optimal decomposition of X : $\sum_{A \in C} \alpha_A \mathbb{I}_A$ with $\alpha_A > 0$, $A \in C$, such that $C' \subseteq C$. In words, Y is leaner than X, if there are optimal decompositions in which X employs every indicator that Y employs.

Proposition 5 (Co-decomposition additivity) *Fix a set of collections $\mathcal{F}$ such that every X has an optimal decomposition (not sub-decomposition) w.r.t. $\mathcal{F}$ for every capacity. Suppose that for every $D, D' \in \mathcal{F}$, whenever there are two different decompositions of the same variable, $\sum_{A \in D} \delta_A \mathbb{I}_A = \sum_{B \in D'} \gamma_B \mathbb{I}_B$, there is $D'' \in \mathcal{F}$ that contains all the A's with $\delta_A > 0$ and all the B's with $\gamma_B > 0$. Then, for every v and every two variables X and Y where Y is leaner than X,*

$$\int_{\mathcal{F}} X \mathrm{d}v + \int_{\mathcal{F}} Y \mathrm{d}v = \int_{\mathcal{F}} (X + Y) \mathrm{d}v. \tag{12}$$

Note that the condition of the proposition is readily satisfied by $\mathcal{F}^{\mathrm{Ch}}$. The reason is (see also the proof of Proposition 1) that every random variable essentially (ignoring indicators whose coefficients are zero) has a unique $\mathcal{F}^{\mathrm{Ch}}$-allowable decomposition. Proposition 5 implies the comonotonic additivity of Choquet integral. Indeed, considering $\mathcal{F}^{\mathrm{Ch}}$ and two comonotonic variables X and Y. Thus, both X and Y can be decomposed using indicators of events taken from the same chain, D. For very $\varepsilon > 0$, the variable $Z_\varepsilon = \sum_{A\in D} \frac{\varepsilon}{n}\mathbb{I}_A$ is smaller than or equal to ε, and moreover, X is leaner than Z_ε and Y is leaner than $X + Z_\varepsilon$ (because the coefficients of all $A \in D$ are positive in the decompositions of Z_ε and of $X + Z_\varepsilon$). Proposition 5 implies that $\int_{\mathcal{F}^{\mathrm{Ch}}} X \mathrm{d}v + \int_{\mathcal{F}^{\mathrm{Ch}}} Z_\varepsilon \mathrm{d}v = \int_{\mathcal{F}^{\mathrm{Ch}}} (X + Z_\varepsilon)\, \mathrm{d}v$ and $\int_{\mathcal{F}^{\mathrm{Ch}}} Y \mathrm{d}v + \int_{\mathcal{F}^{\mathrm{Ch}}} (X + Z_\varepsilon) \mathrm{d}v = \int_{\mathcal{F}^{\mathrm{Ch}}} (Y + X + Z_\varepsilon)\, \mathrm{d}v$. Thus, $\int_{\mathcal{F}^{\mathrm{Ch}}} Y \mathrm{d}v + \int_{\mathcal{F}^{\mathrm{Ch}}} X \mathrm{d}v + \int_{\mathcal{F}^{\mathrm{Ch}}} Z_\varepsilon \mathrm{d}v = \int_{\mathcal{F}^{\mathrm{Ch}}} (Y + X + Z_\varepsilon)\, \mathrm{d}v$. As ε shrinks to 0 we obtain, $\int_{\mathcal{F}^{\mathrm{Ch}}} Y \mathrm{d}v + \int_{\mathcal{F}^{\mathrm{Ch}}} X \mathrm{d}v = \int_{\mathcal{F}^{\mathrm{Ch}}} (Y + X)\, \mathrm{d}v$, which is comonotonic additivity.

Proposition 5 implies also that whenever $\mathcal{F}$ consists of only one D, like $\mathcal{F}^{\mathrm{cav}}$, its decomposition-integral respects the additivity property stated in Eq. (12). Therefore, if X and Y are leaner than each other (i.e., the same indicators possess positive coefficients in their optimal decompositions), then Eq. (12) holds true. In particular, the concave integral is linear over those variables that use the same indicators in their optimal decompositions.

The additivity of Choquet integral, as expressed in Eq. (12), does not depend on the underlying capacity. Two random variables are comonotone regardless of the capacity v, and for such variables, Eq. (12) would always be true. On the other hand, whether or not Eq. (12) applies to the concave integral, does depend on v. The reason for this difference between the integrals is that in Choquet integral, the optimal decomposition does not depend on v (it always uses the same chain for every v), while it does depend on v when it comes to the concave integral.

Proof of Proposition 5 Fix v and suppose that $X = \sum_{A\in C} \alpha_A \mathbb{I}_A$ with $\alpha_A > 0$, $A \in C$ is an $\mathcal{F}$-allowable optimal decomposition of X and $\sum_{A\in C} \beta_A \mathbb{I}_A$ with $\beta_A \geq 0$, $A \in C$ is an optimal decomposition of Y. In particular, Y is leaner than X. We show Eq. (12).

Let $\sum_{B\in D'} \gamma_B \mathbb{I}_B$ be an optimal decomposition of $X + Y$. If this decomposition equals $\sum_{A\in C} (\alpha_A + \beta_A)\mathbb{I}_A$, then Eq. (12) is true. Otherwise, $\sum_{A\in C} (\alpha_A + \beta_A)\mathbb{I}_A$ and $\sum_{B\in D'} \gamma_B \mathbb{I}_B$ are two different decompositions of $X + Y$. This implies that $\int_{\mathcal{F}} X \mathrm{d}v + \int_{\mathcal{F}} Y \mathrm{d}v \leq \int_{\mathcal{F}} (X + Y)\, \mathrm{d}v$. By assumption, there is D'' that contains all the A's with $\alpha_A + \beta_A > 0$ and all the B's with $\gamma_B > 0$. Thus, X, Y and $X + Y$ all have D'' optimal decompositions (i.e., optimal decompositions of X, Y and $X + Y$ that use only members in D'').

Suppose, to the contrary of the proposition, that $\int_{\mathcal{F}} X \mathrm{d}v + \int_{\mathcal{F}} Y \mathrm{d}v < \int_{\mathcal{F}} (X + Y)\, \mathrm{d}v$. Recall that in the optimal decomposition of X, $\sum_{A\in C} \alpha_A \mathbb{I}_A$, all the coefficients α_A are strictly positive. Thus, for $\varepsilon > 0$ sufficiently small, $\sum_{A\in C} \alpha_A \mathbb{I}_A - \varepsilon \sum_{A\in C} (\alpha_A + \beta_A)\mathbb{I}_A + \varepsilon \sum_{B\in D'} \gamma_B \mathbb{I}_B$ is a D''-decomposition of X (that is, all the coefficients are non-negative). Thus,

$$\int_{\mathcal{F}} X \mathrm{d}v \geq \sum_{A \in C} \alpha_A v(A) - \varepsilon \sum_{A \in C} (\alpha_A + \beta_A) v(A) + \varepsilon \sum_{B \in D'} \gamma_B v(B)$$

$$> \int_{\mathcal{F}} X \mathrm{d}v - \varepsilon \int_{\mathcal{F}} X \mathrm{d}v - \varepsilon \int_{\mathcal{F}} Y \mathrm{d}v + \varepsilon \left(\int_{\mathcal{F}} X \mathrm{d}v + \int_{\mathcal{F}} Y \mathrm{d}v \right) = \int_{\mathcal{F}} X \mathrm{d}v.$$

Since this is a contradiction, Eq. (12) is proven. □

Definition 4 Two sets of collections $\mathcal{F}$ and $\mathcal{F}'$ are *equivalent* if they induce the same integral. That is, for every v, $\int_{\mathcal{F}} \cdot \mathrm{d}v = \int_{\mathcal{F}'} \cdot \mathrm{d}v$.

5 Three essential properties

In this section, we state and prove three theorems that deal with essential properties: concavity, monotonicity w.r.t. stochastic dominance and translation-covariance. We characterize the sets of collections corresponding to decomposition-integrals that maintain each of these properties. Among the known integrals we discussed, Choquet integral maintains monotonicity w.r.t. stochastic dominance and translation-covariance, but does not maintain concavity. The concave integral, on the other hand, maintains concavity, but does not maintain monotonicity w.r.t. stochastic dominance and translation-covariance. As one can see, there is a trade-off between the different properties, meaning that if we want the integral to maintain concavity, we have to give up monotonicity w.r.t. stochastic dominance, for instance, and vice versa. This conclusion may help a decision maker to use an adequate integration method depending on the problem under consideration.

Since the concave integral respects concavity, under this integral, the output of two combined groups of workers is typically greater than the sum of their outputs when working separately. Choquet integral, on the other hand, typically fails to exhibit synergetic effects of this type. This difference between the two integrals is one of the reasons why the concave integral is more suitable for measuring productivity of groups of workers than the Choquet integral. However, in cases where monotonicity w.r.t. stochastic dominance is indispensable, another method, such as Choquet integral, would be more suitable than the concave integral.

5.1 Concavity

Concavity is an essential property of an integral when it comes to decision making under uncertainty [see Schmeidler's ambiguity aversion (1989)]. We say that $\int_{\mathcal{F}} \cdot \mathrm{d}v$ is *concave* if for every two variables X and Y, and $\gamma \in [0, 1]$, the following inequality holds true,

$$\int_{\mathcal{F}} (\gamma X + (1-\gamma)Y) \mathrm{d}v \geq \gamma \int_{\mathcal{F}} X \mathrm{d}v + (1-\gamma) \int_{\mathcal{F}} Y \mathrm{d}v.$$

In this section, we characterize the sets of collections $\mathcal{F}$ that $\int_{\mathcal{F}} \cdot \mathrm{d}v$ is concave.

5.1.1 Decomposition-integrals that are concave

Theorem 1 *The decomposition-integral $\int_{\mathcal{F}} \cdot \, \mathrm{d}v$ is concave for every v, if and only if there exists a set of collections $\mathcal{F}'$ containing only one collection such that $\int_{\mathcal{F}} \cdot \, \mathrm{d}v = \int_{\mathcal{F}'} \cdot \, \mathrm{d}v$.*

Obviously, the concave integral maintains the condition of this theorem, since the set of collections inducing it is a singleton—it includes only the power set of N. The following lemma also refers to a set that consists of only one collection.

Lemma 2 *A set of collections $\mathcal{F}$ is equivalent to a singleton set of collections $\mathcal{F}'$ if and only if for every independent collection*[5] *$C \subseteq \cup\mathcal{F}$ there exists $D \in \mathcal{F}$ such that $C \subseteq D$.*

The proof is postponed to the "Appendix".

Proof of Theorem 1 Suppose there exists a set of collections $\mathcal{F}'$ containing only one D' ($D' \subseteq 2^N$) such that $\int_{\mathcal{F}} \cdot \, \mathrm{d}v = \int_{\mathcal{F}'} \cdot \, \mathrm{d}v$. Fix two variables X and Y and $\gamma \in [0, 1]$. Consider $\mathcal{F}'$ and denote the optimal sub-decompositions of X and Y by, $\sum_{A \in D'} \alpha_A \mathbb{I}_A$ and $\sum_{A \in D'} \beta_A \mathbb{I}_A$, respectively. The combination $\gamma \sum_{A \in D'} \alpha_A \mathbb{I}_A + (1-\gamma) \sum_{A \in D'} \beta_A \mathbb{I}_A$ is a sub-decomposition of $\gamma X + (1-\gamma)Y$, and its value is $\gamma \int_{\mathcal{F}'} X \mathrm{d}v + (1-\gamma) \int_{\mathcal{F}'} Y \mathrm{d}v$. Thus, $\int_{\mathcal{F}'} (\gamma X + (1-\gamma)Y) \mathrm{d}v$ is greater than, or equal to, this value. Since, $\int_{\mathcal{F}} \cdot \, \mathrm{d}v = \int_{\mathcal{F}'} \cdot \, \mathrm{d}v$, we obtain that $\int_{\mathcal{F}} \cdot \, \mathrm{d}v$ is concave.

As for the inverse direction, assume that $\int_{\mathcal{F}} \cdot \, \mathrm{d}v$ is concave and, in a way of contradiction, that for every $\mathcal{F}'$ that includes only one D, from Lemma 2, there is an independent collection $C \subseteq \cup\mathcal{F}$, with no $D \in \mathcal{F}$, such that $C \subseteq D$. This ensures the existence of two disjoint subsets of C, say C_1 and C_2, each contained in a different collection in $\mathcal{F}$ (i.e., $C_i \subseteq D_i \in \mathcal{F}, i = 1, 2$) and that no other $D \in \mathcal{F}$ contains both. Since C is an independent collection, so are C_1, C_2 and $C_1 \cup C_2$.

We construct two variables X, Y, and a capacity v, and find $0 < \gamma < 1$ such that $\int_{\mathcal{F}} \gamma X \mathrm{d}v + \int_{\mathcal{F}} (1-\gamma) Y \mathrm{d}v > \int_{\mathcal{F}} (\gamma X + (1-\gamma)Y) \mathrm{d}v$. Define $X = \sum_{A \in C_1} \mathbb{I}_A$ and $Y = \sum_{A \in C_2} \mathbb{I}_A$. Consider the smallest capacity such that $v(A) = \frac{|A|}{|N|}$ for every $A \in C_1 \cup C_2$. That is, $v(B) = 0$ unless $A \subseteq B$ for some $A \in C_1 \cup C_2$, in which case $v(B) = \frac{|A|}{|N|}$, where A is the largest set in $C_1 \cup C_2$, such that $A \subseteq B$. From the definition of v, we have obtained that $\int_{\mathcal{F}} X \mathrm{d}v \geq \sum_{A \in C_1} v(A)$ and $\int_{\mathcal{F}} Y \mathrm{d}v \geq \sum_{A \in C_2} v(A)$.

Fix $0 < \gamma < 1$ and suppose that the optimal sub-decomposition of $\gamma X + (1-\gamma) Y$ is $\sum_{E \in D} \beta_E \mathbb{I}_E$. Thus, $\int_{\mathcal{F}} (\gamma X + (1-\gamma) Y) \mathrm{d}v = \sum_{E \in D} \beta_E v(E)$, where $D \in \mathcal{F}$. We can assume that each $E \in D$, whose β_E is strictly positive, contains at least one $A \in C_1 \cup C_2$ as a subset (since otherwise, $v(E) = 0$). Denote $A(E)$ the largest set in $C_1 \cup C_2$ that is a subset of E, $E \in D$. Thus, $\sum_{E \in D} \beta_E v(E) = \sum_{E \in D} \beta_E v(A(E))$. There exist two cases. The first is when every $A \in C_1 \cup C_2$ has $E \in D$ such that $A = A(E)$. Since D does not contain $C_1 \cup C_2$, there is at least one E with $\beta_E > 0$ such that $A(E) \subsetneqq E$. Thus, $\sum_{E \in D} \beta_E \mathbb{I}_{A(E)}$ is not a decomposition of $\gamma X + (1-\gamma) Y$ but rather a sub-decomposition of it, implying that

[5] $\cup\mathcal{F}$ is a set that contains all $D \in \mathcal{F}$. That is, $\cup\mathcal{F} = \{A \mid A \in D \in \mathcal{F}\}$.

$$\int_{\mathcal{F}} (\gamma X \mathrm{d}v + (1-\gamma) Y) \mathrm{d}v = \sum_{E \in D} \beta_E v(A(E))$$
$$< \gamma \sum_{A \in C_1} v(A) + (1-\gamma) \sum_{A \in C_2} v(A) \leq \gamma \int_{\mathcal{F}} X \mathrm{d}v + (1-\gamma) \int_{\mathcal{F}} Y \mathrm{d}v,$$

which contradicts concavity. The second case is where not every $A \in C_1 \cup C_2$ has $E \in D$ such that $A = A(E)$. Since $C_1 \cup C_2$ is an independent collection, there are no two $C_1 \cup C_2$-decompositions of X. This means that $\sum_{E \in D} \beta_E \mathbb{I}_{A(E)}$ is not a decomposition of $\gamma X + (1-\gamma) Y$. As in the previous case, this implies that $\int_{\mathcal{F}} (\gamma X \mathrm{d}v + (1-\gamma) Y) \mathrm{d}v < \gamma \int_{\mathcal{F}} X \mathrm{d}v + (1-\gamma) \int_{\mathcal{F}} Y \mathrm{d}v$, which contradicts concavity. □

5.1.2 An alternative characterization of the concave integral

Another contribution of the decomposition approach is that it provides a new characterization to the concave integrals. The following characterization is a corollary of Theorem 1. The first condition states that for every event A, there is a $D \in \mathcal{F}$ such that $A \in D$, while the second simply requires concavity.

Corollary 1 *A decomposition-integral* $\int_{\mathcal{F}} \cdot \, \mathrm{d}v$ *satisfies (i)* $\int_{\mathcal{F}} \mathbb{I}_A \mathrm{d}v \geq v(A)$ *for every event A and capacity v; and (ii)* $\int_{\mathcal{F}} \cdot \, \mathrm{d}v$ *is concave, if and only if* $\int_{\mathcal{F}} \cdot \, \mathrm{d}v = \int^{\mathrm{cav}} \cdot \, \mathrm{d}v$.

5.2 Monotonicity w.r.t. stochastic dominance

In this section, we characterize $\mathcal{F}$ for which $\int_{\mathcal{F}} \cdot \, \mathrm{d}v$ is monotonic w.r.t. stochastic dominance.

Definition 5 (i) Let v be a capacity and X, Y be two variables over N. We say that X *stochastically dominates* Y w.r.t. v (denoted $X \succeq^v Y$), if for every number $t \in \mathbb{R}$, $v(X \geq t) \geq v(Y \geq t)$.
(ii) In case $X \succeq^v Y$ and $Y \succeq^v X$, we say that X and Y are stochastically equivalent and denote it $X \sim^v Y$.
(iii) We say that $\int_{\mathcal{F}} \cdot \, \mathrm{d}v$ is *monotonic w.r.t. first-order stochastic dominance* (or simply, *monotonic w.r.t. stochastic dominance*) if $X \succeq^v Y$ implies $\int_{\mathcal{F}} X \mathrm{d}v \geq \int_{\mathcal{F}} Y \mathrm{d}v$.

The following definition is important only for the proof and bears no conceptual significance.

Definition 6 We say that two chains of size k are *similar* if there is a size-preserving one-to-one map between them. Formally, the chains D and G are similar if there is one-to-one map $\phi : D \to G$, such that for every $A \in D$, $|\phi(A)| = |A|$.

The following example demonstrates Definitions 5 and 6 and an idea that appears in the proof of Theorem 2.

Example 3 Let $N = \{1, 2, 3, 4\}$ and $D = \{(1234), (124)\}$. Consider $X = \sum_{T\in D} \mathbb{I}_T = (2, 2, 1, 2)$. We complete D to a chain of size 4: $D' = \{(1), (12), (124), (1234)\}$. Define $G' = \{(3), (34), (234), (1234)\}$. Notice that G' is the complementary chain of D' in the sense that for every $A \in D'$ which is not N, the event $N \setminus A$ belongs to G'.

Let G be the sub-chain of G' that is similar to D. Thus, $G = \{(234), (1234)\}$. Consider $Y = \sum_{B\in G} \mathbb{I}_B = (1, 2, 2, 2)$. In order to define v, we start with the events in D' or in G'. For every $A \in D' \cup G'$, define $v(A) = \frac{|A|}{4}$. This definition makes X and Y stochastically equivalent ($X \sim^v Y$). On every $B \notin D' \cup G'$, define v to be the minimum possible, while maintaining monotonicity w.r.t. inclusion. That is, $v(B) = \max\{v(A); A \in D' \cup G' \text{ and } A \subseteq B\}$.

Now suppose that the set of collections $\mathcal{F}$ includes only D (i.e., $\mathcal{F} = \{D\}$). Then, the optimal sub-decomposition of X is in fact a decomposition: $X = \sum_{T\in D} \mathbb{I}_T$. Thus, $\int_{\mathcal{F}} X\mathrm{d}v = \sum_{T\in D} v(T) = 3/4 + 1 = \frac{7}{4}$. On the other hand, the optimal sub-decomposition of Y is $\mathbb{I}_N$. Therefore, $\int_{\mathcal{F}} Y\mathrm{d}v = v(N) = 1$. We obtain that although $X \sim^v Y$, $\int_{\mathcal{F}} X\mathrm{d}v > \int_{\mathcal{F}} Y\mathrm{d}v$ and hence, $\int_{\mathcal{F}} \cdot\,\mathrm{d}v$ is not monotonic w.r.t. first-order stochastic dominance.

Notice that $\mathcal{F}$ consists only of chains. Theorem 2 states that in order for $\int_{\mathcal{F}} \cdot\mathrm{d}v$ to be monotonic w.r.t. first-order stochastic dominance, $\mathcal{F}$ must include only chains, and furthermore, must include all chains that have the maximal size. Thus, the reason for the lack of monotonicity w.r.t. first-order stochastic dominance in this example is that $\mathcal{F}$ does not include all chains whose size is the same as that of D.

5.2.1 The decomposition-integrals that are monotonic w.r.t. stochastic dominance

Theorem 2 *The decomposition-integral $\int_{\mathcal{F}} \cdot\,\mathrm{d}v$ is monotonic w.r.t. stochastic dominance, if and only if there exists k ($k \in \mathbb{N}$) such that $\mathcal{F}$ is a set of chains not longer than k and contains all chains of size k.*

Proof We show first that if there exists k ($k \in \mathbb{N}$) such that $\mathcal{F}$ is a set of chains not longer than k and contains all chains of size k, then $\mathcal{F}$ is monotonic w.r.t. stochastic dominance. Suppose that $X \succeq^v Y$, and let $\sum_{i=1}^{\ell} \alpha_i \mathbb{I}_{C_i}$ be an optimal $\mathcal{F}$-allowable sub-decomposition of Y, where $C_1 \subseteq C_2 \subseteq \cdots \subseteq C_\ell$ and $\ell \leq k$. In particular, $\sum_{i=1}^{\ell} \alpha_i v(C_i) = \int_{\mathcal{F}} Y\mathrm{d}v$.

We now construct an $\mathcal{F}$-allowable sub-decomposition of X. Define $B_i = \{j \in N;\ X(j) \geq \sum_{m=1}^{i} \alpha_m\}$ for every $i = 1, \ldots, \ell$. Clearly, $\sum_{i=1}^{\ell} \alpha_i \mathbb{I}_{B_i}$ is a sub-decomposition of X. Moreover, $B_1 \subseteq B_2 \subseteq \cdots \subseteq B_\ell$ is a chain whose length is at most k. In particular, it is a sub-chain of a chain whose length is precisely k. Since $\mathcal{F}$ contains all chains of length k, $\sum_{i=1}^{\ell} \alpha_i \mathbb{I}_{B_i}$ is an $\mathcal{F}$-allowable sub-decomposition of X. Thus, $\int_{\mathcal{F}} X\mathrm{d}v \geq \sum_{i=1}^{\ell} \alpha_i v(B_i)$. However, by assumption, $X \succeq^v Y$ which implies in particular that for every $i = 1, \ldots, \ell$, $v(B_i) \geq v(C_i)$. We therefore obtain that $\int_{\mathcal{F}} X\mathrm{d}v \geq \sum_{i=1}^{\ell} \alpha_i v(B_i) \geq \sum_{i=1}^{\ell} \alpha_i v(C_i) = \int_{\mathcal{F}} Y\mathrm{d}v$, as desired.

In order to prove the inverse direction, we first show that $\mathcal{F}$ consists only of chains. Assume to the contrary that $\mathcal{F}$ includes at least one D_0 that is not a chain. We construct two variables X, Y, and a capacity v, such that $X \sim^v Y$, but $\int_{\mathcal{F}} X\mathrm{d}v > \int_{\mathcal{F}} Y\mathrm{d}v$. Since D_0 is not a chain, there are at least two events A, B that are not nested. There are two possible cases. First, $A \cap B = \emptyset$. Define $X = \mathbb{I}_N$ and $Y = \mathbb{I}_A$. Consider the smallest capacity such that $v(A) = v(B) = v(N) = 1$. Obviously, $X \sim^v Y$, but $\int_{\mathcal{F}} X\mathrm{d}v = v(A) + v(B) = 2$ and $\int_{\mathcal{F}} Y\mathrm{d}v = v(A) = 1$.

The second case is where $A \cap B \neq \emptyset$. Define $X = \mathbb{I}_A + \mathbb{I}_B$ and $Y = \mathbb{I}_N$. With the v before, $X \sim^v Y$, but $\int_{\mathcal{F}} X\mathrm{d}v = v(A) + v(B) = 2$ and $\int_{\mathcal{F}} Y\mathrm{d}v = v(N) = 1$.

Suppose that the longest chain (i.e., one that includes the maximal number of events) in $\mathcal{F}$ is of size k. Next, we prove that $\mathcal{F}$ must include *all* chains of size k. Assume the opposite. We start with similar chains of this size. Assume that D and G are similar chains of size k, $D \in \mathcal{F}$ while $G \notin \mathcal{F}$. Define $X = \sum_{A\in D} \mathbb{I}_A$ and $Y = \sum_{C\in G} \mathbb{I}_C$, and v as the uniform additive probability. We obtain $X \sim^v Y$ and therefore $\mathbb{E}_v(Y) = \mathbb{E}_v(X)$. Since we proved that the set of collections $\mathcal{F}$ consists of chains only, the optimal sub-decomposition of Y is a chain. Ignoring zero coefficients, the variable Y (see the proof of Proposition 1) has exactly one chain decomposition. However, the chain G that was used in this decomposition is not in $\mathcal{F}$. Thus, Y has only an optimal sub-decomposition, which is not a decomposition. Suppose that this sub-decomposition is $\sum_{B\in G'} \beta_B \mathbb{I}_B$, where $G' \in \mathcal{F}$. Thus, $\sum_{B\in G'} \beta_B \mathbb{I}_B \leq Y$ and $\sum_{B\in G'} \beta_B \mathbb{I}_B \neq Y$. Since v is the uniform distribution, $\int_{\mathcal{F}} Y\mathrm{d}v = \sum_{B\in G'} \beta_B v(B) < \mathbb{E}_p(Y) = \mathbb{E}_p(X) = \int_{\mathcal{F}} X\mathrm{d}v$. Therefore, $\int_{\mathcal{F}} X\mathrm{d}v > \int_{\mathcal{F}} Y\mathrm{d}v$, which contradicts monotonicity w.r.t. first-order stochastic dominance. We therefore conclude that similar chains of size k are either all in $\mathcal{F}$, or all out of $\mathcal{F}$.

We now show that all chains of size k are in $\mathcal{F}$. Suppose that $D = \{A_1, \ldots, A_k\} \in \mathcal{F}$, where $A_1 \subseteq \cdots \subseteq A_k$ (i.e., D a chain of size k). We complete D to a chain of size n (in an arbitrary way), say $D_1 = \{B_1, \ldots, B_n\}$, where $B_1 \subset B_2 \subset \cdots \subset B_n$. Thus, D_1 is a chain of size n that contains D. Define G_1 to be the chain that includes $E_n = N$ and $E_j = N \setminus B_{n-j}$, $j = 1, \ldots, n-1$. In a sense, G_1 is the complementary chain of D_1.

Let $G = \{C_1, \ldots, C_k\}$, where $C_1 \subseteq \cdots \subseteq C_k$, be a sub-chain of G_1 of size k, such that $G \notin \mathcal{F}$. As before, define $X = \sum_{A_i\in D} \mathbb{I}_{A_i}$ and $Y = \sum_{C_i\in G} \mathbb{I}_{C_i}$. Consider the smallest capacity such that $v(A_i) = v(C_i) = \frac{k-i+1}{n}$ for every $1 \leq i \leq k$. By construction, every $A_i \in D$ and $C_i \in G$ are not nested unless $A_i = N$ or $C_i = N$. Thus, the definition of v does not violate monotonicity w.r.t. inclusion and is therefore well defined. Furthermore, $X \sim^v Y$. An important feature of v is that for every i (letting C_0 be an arbitrary set larger than N) and $T \subsetneq C_{i-1}$, $v(T) \leq v(C_i) = \frac{k-i+1}{n}$.

Since (a) $G \notin \mathcal{F}$, (b) there is no chain in $\mathcal{F}$ longer than k that contains G and (c) there is only one chain decomposition of Y (ignoring zero coefficients), the integral of Y is attained by a sub-decomposition (not decomposition), say $\sum_{m=1}^{\ell} \alpha_m \mathbb{I}_{T_m}$, where $T_1 \subseteq T_2 \subseteq \cdots \subseteq T_\ell$ and $\ell \leq k$. This implies that T_m and C_i are nested for every $m = 1, \ldots, \ell$ and $i = 1, \ldots, k$. Since $\sum_{m=1}^{\ell} \alpha_m \mathbb{I}_{T_m} \leq \sum_{i=1}^{k} \mathbb{I}_{C_i}$, for every $i = 1, \ldots, k$,

$$\sum_{m;\, C_i \subseteq T_m} \alpha_m \leq i. \tag{13}$$

Moreover, as $\sum_{m=1}^{\ell} \alpha_m \mathbb{I}_{T_m}$ is a sub-decomposition and not a decomposition of Y, for at least one i, the inequality (13) is strict. Thus,

$$\int_{\mathcal{F}} Y \mathrm{d}v = \sum_{m=1}^{\ell} \alpha_m v(T_m) = \sum_{i=1}^{k} \sum_{m; C_i \subseteq T_m \subsetneq C_{i-1}}^{\ell} \alpha_m v(T_m)$$
$$\leq \sum_{i=1}^{k} \left(\sum_{m; C_i \subseteq T_m \subsetneq C_{i-1}}^{\ell} \alpha_m \right) v(C_i) < \sum_{i=1}^{k} \frac{k-i+1}{n} = \int_{\mathcal{F}} X \mathrm{d}v.$$

The last inequality is due to (13) which holds with strict inequality for at least one index i. This contradicts monotonicity w.r.t. first-order stochastic dominance. Thus, G must be in $\mathcal{F}$. We conclude that any sub-chain of G_1 whose size is k belongs to $\mathcal{F}$.

Note that any chain of size k is similar to a sub-chain of G_1. Since we proved that all similar chains of the same size are either all in or all out of $\mathcal{F}$, we conclude that all chains of size k are in $\mathcal{F}$, which completes the proof. □

5.2.2 A new characterization of Choquet integral

Using the notion of decomposition-integral, Theorem 2 provides a new characterization of Choquet integral, one that does not use comonotonic additivity. Alongside with the requirement that every variable X has a decomposition, which implies that $k = n$ in Theorem 2, (or alternatively, by Proposition 3, that $\mathbb{E}_P(X) = \int_{\mathcal{F}} X dP$ for every variable X and P additive), we get the following corollary.

Corollary 2 *A decomposition-integral $\int_{\mathcal{F}} \cdot \, \mathrm{d}v$ satisfies (i) $\int_{\mathcal{F}} \cdot \, dP = \mathbb{E}_P(\cdot)$ for every probability P; and (ii) it is monotonic w.r.t. stochastic dominance for every v, if and only if $\int_{\mathcal{F}} \cdot \, \mathrm{d}v = \int^{\mathrm{Ch}} \cdot \, \mathrm{d}v$.*

5.3 Translation-covariance

This section provides a characterization of those sets of collections $\mathcal{F}$ that induce a decomposition-integral, which is translation-covariant for every v: for every $c > 0$, $\int_{\mathcal{F}} (X + c) \mathrm{d}v = \int_{\mathcal{F}} X \mathrm{d}v + c$, when $v(N) = 1$. The following illustrates an example where the integral is not translation-covariant.

Example 4 Let $N = \{1, 2, 3\}$. Consider $\mathcal{F}$ defined as follows. $\mathcal{F} = \{\{(12), (23), (123)\}\}$. Define $X = (2, 4, 1)$ and $c = 1$, $v(N) = 1$, $v(12) = v(23) = 2/3$. Then $\int_{\mathcal{F}} X \mathrm{d}v = 2 \cdot (2/3) + 1 \cdot (2/3) = 2$ while $\int_{\mathcal{F}} (X + 1) \mathrm{d}v = 3 \cdot (2/3) + 2 \cdot (2/3) = \frac{10}{3} > 2 + 1$.

5.3.1 The decomposition-integrals that respect translation-covariance

The following theorem characterizes the sets of collections that always induce an integral which is translation-covariant, regardless of v.

Theorem 3 *The integral $\int_{\mathcal{F}} \cdot \, \mathrm{d}v$ is translation-covariant for every v, if and only if the set of collections $\mathcal{F}$ is (i) composed of chains; and (ii) any $D \in \mathcal{F}$ is contained in $D' \in \mathcal{F}$ such that $N \in D'$.*

Proof Fix v such that $v(N) = 1$. Suppose that the set of collections $\mathcal{F}$ is (i) composed of chains; and (ii) any $D \in \mathcal{F}$ is contained in $D' \in \mathcal{F}$ such that $N \in D'$. Fix X. Due to assumption (ii), one can assume that the optimal sub-decomposition of X is obtained in $D' \in \mathcal{F}$ that contains N. Thus, if $\sum_{E \in D'} \alpha_E \mathbb{I}_E$ is an optimal $\mathcal{F}$-allowable sub-decomposition of X, then $\sum_{E \in D'} \alpha_E \mathbb{I}_E + c\mathbb{I}_N$ is an $\mathcal{F}$-allowable sub-decomposition of $X + c$, and therefore, $\int_{\mathcal{F}} (X + c)\mathrm{d}v \geq \int_{\mathcal{F}} X\mathrm{d}v + c$ for every $c > 0$.

We show now the inverse inequality. Let

$$\sum_{E \in D,\, E \subsetneq N} \alpha_E \mathbb{I}_E + \alpha_N \mathbb{I}_N \tag{14}$$

be an $\mathcal{F}$-allowable optimal sub-decomposition of $X + c$. We show by induction on the number of positive coefficients in Eq. (14) that $\sum_{E \in D,\, E \subsetneq N} \alpha_E v(E) + cv(N) \leq \int_{\mathcal{F}} X\mathrm{d}v + c$. Suppose first that the number of positive coefficients in Eq. (14) is 1. Then, $\alpha_E \mathbb{I}_E$ is the sub-decomposition of $X + c$ for $E \in D \in \mathcal{F}$. We can assume that $\alpha_E \geq c$, because otherwise we could replace α_E by c and have $c\mathbb{I}_E$ as an optimal sub-decomposition of $X + c$. By assumption, there is $D' \in \mathcal{F}$ such that $E, N \in D'$. Thus, $(\alpha_E - c)\mathbb{I}_E + c\mathbb{I}_N$ is an $\mathcal{F}$-allowable sub-decomposition of $X + c$, implying that $(\alpha_E - c)\mathbb{I}_E$ is an $\mathcal{F}$-allowable sub-decomposition of X. Therefore, $(\alpha_E - c)v(E) \leq \int_{\mathcal{F}} X\mathrm{d}v$. Consequently, $\alpha_E v(E) \leq \int_{\mathcal{F}} X\mathrm{d}v + cv(E) \leq \int_{\mathcal{F}} X\mathrm{d}v + c$.

Assume now that whenever the number of positive coefficients in Eq. (14) is less than or equal to k, for every $c > 0$, $\sum_{E \in D,\, E \subsetneq N} \alpha_E v(E) + cv(N) \leq \int_{\mathcal{F}} X\mathrm{d}v + c$. Based on this assumption, we show the same assertion for $k + 1$ positive coefficients.

Let an $\mathcal{F}$-allowable optimal sub-decomposition of $X + c$, as in Eq. (14), have $k + 1$ positive coefficients. We divide the argument into three cases. Case (i): $\alpha_N \geq c$. In this case $\sum_{E \in D,\, E \subsetneq N} \alpha_E \mathbb{I}_E + (\alpha_N - c)\mathbb{I}_N$ is an $\mathcal{F}$-allowable sub-decomposition of X, implying that $\sum_{E \in D,\, E \subsetneq N} \alpha_E v(E) + (\alpha_N - c)v(N) \leq \int_{\mathcal{F}} X\mathrm{d}v$. Thus, $\sum_{E \in D,\, E \subsetneq N} \alpha_E v(E) + \alpha_N v(N) \leq \int_{\mathcal{F}} X\mathrm{d}v + c$, as desired.

Case (ii): $c > \alpha_N > 0$. Here, $\sum_{E \in D,\, E \subsetneq N} \alpha_E \mathbb{I}_E$ is an $\mathcal{F}$-allowable sub-decomposition of $X + (c - \alpha_N)$. We use now the induction hypothesis. Since there are k positive coefficients in $\sum_{E \in D,\, E \subsetneq N} \alpha_E \mathbb{I}_E$, we conclude that $\sum_{E \in D,\, E \subsetneq N} \alpha_E v(E) \leq \int_{\mathcal{F}} X\mathrm{d}v + (c - \alpha_N)$, implying that $\sum_{E \in D,\, E \subsetneq N} \alpha_E v(E) + \alpha_N v(N) \leq \int_{\mathcal{F}} X\mathrm{d}v + c$.

Case (iii): $\alpha_N = 0$. Since D used in Eq. (14) is a chain, there is a largest E whose coefficient α_E is positive. Denote this event by E'. Thus, either $\sum_{E \in D,\, E \subsetneq E'} \alpha_E \mathbb{I}_E + \alpha_{E'} \mathbb{I}_N$ (when $\alpha_{E'} < c$) or $\sum_{E \in D,\, E \subsetneq E'} \alpha_E \mathbb{I}_E + (\alpha_{E'} - c)\mathbb{I}_{E'} + c\mathbb{I}_N$ (when $\alpha_{E'} \geq c$) is an $\mathcal{F}$-allowable sub-decomposition of $X + c$. In either case, we find ourselves in one of the cases discussed above and therefore obtain that $\int_{\mathcal{F}} \cdot \, \mathrm{d}v$ is translation-covariant.

As for the inverse direction, assume that $\int_{\mathcal{F}} \cdot \, \mathrm{d}v$ is translation-covariant for every v. We show first that every $D \in \mathcal{F}$ must be a chain. Else, there is D which contains two non-nested events, say A and B. Let v be the smallest capacity such that $v(A) = v(B) = 2/3$ and $v(N) = 1$. We divide the proof into two cases. The first case is when $A \cap B = \emptyset$. By the definition of v, $\mathbb{I}_A + \mathbb{I}_B$ is an optimal sub-decomposition of

$X = \mathbb{I}_N$. Thus, $\int_{\mathcal{F}} X \,\mathrm{d}v = 4/3$. But then, $\int_{\mathcal{F}}(X+1)\,\mathrm{d}v = \int_{\mathcal{F}} 2X \,\mathrm{d}v = 2\int_{\mathcal{F}} X \,\mathrm{d}v$, which implies that $\int_{\mathcal{F}}(X+1)\,\mathrm{d}v > \int_{\mathcal{F}} X \,\mathrm{d}v + 1$, a contradiction.

It remains to show that in the second case, where $A \cap B \neq \emptyset$, there is also a violation of translation-covariance. Consider the variable X defined by (similar to Example 4),

$$X(s) = \begin{cases} 2, & \text{if } s \in A \setminus B \\ 4, & \text{if } s \in A \cap B \\ 1, & \text{if } s \in B \setminus A \end{cases}$$

The sum $2\mathbb{I}_A + \mathbb{I}_B$ is an optimal sub-decomposition of X and therefore $\int_{\mathcal{F}} X \,\mathrm{d}v = 2$. However, $3\mathbb{I}_A + 2\mathbb{I}_B$ is a decomposition of $X+1$ and therefore, $\int_{\mathcal{F}}(X+1)\,\mathrm{d}v \geq 5(2/3) > \int_{\mathcal{F}} X \,\mathrm{d}v + 1$. Thus, $\int_{\mathcal{F}} \cdot \,\mathrm{d}v$ is not translation-covariant. We therefore conclude that $\mathcal{F}$ is composed of chains. We now show that any $D \in \mathcal{F}$ is contained in $D' \in \mathcal{F}$ such that $N \in D'$. Suppose, in a way of contradiction, that there exists $D \in \mathcal{F}$ and no $D' \in \mathcal{F}$ includes N and contains D (as a subset). Define $X = \sum_{T \in D} \mathbb{I}_T$ and v as the uniform additive probability. Thus, $\int_{\mathcal{F}} X \,\mathrm{d}v = \mathbb{E}_v(X)$. The variable $X+1$ has exactly one chain decomposition—the one that uses $D \cup \{N\}$. By our assumption, $G = D \cup \{N\} \notin \mathcal{F}$, thus, X+1 has only an optimal sub-decomposition, which is not a decomposition: $\sum_{B \in G'} \beta_B \mathbb{I}_B \leq X+1$ that satisfies $\sum_{B \in G'} \beta_B \mathbb{I}_B \neq X+1$. Since v is the uniform distribution, $\int_{\mathcal{F}}(X+1)\mathrm{d}v = \sum_{B \in G'} \beta_B v(B) < \mathbb{E}_v(X+1) = \mathbb{E}_v(X)+1$, contradicting translation-covariance. □

5.3.2 *Choquet integral as a decomposition-integral that satisfies translation-covariance*

Together with Proposition 3, Theorem 3 provides another characterization of Choquet integral.

Corollary 3 *A decomposition-integral $\int_{\mathcal{F}} \cdot \,\mathrm{d}v$ satisfies (i) $\int_{\mathcal{F}} \cdot \, dP = \mathbb{E}_P(\cdot)$ for every probability P and (ii) it is translation-covariant for every v, if and only if $\mathcal{F} = \mathcal{F}^{\mathrm{Ch}}$.*

6 A final comment: the dual approach

In this paper, we introduced the notion of decomposition-integral that depends on a set of collections in a specific way. Recall Definition 2 in which the evaluation of a non-negative random variable X is determined by the maximal approximation of X from below. That is, the decomposition-integral of X is equal to the value of the optimal $\mathcal{F}$-allowable sub-decomposition of X. It turns out that this approach unifies many well-established and widely accepted integral schemes. One could think, however, on a dual approach to decomposition-integrals. Instead of approximating a variable X from below, it is as plausible to approximate X from above. Furthermore, instead of evaluating X as the value of its optimal sub-decomposition, one could, as plausibly, evaluate X as the value of its closest super-decomposition. This would be the minimum over all super-decompositions of X.

Formally, define the dual decomposition-integral as follows.

Definition 7 The *dual decomposition-integral w.r.t.* $\mathcal{F}$ is defined as,

$$\int_{\mathcal{F}}^{*} X \mathrm{d}v = \min\left\{\sum_{A\in D}\alpha_A v(A);\ D\in\mathcal{F} \text{ and } \sum_{A\in D}\alpha_A \mathbb{I}_A \geq X, \text{ where } \alpha_A \geq 0,\ A\in D\right\}.$$

Both the decomposition-integral and its dual were considered in the context of partially specified probabilities in Lehrer and Teper (2011). It turns out, however, that Šipoš integral (1979) predated Lehrer and Teper (2011) in using both approaches.

Issues related to the dual decomposition-integral are left for future research.

7 The literature

7.1 Choquet and the concave integrals in the literature

Schmeidler (1986, 1989) was the first to make the connection between Choquet integral and decisions under uncertainty. Schmeidler provides an axiomatization for Choquet expected utility maximization. Among the follow-ups on Schmeidler's work, one can find Gilboa (1987) who axiomatized Choquet expected utility maximization in Savage (1954) framework, Wakker (1989) and Nakamura (1990) who examine a finite states space. Wakker (1990) characterizes optimistic and pessimistic risk attitudes in Schmeidler's model using the Choquet integral.

Dow and Werlang (1994) and Lo (1996) use Choquet expected utility maximization in multi-agent models. They extend the notion of Nash equilibrium to cases where the beliefs of players about others' strategies are represented by capacities. The Choquet integral is also used for pricing insurance contracts and financial assets (see Chateauneuf et al. 1996; Waegenaere and Walker 2001; Wang et al. 1997 and others). Waegenaere et al. (2003) show that the Choquet pricing is consistent with a general equilibrium.

Choquet integral is also used in multi-criteria decision making and game theory (see Grabisch and Labreuche 2010 for a summary on this subject). Marichal (2000) uses the Choquet integral as a tool to aggregate interacting criteria. Chiang (1999) uses Choquet integral in network implementation for decision analysis. Grabisch and Labreuche (2005a,b) introduced the notion of bicapacity, which is consonant with prospect theory of Kahneman and Tversky (1979). Bicapacities reflect different attitudes of decision makers toward gains and losses. Lehrer (2012) uses the concave integral in a model of decision making and in games with partially specified probabilities. Lehrer and Teper (2011) use the concave integral in a context of decision makers' growing awareness. Araujo et al. (2012) used the concave integral for defining a pricing rule of bets.

Faigle and Grabisch (2011) defined an integral based on one collection of sets whose union is N. Let D be a collection such that $\cup D = N$. Define the singleton $\mathcal{F} = \{D\}$. The integral defined by Faigle and Grabisch is $\int_{\mathcal{F}} X\mathrm{d}v$. This integral bears similarities with both the concave and the Choquet integrals. Like the concave integral, the set of collections used by Faigle and Grabisch is a singleton, and by Theorem 1, it is concave.

7.2 Faigle and Grabisch integral

A closely related concept to ours is that of Faigle and Grabisch (2011). They defined an integral based on one collection of sets whose union is N. Let D be a collection such that $\cup D = N$. Define the singleton $\mathcal{F} = \{D\}$. The integral defined by Faigle and Grabisch is $\int_{\mathcal{F}} X \mathrm{d}v$. This integral bears similarities with both the concave and the Choquet integrals. Like the concave integral, the set of collections used is a singleton.

Faigle and Grabisch actually define $\int_{\mathcal{F}} X \mathrm{d}v$ as a solution of a maximization problem in the style of Eq. (5) only for belief functions. They then extend the definition to a general capacity using the fact that any capacity is a difference between two belief functions. The resulting integral, like the Choquet integral, is concave if and only if the underlying capacity is supermodular.

8 Appendix

Proof of Proposition 1 We show first that the transition from Eq. (3) to Eq. (4) is true. Let X be a variable and let $Y = \sum_{i=1}^{k} \alpha_i \mathbb{I}_{A_i}$ have the following properties: (a) it is $\mathcal{F}^{\mathrm{Ch}}$-allowable sub-decomposition of X (satisfying $A_1 \subseteq \cdots \subseteq A_k$) that achieves the maximum of the RHS of Eq. (3); (b) $\alpha_i > 0$ for every $i = 1, \ldots, k$. We can assume that there is no other sub-decomposition that satisfies (a) and (b) and (weakly) dominates Y. That is, for every variable $Z \neq Y$ that satisfies (a) and (b), there is $\ell \in N$ such that $Y(\ell) > Z(\ell)$. We show that Y is actually a decomposition of X.

Assume, on the contrary, that there is $j \in N$ such that $Y(j) < X(j)$. If $j \in A_i$ for every $i = 1, \ldots, k$, then the set $\{A_1, \ldots, A_k, \{j\}\}$ is a chain and $Y' = Y + (X(j) - Y(j))\mathbb{I}_{\{j\}}$ satisfies (a) and (b). Moreover, $Y' \neq Y$ and $Y'(\ell) \geq Y(\ell)$ for every $\ell \in N$, which contradicts the choice of Y. We may therefore assume that there is an index i such that $j \notin A_i$. Let i_0 be the smallest index that $j \notin A_{i_0}$. The set $\{A_1, \ldots, A_{i_0} \cup \{j\}, A_{i_0}, A_{i_0+1}, \ldots, A_k\}$ is a chain. Furthermore, $Y' = \sum_{i \neq i_0} \alpha_i \mathbb{I}_{A_i} + \beta \mathbb{I}_{A_{i_0} \cup \{j\}} + (\alpha_{i_0} - \beta)\mathbb{I}_{A_{i_0}}$, where $\beta = \min[X(j) - Y(j), \alpha_{i_0}]$, satisfies (a) and (b). Since $Y' \neq Y$ and $Y'(\ell) \geq Y(\ell)$ for every $\ell \in N$, we obtain a contradiction to the choice of Y. We thus conclude that Y is a decomposition of X (i.e., $X = Y$).

It remains to show that, ignoring indicators whose coefficients are zero, X has only one decomposition. That is, $\{A_1, \ldots, A_k\} = \{A_1(X), \ldots, A_n(X)\}$. By definition, $A_i(X) = \{j \in N;\ X(j) \geq X_{\sigma(i)}\}$. Thus, it is enough to show that if $m, \ell \in N$ satisfy $X(m) = X(\ell)$, then $m \in A_i \Leftrightarrow \ell \in A_i$. Let $m, \ell \in N$ satisfy $X(m) = X(\ell)$. If there is i_0 such that $\ell \in A_{i_0}$ and $m \notin A_{i_0}$, then due to property (b) and the fact that $X = Y,\ X(m) = Y(m) < Y(\ell) = X(\ell)$, in contradiction with the choice of m and ℓ. □

Proof of Lemma 1 Suppose that there exists an optimal sub-decomposition of X w.r.t. $\mathcal{F}$ is obtained by a D-sub-decomposition ($D \in \mathcal{F}$), $\sum_{A \in D} \alpha_A \mathbb{I}_A$. Define the set $D_X = \{A \in D \mid \alpha_A > 0\}$. We may choose a sub-decomposition such that $|D_X|$ is minimal. If D_X is an independent collection, the proof is complete. Otherwise, the variables $\mathbb{I}_A, A \in D_X$ are linearly dependent. Meaning that there is a linear combination $\sum_{A \in D_X} \delta_A \mathbb{I}_A = 0$ where at least one $\delta_A \neq 0$. Without loss of generality, we may assume that $\sum_{A \in D_X} \delta_A v(A) \leq 0$. Otherwise we could consider $\sum_{A \in D_X} (-\delta_A)\mathbb{I}_A$

instead. Since all $\mathbb{I}_A$ are non-negative, the fact that at least one $\delta_A \neq 0$ implies that there is at least one $\delta_A > 0$. Let $\varepsilon = \min_{A \in D_X, \delta_A > 0} \frac{\alpha_A}{\delta_A}$. Since all the coefficients $\alpha_A - \varepsilon\delta_A$ are greater than or equal to 0, $\sum_{A \in D} \alpha_A \mathbb{I}_A - \varepsilon \sum_{A \in D_X} \delta_A \mathbb{I}_A = \sum)A \in D(\alpha_A - \varepsilon\delta_A)\mathbb{I}_A$ is a sub-decomposition of X. As for optimality of this sub-decomposition, $\sum_{A \in D}(\alpha_A - \varepsilon\delta_A)v(A) = \sum_{A \in D} \alpha_A v(A) - \varepsilon \sum_{A \in D_X} \delta_A v(A) \geq \sum_{A \in D} \alpha_A v(A)$. Therefore, this is an optimal sub-decomposition. Moreover, for at least one $A \in D_X$, the coefficient $\alpha_A - \varepsilon\delta_A = 0$, implying that $\sum_{A \in D}(\alpha_A - \varepsilon \cdot \delta_A)\mathbb{I}_A$ is an optimal sub-decomposition of X that involves a smaller number of indicators than does D_X, contradicting the choice of D_X. It implies that D_X is indeed an independent collection. □

Proof of Lemma 2 Suppose that for every independent collection $C \subseteq \cup\mathcal{F}$, there exists $D \in \mathcal{F}$ such that $C \subseteq D$. Define the following set consisting of one collection: $\mathcal{F}' = \{\cup\mathcal{F}\}$. By assumption, for every $D' \in \mathcal{F}'$ and every independent collection $C \subseteq D'$ (i.e., for every independent $C \subseteq \cup\mathcal{F}$), there is $D \in \mathcal{F}$ such that $C \subseteq D$. Thus, from Proposition 4, $\int_{\mathcal{F}'} \cdot \, \mathrm{d}v \leq \int_{\mathcal{F}} \cdot \, \mathrm{d}v$. On the other hand, from the definition of $\mathcal{F}'$, for every $D \in \mathcal{F}$ and every independent collection $C \subseteq D$, there is $D' \in \mathcal{F}'$ such that $C \subseteq D'$. Thus, again, due to Proposition 4, $\int_{\mathcal{F}} \cdot \, \mathrm{d}v \leq \int_{\mathcal{F}'} \cdot \, \mathrm{d}v$, which leads us to conclude that $\int_{\mathcal{F}} \cdot \, \mathrm{d}v = \int_{\mathcal{F}'} \cdot \, \mathrm{d}v$.

As for the inverse direction, suppose $\int_{\mathcal{F}'} \cdot \, \mathrm{d}v = \int_{\mathcal{F}} \cdot \, \mathrm{d}v$, and $\mathcal{F}' = \{D'\}$ (i.e., $\mathcal{F}'$ is a singleton). We show that $\cup\mathcal{F} \subseteq D'$. Assume to the contrary that $\cup\mathcal{F} \not\subseteq D'$. Then, there exists $D_1 \in \mathcal{F}$ such that $D_1 \not\subseteq D'$. By assumption, $\int_{\mathcal{F}'} \cdot \, \mathrm{d}v \geq \int_{\mathcal{F}} \cdot \, \mathrm{d}v$, and from Proposition 4, we infer that for every independent collection $C \subseteq D_1$, there is $D \in \mathcal{F}'$ such that $C \subseteq D$. Any event in D_1 is an independent collection; thus, D' must include any event in D_1 and thus must contain D_1 itself (i.e., $D_1 \subseteq D'$).

Finally, consider an independent $C \subseteq \cup\mathcal{F}$. By the previous argument $C \subseteq D'$. Since $\int_{\mathcal{F}'} \cdot \, \mathrm{d}v \leq \int_{\mathcal{F}} \cdot \, \mathrm{d}v$, we obtain from Proposition 4 that there exists $D \in \mathcal{F}$ such that $C \subseteq D$, which completes the proof. □

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