Journal of Mathematical Economics 47 (2011) 48–53

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

Equilibrium payoffs of finite games

Ehud Lehrer[a], Eilon Solan[a], Yannick Viossat[b,*]

[a] School of Mathematical Sciences, Tel-Aviv University, Israel
[b] CEREMADE, Université Paris-Dauphine, France

ARTICLE INFO

Article history:
Received 9 July 2009
Received in revised form 5 October 2010
Accepted 24 October 2010
Available online 21 December 2010

JEL classification:
C72

Keywords:
Equilibrium payoffs
Correlated equilibrium

ABSTRACT

We study the structure of the set of equilibrium payoffs in finite games, both for Nash and correlated equilibria. In the two-player case, we obtain a full characterization: if U and P are subsets of $\mathbb{R}^2$, then there exists a bimatrix game whose sets of Nash and correlated equilibrium payoffs are, respectively, U and P, if and only if U is a finite union of rectangles, P is a polytope, and P contains U. The n-player case and the robustness of the result to perturbation of the payoff matrices are also studied. We show that arbitrarily close games may have arbitrarily different sets of equilibrium payoffs. All existence proofs are constructive.

1. Introduction

This paper studies equilibrium payoffs of finite games, both Nash and correlated equilibria. It addresses the following questions: what are the possible Nash and correlated equilibrium payoffs of a finite game? Given the set of Nash equilibrium payoffs of a game, what can we infer about its set of correlated equilibrium payoffs (and vice-versa)? And finally, for which subsets U and P of $\mathbb{R}^n$ is there a game whose sets of Nash and correlated equilibrium payoffs are, respectively, U and P?

For bimatrix games, we fully answer these questions. First, a subset of $\mathbb{R}^2$ is the set of Nash equilibrium payoffs of a bimatrix game if and only if it is a finite union of rectangles.[1] Second, any polytope in $\mathbb{R}^2$ is the set of correlated equilibrium payoffs of a bimatrix game (the converse is known). Third, for any finite union of rectangles U and any polytope $P \subset \mathbb{R}^2$ containing U, there exists a bimatrix game whose sets of Nash and correlated equilibrium payoffs are, respectively, U and P. This implies that for any bimatrix game G and any polytope P containing the *Nash* equilibrium payoffs of G, there exists a game G' with the same set of Nash equilibrium payoffs as G, and P as set of correlated equilibrium payoffs (even though P need not contain the correlated equilibrium payoffs of G).

For n-player games, we obtain partial results. It is still true that any polytope in $\mathbb{R}^n$ is the set of Nash equilibrium payoffs of an n-player game. Furthermore, for any n-player game G and any polytope $P \subset \mathbb{R}^n$ containing the *correlated* equilibrium payoffs of G, there exists a game with the same set of Nash equilibrium payoffs as G, and P as set of correlated equilibrium payoffs. The structure of the set of Nash equilibrium payoffs of n-player games is still unknown.

The games that we use to prove these results are highly non-generic, and we therefore study the robustness of the results to perturbations of the payoff matrices. Since almost all games have a finite set of Nash equilibria, the best one can hope to show is that, for any finite set $U \subset \mathbb{R}^n$ and any polytope $P \subset \mathbb{R}^n$ containing U, there exists an open set of n-player games whose set of Nash and correlated equilibrium payoffs are arbitrarily close to U and P, respectively. This is indeed what we show.

Finally, we show that arbitrarily close games may have arbitrarily different equilibrium payoffs. That is, for any n-player games G and G', there are arbitrarily close games Γ and Γ' such that Γ and Γ' have the same Nash and correlated equilibrium payoffs as, respectively, G and G'.

The remainder of this article is organized as follows. Section 2 introduces notation and definitions. In section 3 , the main results are stated, and proved for bimatrix games. Section 4 studies the robustness of the results to perturbations of the payoff matrices. Section 5 shows that arbitrarily close games may have arbitrarily different sets of equilibrium payoffs. Appendix A deals with n-player games.

* Corresponding author.
E-mail addresses: lehrer@post.tau.ac.il (E. Lehrer), eilons@post.tau.ac.il (E. Solan), viossat@ceremade.dauphine.fr (Y. Viossat).

[1] To our knowledge, the "only if" direction was never formally stated before, but it follows from standard results; the "if" direction is new.

0304-4068/$ – see front matter
doi:10.1016/j.jmateco.2010.10.007

2. Notation and definitions

Let G be an n-player game in strategic form with S_k being the pure strategy set of player k. Let $S := \times_{1\leq k\leq n} S_k$ and $S_{-k} := \times_{j\neq k} S_j$. Player k's payoff function is $u_k : S \to \mathbb{R}$. As usual, if $s\in S$, we may write $s = (s_k, s_{-k})$. A *correlated equilibrium* of G (Aumann, 1974, 1987) is a probability distribution μ on the set S of pure strategy profiles such that, for every player $k \in \{1, .., n\}$ and every pure strategy $s_k \in S_k$:

$$\forall t_k \in S_k, \quad \sum_{s_{-k}\in S_{-k}} \mu(s_k, s_{-k})[u_k(s_k, s_{-k}) - u_k(t_k, s_{-k})] \geq 0. \tag{1}$$

The set of correlated equilibria of G is a polytope, which contains the Nash equilibria. An *extreme correlated equilibrium* is an extreme point of this polytope.

Let $u_k(\mu) := \sum_{s\in S}\mu(s)u_k(s)$ denote the average payoff of player k in the correlated equilibrium μ. We denote by *CEP*(G) the set of *correlated equilibrium payoffs* of G. That is, the set of n-tuples $(u_1(\mu), \ldots, u_n(\mu))$ where μ is a correlated equilibrium of G. This is a polytope in $\mathbb{R}^n$. Similarly, *NEP*(G) denotes the set of Nash equilibrium payoffs of G. We sometimes write NE and CE for "Nash equilibrium" and "correlated equilibrium", respectively. If A is a subset of $\mathbb{R}^n$ then *Conv*(A) denotes its convex hull. If B is a finite set, then $|B|$ denotes its number of elements.

Throughout, when we write that a set is in $\mathbb{R}^n$, we mean that it is a subset of $\mathbb{R}^n$, and when we say "game" we mean "finite game".

3. Main results

The main focus of this paper is the structure of *CEP*(G) and *NEP*(G). It is clear that for any game *NEP*(G) ⊆ *CEP*(G). The following proposition states that any polytope P is the set of correlated equilibrium payoffs for some G. Furthermore, G may be chosen so that P is the convex hull of the set of NE payoffs, *NEP*(G).

Proposition 1. *For any polytope P in $\mathbb{R}^n$ there exists an n-player game G such that CEP(G) = Conv(NEP(G)) = P.*

Proof. We prove the result for bimatrix games (see the Appendix A for the n-player case). Let P be a polytope in $\mathbb{R}^2$ and $(x_1, y_1), \ldots, (x_m, y_m)$ be its extreme points. Assume that for every $i \in \{1, 2, .., m\}$, x_i and y_i are strictly positive. This is without loss of generality, because adding a constant to all payoffs of a game does not change its Nash equilibria nor its correlated equilibria. Consider the $m \times m$ game with payoff matrix $(a_{ij}, b_{ij})_{1\leq i,j\leq m}$ such that, for every i in $\{1, 2, \ldots, m\}$, $a_{ii} = a_{mi} = x_i$, $b_{ii} = b_{im} = y_i$, and all other payoffs are zero. For $m = 4$ this gives:

$$\begin{pmatrix} x_1, y_1 & 0, 0 & 0, 0 & 0, y_1 \\ 0, 0 & x_2, y_2 & 0, 0 & 0, y_2 \\ 0, 0 & 0, 0 & x_3, y_3 & 0, y_3 \\ x_1, 0 & x_2, 0 & x_3, 0 & x_4, y_4 \end{pmatrix}. \tag{2}$$

Clearly, any diagonal cell corresponds to a pure Nash equilibrium. Furthermore, the last row (column) is a weakly dominant strategy for the row (column) player, and it gives a strictly higher payoff than choosing row (column) $i \neq m$ whenever the column (row) player does not choose column (row) i. It follows that in every correlated equilibrium, the probability of every off-diagonal cell is zero, hence there are no Nash equilibria or extreme correlated equilibria other than the diagonal cells. Therefore, the set of correlated equilibrium payoffs is equal to P. Note that the convex hull of the set of Nash equilibrium payoffs is also equal to P. □

Remark. The game (2) is similar to the game used in (Viossat, 2008) to show that the set of games with at most k Nash equilibria is not open.

To state our next result, we first need a definition. A subset of $\mathbb{R}^2$ is a *rectangle* if it is of the form $[a, b] \times [c, d]$, for some real numbers a, b, c, d, with $a \leq b$, $c \leq d$.

Proposition 2.

(a) *In any bimatrix game, the set of Nash equilibrium payoffs is a finite union of rectangles.*
(b) *for any nonempty finite union of rectangles U, there exists a bimatrix game whose set of Nash equilibrium payoffs is U and whose set of correlated equilibrium payoffs is Conv(U).*
(c) *for any bimatrix gameG , there exists a bimatrix game G′ such that NEP(G′) = NEP(G) and CEP(G′) = Conv(NEP(G)).*

Proof. Proof of (a): Consider a bimatrix game with S_1 and S_2 being the pure strategy sets. For any $S'_1 \subset S_1$, $S'_2 \subset S_2$ and any k in $\{1, 2\}$, denote by $NEP(S'_1, S'_2)$ (resp. $NEP_k(S'_1, S'_2)$) the set of payoffs (resp. player k's payoffs) associated with Nash equilibria whose support is $S'_1 \times S'_2$ and denote by $\overline{NEP(S'_1, S'_2)}$ its closure. Since the set of best responses of player k to any fixed strategy profile of the other players is convex, and since the Nash equilibria of bimatrix games with the same support are exchangeable,[2] it follows that $NEP_k(S'_1, S'_2)$ is an interval and that

$$NEP(S'_1, S'_2) = NEP_1(S'_1, S'_2) \times NEP_2(S'_1, S'_2).$$

Therefore $\overline{NEP(S'_1, S'_2)}$ is either empty or a rectangle. Moreover,

$$NEP = \bigcup_{S'_1\subset S_1, S'_2\subset S_2} NEP(S'_1, S'_2) = \bigcup_{S'_1\subset S_1, S'_2\subset S_2} \overline{NEP(S'_1, S'_2)},$$

where the second equality holds because *NEP* is closed. Thus, *NEP* is a finite union of rectangles.

Note that (a) also follows, with the same argument, from any result showing that the set of Nash equilibria of a bimatrix game is a finite union of products $C_1 \times C_2$ of convex subsets C_1 of $\Delta(S_1)$ and C_2 of $\Delta(S_2)$ such that equilibria in $C_1 \times C_2$ are exchangeable. In particular, (a) follows from the fact that the set of Nash equilibria of a bimatrix game is a finite union of maximal Nash subsets (see Millham, 1974; Winkels, 1979; Jansen, 1981, and for recent references, see von Stengel, 2002, and Avis et al., 2010, Proposition 4).[3]

Proof of (b): Let $m \in \mathbb{N}$ and for $1 \leq i \leq m$, let a_i, b_i, c_i, d_i be real numbers. Let $U = \bigcup_{1\leq i\leq m}[a_i, b_i] \times [c_i, d_i]$. Assuming w.l.o.g. that the numbers a_i, b_i, c_i, d_i are all positive, we build below a bimatrix game with U as set of Nash equilibrium payoffs and *Conv*(U) as set of correlated equilibrium payoffs. Consider first the game with payoff matrices:

$$(A_i, B_i) = \begin{pmatrix} a_i, c_i & b_i, c_i \\ a_i, d_i & b_i, d_i \end{pmatrix}.$$

In this game, a player does not influence its own payoffs and the set of Nash equilibrium payoffs is $[a_i, b_i] \times [c_i, d_i]$. Let

$$(A_i, 0) = \begin{pmatrix} a_i, 0 & b_i, 0 \\ a_i, 0 & b_i, 0 \end{pmatrix} \quad , \quad (0, B_i) = \begin{pmatrix} 0, c_i & 0, c_i \\ 0, d_i & 0, d_i \end{pmatrix},$$

[2] We say that the two NE (p, q) and (p', q') are exchangeable, if (p, q') and (p', q) are also NE.

[3] We find our proof above more elementary than the one using the last statement.

and consider the game built by blocks:

$$\begin{pmatrix} (A_1,B_1) & 0 & .. & 0 & (0,B_1) \\ 0 & (A_2,B_2) & .. & 0 & (0,B_2) \\ .. & .. & .. & .. & .. \\ 0 & 0 & .. & (A_{m-1},B_{m-1}) & (0,B_{m-1}) \\ (A_1,0) & (A_2,0) & .. & (A_{m-1},0) & (A_m,B_m) \end{pmatrix}, \quad (3)$$

where an isolated 0 represents a 2×2 block of payoffs (0, 0). This game has the same structure as (2), where the payoffs x_i and y_i have been replaced by the blocks A_i and B_i, respectively. Any mixed strategy profile with support in one of the blocks (A_i, B_i) is a Nash equilibrium. Furthermore, it is easy to prove along the lines of the proof of Proposition 1 that there are no other Nash equilibria and that the set of correlated equilibria is the convex hull of the set of Nash equilibria. It follows first, that the set of NE payoffs of (3) is equal to U, which shows that any finite union of rectangles is the set of NE payoffs of a bimatrix game; and second, that the set of CE payoffs of (3) is equal to $Conv(U)$.

Proof of (c): Apply (b) with $U = NEP(G)$, which is a finite union of rectangles by (a). Note that G' is not a transformation of G, but a transformation of a game having the same Nash equilibrium payoffs as G. □

Lemma 3. *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. For any n-player game G, there exists a n-player game with the same set of Nash equilibrium payoffs as G and whose set of correlated equilibrium payoffs is the convex hull of $(x_1, x_2, \ldots, x_n)$ and of the set of correlated equilibrium payoffs of G.*

Proof. We prove the result in the two-player case. For the n-player case, Appendix A . Let G be a two-player $m_1 \times m_2$ game and $(x, y) \in \mathbb{R}^2$. Assume w.l.o.g. that x and y are strictly greater than 1 and that all the payoffs in G are positive. Consider the $(3 + m_1) \times (3 + m_2)$ game

$$\Gamma = \left(\begin{array}{ccc|c} 0{,}0 & x+1, y-1 & x-1, y+1 & \\ x-1, y+1 & 0{,}0 & x+1, y-1 & [0, y] \\ x+1, y-1 & x-1, y+1 & 0{,}0 & \\ \hline & [x{,}0] & & G \end{array} \right), \quad (4)$$

where $[x, 0]$ (resp. $[0, y]$) denotes a block of payoffs $(x, 0)$ (resp. $(0, y)$) of appropriate size (same notation for player 2). The top-left block is similar to Moulin and Vial's (1978) example of a game with a correlated equilibrium payoff that Pareto dominates all Nash equilibrium payoffs. Let ν denote the correlated strategy putting probability 1/6 on every off-diagonal square of the top-left block, and probability 0 on every other square of the whole payoff matrix. Clearly, ν is a correlated equilibrium of Γ, with payoff (x, y), and every correlated equilibrium of G induces a correlated equilibrium of Γ. We claim that any correlated equilibrium of Γ is a convex combination of ν and of a correlated equilibrium of G.

To see this, let μ be a correlated equilibrium of Γ. Clearly, $\mu_{12} \geq \mu_{13}$, otherwise player 1 would have an incentive to deviate from his first to his last strategy (recall that all payoffs in G are positive). Repeating this reasoning with other strategies and with player 2 leads to the chain of inequalities

$$\mu_{12} \geq \mu_{13} \geq \mu_{23} \geq \mu_{21} \geq \mu_{31} \geq \mu_{32} \geq \mu_{12}.$$

Since the first and last terms are equal, this is a chain of equalities, hence μ puts equal weight on all off-diagonal squares of the top-left block. It is then easy to see that μ puts probability zero on the diagonal of the top-left block as well as on the top-right and bottom-left blocks. This implies that μ is a convex combination of ν and of a correlated equilibrium of G, proving the claim.

It follows that (i) any CE payoff of Γ is a convex combination of the payoff of ν and of a correlated equilibrium payoff of G; and (ii) Γ and G have the same set of Nash equilibria, hence the same NE payoffs. This concludes the proof. □

Proposition 4. *For any n-player game G and for any polytope $P \subset \mathbb{R}^n$ containing CEP(G), there exists a game G′ such that NEP(G′) = NEP(G) and CEP(G′) = P. If n = 2 the set P can be any polytope that contains NEP(G).*

Proof. Let P be a polytope containing $CEP(G)$, with q extreme points. Applying iteratively Lemma 3 (q times), we obtain a game with the same NE payoffs as G and whose set of CE payoffs is the convex hull of P and $CEP(G)$, hence is equal to P.

If $n = 2$, then by Proposition 2 there exists a game G' with $NEP(G') = NEP(G)$ and $CEP(G') = Conv(NEP(G))$. Applying the first part of Proposition 4 to G' gives the result. □

Propositions 2 and 4 imply the following: for any subsets U and P of $\mathbb{R}^2$, there exists a bimatrix game G such that $U = NEP(G)$ and $P = CEP(G)$ if and only if U is a finite union of rectangles, P is a polytope, and P contains U.

Remark. In Proposition 4, the proof of the stronger results for bimatrix games is indirect, in that the game G' is not built by transforming G, but by transforming a game that has the same Nash equilibrium payoffs as G. This requires a good understanding of the structure of the set of Nash equilibrium payoffs of bimatrix games. Such an understanding is lacking for n-player games, hence our weaker results.

4. Robustness

The games used above are highly non-generic. For instance, a small perturbation of the payoffs of (2) is enough to eliminate all its Nash and correlated equilibria but one. This raises the issue of the robustness of our results. Ideally, to show that, for instance, Proposition 4 is robust, one would like to show that for any nonempty finite union of rectangles U and for any polytope P in $\mathbb{R}^2$ containing U, there exists an open set of games whose set of Nash equilibrium payoffs is "close" to U and whose set of correlated equilibrium payoffs is "close" to P. This is hopeless however, since almost all games have a finite set of Nash equilibria. Thus, the best one can hope to prove is the same result when U is a finite set. This is the object of this section.

We first need some definitions. Let $\varepsilon > 0$. For any $\mathbf{x}$ in $\mathbb{R}^n$, let $\|\mathbf{x}\| = \max_{1 \leq i \leq n} |x_i|$. Let A and A' be subsets of $\mathbb{R}^n$. Recall that A and A' are ε-close in the Hausdorff distance sense if

$$\forall \mathbf{x} \in A, \exists \mathbf{x}' \in A', \|\mathbf{x} - \mathbf{x}'\| < \varepsilon, \quad (5)$$

and

$$\forall \mathbf{x}' \in A', \exists \mathbf{x} \in A, \|\mathbf{x} - \mathbf{x}'\| < \varepsilon. \quad (6)$$

Let G be a finite game with payoff function u_k for player k and let $\alpha > 0$. The open ball of center G and radius α, denoted by $B(G, \alpha)$, is the set of all games G' with the same sets of players and strategies as in G and such that $|u'_k(s) - u_k(s)| < \alpha$ for every player k and every pure strategy profile s, where u'_k is the payoff function of player k in G'. A set of games Σ is open if for every game G in Σ, Σ contains an open ball of center G and positive radius.

Because the NE correspondence and the CE correspondence are upper-semi-continuous, it follows that for every game G, and every sequence $(G_n)_{n \in \mathbb{N}}$ of games that converges to G, the limit sets of NE payoffs and CE payoffs of G_n as n goes to infinity are respectively subsets of the sets of NE payoffs and CE payoffs of G. The following Proposition strengthens this observation.

Proposition 5. *Let U be a finite set in $\mathbb{R}^n$. Let $P \subset \mathbb{R}^n$ be a polytope containing U. For every $\varepsilon > 0$, there exists a nonempty open set of n-player games whose set of Nash equilibrium payoffs is ε-close to U and whose set of correlated equilibrium payoffs is ε-close to P.*

Proof. We prove the result for bimatrix games. The proof for n-player games is similar (Appendix A). Let $U = \{(x_1, y_1), \ldots, (x_m, y_m)\}$, let $P \subset \mathbb{R}^2$ be a polytope containing U, with vertices $(x'_1, y'_1), \ldots, (x'_q, y'_q)$. Assume w.l.o.g. that, for all i in $\{1, .., q\}$, x_i and y_i are positive, and x'_i and y'_i strictly greater than 1. For $\alpha \geq 0$ let G_α denote the $m \times m$ game with payoff matrix $(a_{ij}, b_{ij})_{1 \leq i,j \leq m}$ such that: for every i in $\{1, 2, \ldots, m\}$, $a_{ii} = x_i$ and $b_{ii} = y_i$; for every i in $\{1, 2, \ldots, m-1\}$, $a_{mi} = x_i - \alpha$ and $b_{im} = y_i - \alpha$; and all other payoffs are zero. For $m = 4$ this gives:

$$G_\alpha = \begin{pmatrix} x_1, y_1 & 0,0 & 0,0 & 0, y_1 - \alpha \\ 0,0 & x_2, y_2 & 0,0 & 0, y_2 - \alpha \\ 0,0 & 0,0 & x_3, y_3 & 0, y_3 - \alpha \\ x_1 - \alpha, 0 & x_2 - \alpha, 0 & x_3 - \alpha, 0 & x_4, y_4 \end{pmatrix}. \quad (7)$$

Thus, G_0 is the game used in the proof of Proposition 1 and $NEP(G_0) = U$. For every (x, y) in $\mathbb{R}^2$, let $C(x, y)$ denote the game corresponding to the top-left block of the game given in (4).

$$C(x,y) = \begin{pmatrix} 0,0 & x+1, y-1 & x-1, y+1 \\ x-1, y+1 & 0,0 & x+1, y-1 \\ x+1, y-1 & x-1, y+1 & 0,0 \end{pmatrix}. \quad (8)$$

Finally, let Γ_α denote the following game:

$$\Gamma_\alpha = \left(\begin{array}{cccc|c} C(x'_1, y'_1) & 0 & \cdots & 0 & [0, y'_1 - \alpha] \\ 0 & C(x'_2, y'_2) & \cdots & 0 & [0, y'_2 - \alpha] \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \cdots & C(x'_q, y'_q) & [0, y'_q - \alpha] \\ \hline [x'_1 - \alpha, 0] & [x'_2 - \alpha, 0] & \ldots & [x'_q - \alpha, 0] & G_\alpha \end{array}\right),$$

where $[x'_i - \alpha, 0]$ (resp. $[0, y'_i - \alpha]$) means a block of payoffs $(x'_i - \alpha, 0)$ (resp. $(0, y'_i - \alpha)$) of appropriate size. Γ_0 is a slight modification of the game obtained from G_0 by iterative applications (q times) of the method of Lemma 3. Along the lines of the proof of Lemma 3, it is easy to show that the Nash equilibria of Γ_0 correspond to the Nash equilibria of G_0 and that its extreme correlated equilibria are: (i) its Nash equilibria, and (ii) the probability distributions with support in one of the blocks $C(x'_i, y'_i)$ and putting probability 1/6 on every off-diagonal square of this block. It follows that $NEP(\Gamma_0) = U$ and $CEP(\Gamma_0) = P$.

Moreover, for any $\alpha > 0$ small enough and any game Γ in $B(\Gamma_\alpha, \alpha/2)$, every Nash equilibrium of Γ_0 is a Nash equilibrium of Γ; therefore,

$$\forall (a_0, b_0) \in NEP(\Gamma_0), \exists (a, b) \in NEP(\Gamma), \|(a, b) - (a_0, b_0)\| < 3\alpha/2 \quad (9)$$

(a closer look shows that we may replace $3\alpha/2$ by $\alpha/2$ in the RHS, but this is not needed). Finally, let $\varepsilon > 0$. By upper semi-continuity of the Nash equilibrium correspondence, for $\alpha > 0$ small enough, for any game Γ in $B(\Gamma_0, 3\alpha/2)$,

$$\forall (a, b) \in NEP(\Gamma), \exists (a_0, b_0) \in NEP(\Gamma_0), \|(a_0, b_0) - (a, b)\| < \varepsilon. \quad (10)$$

It follows from (9) and (10) that, for any $\alpha > 0$ small enough and any game Γ in $B(\Gamma_\alpha, \alpha/2)$, $NEP(\Gamma)$ and $NEP(\Gamma_0)$ are ε-close. The same argument (up to replacement of Nash equilibrium by correlated equilibrium everywhere) shows that for every α small enough and for every game Γ in $B(\Gamma_\alpha, \alpha/2)$, $CEP(\Gamma)$ and $CEP(\Gamma_0)$ are ε-close. Recalling that $NEP(\Gamma_0) = U$ and $CEP(\Gamma_0) = P$, this completes the proof. □

Note that, in the above proof, for every $\alpha > 0$ small enough, for every game Γ in $B(\Gamma_\alpha, \alpha/2)$:

(i) any Nash equilibrium of Γ_0 is a strict Nash equilibrium
(ii) any extreme correlated equilibrium of Γ_0 is a strict correlated equilibrium of Γ (a correlated equilibrium μ is *strict* if for every pure strategy s_i with positive marginal probability under μ, the inequalities in (1) are strict.)

Furthermore, taking convex hulls and because the set of strict correlated equilibria of a game is convex, (ii) implies that any correlated equilibrium of Γ_0 is a strict correlated equilibrium of Γ. It follows that for every finite set U in $\mathbb{R}^n$, every polytope P containing U and every $\varepsilon > 0$, there exists an open set of games Γ such that: first, both the set of Nash equilibrium payoffs and the set of strict Nash equilibrium payoffs of Γ are ε-close to U; second, both the set of correlated equilibrium payoffs and the set of strict correlated equilibrium payoffs of Γ are ε-close to P.

Non pure equilibria For any $(x, y) \in \mathbb{R}^2$, let $MP(x, y)$ denote the game obtained by adding (x, y) to all the payoffs of Matching Pennies:

$$MP(x,y) = \begin{pmatrix} x+1, x-1 & x-1, y+1 \\ x-1, y+1 & x+1, y-1 \end{pmatrix}$$

This game has a unique Nash equilibrium (and also a unique correlated equilibrium), and this equilibrium's payoff is (x, y).

Using a similar construction as in the proof of Proposition 5, it may be shown that for any $\varepsilon > 0$, any polytope P in $\mathbb{R}^2$ and any finite set $U \subset P$, there is an open set of bimatrix games such that for any game G in this set, $NEP(G)$ and $CEP(G)$ are, respectively, ε-close to U and P, and furthermore, none of the Nash equilibria of G is pure. It suffices to replace the payoffs (x_i, y_i) by the 2×2 game $MP(x_i, y_i)$ and the payoffs $(x_i - \alpha, 0)$ and $(0, y_i - \alpha)$ by blocks of such payoffs. For instance, for $U = \{(x_1, y_1), (x_2, y_2)\}$, the equivalent of the game G_α would be:

$$\left(\begin{array}{c|c} MP(x_1, y_1) & [0, y_1 - \alpha] \\ \hline [x_1 - \alpha, 0] & MP(x_2, y_2) \end{array}\right)$$

where $[x_1 - \alpha, 0]$ and $[0, y_1 - \alpha]$ denote 2×2 blocks of payoffs $(x_1 - \alpha, 0)$ and $(0, y_1 - \alpha)$, respectively.

An open question When $\alpha > 0$ is small, in the game G_α that was defined in the proof of Proposition 5, each $(x_i, y_i)_{1 \leq i \leq m}$ is a pure NE payoff, and there are mixed NE payoffs that are close to $(x_i, y_i)_{i \neq m}$. This is linked to the fact that in the game G_0, the index of the equilibria with payoffs $(x_i, y_i)_{i \neq m}$ is zero. An interesting question[4] is whether one can find necessary and sufficient conditions on the set U that would ensure the existence of a game G such that, the set of NE payoffs of any perturbation of G is close to the set of NE payoffs of G, and, moreover, the number of NE payoffs of any perturbation of G coincides with the number of NE payoffs of G. Note that a necessary condition is that the number of equilibria be odd.

5. Arbitrarily close games with arbitrarily far equilibrium payoffs:

For $\varepsilon > 0$, two games G and G' are *ε-close* if G' belongs to the open ball of center G and radius ε, as defined in Section 4 . This

[4] We thank an anonymous referee for raising this issue.

section shows that arbitrarily close games may have arbitrarily far sets of equilibrium payoffs. For any game G, let $EP(G)=(NEP(G), CEP(G))$.

Proposition 6. *Let G and G' be two n-player games.*

(a) *For any $\varepsilon>0$, there exist n-player games Γ and Γ' that are ε-close, and such that $EP(\Gamma)=EP(G)$ and $EP(\Gamma')=EP(G')$.*
(b) *If $NEP(G)\subset NEP(G')$ and $CEP(G)\subset CEP(G')$, then there exists a sequence of n-player games $(\Gamma_k)_{k\in\mathbb{N}}$ that converges to a game Γ, and such that $EP(\Gamma_k)=EP(G)$ for all k in $\mathbb{N}$ and $EP(\Gamma)=EP(G')$.*

Proof. We first prove the result for bimatrix games. Let S_k and u_k (resp. S'_k and u'_k) denote player k's strategy set and payoff function in G (resp. G'). Let $u=(u_1,u_2)$ and $u'=(u'_1,u'_2)$ denote the joint payoff functions in G and G'. Up to duplicating rows or columns in G or G', which does not affect equilibrium payoffs, we may assume that G and G' have the same size: $|S_1|=|S'_1|$ and $|S_2|=|S'_2|$. We also assume w.l.o.g. that all the payoffs in G and G' are strictly positive.

Proof of (a): Let ε be a real number, which may be negative. Let Γ_ε denote the game with payoff function v_k and strategy set $\{1,\ldots,|S_k|,|S_k|+1,\ldots,2|S_k|\}$ for player k such that, letting $v=(v_1,v_2)$: for every $1\le i\le|S_1|$ and $1\le j\le|S_2|$, first, $v(i,j)=u'(i,j)$ and $v(i+|S_1|,j+|S_2|)=u(i,j)$, second, $v(i+|S_1|,j)=(u'_1(i,j)+\varepsilon,0)$ and $v(i,j+|S_2|)=(u_1(i,j)-\varepsilon,0)$. The graphical description of this game appears below.

$$\Gamma_\varepsilon=\left(\begin{array}{c|c} G' & (G_1-\varepsilon,0) \\ \hline (G'_1+\varepsilon,0) & G \end{array}\right) \tag{11}$$

In (11), the notation $(G'_1+\varepsilon,0)$ means a game of the same size as G', and in which the payoffs of player 1 are as in G', plus ε, and those of player 2 are zero.

By iterative elimination of strictly dominated strategies, we get that for every $\varepsilon>0$, $EP(\Gamma_\varepsilon)=EP(G)$, and for every $\varepsilon<0$, $EP(\Gamma_\varepsilon)=EP(G')$. The result follows.

In the 3-player case, we may again assume that G and G' have the same size, and the generalization of Γ_ε is the $2|S_1|\times 2|S_2|\times|S_3|$ game such that for every $1\le i\le|S_1|$, $1\le j\le|S_2|$ and $1\le l\le|S_3|$, $v(i,j,l)=u'(i,j,l)$, $v(i+|S_1|,j+|S_2|,l)=u(i,j,l)$, $v(i+|S_1|,j,l)=(u'_1(i,j)+\varepsilon,0,0)$ and $v(i,j+|S_2|,l)=(u_1(i,j)-\varepsilon,0,0)$. The n-player case is similar.

$$\Gamma=\left(\begin{array}{c|c} C(x,y,z) & [0,y,0] \\ \hline [x,0,0] & [x,y,0] \end{array}\right)\left(\begin{array}{c|c} [0,0,z] & [0,y,z] \\ \hline [x,0,z] & G(s_3) \end{array}\right)\left(\begin{array}{c|c} [0,0,z] & [0,y,z] \\ \hline [x,0,z] & G(t_3) \end{array}\right) \tag{14}$$

Proof of (b): For $\varepsilon\ge0$, let $\tilde{\Gamma}_\varepsilon$ denote the $2|S_1|\times2|S_2|$ game depicted below.

$$\tilde{\Gamma}_\varepsilon=\left(\begin{array}{c|c} G' & (0,G'_2) \\ \hline (G'_1+\varepsilon,0) & G \end{array}\right) \tag{12}$$

The payoffs are as in (11) with the following exception: letting $\tilde{v}_k$ denote player k's payoff function and $\tilde{v}=(\tilde{v}_1,\tilde{v}_2)$, for every $1\le i\le|S_1|$ and $1\le j\le|S_2|$, $\tilde{v}(i,j+|S_2|)=(0,u'_2(i,j))$.

It follows from iterative elimination of strictly dominated strategies that, for every $\varepsilon>0$, $EP(\tilde{\Gamma}_\varepsilon)=EP(G)$; but, by a reasoning similar to the proof of Proposition 1, $NEP(\tilde{\Gamma}_0)=NEP(G)\cup NEP(G')$ and $CEP(\tilde{\Gamma}_0)=Conv(CEP(G),CEP(G'))$. In particular, if $NEP(G)\subset NEP(G')$ and $CEP(G)\subset CEP(G')$, then $\tilde{\Gamma}_0$ and G' have the same equilibrium payoffs. The result follows. The generalization to the n-player case is as for point (a). □

Together with Propositions 2 and 4 , Proposition 6 implies the following: for any polytopes P and P' in $\mathbb{R}^2$, for any finite union of rectangles $U\subset P$ and $U'\subset P'$, and for any $\varepsilon>0$, there are ε-close games Γ and Γ' such that $EP(\Gamma)=(U,P)$ and $EP(\Gamma')=(U',P')$.

Acknowledgements

Viossat thanks Bernhard von Stengel, participants to the game theory seminar of the Institut Henri Poincaré, Paris, and to the "Communication and networks in games" workshop in Valencia. He particularly thanks Roberto Lucchetti, Sergiu Hart and John Levy for questions which lead to Sections 4 and 5. PICASSO funding, and the support of the GIP ANR (Croyances Project) and of the Risk Foundation (Groupama Chair) are gratefully acknowledged by Viossat. The work of Solan was partially supported by ISF grant 212/09.

Appendix A. Proofs of Lemma 3 and Propositions 1 and 5 in the *n*-player case

Proof of Lemma 3. Let G be a three-player game and let $(x,y,z)\in\mathbb{R}^3$. Assume w.l.o.g. that x and y are strictly greater than 1, and that the payoffs in G are positive. Let $C(x,y,z)$ denote the $3\times3\times1$ game (player 3 is a dummy) where player 1 chooses a row, player 2 chooses a column, the payoffs of players 1 and 2 are as in (8), and the payoff of player 3 is always z.

$$C(x,y,z)=\begin{pmatrix} 0,0,z & x+1,y-1,z & x-1,y+1,z \\ x-1,y+1,z & 0,0,z & x+1,y-1,z \\ x+1,y-1,z & x-1,y+1,z & 0,0,z \end{pmatrix}. \tag{13}$$

Let Γ be the following three-player game. The pure strategy set of each player k is $S'_k\sqcup S_k$ (disjoint union), where S_k and S'_k are respectively the pure strategy set of player k in G and in $C(x,y,z)$. If for every k, player k chooses a strategy in S_k (resp. S'_k), then the payoffs are as in G (resp. as in $C(x,y,z)$). If there exist players i and l such that k chooses a strategy in S_k and l a strategy in S'_l, then the payoff of player 1 (resp. 2, 3) is x (resp. y, z) if he chooses a strategy in S_1 (resp. S_2, S_3) and 0 otherwise. The game is thus of size $(3+|S_1|)\times(3+|S_2|)\times(1+|S_3|)$.

The graphical description of the game appears in (14), when $S_3=\{s_3,t_3\}$, i.e. player 3 has two pure strategies in G. The top rows (resp. left columns, left matrix) correspond to pure strategies in S'_1 (resp. in S'_2, in S'_3), while the bottom rows (resp. right columns, middle and right matrices) correspond to pure strategies in S_1 (resp. in S_2, in S_3).

In (14), $G(s_3)$ (resp. $G(t_3)$) denotes the payoffs of G when player 3 chooses s_3 (resp. t_3). The bracket $[x,0,0]$ denotes a $|S_1|\times3$ block of payoffs $(x,0,0)$. The notation $[0,y,0]$, $[0,0,z]$, $[0,y,z]$, $[x,y,0]$ and $[x,0,z]$ should be interpreted analogously.

The same proof as in the two-player case shows that Γ has the same set of NE payoffs as G and that its set of CE payoffs is the convex hull of (x,y,z) and of the set of CE payoffs of G.

In the n-player case, the generalization of $C(x,y,z)$ simply consists in adding more dummy players with constant payoff. The generalization of Γ should be clear from the verbal description of the three-player case. □

Proofs of Propositions 1 and 5. Proposition 1 can be proved by fixing a polytope P and applying Proposition 4 to a game with constant payoffs included in P. This is also true for bimatrix games. However, it is more instructive to provide a direct proof. Let $U=\{(x_i,y_i,z_i),1\le i\le m\}$ and let P be the convex hull of $\{(x'_i,y'_i,z'_i),1\le i\le q\}$.

Assume w.l.o.g. that x_i, y_i, z_i and z'_i are positive, and x'_i and y'_i strictly greater than 1, for all i. Let G_α denote the $m \times m \times 1$ game obtained from the game described above Eq. (7) by adding a dummy player with payoffs $z_1, \ldots, z_m$ on the diagonal, and 0 elsewhere. For $m = 4$ this gives:

$$G_\alpha = \begin{pmatrix} x_1, y_1, z_1 & 0,0,0 & 0,0,0 & 0, y_1 - \alpha, 0 \\ 0,0,0 & x_2, y_2, z_2 & 0,0,0 & 0, y_2 - \alpha, 0 \\ 0,0,0 & 0,0,0 & x_3, y_3, z_3 & 0, y_3 - \alpha, 0 \\ x_1 - \alpha, 0, 0 & x_2 - \alpha, 0, 0 & x_3 - \alpha, 0, 0 & x_4, y_4, z_4 \end{pmatrix}.$$

The same argument as in the two-player case shows that the Nash equilibria of G_0 are equal to its extreme correlated equilibria and correspond to the diagonal squares. It follows that $NEP(G_0) = U$ and that $CEP(G_0)$ is the convex hull of U. This proves Proposition 1 in the three-player case (for the n-player case, just add more dummy players).

Now recall (13) and let G' denote the following $3q \times 3q \times 1$ game with block diagonal payoff matrix

$$G' = \begin{pmatrix} C(x'_1, y'_1, z'_1) & & 0 \\ & \ddots & \\ 0 & & C(x'_q, y'_q, z'_q) \end{pmatrix}.$$

Let x, y, z be positive real numbers. Let Γ_α denote the following $(3q + m) \times (3q + m) \times 2$ game:

$$\left(\begin{array}{c|c} G' & [0, y'_i - \alpha, 0] \\ \hline [x'_j - \alpha, 0, 0] & [x, y, 0] \end{array} \right) \left(\begin{array}{c|c} [0, 0, z'_i - \alpha] & [0, y, z] \\ \hline [x, 0, z] & G_\alpha \end{array} \right)$$

This should be read as follows: if player 1 chooses row $i > 3q$ and player 2 chooses column $j \le 3q$ with $3p + 1 \le j \le 3p + 3$, then the payoffs are $(x'_p - \alpha, 0, 0)$ if player 3 chooses the left matrix and $(x, 0, z)$ if player 3 chooses the matrix on the right.

Fix $\varepsilon > 0$ and assume that P contains U. The same arguments as in the two-player case show that, for every α small enough, and for every game Γ in $B(G_\alpha, \alpha/2)$, $NEP(\Gamma)$ is ε-close to U and $CEP(\Gamma)$ is ε-close to P. The n-player case is similar. This proves Proposition 5. Note that, instead of $C(x, y, z)$, Γ, G_α, Γ_α, and their n-player versions, it is possible to use games in which the roles of the players are symmetric (no dummies), but this is less parsimonious. □

References

Aumann, R., 1974. Subjectivity and correlation in randomized strategies. Journal of Mathematical Economics 1, 67–96.

Aumann, R.J., 1987. Correlated equilibria as an expression of Bayesian rationality. Econometrica 55, 1–18.

Avis, D., Rosenberg, G.D., Savani, R., von Stengel, B., 2010. Enumeration of Nash equilibria for two-player games, Economic Theory 42, 9–37.

Jansen, M.J.M., 1981. Maximal Nash Subsets for Bimatrix Games. Naval research logistics quarterly 28, 147–152.

Millham, C.B., 1974. On Nash subsets of bimatrix games. Naval Research Logistics Quarterly 21, 307–317.

Moulin, H., Vial, J-P., 1978. Strategically zero-sum games: the class of games whose completely mixed equilibria cannot be improved upon. International Journal of Game Theory 15, 133–154.

von Stengel, B., 2002. Computing equilibria for two-person games. In: Aumann, R.J., Hart, S. (Eds.), In: Handbook of Game Theory, vol. 3. Elsevier Science Publishers (North Holland), pp. 1723–1759, chap. 45.

Viossat, Y., 2008. Is having a unique equilibrium robust? Journal of Mathematical Economics 44, 1152–1160.

Winkels, H.M., 1979. An algorithm to determine all equilibrium points in a bimatrix game. In: Moeschlin, O., Pallaschke, D. (Eds.), Game Theory and Related Topics. North-Holland, Amsterdam, pp. 137–148.