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A note on the evaluation of information in zero-sum repeated games[☆]

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ABSTRACT

Two players play a zero-sum repeated game with incomplete information. Before the game starts one player receives a private signal that depends on the realized state of nature. The rules that govern the choice of the signal are determined by the information structure of the game. Different information structures induce different values. The value-of-information function of a game associates every information structure with the value it induces. We characterize those functions that are value-of-information functions for some zero-sum repeated game with incomplete information.

1. Introduction

In Bayesian games with information structures players have a prior distribution of and receive a partial information on the game actually played. This information affects the players' posterior distribution and thereby the equilibrium payoffs of the game.

An outside observer, who does not observe the prior distribution nor the players' actions, collects data about the outcomes associated with different information structures. The question arises as to which observations can refute the model of rational players playing a Bayesian game. This issue was first addressed by Gilboa and Lehrer (1991) who dealt with a decision maker who knows the cell of a partition containing the realized state (as in Aumann, 1976). We consider a similar question in the framework of repeated zero-sum games with incomplete information that were introduced and studied by Aumann and Maschler (1995). The first advantage of zero-sum games is that they have a unique equilibrium payoff, the value. The second advantage is that in these games receiving more information can never be harmful. The latter property is particular to zero-sum games, as shown first by Hirshleifer (1971) and later by Kamien et al. (1990) and Bassan et al. (1999). Gossner and Mertens (2001) compared different information structures in zero-sum repeated games with incomplete information.

Prior to starting the game one player is informed of the cell of a partition that contains the realized state. Different partitions induce different repeated games and typically result in different values. We called a *value-of-information* function the one that associates partitions with the corresponding values in the induced Bayesian games. We prove that any monotonic function over partitions is a value-of-information function of some repeated game with incomplete information

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on one side. Hence, essentially no observed outcomes with respect to information can disprove the assertion that agents are Bayesian rational. This result stands in sharp contrast to one-player decision problems where an additional condition beyond monotonicity is needed.

A companion paper (Lehrer and Rosenberg, 2006) deals with value-of-information functions of *one-shot* zero-sum games. While in one-shot games it is always optimal to use all available information, in repeated games a full use of the information can be sub-optimal. One might therefore expect that receiving more information in repeated games is less beneficial than in one-shot games and that the impact of information on the value in one-shot games and in repeated games is different. It turns out that this intuition is misleading and the *sets* of the value-of-information functions for one-shot and repeated games coincide. While the problems treated here and in Lehrer and Rosenberg (2006) are similar, the proofs in the two papers use totally different techniques.

The proof uses a few properties of the value function of one-shot zero-sum Bayesian games in which players know only the prior.[1] As a by-product we characterize these functions in the case of two states and leave the general problem open.

This paper is organized as follows. The next section presents the model of information structures in repeated games with one-sided information. Section 3 describes the main result. A sketch of the proof is given in Section 4, and the paper ends with final comments and open problems.

2. The model

2.1. Information structures

We consider two-player games with incomplete information. A state k is randomly selected from a finite state space K according to a common prior p. *One-sided partitional information structure* (or simply *information structure*) is represented by a partition $\mathcal{Q}$ of the set K. Denote by $\mathcal{Q}(k)$ the cell of $\mathcal{Q}$ that contains k. When k is realized, player 1 gets to know $\mathcal{Q}(k)$ and player 2 knows nothing about k beyond the prior p. Note that any game has finitely many partitional information structures.

2.2. The repeated game

The n-stage game with one-sided information, denoted by $\Gamma_n(p, \mathcal{Q})$, is defined by an integer n, an information structure $\mathcal{Q}$, a probability p over K, a finite set of actions for each player $i \in \{1, 2\}$, A_i, and a payoff function g from $K \times A_1 \times A_2$ to the reals. The payoff associated with (k, a_1, a_2) is denoted by $g_k(a_1, a_2)$.

The game is played as follows: At stage 0, nature chooses an element k of K with probability p, player 1 is then informed of $\mathcal{Q}(k)$. The game is played in n stages. At stage $m = 1, \ldots, n$, players 1 and 2 simultaneously choose actions according to probability distributions that may depend on the history of previous actions and signals. If the realized state is k and the pair of chosen actions is (a_1^m, a_2^m) the payoff at stage m is $g_k(a_1^m, a_2^m)$. The pair of chosen actions (and not the payoff) is then announced to both players and the game proceeds to the next stage. The payoff in $\Gamma_n(p, \mathcal{Q})$ is the expected average of the n-stage payoffs received during the game.

A *behavior strategy* of player 1 is a sequence $\tau_1 = (\tau_1^1, \tau_1^2, \ldots, \tau_1^m, \ldots)$, where τ_1^m is a function from his information at stage m, $\mathcal{Q} \times (A_1 \times A_2)^{m-1}$ to the set[2] $\Delta(A_1)$ of probability distributions over his set of actions. A behavior strategy of player 2 is a sequence $\tau_2 = (\tau_2^1, \tau_2^2, \ldots, \tau_2^m, \ldots)$, where τ_2^m is a function from his information at stage m, $(A_1 \times A_2)^{m-1}$ to the set $\Delta(A_2)$. When applied to the game $\Gamma_n(p, \mathcal{Q})$, all τ_i^m, $m > n$, are payoff irrelevant.

The probability distribution p, the partition $\mathcal{Q}$ and the pair of strategies τ_1, τ_2 induce a probability over the set of histories of length n, $H^n = K \times (A_1 \times A_2)^n$. The expectation with respect to this probability will be denoted by $\mathbb{E}_{\tau_1\tau_2}^{p,\mathcal{Q}}$ or simply by $\mathbb{E}$ when no confusion can arise. If the players use the strategies τ_1, τ_2, the associated payoff in the n-stage game is $\gamma_n^{\mathcal{Q}}(\tau_1, \tau_2, p) = \mathbb{E}_{\tau_1\tau_2}^{p,\mathcal{Q}}[1/n\sum_{m=1}^{n} g_k(a_1^m, a_2^m)]$. This is the expected average payoff received along the n stages of the game.

The game $\Gamma_n(p, \mathcal{Q})$ is a finite game and therefore, by the minmax theorem, has a value denoted by $v_n^{\mathcal{Q}}(p, (g_k)_{k \in K})$. The following proposition whose proof is omitted states that this sequence converges.

Proposition 1. The sequence $v_n^{\mathcal{Q}}(p, (g_k)_{k \in K})$ has a limit denoted by $v^{\mathcal{Q}}(p, (g_k)_{k \in K})$.

The function $v^{\mathcal{Q}}(p, (g_k)_{k \in K})$ will be referred to as the long-run value of the game. This is an approximation of the equilibrium payoff in long games. A long duration of the games implies that player 1 might want to limit his use of available information because it may give a relative informational advantage to player 2. Excessive use of information can increase player 1's payoffs in the short-run but might be harmful to him in the long run.

[1] See Aumann and Maschler (1995) and Mertens and Zamir (1971).

[2] Throughout this paper $\Delta(X)$ denotes the set of probability distributions over a set X.

2.3. Value-of-information functions

We focus on the long-run value of the game viewed as a function of the information structure. In the one-sided information case, let $(g_k)_{k\in K}$ be a payoff function and let p be a distribution over K. When the information is given through $\mathcal{Q}$ the value of the induced game is denoted $v^{\mathcal{Q}}(p,(g_k)_{k\in K})$.

Definition 1. A function f defined over all partitions of K is the *value-of-information* of an incomplete information game with one-sided information and partitional information structure if there exist (i) a distribution p over K and (ii) payoff functions $(g_k)_{k\in K}$ such that for any partition $\mathcal{Q}$ over K, $f(\mathcal{Q}) = v^{\mathcal{Q}}(p,(g_k)_{k\in K})$.

3. The result

Describing the properties that value-of-information functions must owe in repeated zero-sum games is the main theme of this note. Since in zero-sum games it is always beneficial for a player to have more information, the following monotonicity condition is clearly necessary for a value-of-information function.

Definition 2. A function v from the set of partitions of a finite set K to the real numbers is said to be *monotonic* if for any two partitions $\mathcal{Q}$ and $\mathcal{Q}'$, the fact that $\mathcal{Q}$ refines $\mathcal{Q}'$ (i.e., for any $T\in\mathcal{Q}$ there is an $T'\in\mathcal{Q}'$ such that $T\subset T'$) implies $v(\mathcal{Q})\geq v(\mathcal{Q}')$.

The following theorem states that this condition is also sufficient.

Theorem 1. Let V be a function from the set of partitions of K to the real numbers. The function V is a value-of-information function of a repeated game with state space K, one-sided information, and partitional signaling if and only if it is monotonic.

The sketch of the proof of this theorem is given in Section 4 and a detailed proof is postponed to the Appendix A.

The main implication of this theorem is that even when one takes into account the strategic use of information the Bayesian paradigm imposes no restriction on value-of-information functions beyond monotonicity. It implies, for instance, that there is no restriction on the first derivative of the value, which could mean that the effect of a fixed addition of information on the value may go down or up with the information that already exists.

Lehrer and Rosenberg (2006) proved that the same characterization holds for one-shot games. Therefore, repetition does not affect the *set* of value-of-information functions, although it might affect the value of a particular game.

4. The proof

In this section, we provide a sketch of Theorem 1's proof. An elaborate proof of this theorem is deferred to Appendix A. The proof employs a few tools that will be described in the following subsections.

4.1. The one-shot game

The first tool we need is the value of the one-shot Bayesian game.

Suppose that $p=(p_k)\in\Delta(K)$. The value of the one-shot Bayesian game with null information (i.e., no player obtains additional information about the realized state beyond the prior p) is denoted $u(p)$. Formally,

Notation 1. $u(p)$ is the value of the game defined by the action sets A_1, A_2 and the payoff $\sum_{k\in K} p(k)g_k(a_1,a_2)$, when the pair of actions (a_1,a_2) is played.

This function also plays a central role in the theory of repeated games with incomplete information (see Aumann and Maschler, 1995).

Definition 3. A function u defined on $\Delta(K)$ is *realizable* if there are games G_k, $k\in K$, with the same action sets and payoff functions g_k, such that the value of $\bar{G}(p)=\sum_{k\in K} p(k)G_k$ is $u(p)$ for every p in $\Delta(K)$, where $\bar{G}(p)$ is the game with the same action sets and payoff function $\sum_{k\in K} p(k)g_k$.

The proof of Theorem 1 relies on the fact that the set of realizable functions is large. Mertens and Zamir (1971) proved that the set of realizable functions is dense in the set of continuous functions. This is not sufficient for our purposes since we need a precise realization and not merely an approximation. The following proposition is what we need and is specific to repeated games in which the long-run value is characterized by the value of the game with no information.[3]

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[3] It is worth emphasizing that the proof of the main result hinges on Proposition 2 which bears no relation to the proof of a similar result for one-shot games that builds on Gilboa and Lehrer (1991).

Proposition 2.

(i) Given a finite number of pairs $(x_\ell, y_\ell) \in \Delta(K) \times IR$, $\ell = 1, \ldots, L$, there is a realizable function u such that $u(x_\ell) = y_\ell$, $\ell = 1, \ldots, L$.
(ii) If C_1 is finite, C_2 is a union of closed polygons, $C_1 \cap C_2 = \varnothing$ and c_1 and c_2 are two numbers such that $c_1 > c_2$, then there is a realizable function u that satisfies $u(x) \le c_2$ when $x \in C_2$, $u(x) = c_1$ when $x \in C_1$, and $u(x) \le c_1$ otherwise.
(iii) Let C_1 and C_2 be two disjoint closed semi-algebraic sets, and f_1 and f_2 two realizable functions. Then, there is a realizable function u that satisfies $u(x) = f_1(x)$ when $x \in C_1$, and $u(x) = f_2(x)$ when $x \in C_2$.

4.2. The long-run value

Let f be a real-valued function defined on $\Delta(K)$ and cav(f) the minimal concave function which is greater than or equal to f. For a partition $\mathcal{Q}$ and $B \in \mathcal{Q}$ let $p(B) = \sum_{k \in B} p(k)$. Denote by $\pi_p(\cdot|B)$ the conditional probability over K, given $B \subset K$ (i.e., $\pi_p(k|B) = p(k)/p(B)$, $\forall k \in B$).

Notation 2. For a partition $\mathcal{Q}$ denote by $\mathcal{M}(\mathcal{Q})$ the set of matrices $M = (m_{iB})_{\substack{i \le D \\ B \in \mathcal{Q}}}$ with D lines (D can be any positive integer) and $|\mathcal{Q}|$ columns such that for any i, B, $m_{iB} \ge 0$, and for any B, $\sum_{i \le D} m_{iB} = p(B)$. For such a matrix we denote by m_i the quantity $\sum_{B \in \mathcal{Q}} m_{iB}$, and by $p_i(M)$ the probability distribution over K defined by

$$p_i(M)(k) = \frac{\sum_{B \in \mathcal{Q}} m_{iB} \pi_p(k|B)}{m_i}.$$

The following proposition extends Aumann and Maschler (1995) that states that when player 1 is fully informed of k the long-run value is cav(u)(p).

Proposition 3. The value of the game with one-sided information is

$$v^{\mathcal{Q}}(p, (g_k)_{k \in K}) = \max\left\{ \sum_{i \le D} m_i u(p_i(M)) | M \in \mathcal{M}(\mathcal{Q}) \right\}. \tag{1}$$

The concavification of u can also be defined as $\mathrm{cav}(u)(p) = \max\{\sum_{i \le D} m_i u(p_i) | m_i \ge 0, \sum_i m_i = 1, \sum_i m_i p_i = p\}$. Proposition 3 states that the long-run value is the maximum over a smaller class of possible (m_i, p_i), $i \le D$. In other words, $v^{\mathcal{Q}}(p, (g_k)_{k \in K})$ is a kind of a local concavification of the function u.

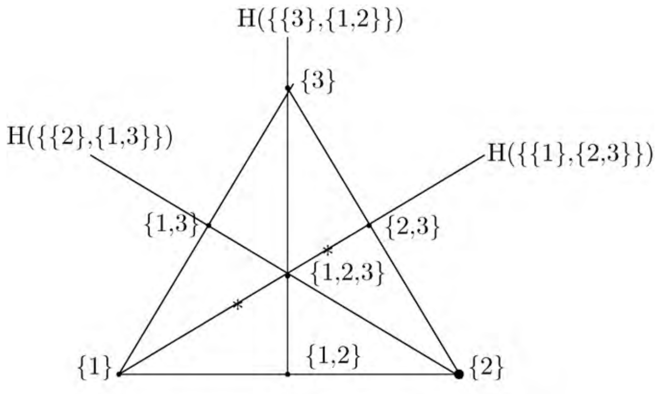
4.3. A sketch of the proof of Theorem 1

Let V be a monotonic function of partitions. Our goal is to construct a game for which the long-run value associated with the partition $\mathcal{Q}$ is equal to $V(\mathcal{Q})$. To this end we construct a realizable function u such that $V(\mathcal{Q}) = v^{\mathcal{Q}}(p, (g_k)_{k \in K})$ as defined by Eq. (1).

Let p be the uniform probability $(1/|K|, \ldots, 1/|K|)$. Note that in order to prove that V is a value-of-information function we need the existence of a probability p, which can therefore be chosen to be the uniform one. A probability distribution over the state space is represented by a point in the simplex. A subset B of K is represented by the point in the simplex associated with the conditional probability $\pi(\cdot|B)$. A partition is represented by a linear subspace, denoted by $H(\mathcal{Q})$, spanned by $\pi(\cdot|B)$, where B runs over all the cells of the partition. Note that $H(\mathcal{Q}) \subset H(\mathcal{Q}')$ if and only if $\mathcal{Q}'$ refines $\mathcal{Q}$. Note, moreover, that for every partition $\mathcal{Q}$, $p \in H(\mathcal{Q})$ (see the figure below).

To satisfy Eq. (1) it is sufficient to find u that satisfies the following two conditions: (i) If $\mathcal{Q}'$ refines $\mathcal{Q}$, then u is less than or equal to $V(\mathcal{Q}')$ on $H(\mathcal{Q})$; and (ii) for each partition $\mathcal{Q}$ there exist points $p_1, \ldots, p_{|\mathcal{Q}|}$ in $H(\mathcal{Q})$, such that $u(p_\ell) = V(\mathcal{Q})$ and p can be written as a convex combination of $p_1, \ldots, p_{|\mathcal{Q}|}$. Such a realizable function exists by Proposition 2.

The following figure illustrates the case of $K = \{1, 2, 3\}$. The partition $\mathcal{Q} = \{1, 2, 3\}$ is represented by the 0-dimensional space $H(\{1, 2, 3\})$, which is the center of the triangle, denoted in the figure as $\{1, 2, 3\}$; the partitions of K into two sets are represented by lines and $H(\{\{1\}, \{2\}, \{3\}\})$ is the whole simplex. The points marked $*$ are on $H(\{\{1\}, \{2, 3\}\})$ and the center of the triangle is a convex combination of them.



The detailed proof, provided in the appendix, shows that if V is monotonic, then there is a realizable function u such that the value u attains at the center is less than or equal to $V(\{\{1\},\{2,3\}\})$, while the values u attains at the points marked $*$ are equal to it.

5. Comments and open problems

5.1. General information structures

The main theorem discusses partitional structures. It would be interesting to study value-of-information functions defined over more general signaling structures.

5.2. Repeated games with lack of information on both sides

The model studied here can be extended to situations where player 1 is informed through a partition $\mathcal{P}$ and player 2 through a partition $\mathcal{Q}$. Mertens and Zamir (1971) proved the existence of the long-run value of the game (denoted by $v^{\mathcal{P},\mathcal{Q}}(p,g)$) in this case (defined as the limit of the values of n-stage games as n goes to infinity, or of λ-discounted games as λ goes to 0). Characterizing the value-of-information functions in this case is an open problem.

5.3. The general null information case

Null information is the case where player 2 gets no information at all on the state. One can fully characterize the value of games with null information and only two states. Indeed, along the same lines as the one provided in the appendix, one can prove that when there are two states, if f is a real polynomial that does not vanish, then $1/f$ is realizable. This in turn implies that a function is realizable if and only if it is piecewise rational.

In the case of more than two states we could prove that every polynomial is realizable, and that, for every continuous and piecewise rational function q and every $\varepsilon > 0$, there is a realizable function u that coincides with q over a set that occupies (in the sense of Lesbegue measure) $1-\varepsilon$ of $\Delta(K)$. We conjecture that every continuous and piecewise rational function over $\Delta(K)$, as in the two-state case, is realizable.

Appendix A.

The proof of Proposition 2 is broken into three parts, one for each item.

Proof of Proposition 2 (i)

Lemma A.1. If u and v are realizable, then so is $\min(u,v)$.

If the game $G_1,\ldots,G_{|K|}$ realizes u and $G'_1,\ldots,G'_{|K|}$ realizes v then the following two stage game realizes $\min(u,v)$: = player 2 first chooses action L or R, player 1 sees the choice and then the game is $G_1,\ldots,G_{|K|}$ if the choice was L and $G'_1,\ldots,G'_{|K|}$ if it was R.

Lemma A.2. If u is realizable, then so is u^2.

Proof. If u is realizable by the game $G_1,\ldots,G_{|K|}$, then the game whose i-th matrix game is $\begin{pmatrix} G_i & 0 \\ 2G_i-1 & G_i \end{pmatrix}$ realizes u^2. □

Lemma A.3. Any polynomial is realizable.

Proof. Mertens and Zamir (1971) showed that if u and v are realizable then so is $u+v$. By Lemma A.2, $(u+v)^2$ is also realizable as are u^2 and v^2. Therefore, $uv = \frac{1}{2}\left((u+v)^2 - u^2 - v^2\right)$ is also realizable.

Moreover, any constant function is realizable. The game $G_1, \ldots, G_{|K|}$ all of whose matrices are identically 0, except for the i-th one which is identically 1, realizes the polynomial $f_i(p) = p_i$. Any polynomial is therefore realizable by iteratively adding and multiplying constants and the polynomials f_i. □

Proposition 2 (i) follows from the fact that for every $(x_\ell, y_\ell) \in \Delta(K) \times IR$, $\ell = 1, \ldots, L$, there is a polynomial u such that $u(x_\ell) = y_\ell$, $\ell = 1, \ldots, L$.

Proof of Proposition 2 (ii)

Lemma A.4. Let C be a closed semi-algebraic set. Then there is a realizable function u that satisfies $u(x) \le 0$ when $x \in C$, and $u(x) > 0$, otherwise.

Proof. Let D be a set of the form

$$\{x \in \mathbb{R}^k; f_1(x) = f_2(x) = \ldots = f_\ell(x) = 0, r_1(x) \ge 0, r_2(x) \ge 0, \ldots, r_m(x) \ge 0\},$$

where $f_1, \ldots, f_\ell, r_1, \ldots, r_m$ are polynomials. By Proposition 3, $f_1, \ldots, f_\ell, r_1, \ldots, r_m$ are realizable and by Mertens and Zamir (1971), $u_D = \min\{f_1, \ldots, f_\ell, -f_1, \ldots, -f_\ell, r_1, \ldots, r_m\}$ is realizable. Clearly, $u_D(x) \ge 0$ when $x \in C$ and $u_D(x) < 0$, otherwise. Since any closed semi-algebraic set is a finite union of such D's (see, Bochnak et al., 1988, p. 46), the desired u is $u = -\max\{u_D\}_D$, which is also realizable. □

A union of closed polygons is a closed semi-algebraic set. By Lemma A.4, there is a realizable function u which is less than or equal to 0 on C_2 and greater than 0 otherwise. Let c be the minimum of u over the set C_1. By multiplying u with the constant $c_1 - c_2/c$ and adding c_2 one obtains a realizable function u' that satisfies $u'(x) \le c_2$ when $x \in C_2$ and $u'(x) \ge c_1$ when $x \in C_1$. Taking the minimum of u' and c_1 would yield the realizable function needed for Proposition 2 (ii).

Proof of Proposition 2 (iii) By Lemma A.4, for $i = 1, 2$, there is u'_i which is at least 0 on C_i and less than 0 otherwise. Consider $u_i = \max\{\min\{u'_i + 1, 1\}, 0\}$. u_i is realizable, bounded between 0 and 1, and equal to 1 on C_i, and less than 1 otherwise.

For any integer ℓ_i, $u_i^{\ell_i}$ is also realizable. It is bounded between 0 and 1, and is equal to 1 on C_i and less than 1 otherwise. By adding a large positive number, say M, we may assume that the functions f_1 and f_2 are positive. There are sufficiently large ℓ_i's such that $f_j > f_i u_i^{\ell_i}$, $i \ne j$, on C_j. Thus, $\max\{f_1 u_1^{\ell_1}, f_2 u_2^{\ell_2}\}$ is realizable and (subtracting M, if necessary) satisfies Proposition 2 (iii).

Proof of Proposition 3. We first define an auxiliary game with one-sided information with $\mathcal{Q}$ being the set of states, player 1 is fully informed of the state, the sets of the players' actions remain unchanged, and the payoff associated with a pair of actions a_1, a_2 and state B is $\bar{g}_B(a_1, a_2) = \sum_{k \in K} \pi_p(k|B) g_k(a_1, a_2)$. The probability of the state B is $p(B)$. Denote this probability over $\mathcal{Q}$ by r. By Aumann and Maschler (1995) the long-run value of this game is $\mathrm{cav}(\bar{u})(r)$, where $\bar{u}$ is the value of the auxiliary game played once with player 1's information being trivial.

For any q in $\Delta(\mathcal{Q})$, let $p_q \in \Delta(K)$ satisfy $p_q(k) = \sum_{B \in \mathcal{Q}} q(B) \pi_p(k|B)$. Note that $\bar{u}(q) = u(p_q)$.

Hence,

$$\mathrm{cav}(\bar{u})(r) = \max\left\{ \sum_i m_i \bar{u}(q_i) \middle| \begin{array}{c} m_i \ge 0, \sum_i m_i = 1 \\ q_i \in \Delta(\mathcal{Q}), \sum_i m_i q_i = r \end{array} \right\}.$$

For any q_i in $\Delta(\mathcal{Q})$, denote $m_{iB} = m_i q_i(B)$ and define the matrix $M = (m_{iB})$. Since, $\sum_{i,B} m_{iB} = \sum_i m_i = 1$ and $\sum_i m_{iB} = r(B)$, we conclude that $M \in \mathcal{M}(\mathcal{Q})$. Moreover, $\bar{u}(q_i) = u(p_i(M))$ and $\mathrm{cav}(\bar{u})(r) \le \max\left\{\sum_{i \le D} m_i u(p_i(M)) | M \in \mathcal{M}(\mathcal{Q})\right\}$.

On the other hand, for any $M \in \mathcal{M}(\mathcal{Q})$, $\sum_i m_i u(p_i(M)) = \sum_i m_i \bar{u}(q_i)$, with $q_i(B) = m_{iB}/m_i$. Therefore, $u(p_i(M)) = \bar{u}(q_i)$ implying $\sum_{i \le D} m_i u(p_i(M)) \le \mathrm{cav}(\bar{u})(r)$. The last two inequalities yield the desired result.

A detailed proof of Theorem 1.

Let V be a monotonic function over partitions. Let p be $(1/|K|, \ldots, 1/|K|)$. Order all partitions of K: $\mathcal{P}_1, \ldots, \mathcal{P}_\ell$, so that if $j < i$ then $V(\mathcal{P}_j) \le V(\mathcal{P}_i)$, which implies that $\mathcal{P}_j$ does not refine $\mathcal{P}_i$.

Notation 3. Let $\mathcal{P}$ be a partition. Denote by $H(\mathcal{P})$ the space in $\mathbb{R}^{|K|}$ spanned by $\{1l_T; T \in \mathcal{P}\}$, where $1l_T$ denotes the indicator vector of the set T.

Note that the partition $\mathcal{P}$ refines $\mathcal{P}'$ if and only if $H(\mathcal{P}')$ is a subspace of $H(\mathcal{P})$. This implies that if $i > j$, then[4] $\dim\left[H(\mathcal{P}_j) \cap H(\mathcal{P}_i)\right] < \dim H(\mathcal{P}_i)$.

We prove that for any $i \le \ell$ the following property, denoted by E_i, holds: there is a realizable function u such that for any $j \le i$ and every $p \in H(\mathcal{P}_j)$, $u(p) \le V(\mathcal{P}_j)$. Furthermore, for any $j \le i$ there is a $D_j \times |\mathcal{P}_j|$ matrix[5] $M_j \in \mathcal{M}(\mathcal{P}_j)$ such that for any $r \le D_j$, $u(p_r(M_j)) = V(\mathcal{P}_j)$.

[4] $\dim(H)$ denotes the dimension of H.

[5] Recall Notation 2. $\mathcal{P}_j$ in $\mathcal{M}(\mathcal{P}_j)$ stands for the information structure that corresponds to the partition $\mathcal{P}_j$.

We proceed by induction over i. Let $i = 1$. The partition $\mathcal{P}_1$ is the trivial partition and u can be taken to be the constant function $V(\mathcal{P}_1)$. Now assume that E_{i-1} holds and denote by u_{i-1} the corresponding realizable function.

Step 1: Definition of a class of matrices.

Let $D = |\mathcal{P}_i|$ and consider the square matrix M_1 with D columns, all of whose off-diagonal entries are zero. The diagonal entry corresponding to $T \in \mathcal{P}_i$ is $p(T)$. Note that for every row r of M_1, $p_r(M_1) \neq p$. Let M_2 be a matrix of the same dimension whose entries in the column corresponding to the cell $T \in \mathcal{P}_i$ are all equal to $p(T)/D$. Obviously, $M_1, M_2 \in \mathcal{M}(\mathcal{P}_i)$. Define $M = \alpha M_1 + (1-\alpha)M_2$. If α is positive and sufficiently small, then $M \in \mathcal{M}(\mathcal{P}_i)$ and for every row r of M, $p_r(M) \neq p$. Furthermore, all entries of M are strictly positive.

Define $H(\mathcal{P}_i)^0 = \{v; v = \sum \varepsilon_T \pi_p(\cdot|T), \sum_{T \in \mathcal{P}_i} \varepsilon_T = 0\}$. For every $v \in H(\mathcal{P}_i)^0$ let $a_{rT} = m_r \varepsilon_T \langle p_r(M) - p, v\rangle,\ r \leq D,\ T \in \mathcal{P}_i$. Consider the matrix M^v whose (r, T)-th entry is $m_{rT} + a_{rT}$. If the ε_T's that define v are small enough, then the entries of M^v are positive.

We show now that M^v is in $\mathcal{M}(\mathcal{P}_i)$. This is so since $\sum_r m_{rT} + a_{rT} = p(T) + \sum_r m_r \varepsilon_T \langle p_r(M) - p, v\rangle = p(T) + \varepsilon_T \langle \sum_r m_r p_r(M) - p, v\rangle = p(T)$. The last equality is due to the fact that $\sum_r m_r p_r(M) = p$.

Step 2: There is $v \in H(P_i)^0$ **such that for any row** r, $p_r(M^v) \notin \cup_{j<i} H(P_j)$.

Note that $\sum_{T \in \mathcal{P}_i} a_{rT} = \sum_{T \in \mathcal{P}_i} m_r \varepsilon_T \langle p_r(M) - p, v\rangle = m_r \langle p_r(M) - p, v\rangle \sum_{T \in \mathcal{P}_i} \varepsilon_T = 0$. Therefore, $p_r(M^v) = p_r(M) + \langle p_r(M) - p, v\rangle \sum_T \varepsilon_T \pi_p(\cdot|T) = p_r(M) + \langle p_r(M) - p, v\rangle v$. Assume by contradiction that the claim is false. Then there is a neighborhood of $H(\mathcal{P}_i)^0$ around the origin, denoted W, such that for every $v \in W$ there is $j < i$ and a row r such that $p_r(M^v) = p_r(M) + \langle p_r(M) - p, v\rangle v \in H(\mathcal{P}_j)$. Define the set F_{rj} to be the set containing $v \in \bar{W}$ such that $p_r(M) + \langle p_r(M) - p, v\rangle v \in H(\mathcal{P}_j)$, where $\bar{W}$ is the closure (the relative one in $H(\mathcal{P}_i)^0$) of W. F_{rj} is a closed set for every r and j.

By assumption, the union of the closed sets F_{rj} contains $\bar{W}$. As a complete space, $\bar{W}$ is of category II. Thus, at least one of the F_{rj}'s contains an open set. Therefore, there are j and r so that $p_r(M) + \langle p_r(M) - p, v\rangle v \in H(\mathcal{P}_j)$ for v' s in an open (in $\bar{W}$) set.

Note that for every $v \in W$, $p_r(M) + \langle p_r(M) - p, v\rangle v \in H(\mathcal{P}_i) \cap \Delta(K)$. Furthermore, since $p_r(M) - p \neq 0$, the map $v \mapsto p_r(M) + \langle p_r(M) - p, v\rangle v$ is an open map. Thus, $H(\mathcal{P}_j) \cap \Delta(K)$ contains an open set of $H(\mathcal{P}_i) \cap \Delta(K)$. Since both are intersections of linear spaces whose spanning vectors are in $\Delta(K)$ with $\Delta(K)$ itself, it implies that $H(\mathcal{P}_j) \cap \Delta(K)$ contains $H(\mathcal{P}_i) \cap \Delta(K)$. However, when $j < i$, $\dim\left[H(\mathcal{P}_j) \cap H(\mathcal{P}_i)\right] < \dim H(\mathcal{P}_i)$. This implies that $H(\mathcal{P}_i) \cap \Delta(K)$ is not included in $H(\mathcal{P}_j) \cap \Delta(K)$. We therefore conclude that there exists a matrix, denoted M_i, in $\mathcal{M}(\mathcal{P}_i)$ that satisfies $p_r(M_i) \notin \cup_{j<i} H(\mathcal{P}_j)$ for every row r of M_i.

Step 3: Conclusion of the proof.

By Proposition 2 (ii), there is a realizable function f that satisfies the following:

- For every row r of M_i, f attains its maximum, $V(\mathcal{P}_i)$, on $p_r(M_i)$; and
- f is smaller than or equal to $\min_{1 \leq j \leq i-1} V(\mathcal{P}_j)$ on $\cup_{1 \leq j \leq i-1} H(\mathcal{P}_j) \cap \Delta(K)$.

By taking the maximum of f and the function u_{i-1} we get a realizable function u_i that satisfies:

(a) For any $j \leq i$, u_i is smaller than or equal to $V(\mathcal{P}_j)$ on $H(\mathcal{P}_j) \cap \Delta(K)$; and
(b) for any $j \leq i$, there is a $D_j \times l$ matrix $M_j \in \mathcal{M}(\mathcal{P}_j)$ such that for any $1 \leq r \leq D_j$, $u(p_r(M_j)) = V(\mathcal{P}_j)$.

Property E_l is therefore proven by induction. Let $(g_k)_{k \in K}$ be the payoff functions that realize u_l. **(a)** Implies that $v^{\mathcal{P}_i \mathcal{O}}(p, (g_k)_{k \in K}) \leq V(\mathcal{P}_i)$ for every $i \leq l$. Proposition 3 and **(b)** imply that $v^{\mathcal{P}_i \mathcal{O}}(p, (g_k)_{k \in K}) = V(\mathcal{P}_j)$, which completes the proof.

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