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On some families of cooperative fuzzy games

Yaron Azrieli · Ehud Lehrer

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Abstract In a fuzzy cooperative game the players may choose to partially participate in a coalition. A fuzzy coalition consists of a group of participating players along with their participation level. The characteristic function of a fuzzy game specifies the worth of each such coalition. This paper introduces well-known properties of classical cooperative games to the theory of fuzzy games, and studies their interrelations. It deals with convex games, exact games, games with a large core, extendable games and games with a stable core.

Keywords Fuzzy games · Core · Convex games

1 Introduction

In the theory of classical TU cooperative games (classical games from now on), a characteristic function is defined over all the subsets of the set of players. The *core* of a classical game is the set of allocations which can not be blocked by any coalition (subset of players). Aubin (1979, 1981a) suggested to consider *fuzzy coalitions* as well, which are coalitions composed of different "fractions" of the various players. His hope was to achieve a refinement of the core by enlarging the set of constrains that a potential core allocation should satisfy.

There are two main justifications in the literature for the use of fuzzy coalitions. The first, suggested by Aubin himself, is that every player can choose his *level of participation* in a coalition, and not only whether to participate or not. This interpretation

Y. Azrieli (✉) · E. Lehrer
School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel
e-mail: azrieliy@post.tau.ac.il

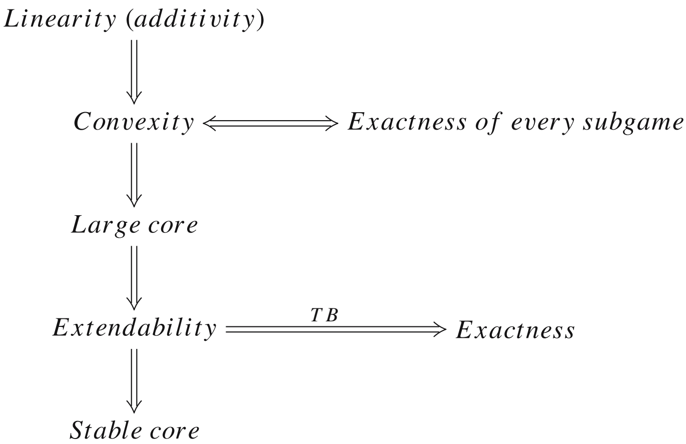
E. Lehrer
e-mail: lehrer@post.tau.ac.il

of fuzzy coalitions is especially appealing when each player has an endowment of some (divisible) private resource (time or money are two instances). In this case, a fuzzy coalition is merely a specification of the amount that each agent invests in the joint project.

The second justification is based on a "large economy" argument (Husseinov 1994). Every finite economy is associated with an infinite non-atomic economy. An agent of the original finite economy is replaced by a continuum of identical agents. This gives a natural way to interpret the meaning of a fuzzy coalition in the finite economy. Indeed, a fuzzy coalition in the finite economy becomes an ordinary coalition in the infinite one. Azrieli and Lehrer (2007) use the relation between fuzzy coalitions of a finite economy and ordinary coalitions in a non-atomic economy in order to characterize market games originated in large economies.

It is quite surprising, therefore, that relatively little attention has been given to the study of fuzzy games.[1] This is in contrast to the vast research in the theory of classical games. The purpose of this paper is to define and analyze some families of fuzzy games. The definition of each of these families is inspired by an analogous definition in the classical theory.

The families we discuss include *convex* games, *exact* games, games with a *large core*, *extendable* games and games with a *stable core*. The following diagram summarizes the inclusion relations between these families when classical games are considered (see Biswas et al. 1999; van Gellekom and Potters 1999). In what follows TB attached to an arrow means that the relation holds only for totally balanced games.



Cooperative games

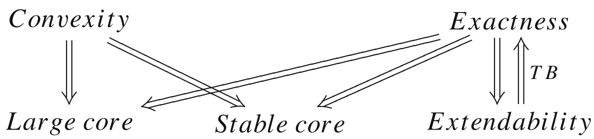
Each of the aforementioned families of classical games has its natural counterpart family of fuzzy games. Convex fuzzy games were first studied by Tsurumi et al. (2001). However, their definition of convexity of a fuzzy game is weaker than the one used in the current paper. Our definition is equivalent to the one used in Branzei et al. (2003). Stable sets of fuzzy games were introduced by Tijs et al. (2004b). The

[1] A recent book of Branzei et al. (2005) provides a comprehensive summary of this literature.

rest of the families are defined in this paper. It turns out that some relations between these families that hold true in classical games are no longer true in fuzzy games. New and somewhat surprising relations appear in this context. These are demonstrated in the following two diagrams.

$$\textit{Convexity and exactness} \Longleftrightarrow \textit{Linearity} \Longleftrightarrow \textit{Exactness of every subgame}$$

Fuzzy games 1



Fuzzy games 2

The fact that convexity implies that the core is stable is due to Tijs et al. (2004b). The rest of the results are proved here.

Comparing the figures above, one can see significant differences between classical and fuzzy games. First, notice that every classical convex game is exact, while convexity does not imply exactness in fuzzy games. Moreover, the only fuzzy games which are both exact and convex are linear games. Second, for totally balanced classical games exactness is a strictly weaker property than either largeness of the core or extendability. However, in fuzzy games, exactness implies both largeness of the core and extendability (for totally balanced fuzzy games exactness and extendability are equivalent).

Another difference appears when considering games whose every sub-game is exact. In cooperative games such a property is equivalent to convexity, while in fuzzy games it is much stronger. The only fuzzy games which have this property are the linear games.

2 Fuzzy games

2.1 Basic definitions

Let $N = \{1, 2, \ldots, n\}$ be a finite set. For every non-negative vector $Q \in \mathbb{R}^n$, let $F(Q)$ be the box $F(Q) = \{c \in \mathbb{R}^n;\ 0 \leq c \leq Q\}$. The point Q is interpreted as the "grand coalition", and every $c \in F(Q)$ is a possible (fuzzy) coalition. The *support* of a coalition $c = (c_1, \ldots, c_n) \in F(Q)$ is the set $\text{supp}(c) = \{i \in N;\ c_i > 0\}$. We denote by $|c|$ the l_1 norm of c, that is $|c| = \sum_{i=1}^{n} |c_i|$. Since we only deal with non-negative vectors, $|c| = \sum_{i=1}^{n} c_i$.

Definition 1 A **fuzzy game** is a pair (Q, v) such that

(i) $Q \in \mathbb{R}^n$ and $Q \geq 0$;

(ii) $v : F(Q) \to \mathbb{R}$ is bounded and satisfies $v(0) = 0$.

We will also be interested in sub-games of a given fuzzy game. These are naturally defined as in the classical theory.

Definition 2 Let (Q, v) be a fuzzy game and fix some $c \in F(Q)$. The **sub-game of (Q,v) with respect to c** is the fuzzy game (c, v_c), where for every $d \in F(c)$, $v_c(d) = v(d)$.

We identify subsets of N with the corresponding extreme points of $F(Q)$. That is, for every $S \subseteq N$, $Q^S \in \mathbb{R}^n$ is the vector defined by $Q_i^S = Q_i$ if $i \in S$, and $Q_i^S = 0$ otherwise. In particular, if Q is the unit vector, then the points $\{Q^S;\ S \subseteq N\}$ are the vertices of the unit cube. The *classical game induced by the fuzzy game* (Q, v) is the restriction of v to the vertices of $F(Q)$. Formally,

Definition 3 The **classical game induced by the fuzzy game (Q,v)** is $(N, \bar{v})$ where $\bar{v} : 2^N \to \mathbb{R}$ is defined by $\bar{v}(S) = v(Q^S)$.

2.2 Balanced and totally balanced games

We review here the definition of the core of a fuzzy game.

Definition 4 (Aubin 1979, 1981a) The **core** of the fuzzy game (Q, v), denoted $Core(Q, v)$, is the set of vectors $x \in \mathbb{R}^n$ such that[2]

(i) $xQ = v(Q)$; and
(ii) $xc \geq v(c)$ for any coalition $c \in F(Q)$.

The Bondareva–Shapley theorem (Bondareva 1962; Shapley 1967) shows that a classical game (Q, v) has a non-empty core iff it is *balanced* and that every sub-game of (Q, v) has a non-empty core iff (Q, v) is *totally balanced*. An analogous result for fuzzy games is given by Sharkey and Telser (1978). In Azrieli and Lehrer (2007) there is a similar characterization, but the method of proof is different. For the sake of completeness, we state here the main result as it appears in Azrieli and Lehrer (2007). This requires the following definition.

Definition 5 Fix a non-negative vector $Q \in \mathbb{R}^n$ and let $v : F(Q) \to \mathbb{R}$.

(i) The **Strong Super-Additive cover** of v is the function $\boldsymbol{SSa}v : F(Q) \to \mathbb{R}$ defined by

$$\boldsymbol{SSa}v(d) = \sup\left\{\sum_{j=1}^{L} \lambda_j v(c_j);\ L \in \mathbb{N},\ \sum_{j=1}^{L} \lambda_j c_j = d,\ \lambda_j \geq 0,\ c_j \in F(d),\ j = 1, \ldots, L\right\}.$$

[2] For two vectors $z, w \in \mathbb{R}^n$, zw denotes the inner product; i.e., $zw = \sum_{i=1}^{n} z_i w_i$.

(ii) v is called **Strongly Super-Additive** if $v = \boldsymbol{SSa}v$ on $F(Q)$.

Theorem 1 (i) *The fuzzy game* (Q, v) *has a non-empty core iff* $\mathbf{SSa}v(Q) = v(Q)$.
(ii) *Every sub-game of the fuzzy game* (Q, v) *has a non-empty core iff* v *is Strongly Super-Additive.*

In the sequel, we keep the terminology used in the classical theory and say that (Q, v) is balanced if $\boldsymbol{SSa}v(Q) = v(Q)$ and that it is totally balanced if v is Strongly Super-Additive.

3 Convex games

Shapley (1971) calls the classical game (N, v) *convex* if $v(S \cup T) + v(S \cap T) \geq v(T) + v(S)$ for any two coalitions $S, T \subseteq N$. There are several possible ways to extend this definition to fuzzy games. In Branzei et al. (2003) a fuzzy game (Q, v) is said to be convex if it satisfies the following two properties:

(i) $v(s \vee t) + v(s \wedge t) \geq v(s) + v(t)$ for every $s, t \in F(Q)$, where $s \vee t$ $(s \wedge t)$ is the vector in $\mathbb{R}^n$ whose i-th coordinate equals $\max\{s_i, t_i\}$ $(\min\{s_i, t_i\})$.
(ii) The function v is coordinate-wise convex on $F(Q)$. That is, for any $i \in N$ and for every s_{-i}, the function $v_{s_{-i}}(s_i) = v(s_i, s_{-i})$ is convex.

We choose to define convexity of a fuzzy game in a slightly different way, which seems to us more natural. Like in Shapley's definition, the idea is that the marginal contribution of any coalition is increasing.

Definition 6 The fuzzy game (Q, v) is **convex** if, whenever s, t, d and $t + d$ are coalitions such that $s \leq t$, it follows that $v(s + d) - v(s) \leq v(t + d) - v(t)$.

Branzei et al. (2003) show that their definition implies ours (see Proposition 4 there). It is not hard to see that the converse is also true.

An important property of convex classical games is that they are balanced (and, of course, totally balanced as sub-games of a convex game are also convex). Branzei et al. (2003) show that this is true for convex fuzzy games as well. Moreover, when (Q, v) is convex, $Core(Q, v)$ coincides with the core of the (convex) classical game $(N, \bar{v})$ (in their paper it is assumed that all the coordinates of Q equal 1). Other properties and characterizations of convex games can be found in Tijs and Branzei (2004a).

In this section we discuss other properties of convex fuzzy games that, to the best of our knowledge, have not appeared elsewhere. The first question which arises is whether convexity of the fuzzy game (Q, v) is equivalent to convexity of v as a function on $F(Q)$. The following proposition provides the answer.

Proposition 1 (i) *If* $n = 1$ *then convexity of* v *on* $[0, Q]$ *is equivalent to convexity of the game* (Q, v).

(ii) *For* $n \geq 2$, *convexity of* v *on* $F(Q)$ *does not imply and is not implied by convexity of the game* (Q, v).

Proof (i) This is a consequence of the equivalence of our definition and that of Branzei et al. (2003).

(ii) We start with an example where the game (Q, v) is convex but v is not convex on $F(Q)$. Let $Q = (1, 1)$ and define $v(c_1, c_2) = -c_1(1 - c_2)$ for $c = (c_1, c_2) \in F(Q)$. Notice that v is not convex on $F(Q)$ since, for example, $v(\frac{1}{2}, \frac{1}{2}) = -\frac{1}{4} > -\frac{1}{2} = \frac{1}{2}v(0, 1) + \frac{1}{2}v(1, 0)$ (actually, v is strictly concave on[3] Δ). However, the game (Q, v) is convex. Indeed, a little bit of algebra gives $v(t + d) - v(t) = t_1 d_2 + d_1 t_2 - d_1 + d_1 d_2$. Similarly, $v(s + d) - v(s) = s_1 d_2 + d_1 s_2 - d_1 + d_1 d_2$. Therefore, if $t \geq s$ then $v(t+d) - v(t) \geq v(s+d) - v(s)$.

On the other hand, let $Q = (2, 2)$ and define $v(c_1, c_2) = (c_1 - c_2)^2$ for any $c \in F(Q)$. Then obviously v is a convex function. Define $s = (0, 1)$, $t = (0, 2)$, $d = (2, 0)$. Then $s \leq t$ but $v(s + d) - v(s) = 0 > -4 = v(t + d) - v(t)$. Therefore, (Q, v) is not convex. □

It is clear that any sub-game of a convex fuzzy game is convex. Thus, any convex fuzzy game is totally balanced. Theorem 1 implies that if (Q, v) is convex, then v is Strongly Super-Additive. A natural question is whether the converse, namely whether Strong Super-Additivity implies convexity, is correct. The following example shows that the answer is no.

Example 1 Let $Q = (1, \ldots, 1)$ be the n-dimensional unit vector. For $c \in F(Q)$ define

$$v(c) = \begin{cases} |c|/2 & \text{if } 0 \leq |c| \leq 1/4; \\ 3|c|/2 - 1/4 & \text{if } 1/4 \leq |c| \leq 1/2; \\ |c| & \text{if } 1/2 \leq |c| \leq n. \end{cases}$$

To check that (Q, v) as well as each one of its sub-games has a non-empty core define for every $c \in F(Q)$ the vector $x(c) \in \mathbb{R}^n$ by $x(c)_i = \frac{v(c)}{|c|}$, $i = 1, \ldots, n$. A simple computation shows that, for any $c \in F(Q)$, $x(c) \in Core(c, v_c)$. Thus, v is Strongly Super-Additive on $F(Q)$. However, (Q, v) is not a convex game since the function v is not coordinate-wise convex.

We move on to discuss continuity of convex fuzzy games. By Proposition 1, convexity of (Q, v) does not imply that v is a convex function. Therefore, a matter of interest is whether convexity of (Q, v) guarantees continuity of v on $F(Q)$. The following example shows that, in general, convex games need not be continuous on the boundary of $F(Q)$.

Example 2 Let $Q = (1, \ldots, 1)$ be the n-dimensional unit vector.

(1) Define $v(c) = -1$ for every $c \in F(Q) \setminus L$, where L is the line connecting 0 with the point $(0, \ldots, 0, 1)$ and $v(c) = 0$ for every $c \in L$. (Q, v) is convex but v is not continuous in every point of L.

(2) Define $v(c) = 0$ for every $c \in F(Q) \setminus L$, where L is the line connecting $(1, \ldots, 1, \frac{1}{2})$ with Q and $v(c) = 1$ for every $c \in L$. (Q, v) is convex but v is not continuous in every point of L.

[3] Δ denotes the $(n - 1)$ dimensional unit simplex in $\mathbb{R}^n$.

Theorem 2 (i) *A convex fuzzy game is continuous in the interior of* $F(Q)$.
(ii) *If a convex fuzzy game is continuous in* 0 *and* Q, *then it is continuous in* $F(Q)$.

Proof The result in (i) can be easily derived from the fact that the characteristic function v of a convex game is coordinate-wise convex (see Rockafellar 1997, p. 89, Theorem 10.7).

As for (ii), notice first that (i) ensures that v is continuous in the relative interior of every facet. The proof of (ii) is by induction on the dimension of the game, n. For $n = 1$ the assertion is trivial. Assume that (ii) holds for every game of dimension less than n and we prove the assertion for n.

Let F be a facet of $F(Q)$. There are c and d such that $F = \{e;\ c \le e \le d\}$. In order to show continuity of v in F it is enough to show continuity of v in c and d. Let e be a coalition such that $c + e \in F$. Due to convexity of (Q, v),

$$v(e) \le v(c + e) - v(c) \le v(Q) - v(Q - e). \tag{1}$$

However, if $|e|$ is sufficiently small, then both sides of (1) are close to zero. This proves continuity at c. Continuity at d is shown in a similar way.

Now, let two coalitions c and d be in the boundary of $F(Q)$. If c and d are sufficiently close to each other, then there is a point e which is in the same facet as c and in the same facet (possibly different) as d. Furthermore, e is close to both c and d. Thus, continuity of v in every facet implies continuity within the boundary of $F(Q)$.

It remains to show that if c is in the boundary and d is in the interior and they are close to each other, then $v(c)$ and $v(d)$ are close. Since c is in the boundary, either $c \wedge d$ or $c \vee d$ are in the boundary. If $q = c \wedge d$ is in the boundary, then

$$v(d - q) \le v(d) - v(q) \le v(Q) - v(Q - d + q). \tag{2}$$

Thus, when d is close to c, $d - q$ is close to zero and the two sides of (2) are close to zero. Therefore, $v(d)$ is close to $v(q)$. Since q and c are in the boundary, $v(q)$ and $v(c)$ are close to each other, which shows that $v(d)$ and $v(c)$ are close to each other.

The proof is similar when $q = c \vee d$ is in the boundary. We conclude that v is continuous, as required. □

4 Exact games

A classical game is called *exact* (Schmeidler 1972) if for every coalition there is a core member which gives this coalition exactly its worth. This definition naturally extends to fuzzy games.

Definition 7 The fuzzy game (Q, v) is **exact** if for every $c \in F(Q)$ there exists $x \in Core(Q, v)$ such that $xc = v(c)$.

The following theorem characterizes exact games.

Theorem 3 *A fuzzy game* (Q, v) *is exact iff the following conditions hold:*

(1) *v is concave on $F(Q)$;*
(2) *v is homogeneous on $F(Q)$;*
(3) *$v(\alpha c + (1-\alpha)Q) = \alpha v(c) + (1-\alpha)v(Q)$ for any $c \in F(Q)$ and for any $\alpha \in (0, 1)$.*

Proof Assume that (1)–(3) hold. Since v is homogeneous we can assume w.l.o.g. that $\Delta \subseteq F(Q)$. For any $q \in \Delta$ denote by q_{-n} the $(n-1)$ dimensional vector consisting of the first $n-1$ coordinates of q. Thus, the n-th coordinate of q, q_n is equal to $1 - |q_{-n}|$. Consider the set $D = \{(q, t);\ q_{-n} \in \Delta,\ t < v(q)\}$. Fix $c \in \Delta$ and define $L = \{\alpha(c_{-n}, v(c)) + (1-\alpha)((\frac{Q}{|Q|})_{-n}, v(\frac{Q}{|Q|});\ \alpha \in [0, 1]\}$. The set L is the line connecting $(c_{-n}, v(c))$ and $((\frac{Q}{|Q|})_{-n}, v(\frac{Q}{|Q|}))$. It is a convex set in $\mathbb{R}^n$, and since v is concave, D is also a convex set. Furthermore, the interior of D is not empty.

Fix $\beta \in (0, 1)$ and let $\alpha \in (0, 1)$ be such that $\beta = \frac{\alpha}{|\alpha c + (1-\alpha)Q|}$ (the existence of such α is guaranteed by the fact that $|c| = 1$). Notice that $1 - \beta = \frac{(1-\alpha)|Q|}{|\alpha c + (1-\alpha)Q|}$. By conditions (2) and (3),

$$v\left(\beta c + (1-\beta)\frac{Q}{|Q|}\right) = v\left(\frac{\alpha c + (1-\alpha)Q}{|\alpha c + (1-\alpha)Q|}\right)$$
$$= \frac{v(\alpha c + (1-\alpha)Q)}{|\alpha c + (1-\alpha)Q|} = \beta v(c) + (1-\beta)v\left(\frac{Q}{|Q|}\right).$$

The above equality implies that the sets D and L are disjoint. Moreover, the line segment L is on the boundary of D. The separation theorem ensures that there is a hyperplane in $\mathbb{R}^n$ that separates L from D, and since L is on the boundary of D, it follows that L is contained in this hyperplane. This implies that there is a vector $x \in \mathbb{R}^n$ such that $x \neq 0$ and $xq \geq v(q)$ for every $q \in \Delta$ with equality for every $q \in L$. In particular $x \in Core(Q, v)$ and $xc = v(c)$. Finally, exactness of (Q, v) follows from homogeneity of v on $F(Q)$.

As for the converse, assume that (Q, v) is exact. Let $c, d \in F(Q)$ and fix $\alpha \in (0, 1)$. Let $x \in Core(Q, v)$ be such that $x(\alpha c + (1-\alpha)d) = v(\alpha c + (1-\alpha)d)$. Then $\alpha v(c) + (1-\alpha)v(d) \leq \alpha xc + (1-\alpha)xd = x(\alpha c + (1-\alpha)d) = v(\alpha c + (1-\alpha)d)$, so v is concave.

Next, we show that v is homogeneous. Fix $c \in F(Q)$ and $\alpha > 0$ such that $\alpha c \in F(Q)$. Let $x, y \in Core(Q, v)$ be such that $xc = v(c)$ and $y\alpha c = v(\alpha c)$. Since both x and y are in the core, $xc = v(c) \leq yc$ and $\alpha xc \geq v(\alpha c) = \alpha yc$. It follows that $v(\alpha c) = \alpha v(c)$.

Finally, for some $c \in F(Q)$ and $\alpha \in (0, 1)$, let $x, y \in Core(Q, v)$ be such that $xc = v(c)$ and $y(\alpha c + (1-\alpha)Q) = v(\alpha c + (1-\alpha)Q)$. Then $\alpha v(c) + (1-\alpha)v(Q) = \alpha xc + (1-\alpha)xQ \geq v(\alpha c + (1-\alpha)Q) = \alpha yc + (1-\alpha)yQ \geq \alpha v(c) + (1-\alpha)v(Q)$, so we have an equality. □

Remark 1 A fuzzy game (Q, v) is exact if and only if $v(c) = \min\{xc;\ x \in Core(Q, v)\}$ for every $c \in F(Q)$. Thus, in order to construct exact games one needs only to take the minimum of a family of linear functions which coincide on the point Q. For instance, let $Q = (1, 1)$ and consider the linear functions defined by $x = (1, -1)$ and

$y = (-1, 1)$. Define $v(c) = \min\{xc, yc\} = \min\{c_1 - c_2, c_2 - c_1\}$ for every $c = (c_1, c_2) \in F(Q)$. Then (Q, v) is an exact fuzzy game.

Remark 2 Aubin (1981b, Sect. 5) studies fuzzy games whose characteristic function is homogeneous and super-additive (and thus concave) on the entire non-negative orthant. Such characteristic function is the point-wise minimum of a set of linear functions, but the restriction of it to some compact cube may not result in an exact game. This is because condition (3) of Theorem 3 is not necessarily satisfied.

It is well known that the vector of marginal contributions (with respect to any order of the players) of a convex cooperative game is in the core. This fact implies that every convex game is exact. A consequence of Theorem 3 is that this is not the case for fuzzy games. In fact, it turns out that a fuzzy game is both convex and exact if and only if it is linear, as stated in the following proposition.

Proposition 2 *A fuzzy game (Q, v) is both exact and convex iff there is $x \in \mathbb{R}^n$ such that $v(c) = xc$ for every $c \in F(Q)$.*

Proof First, it is trivial that if $v(c) = xc$ for some $x \in \mathbb{R}^n$ then (Q, v) is convex and exact. Conversely, assume that (Q, v) is both exact and convex. Fix a coalition $c \neq Q$ with full support. For every $c' \in F(Q)$ there is $\alpha > 0$ such that $\alpha c' \in F(c)$. Since (Q, v) is exact it is homogeneous and, thus, it is enough to show that the sub-game (c, v_c) is linear.

By exactness, there is $x \in Core(Q, v)$ such that $xc = v(c)$. We claim that $xd = v(d)$ for every $d \in F(c)$, meaning that (c, v_c) is linear, as claimed. Otherwise, there is $d \in F(c)$ such that $xd > v(d)$. Let $y \in Core(Q, v)$ be such that $yd = v(d)$. Define $e = \delta(Q - d)$, where $\delta > 0$ is small enough to ensure that $c + e \in F(Q)$. By Theorem 3, $v(d+e)-v(d) = v((1-\delta)d+\delta Q)-v(d) = (1-\delta)v(d)+\delta v(Q)-v(d) = \delta(v(Q)-v(d)) = \delta y(Q-d)$. Also, $v(c+e)-v(c) \leq x(c+e)-xc = xe = \delta x(Q-d)$. Thus, $v(d + e) - v(d) > v(c + e) - v(c)$ which contradicts the convexity of (Q, v). □

Our next goal is to characterize fuzzy games with the property that each one of their sub-games is exact. It turns out that a fuzzy game with such a property is linear. Note that this result stands in strict contrast with the situation in the classical theory where exactness of every sub-game is equivalent to convexity (Biswas et al. 1999, p. 10). We first need the following lemma.

Lemma 1 *Let (Q, v) be a fuzzy game. Suppose that there are n algebraically independent coalitions $c^1, \ldots, c^n$ such that*

(i) *For $i = 1, \ldots, n$, c^i has a full support;*
(ii) *The sub-game (c^i, v_{c^i}) is exact, $i = 1, \ldots, n$; and*
(iii) *$Q = c_1$.*

Then, v is a linear function: $v(d) = xd$ for some $x \in \mathbb{R}^n$.

Proof Since (Q, v) is exact, by Theorem 3, v is a homogenous function on $F(Q)$. Thus, in order to prove the lemma, it is enough to prove that v is linear in Δ (as in the previous proof, one can assume that $\Delta \subseteq F(Q)$).

We first claim that for every $i = 1, \ldots, n$ and for every $d \in \Delta$, v is linear in the interval between $\frac{c^i}{|c^i|}$ and d. Indeed, since c^i has a full support, there is $\varepsilon > 0$ such that $c^i - \varepsilon d \in F(Q)$ and also have a full support. By assumption, the sub-game (c^i, v_{c^i}) is exact and by Theorem 3, for every $\beta \in (0, 1)$, $v\left(\frac{\beta c^i + (1-\beta)\varepsilon d}{|\beta c^i + (1-\beta)\varepsilon d|}\right) = \frac{\beta}{|\beta c^i + (1-\beta)\varepsilon d|} v(c^i) + \frac{1-\beta}{|\beta c^i + (1-\beta)\varepsilon d|} v(\varepsilon d)$. Since v is homogenous the righthand side of the latter equality is equal to $\frac{\beta |c^i|}{|\beta c^i + (1-\beta)\varepsilon d|} v(\frac{c^i}{|c^i|}) + \frac{(1-\beta)\varepsilon}{|\beta c^i + (1-\beta)\varepsilon d|} v(d)$. This implies linearity in the interval between $\frac{c^i}{|c^i|}$ and d because $\frac{\beta |c^i|}{|\beta c_i + (1-\beta)\varepsilon d|}$ is onto $(0, 1)$, as a function of β.

Next we show that $v(\sum_{i=1}^n \alpha_i \frac{c^i}{|c^i|}) = \sum_{i=1}^n \alpha_i v(\frac{c^i}{|c^i|})$, when $\sum_{i=1}^n \alpha^i = 1$ and $\alpha^i \geq 0,\ i = 1, \ldots, n$. This is a simple consequence (by induction) of the previous claim. Thus, v is linear in the convex hull of $\{\frac{c^i}{|c^i|}\}_{i=1}^n$, denoted C.

Now, let $d \in \Delta$. Since $c^1, \ldots, c^n$ are independent, there are coefficients (not necessarily positive) $\gamma_i,\ i = 1, \ldots, n$, such that $d = \sum_{i=1}^n \gamma_i \frac{c^i}{|c^i|}$. Furthermore, this representation of d as a linear combination of the $\frac{c^i}{|c^i|}$'s is unique. Since all $\frac{c^i}{|c^i|}$ are in Δ, the sum of the coefficients is 1.

From here on the proof follows an induction on the number of negative coefficients. If this number is 0, then $d \in C$, a case that has been considered above. Now suppose that if the number of negative coefficients is less than k, then $v(d) = \sum_{i=1}^n \gamma_i v(\frac{c^i}{|c^i|})$. We prove this assertion when the number of negative coefficients is k.

We know that for every $j = 1, \ldots, n$ and $\beta \in (0, 1)$, $v(\beta \frac{c^j}{|c^j|} + (1-\beta)d) = \beta v(\frac{c^j}{|c^j|}) + (1-\beta) v(d)$. However, if $\gamma_j < 0$ and β is sufficiently close to 1, the number of negative coefficients of $\beta \frac{c^j}{|c^j|} + (1-\beta)d = \sum_{i \neq j} (1-\beta)\gamma_i \frac{c^i}{|c^i|} + \big(\beta + (1-\beta)\gamma_j\big) \frac{c^j}{|c^j|}$ is $k-1$. Thus, when β is sufficiently close to 1, $\beta v(\frac{c^j}{|c^j|}) + (1-\beta) v(d) = \sum_{i \neq j} (1-\beta)\gamma_i v(\frac{c^i}{|c^i|}) + \big(\beta + (1-\beta)\gamma_j\big) v(\frac{c^j}{|c^j|})$. By rearranging the last equality, we obtain that $v(d) = \sum_{i=1}^n \gamma_i v(\frac{c^i}{|c^i|})$, as desired. □

Theorem 4 *Every sub-game of a fuzzy game (Q, v) is exact iff v is a linear function on $F(Q)$.*

Proof From the previous lemma it follows that if every sub-game of a fuzzy game (Q, v) is exact, then v is a linear function on $F(Q)$. The converse is trivially true. □

5 Large cores

The concept of a *large core* in the classical theory was first introduced by Sharkey (1982). The classical game (N, v) has a large core if, for every vector $y \in \mathbb{R}^n$ with

$y(S) \geq v(S)$ for every $S \subseteq N$, there is a core member x such that $x \leq y$. His definition (with the necessary minor changes) is suitable for fuzzy games as well.

Definition 8 The fuzzy game (Q, v) has a **large core** if, for every $y \in \mathbb{R}^n$ that satisfies $yc \geq v(c)$ for every $c \in F(Q)$, there is $x \in Core(Q, v)$ such that $x \leq y$.

We are interested in the relations between the family of games with a large core and the families defined in the previous sections. In the classical theory every convex game has a large core, while the converse is false (Sharkey 1982). Also, if a totally balanced game has a large core then it is exact (Sharkey 1982). The converse is false (Biswas et al. 1999).

For fuzzy games, the relations between the various families change. First, we show that every exact fuzzy game has a large core while the converse is false even for totally balanced games. This stands in contrast to the classical games case. The proof of the theorem uses the following lemma, proved in Azrieli and Lehrer (2005).

Lemma 2 *Let $A \subseteq \mathbb{R}^n$ be a convex set and let $f : A \to \mathbb{R}$ be concave. Assume that $H \subseteq \mathbb{R}^n$ has the following two properties*:

(i) *H is closed and convex;*
(ii) *For every $q \in A$ there is $y \in H$ such that $y \cdot q = f(q)$ and $y \cdot q' \geq f(q')$ for every $q' \in A$ (the hyperplanes defined by the vectors in H support the entire graph of f).*

Let q_0 be in the interior of A and assume that $x \in \mathbb{R}^n$ is a linear support for f at q_0. Then, $x \in H$.

Theorem 5 *If (Q, v) is exact then it has a large core. The converse is false even if (Q, v) is totally balanced.*

Proof Assume that (Q, v) is exact and let $y \in \mathbb{R}^n$ be such that $yc \geq v(c)$ for every $c \in F(Q)$. Define the set $D = \{x \in \mathbb{R}^n;\ x \leq y,\ xc \geq v(c) \text{ for every } c \in F(Q)\}$. Since D is compact it has a minimal element, say x'. Obviously, $x'c = v(c)$ for some $c \in F(Q)$. Moreover, there exist such c with a full support. Otherwise, if $c_i = 0$ for some $1 \leq i \leq n$, then we can reduce x'_i, contradicting the minimality of x'.

By Theorem 3, v is homogeneous. Therefore, for $0 < \alpha < 1$, $x'\alpha c = \alpha v(c) = v(\alpha c)$. Since c has a full support, αc is in the interior of $F(Q)$. We now use Lemma 2 where, in the lemma's notation, $A = F(Q),\ f = v,\ H = Core(Q, v)$ and $q_0 = \alpha c$. It follows that $x' \in Core(Q, v)$ so (Q, v) has a large core.

To see that the converse is false consider the game where Q is the unit vector of $\mathbb{R}^n$, $v(Q) = 1$ and $v(c) = 0$ for every $c \neq Q$. Notice first that (Q, v) is totally balanced but not exact. However, it has a large core. Indeed, if $yc \geq v(c)$ for every $c \in F(Q)$ then $y_i \geq 0$ for every $i = 1, \ldots, n$ and $|y| = yQ \geq v(Q) = 1$. For every such y one can find $x \leq y$ which is in $Core(Q, v)$. □

As in the classical theory, convexity implies largeness of the core, as stated in the following theorem.

Theorem 6 *If (Q, v) is convex then it has a large core. The converse is false.*

Proof Let (Q, v) be a convex game. Denote by E the unit vector in $\mathbb{R}^n$ (that is, $E_i = 1,\ i = 1, \ldots, n$). Assume first that $Q = E$. Let $y \in \mathbb{R}^n$ be such that $yc \geq v(c)$ for every $c \in F(Q) = F(E)$. In particular, $yE^S \geq v(E^S)$ for every $S \subseteq N$. By Branzei et al. (2003, Proposition 2), the induced classical game $(N, \bar{v})$ is convex. Therefore, $(N, \bar{v})$ has a large core (Sharkey 1982). It follows that there is $x \leq y$ in the core of $(N, \bar{v})$. By Branzei et al. (2003, Theorem 7), $Core(N, \bar{v}) = Core(E, v)$, so $x \in Core(E, v)$.

Now, let Q be arbitrary. Define the auxiliary game (E, v') by $v'(c) = v(Q_1c_1, \ldots, Q_nc_n)$ for every $c = (c_1, \ldots, c_n) \in F(E)$. It is straightforward to check that (E, v') is convex and, thus, has a large core. If $y \in \mathbb{R}^n$ satisfies $yd \geq v(d)$ for every $d \in F(Q)$ then the vector $y' \in \mathbb{R}^n$ defined by $y'_i = Q_i y_i,\ i = 1, \ldots, n$ satisfies $y'c \geq v'(c)$ for every $c \in F(E)$. Therefore, there is $x' \in Core(E, v')$ such that $x' \leq y'$. Define $x \in \mathbb{R}^n$ by $x_i = \frac{x'_i}{Q_i},\ i = 1, \ldots, n$. Then it is easy to check that $x \leq y$ and $x \in Core(Q, v)$.

To see that the converse is false, consider the game (Q, v) of Example 1. Although (Q, v) is not convex it has a large core. Indeed, if $yc \geq v(c)$ for every $c \in F(Q)$ then in particular $y_i \geq 1$ for every $i = 1, \ldots, n$. This is because one can consider the coalition which equals 1 in its i coordinate and 0 elsewhere. Thus, the vector $x \in \mathbb{R}^n$ which equals 1 in all of its coordinates satisfies $x \leq y$ and $x \in Core(Q, v)$. □

6 Extendability

Let (Q, v) be a fuzzy game and let S be a subset of N. We denote by $Core_S(Q, v)$ the projection of $Core(Q, v)$ to $\mathbb{R}^S$. That is, $Core_S(Q, v) = \{x \in \mathbb{R}^S$; there is $y \in Core(Q, v)$ such that $y_i = x_i$ for every $i \in S\}$. Notice that $Core_N(Q, v) = Core(Q, v)$.

Definition 9 The fuzzy game (Q, v) is **extendable** if $Core(c, v_c) \subseteq Core_{\text{supp}(c)}(Q, v)$ for every coalition $c \in F(Q)$. In other words, (Q, v) is extendable if, for every coalition c, every core element of the sub-game (c, v_c) is, or can be extended to, a core element of the entire game.

The concept of extendability in cooperative games is due to Kikuta and Shapley (1986). In classical games it is known that extendability is a sufficient (but not necessary) condition for core stability. Moreover, for totally balanced games, largeness of the core implies extendability which, in turn, implies exactness. The converse of each of these statements is false. In fuzzy games, however, the situation is different. The next theorem shows that for totally balanced fuzzy games exactness and extendability are equivalent.

Theorem 7 *Every exact game is extendable. For totally balanced games the converse is also true.*

Proof First, notice that if (Q, v) is totally balanced and extendable then it is exact. The proof of the converse is based on Lemma 2. Let (Q, v) be exact and fix a coalition $c \in F(Q)$. Denote $S = \text{supp}(c)$ (it is possible that $S = N$) and assume that $x \in \mathbb{R}^S$ is in $Core(c, v_c)$.

By the homogeneity of v, the linear function defined by x lies above v on $F(Q^S)$. Indeed, if $d \in F(Q^S)$ then for small enough $\delta > 0$, $x\delta d \geq v(\delta d) = \delta v(d)$, which implies $xd \geq v(d)$. This means that the hyperplane defined by x is a linear support for the restriction of v to $F(Q^S)$ at the point c.

In addition, we may assume w.l.o.g. that c is an interior point of $F(Q^S)$. Otherwise, we can consider the coalition δc for some $0 < \delta < 1$. By homogeneity, the cores of (c, v_c) and $(\delta c, v_{\delta c})$ are equal.

The set $Core_S(Q, v)$ is closed and convex. Exactness implies that v is concave on $F(Q^S)$. Moreover, since (Q, v) is exact the hyperplanes defined by the elements of $Core_S(Q, v)$ support the entire graph of v restricted to $F(Q^S)$. Thus, we may apply Lemma 2 and conclude that $x \in Core_S(Q, v)$. □

It is known (Biswas et al. 1999) that there are exact classical games which are not extendable. Let (N, v) be such a game, and denote its core by C. Thus, for every coalition $S \subseteq N$, $v(S) = \min_{x \in C} x(S)$. Define the fuzzy game (Q, u) as follows: Q is the unit vector in $\mathbb{R}^n$ and for every $c \in F(Q)$, $u(c) = \min_{x \in C} xc$. Note that $v(S) = u(Q^S)$ for every $S \subseteq N$.

It is clear that (Q, u) is exact and, therefore, extendable. This means that if y is in the core of the classical sub-game (S, v_S) and y is not extendable, then y is not in the core of the sub-game (Q^S, v_{Q^S}). That is, there is $d \in F(Q^S)$ such that $u(d) = \min_{x \in C} xd > yd$.

7 Stable cores

Stable sets for cooperative games had been extensively studied. One of the most challenging tasks in this subject is to characterize the family of games with a stable core. Many sufficient conditions for core stability are known, but none of them is also necessary.

Stable sets for fuzzy games were first introduced by Tijs et al. (2004b). Their definition of a stable set for a fuzzy game is analogous to the one in classical cooperative games. They prove that the core of a convex fuzzy game is stable.

Here, we prove that exactness is also sufficient for core stability in fuzzy games. The proof is based on the fact that in fuzzy games exactness implies extendability, in contrast to the situation in cooperative games. For completeness, we repeat the definition of a stable set for a fuzzy game.

An *imputation* for the fuzzy game (Q, v) is a vector $x = (x_1, \ldots, x_n)$ such that $xQ = v(Q)$ and $x_i Q_i \geq v(Q_i^{\{i\}})$ for every $i \in N$. The set of all imputations of (Q, v) is denoted by $I(Q, v)$. For $x, y \in I(Q, v)$ and a coalition $c \in F(Q)$, we say that x *dominates* y *via* c (denoted $x \succeq_c y$) if $xc \leq v(c)$ and $x_i > y_i$ for every $i \in supp(c)$. We say that x *dominates* y (denoted $x \succeq y$) if there exist $c \in F(Q)$ such that $x \succeq_c y$.

Definition 10 The set of imputations $D \subseteq I(Q, v)$ is *stable* if the following two conditions hold:

(i) If $x, y \in D$, then it is not true that $x \succeq y$.
(ii) For every imputation $y \notin D$ there is $x \in D$ such that $x \succeq y$.

Theorem 8 *If (Q, v) is exact then $Core(Q, v)$ is a stable set.*

Proof First, assume that $x, y \in Core(Q, v)$. If $x \succeq_c y$ for some coalition $c \in F(Q)$ then $x_i > y_i$ for every $i \in \text{supp}(c)$, which implies $xc > yc$. However, since y is in the core, we get $xc > yc \geq v(c)$ which contradicts the fact that $x \succeq_c y$. Therefore, condition (i) of the definition is satisfied.

As for (ii), assume that y is an imputation outside the core. Define the function $g : \Delta \to \mathbb{R}$ by $g(c) = v(c) - yc$ (as v is homogeneous, we may assume that $\Delta \subseteq F(Q)$). We claim first that there is $c \in \Delta$ such that $g(c) > 0$. Indeed, since $y \notin Core(Q, v)$ there is $d \in F(Q)$ such that $v(d) > yd$. By homogeneity, $v(\frac{d}{|d|}) > y\frac{d}{|d|}$. Denoting $c = \frac{d}{|d|}$ we have $c \in \Delta$ and $g(c) > 0$.

Notice that g is continuous over Δ and thus have a maximum. Let $\bar{c} \in \text{argmax} g(c)$ and let $\delta = g(\bar{c}) > 0$. Let $S = \text{supp}(\bar{c})$ and define the vector $x \in \mathbb{R}^S$ by $x_i = y_i + \delta$ for every $i \in S$.

We show that $x \in Core(\bar{c}, v_{\bar{c}})$. First, $x\bar{c} = y\bar{c} + \delta = y\bar{c} + g(\bar{c}) = v(\bar{c})$. Also, if $d \in F(\bar{c})$ then $\text{supp}(d) \subseteq S$. This means that $xd = \sum_{i \in S} x_i d_i = \sum_{i \in S}(y_i + \delta)d_i = yd + \delta|d|$. Thus, $xd = |d|x\frac{d}{|d|} = |d|(y\frac{d}{|d|} + \delta) \geq |d|(y\frac{d}{|d|} + g(\frac{d}{|d|})) = |d|v(\frac{d}{|d|}) = v(d)$. Finally, by Theorem 7, x can be extended to a core element x' of the entire game (Q, v). It follows that $x' \succeq_{\bar{c}} y$ and the proof is complete. □

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