THE CONCAVE INTEGRAL OVER LARGE SPACES

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Abstract. This paper investigates the concave integral for capacities defined over large spaces. We characterize when the integral with respect to capacity v can be represented as the infimum over all integrals with respect to additive measures that are greater than or equal to v. We introduce the notion of *loose extendability* and study its relation to the concave integral. A non-additive version for the Levi theorem and the Fatou lemma are proven. Finally, we provide several convergence theorems for capacities with large cores.

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1. Introduction

Choquet integral [2] was the first to deal with integration according to a capacity. Choquet introduced an integral which coincides with that of Lebesgue when the capacity is additive. Lehrer [4] presented a concave integral, which differs from the Choquet integral when the capacity is not convex (super modular), and characterized it when the underlying space is finite. This paper investigates the concave integral when the underlying space is large (not necessarily finite).

The connection between the integral with respect to (w.r.t.) capacities and the theory of decision making under uncertainty was made by Schmeidler [9] who proposed an alternative theory to the traditional expected-utility theory of von-Neumann and Morgenstern [7]. According to Schmeidler's model decision makers evaluate their alternatives using non-additive probabilities, namely capacities, and the Choquet integral.

Lehrer [5] proposed a model of decision making with partially-specified probability. This model suggests that decision makers obtain partial information about the governing distribution and they utilize this partial information by resorting a variant of the concave integral.

For a fixed capacity v we define the *loose core* to be the set of all additive measures that are greater than or equal to v. It turns out that, unlike the finite case, in general the loose core could be empty. The first results of this paper concern the possibility of representing the integral w.r.t. a capacity v in a dual fashion, namely as the infimum of integrals w.r.t. measures in the loose core. When the capacity is defined over a finite space, this kind of a representation is always possible and is a consequence of a separation theorem (see Lehrer [4]). In the general case, however, it is not true that the integral can always be represented in this way. We characterize when the loose core is not empty and when the integral coincides with the infimum of integrals w.r.t. measures in the loose core.

A capacity v over a finite space has a *large core* (see Sharkey [10]) iff the core of v is not empty, and for every additive measure $\mu \geq v$, there exist an additive capacity $\mu \geq P \geq v$. Lehrer [4] and Azrieli and Lehrer [1] showed the implication of a large core to the concave integral in the finite case. In case v has a large core, then whenever the integral of a non-negative function f is the infimum of integrals of f w.r.t. measures in the loose core, it is also the infimum of integrals of f w.r.t. measures in the core. Furthermore, the integral is invariant to the addition of a constant. That is, the integral of $f + c$ w.r.t. v, where c is a constant, is equal to the sum of the integral of f with respect to v and c.

It turns out that representing the integral in a dual fashion is tightly connected with a property called *loose extendability*. We say that a capacity v is *loosely extendable* if for any subset, say Y, any additive measure over Y which is greater than or equal to v over Y can be extended to a member in the loose core. The second type of results deals with the connection between loose extendability and the concave integral. For every subset Y of the entire space we define an auxiliary capacity over Y, based on the maximal marginal contribution of $A \subseteq Y$ to any subset in the complement of Y. Some properties that the concave integral w.r.t. this auxiliary capacity might possess determine whether the capacity v is loosely extendable.

Convergence theorems are the last results of the paper. Li and Song [8] formulated a few convergence theorems that refer to the Choquet integral. They proved that whenever an increasing sequence of non-negative measurable functions converges almost everywhere to a measurable function, then the sequence of Choquet integrals converges to the Choquet integral of the limit function. In this analysis the precise definition of what is 'almost everywhere' w.r.t. a capacity is crucial.

When a capacity is a measure, a property is satisfied almost everywhere if it is satisfied over a set whose capacity is 1. According to the definition of Wang and Klir [11], a property is satisfied almost everywhere w.r.t. a capacity if it fails to be satisfied over a set of capacity 0. Using this definition Li and Song [8] proved a monotonic convergence theorem w.r.t. the Choquet integral.

The definition of Wang and Klir [11] does not guarantee a convergence theorem w.r.t. the concave integral. We adopt a stronger definition of 'almost everywhere'. Assuming the stronger definition, we establish integral convergence theorems. Specifically, we prove versions of Levi monotonic convergence theorem and the Fatou lemma for capacities.

Making use of the integral's representation, we strengthen these results for integrals w.r.t. capacities that have a large core, and we prove a non-additive version for the dominated convergence theorem.

The paper is organized as follows. Section 2 introduces the integral. Section 3 compares the concave integral and the Choquet integral. Section 4 characterizes the representation of the integral. Connections between the concave integral and the notion of extendability appear in Section 5. Section 6 deals further with representation of integrals w.r.t.

capacities with large cores. Section 7 presents integral convergence theorems, and we conclude with some final comments in Section 8.

2. The concave integral for capacities

Let X be some set, $\mathcal{F}$ be a σ-algebra over X.

Definition 1. *Consider an extended*[1] *set function (or simply, set function)* $\nu : \mathcal{F} \to [0, \infty]$ *such that* $\nu(\emptyset) = 0$*.*

- ν *is* monotone *iff* $\nu(A) \leq \nu(B)$ *for all* $A \subseteq B$ *where* $A, B \in \mathcal{F}$*.*
- ν *is an* additive measure *iff* ν *is finite, that is* $\nu(X) < \infty$*, and* $\nu(A \cup B) = \nu(A) + \nu(B)$ *whenever* $A, B \in \mathcal{F}$ *are disjoint.*
- ν *is a* capacity *iff it is monotone and* $\nu(X) = 1$*.*

Let $\mathcal{M}$ denote the collection of all non-negative measurable functions from X to $\mathbb{R}_+$.[2] An extended function $H : \mathcal{M} \to [0, \infty]$ is concave iff $H(\alpha f + (1-\alpha)g) \geq \alpha H(f) + (1-\alpha)H(G)$ for every $\alpha \in (0,1)$ and $f, g \in \mathcal{M}$, and it is positive homogeneous iff $H(\alpha f) = \alpha H(f)$ for every $\alpha \geq 0$ and $f \in \mathcal{M}$.

Fix a capacity v and $f \in \mathcal{M}$.

Definition 2 (Lehrer [4])**.** *The* concave integral *of* f *over* A *w.r.t.* v *is defined by*

$$\int_A^{Cav} f dv := \inf\{H(f \cdot \mathbb{1}_A)\}, \tag{1}$$

where the infimum is taken over all concave and positive homogeneous extended functions $H : \mathcal{M} \to [0, \infty]$ *that satisfy* $H(\mathbb{1}_E) \geq v(E)$ *for all* $E \in \mathcal{F}$*, with* $\mathbb{1}_E$ *being the indicator function of* E*.*

Remark 1. *By considering concave and positive homogeneous functions to be extended functions (i.e., that might infinite) we allow, just as in the additive case, for non-integrable functions. This guarantees that the infimum in eq. (1) is not taken over an empty set of functions.*

Given a monotone extended set function ν, the concave integral w.r.t. ν is defined in the same manner.

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[1]The word extended signifies that the function may take the value infinity.

[2]A real function f is measurable iff $f^{-1}(B) \in \mathcal{F}$ for every Borel set B of real numbers.

We say that f is *integrable over* $A \in \mathcal{F}$ if $\int_A^{Cav} f dv$ is finite. If f is integrable over X we say that it is integrable.

Proposition 1. *For every capacity v and a measurable non-negative function f,*

$$\int_A^{Cav} f dv := \sup \Big\{ \sum_{i=1}^{N} \lambda_i v(A_i);\ \sum_{i=1}^{N} \lambda_i \mathbb{1}_{A_i} \leq f \cdot \mathbb{1}_A,\ A_1, ..., A_N \in \mathcal{F} \text{ and } \lambda_i \geq 0, N \in \mathbb{N} \Big\}.$$

Remark 2. *a. If v is a measure, then the concave integral w.r.t. v is the usual Lebesgue integral.*
b. Lehrer [4] *proved proposition 1 in the case where X is finite.*

Proof. Define $W_A(f) = \sup \big\{ \sum_{i=1}^{N} \lambda_i v(A_i);\ \sum_{i=1}^{N} \lambda_i \mathbb{1}_{A_i} \leq f \cdot \mathbb{1}_A,\ A_1, ..., A_N \in \mathcal{F}$ and $\lambda_i \geq 0, N \in \mathbb{N} \big\}$. We show first that $W_A(f) \geq \int_A^{Cav} f dv$ for every $f \in \mathcal{M}$ and $A \in \mathcal{F}$. Let $g, h \in \mathcal{M}$. If $\sum_{i \leq N} \lambda_i \mathbb{1}_{A_i} \leq h \cdot \mathbb{1}_A$ and $\sum_{j \leq M} \alpha_j \mathbb{1}_{B_j} \leq g \cdot \mathbb{1}_A$, then $\sum_{i \leq N} \lambda_i \mathbb{1}_{A_i} + \sum_{j \leq M} \alpha_j \mathbb{1}_{B_j} \leq (g+h) \cdot \mathbb{1}_A$. Thus, $W_A(\cdot)$ is super additive. That is, $W_A(g) + W_A(h) \leq W_A(g+h)$. Since W_A is also homogeneous, it is concave. Finally, since $W_A(\mathbb{1}_A) \geq v(A)$ for all $A \in \mathcal{F}$, we get the desired inequality.

To prove the inverse inequality, fix a concave and homogeneous $H : \mathcal{M} \to \mathbb{R}$ satisfying $H(\mathbb{1}_B) \geq v(B)$ for all $B \in \mathcal{F}$. Such a function H is super additive. Indeed, for $g, h \in \mathcal{M}$,

$$H(h+g) = H\bigg(2\Big(\frac{h}{2} + \frac{g}{2}\Big)\bigg) = 2H\Big(\frac{h}{2} + \frac{g}{2}\Big) \geq 2\Big(\frac{1}{2}H(h) + \frac{1}{2}H(g)\Big) = H(h) + H(g).$$

Now, for every $\sum_{i \leq N} \lambda_i \mathbb{1}_{A_i} \leq f \cdot \mathbb{1}_A$,

$$H(f) \geq H\Big(\sum_{i \leq N} \lambda_i \mathbb{1}_{A_i}\Big) \geq \sum_{i \leq N} \lambda_i H(\mathbb{1}_{A_i}) \geq \sum_{i \leq N} \lambda_i v(A_i).$$

Thus,

$$H(f) \geq W_A(f).$$

Since H is arbitrary, we conclude that $\int_X^{Cav} f dv \geq W_A(f)$. □

Notice that the proof also shows that the infimum in eq. (1) can be replaced by minimum and the latter is obtained at $H = W_A$.

From this point on we denote this integral of f w.r.t. v by $\int_X f dv$, omitting the notation 'Cav'.

Whenever v is a measure than $\int_X \mathbb{1}_A dv = v(A)$ for every $A \in \mathcal{F}$. In the non-additive case, if the latter equality holds for every $A \in \mathcal{F}$ we say that v is *totally ballanced (TB)*. However, it is not hard to construct an example that $\int_X \mathbb{1}_A dv > v(A)$ for some $A \in \mathcal{F}$.

The following lemma shows that in the view of the concave integral all capacities are totally balanced.

Lemma 1. *Given a capacity v over $\mathcal{F}$, the capacity over $\mathcal{F}$ defined by*

$$\hat{v}(A) := \int_X 1\!\!1_A dv$$

satisfies the following properties:
(i) $\hat{v} \geq v$;
(ii) $\int_X f d\hat{v} = \int_X f dv$ *for every* $f \in \mathcal{M}$; *and*
(iii) $\hat{v}$ *is TB.*

Proof. (i) By definition of the concave integral $v(A) \leq \int_X 1\!\!1_A dv$ for all $A \in \mathcal{F}$, therefore $v \leq \hat{v}$.
(ii) Let f be some non-negative integrable function. Since $\hat{v} \geq v$, $\int_X f d\hat{v} \geq \int_X f dv$. Fix $\varepsilon > 0$. There exists $\sum_{k=1}^K \lambda_k 1\!\!1_{A_k} \leq f$ such that $\int_X f d\hat{v} \leq \sum_{k=1}^K \lambda_k \hat{v}(A_k) + \varepsilon$. Now,

$$\int_X f d\hat{v} \leq \sum_{k=1}^K \lambda_k \hat{v}(A_k) + \varepsilon = \sum_{k=1}^K \lambda_k \int_X 1\!\!1_{A_k} dv + \varepsilon \leq \int_X f dv + \varepsilon.$$

If f is not integrable then by (i) we have the desired result.

Notice that this means that $\hat{v}(A) = \int_X 1\!\!1_A dv$ for every $A \in \mathcal{F}$. Therefore $\hat{v}$ is TB and we have (iii). □

$\hat{v}$ is called the *totally balanced cover* of v.

3. The concave integral and the Choquet integral

Given a capacity v and a non-negative measurable function f, the Choquet integral of f w.r.t. v over $A \in \mathcal{F}$ is defined by

$$\int_A^{Cho} f dv := \int_0^\infty v(\{s \in A;\ f(s) \geq t\}) dt, \tag{2}$$

where the latter integral is an extended Reimann integral. by the definition of the Reimann integral

$$\int_A^{Cho} f dv = \sup\left\{ \sum_{i=1}^N \lambda_i v(A_i);\ \sum_{i=1}^N \lambda_i 1\!\!1_{A_i} \leq f \cdot 1\!\!1_A,\ \{A_i\}_{i=1}^N \subset \mathcal{F} \text{ is decreasing, } \lambda_i \geq 0, N \in \mathbb{N} \right\},$$

where by decreasing we mean that $A_{i+1} \subseteq A_i$ for every $1 \leq i \leq N-1$.

A summation $\sum_{i=1}^{N} \lambda_i \mathbb{1}_{A_i}$ is a *lower evaluation* for $f \in \mathcal{M}$ if it is less than or equal to f. The *value* w.r.t. v of such a lower evaluation is $\sum_{i=1}^{N} \lambda_i v(A_i)$. The concave integral of f w.r.t. v is the supremum of values over all lower evaluations for f. We have seen that Choquet integral is the supremum of values over a particular partial collection of lower evaluations. In particular, $\int_A f dv \geq \int_X^{Cho} f dv$ for every f and $A \in \mathcal{F}$.

The following example shows that the two integrals do not always coincide.

Example 1. *Let $X = [0,1]$ endowed with the Borel σ-algebra. Define a capacity v as follows. For every $A \in \mathcal{F}$*

$$v(A) := \begin{cases} 1, & \{0\} \subsetneq A, \\ 0, & \textit{otherwise}. \end{cases}$$

Let $f = \mathbb{1}_X + \mathbb{1}_{\{1\}}$ and $g = \mathbb{1}_X + \mathbb{1}_{\{0\}}$. *Now, $v(\{f \geq x\}) = v(\{g \geq x\})$ for every $x \in [0,1]$, thus $\int_{[0,1]}^{Cho} f dv = \int_{[0,1]}^{Cho} g dv$. Moreover, both integrals are equal to 1. On the other hand, $\int_{[0,1]} f dv = 1$, whereas $\int_{[0,1]} g dv = v\big([0, \frac{1}{2}]\big) + v\big(\{0\} \cup (\frac{1}{2}, 1]\big) = 2$. Note that the two functions differ only at 0 and 1, but 0 is more likely than 1 in the sense that $1 = v(\{0\} \cup A) > v(\{1\} \cup A) = 0$ for every $A \in \mathcal{F}$ that does not contain 0 or 1. As oppose to the Choquet integral, the concave integral takes into account these differences and as a result valuates g more than f.*

The question arises as to when the two integrals coincide. A capacity v is *convex* iff $v(A) + v(B) \leq v(A \cup B) + v(A \cap B)$ for all $A, B \in \mathcal{F}$.

Proposition 2. *The concave integral coincides with the Choquet integral iff v is convex.*

Proof. The first implication is simple. Indeed, if a capacity v is not convex, then there exist $A, B \in \mathcal{F}$ such that $v(A) + v(B) > v(A \cup B) + v(A \cap B)$. Thus

$$\int_X (\mathbb{1}_A + \mathbb{1}_B) dv \geq v(A) + v(B) > v(A \cup B) + v(A \cap B) = \int_X^{Cho} (\mathbb{1}_A + \mathbb{1}_B) dv.$$

Now, assume that v is convex. Let $f \in \mathcal{M}$ be such that $\int_X f dv < \infty$, then for every $\varepsilon > 0$ there exist $\sum_{i=1}^{N} \lambda_i \mathbb{1}_{A_i} \leq f$ such that

$$\int_X f dv \leq \sum_{i=1}^{N} \lambda_i v(A_i) + \varepsilon \leq \int_X^{Cho} \left(\sum_{i=1}^{N} \lambda_i \mathbb{1}_{A_i} dv \right) + \varepsilon \leq \int_X^{Cho} f dv + \varepsilon,$$

where the second inequality holds due to Lovasz [6] and Azrieli and Lehrer [1] (given such a lower evaluation of f we can reduce ourselves to the finite case). Since ε is arbitrarily small we have that $\int_X f dv \leq \int_X^{Cho} f dv$. The other inequality always holds therefore $\int_X f dv = \int_X^{Cho} f dv$.

If $\int_X f dv = \infty$ then for every large L there exist $\sum_{i=1}^N \lambda_i \mathbb{1}_{A_i} \leq f$ such that $\sum_{i=1}^N \lambda_i v(A_i) > L$. The proof from this point is similar to the one above. □

4. Representation of the integral

The main result of this paper concerns the representation of the integral.

Let $\mathcal{F}_\infty(X)$ be the Banach space of all measurable bounded functions over X, endowed with the sup norm.[3] Denote by $\mathcal{F}_\infty^+(X)$ the closed convex subset of $\mathcal{F}_\infty(X)$ containing all non-negative functions.

We note first that, as a functional, the integral w.r.t. a capacity need not be continuous over $\mathcal{F}_\infty^+(X)$.

Example 2. *Consider the capacity v over $\mathbb{N}$ such that for every $A \subseteq \mathbb{N}$, $v(A) := \max\left\{\frac{1}{n};\ n \in A\right\}$.*

For every $n \in \mathbb{N}$ let,

$$f_n := \frac{\mathbb{1}_{\{1,\ldots,n\}}}{\sum_{i \leq n} v(\{i\})}.$$

$\int_{\mathbb{N}} f_n dv = 1$, while $\{f_n\}_{n\in\mathbb{N}}$ converges to 0 in the norm.

For $f \in \mathcal{F}_\infty^+(X)$ we define, with abuse of notation,

$$\hat{v}(f) := \inf_{\varepsilon>0} \int_X (f+\varepsilon) dv.$$

It is always true that $\hat{v}(f) \geq \int f dv$.

Definition 3. *Let ν be a monotone set function.*
(1) The loose core *of ν, denoted by $LsCore(\nu)$, is the set of all additive measures that are greater than or equal to ν.*
(2) The core *of ν, denoted by $Core(\nu)$, is the set of all additive measures $P \in LsCore(\nu)$ such that $P(X) = \nu(X)$.*

The capacity in Example 2 has an empty loose core.

Theorem 1. *Let v be a capacity and let $f \in \mathcal{F}_\infty^+(X)$ such that $\hat{v}(f) < \infty$. $\hat{v}(f) = \int_X f dv$ iff*

$$\int_X f dv = \inf_{\mu \in LsCore(v)} \int_X f d\mu.$$

[3] That is for every $f \in \mathcal{F}_\infty(X)$, $||f|| = \sup_{x\in X} |f(x)|$. Note that this is not the usual essential supremum norm of almost everywhere bounded measurable functions.

Proof. Assume first that $\int_X f dv = \inf_{\mu \in LsCore(v)} \int_X f d\mu$. Then, $\hat{v}(f) = \inf_{\varepsilon>0} \int_X (f + \varepsilon) dv \leq \inf_{\varepsilon>0} \int_X (f+\varepsilon) d\mu$ for every finite and additive measure $\mu \in LsCore(v)$. The last term is equal to $\inf_{\varepsilon>0} \int_X f d\mu + \varepsilon \cdot \mu(X) = \int_X f d\mu$, which proves the 'if' direction.

As for the inverse direction, assume that $\hat{v}(f) = \int_X f dv$. $\hat{v}$ is upper semicontinuous[4] over $\mathcal{F}^+_\infty(X)$. Indeed, consider a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^+_\infty(X)$ that converges in the norm to $f \in \mathcal{F}^+_\infty(X)$. Fix $\varepsilon > 0$. There exist $\delta > 0$ and $N \in \mathbb{N}$ such that $\hat{v}(f) \geq \int_X f + 2\delta dv - \varepsilon \geq \int_X f_n + \delta dv - \varepsilon \geq \hat{v}(f_n) - \varepsilon$, for all $n > N$. Since ε is arbitrary small we have that $\hat{v}(f) \geq \lim_{n\to\infty} \hat{v}(f_n)$. In the same way, applying the integral's concavity, we get that $\hat{v}$ is concave over $\mathcal{F}^+_\infty(X)$.

We obtained that $\hat{v}$ is an upper semicontinuous concave function defined over $\mathcal{F}^+_\infty(X)$, which is a closed convex subset of a Banach space $\mathcal{F}_\infty(X)$. Thus, $\hat{v}$ is the infimum over all affine functions greater than or equal to $\hat{v}$ over $\mathcal{F}^+_\infty(X)$ (see e.g., Ekeland and Temam [3]).

Now fix $f \in \mathcal{F}^+_\infty(X)$ and $\varepsilon > 0$. There is a linear function φ and a constant c such that $\hat{v}(f) \leq \varphi(f) + c \leq \hat{v}(f) + \varepsilon$ and $\hat{v}(g) \leq \varphi(g) + c$ for every $g \in \mathcal{F}^+_\infty(X)$. Let k be a positive number and apply the previous inequality to $g = kf$. We get, $\hat{v}(kf) \leq \varphi(kf) + c$. Since both φ and $\hat{v}$ are homogenous, $k\hat{v}(f) \leq k\varphi(f) + c$. Thus, $\hat{v}(f) \leq \varphi(f) + \frac{c}{k}$. On one hand, k can be arbitrary close to zero, meaning that $c \geq 0$. On the other hand, k can be arbitrary large which means that $\hat{v}(f) \leq \varphi(f) \leq \hat{v}(f) + \varepsilon$. We conclude that $\hat{v}$ is the infimum over all *linear* functions that are greater than or equal to $\hat{v}$ over $\mathcal{F}^+_\infty(X)$.

Any linear φ which is greater than or equal to $\hat{v}$ over $\mathcal{F}^+_\infty(X)$ induces a finite measure: $\hat{\varphi}(A) = \varphi(\mathbb{1}_A)$ for every $A \in \mathcal{F}$. Since $\hat{v}(\mathbb{1}_A) \geq v(A)$, $\hat{\varphi}(A) \geq v(A)$. Moreover, $\varphi(f) \geq \int_X f d\hat{\varphi}$. Indeed, if $\sum_{i=1}^N \lambda_i \mathbb{1}_{A_i} \leq f$, then

$$\sum_{i=1}^N \lambda_i \hat{\varphi}(A_i) = \sum_{i=1}^N \lambda_i \varphi(\mathbb{1}_{A_i}) = \varphi\left(\sum_{i=1}^N \lambda_i \mathbb{1}_{A_i}\right) \leq \varphi(f).$$

The last inequality is due to the fact that $f - \lambda_i \mathbb{1}_{A_i} \in \mathcal{F}^+_\infty(X)$ and $\varphi(f - \lambda_i \mathbb{1}_{A_i}) \geq \hat{v}(f - \lambda_i \mathbb{1}_{A_i}) \geq 0$. Thus, $\int_X f d\hat{\varphi} \leq \varphi(f)$ and the proof is complete. □

The following example illustrates a case where $\hat{v}(f) > \int_X f dv$.

Example 3. *Let $X = (0, 1)$ endowed with the Borel σ-algebra, and let $v(A) = 1$ if A contains an open neighborhood of 1 and if 0 is an accumulation point of A; otherwise,*

[4]A function $H : X \to \mathbb{R}$ is upper semicontinuous iff for every $x_n \to x$ (in the norm), $\limsup H(x_n) \leq H(x)$.

$v(A)=0$. Consider $f=\mathbb{1}_{[\frac{1}{2},1)}$. It is easy to check that $\int_X f dv=0$, whereas $\int_X (f+\varepsilon)dv=1$ for all $\varepsilon>0$. Thus, $\hat{v}(f)>\int_X f dv$.

Corollary 1. *The following are equivalent:*
(i) $LsCore(v)\neq\emptyset$;
(ii) $\int_X dv<\infty$;
(iii) $\int_X f dv<\infty$ for every $f\in\mathcal{F}^+_\infty(X)$;
(iv) $\hat{v}(f)<\infty$ for every $f\in\mathcal{F}^+_\infty(X)$; and
(v) The set of all concave and positive homogeneous functions $H:\mathcal{F}^+_\infty(X)\to[0,\infty)$ that satisfy $H(\mathbb{1}_E)\geq v(E)$ for all $E\in\mathcal{F}$ is not empty.

Proof. By the monotonicity and homogeneity of the integral, we have that (ii),(iii) and (iv) are equivalent and that (i) implies (ii). Now, assume (ii), that is $\hat{v}(\mathbb{1}_X)=\inf_{\varepsilon>0}\int_X(1+\varepsilon)dv=\inf_{\varepsilon>0}(1+\varepsilon)\int_X dv=\int_X dv<\infty$. Theorem 1 holds for $\mathbb{1}_X$, in particular $LsCore(v)$ is not empty, and we have (i). Finally, it is clear that (ii) and (v) are equivalent. □

Corollary 2. *Let $f\in\mathcal{F}^+_\infty(X)$ be a function such that $\inf\{f(x);\ x\in X\}>0$ and $\hat{v}(f)<\infty$. Then,*

$$\int_X f dv=\inf_{\mu\in LsCore(v)}\int_X f d\mu. \tag{3}$$

Proof. We prove that $\hat{v}(f)=\int_X f dv$ by showing that the integral w.r.t. v is continuous at f. Fix $\varepsilon>0$. Since $\inf\{f(x);\ x\in X\}>0$, there is $\delta>0$ such that $(1-\delta)f\leq f_\varepsilon\leq(1+\delta)f$ for every non-negative f_ε in the open ball centered at f with radius ε. Define,

$$\hat{\delta}=\inf\Big\{\delta>0;\ (1-\delta)f\leq f_\varepsilon\leq(1+\delta)f \text{ for all } f_\varepsilon\in B_\varepsilon(f)\Big\}.$$

Note that $\hat{\delta}\to 0$ as $\varepsilon\to 0$.
Since the integral is homogenous and monotonic, we obtain

$$(1-\hat{\delta})\int_X f dv\leq\int_X f_\varepsilon dv\leq(1+\hat{\delta})\int_X f dv.$$

By letting $\varepsilon\to 0$, we conclude that $\int_X f_\varepsilon dv\to\int_X f dv$. □

Let ν be a monotone set function and $A\in\mathcal{F}$. $\mathcal{F}_A:=\{F\subseteq A; F\in\mathcal{F}\}$. Define $\nu_A:=\nu|_{\mathcal{F}_A}$, that is $\nu_A(F)=\nu(F)$ for all $F\in\mathcal{F}_A$.

For $f\in\mathcal{F}^+_\infty(X)$ define $PD(f):=\{x\in X;\ f(x)>0\}$. The next example shows that in general the conclusion of Corollary 2 is incorrect.

Example 4. *Recall Example 3. There,* $PD(f) = [\frac{1}{2}, 1)$ *and the infimum of* f *over* $PD(f)$ *is greater than* 0*. Corollary 2 states that eq. (3) holds. Indeed,* $\int_X f dv = \int_X f d\mu_0$*, where* μ_0 *is identically zero over* $\mathcal{F}_{PD(f)}$*. Let* P *be the additive capacity assigning* 1 *to sets that contain an open neighborhood of* 1*, and* 0 *otherwise.* $LsCore(v)$ *consists only of additive measures of the form* cP*, where* $c \geq 1$*. The integral of* f *w.r.t. any member of* $LsCore(v)$ *is greater than* 1*. Thus, the integral of* f *w.r.t.* v *cannot be expressed as eq. (3).*

Now, for $g \in \mathcal{F}^+_\infty(X)$, similar to $\hat{v}$ we define $\hat{v}_{PD(f)}(g) := \int_X (g + \varepsilon \cdot \mathbb{1}_{PD(f)}) dv$. When $\inf\{f(x);\ x \in PD(f)\} > 0$ we can prove, just as in Corollary 2, that

$$\int_{PD(f)} f dv = \inf_{\mu \in LsCore(v_{PD(f)})} \int_{PD(f)} f d\mu. \tag{4}$$

5. Extendability

In eq. (4) the infimum is taken over all additive measures μ in the loose core of $v_{PD(f)}$. If any such additive measure can be extended to the whole space while being above v, then the integral of f could be expressed as in eq. (3).

Definition 4. *Let* ν *be a monotonic set function.*

(1) An additive measure $\mu \in LsCore(\nu_A)$ *is* extendable *iff there exists an additive measure* $m \in LsCore(\nu)$ *such that* $m_A = \mu$*.*
(2) ν *is* loosely extendable *iff for every* $A \in \mathcal{F}$*, every additive measure in* $LsCore(\nu_A)$ *is extendable to an additive measure in* $LsCore(\nu)$*.*

An obvious consequence of Corollary 2 is,

Corollary 3. *If* v *is loosely extendable and* $\hat{v}(f) < \infty$*, then for every* $A \in \mathcal{F}$*,*

$$\int_X \mathbb{1}_A dv = \inf_{\mu \in LsCore(v)} \int_X \mathbb{1}_A d\mu. \tag{5}$$

Note that in Example 4, the measure μ_0 defined over $PD(f)$ is not extendable and therefore v is not loosely extendable. In this example, as well as in any other example that we could construct, when the integral of f cannot be expressed as in eq. (3), v is not loosely extendable. We are unable to answer the question whether when v is loosely extendable, eq. (3) holds true for every $f \in \mathcal{F}^+_\infty(X)$.

The rest of this section is devoted to presenting results that connect the concave integral and extendability.

Let $E \in \mathcal{F}$. For any monotonic set function ν over $\mathcal{F}_{E^c}$, define a new monotonic set function $\varphi_{(E,\nu)}$ over $\mathcal{F}_E$ as follows:

$$\varphi_{(E,\nu)}(A) := \sup_{B \subseteq E^c} v(A \cup B) - \nu(B).$$

Lemma 2. *v is loosely extendable iff $\int_E d\varphi_{(E,\mu)} < \infty$, for every $E \in \mathcal{F}$ and $\mu \in LsCore(v_{E^c})$.*

Proof. Fix some $E \in \mathcal{F}$ and $\mu \in LsCore(v_{E^c})$. Suppose first that $\int_E d\varphi_{(E,\mu)} < \infty$. By Corollary 1 there exists $\mu_0 \in LsCore(\varphi_{(E,\mu)})$. In particular, $\mu_0(A) \geq \varphi_{(E,\mu)}(A) \geq v(A \cup B) - \mu(B)$ for every $A \subseteq E$ and $B \subseteq E^c$. Therefore, $\mu_0(A) + \mu(B) \geq v(A \cup B)$. Thus, μ_1 defined as $\mu_1(A \cup B) = \mu_0(A) + \mu(B)$ for every $A \subseteq E$ and $B \subseteq E^c$, is in $LsCore(v)$ and it extends μ.

As for the inverse direction, fix $\varepsilon > 0$. Let $\mu_1 \in LsCore(v)$ be an extension of μ. Then, for every $\sum_{i=1}^N \lambda_i \mathbb{1}_{A_i} \leq \mathbb{1}_E$, there exist $B_{A_i} \subseteq E^c$ for all $i \leq N$ such that $\sum_{i=1}^N \lambda_i \varphi_{(E,\mu)}(A_i) \leq \sum_{i=1}^N \lambda_i \big[v(A_i \cup B_{A_i}) - \mu(B_{A_i})\big] + \varepsilon$. The last expression is less than or equal to $\sum_{i=1}^N \lambda_i \big[\mu_1(A_i) + \mu_1(B_{A_i}) - \mu(B_{A_i})\big] + \varepsilon = \sum_{i=1}^N \lambda_i \mu_1(A_i) + \varepsilon \leq \mu_1(E) + \varepsilon$. Thus, $\int_E d\varphi_{(E,\mu)} < \mu_1(E) + 1$. □

For every $E \in \mathcal{F}$ and $\varepsilon > 0$, define $\psi_E(\varepsilon) = \sup \int_X (f + \varepsilon \mathbb{1}_E) dv - \int_X f dv$, where the supremum is taken over all $f \in \mathcal{F}^+_\infty(X)$ such that $PD(f) \subseteq E^c$.

Lemma 3. *Let $E \in \mathcal{F}$. If $\int_E d\varphi_{(E,\hat{v})} < \infty$, then $\lim_{\varepsilon \to 0} \psi_E(\varepsilon) = 0$.*

Proof. Fix $f \in \mathcal{F}^+_\infty(X)$ with $PD(f) \subseteq E^c$ and some $\varepsilon > 0$. For every $\delta > 0$ there exist $\sum_{i=1}^N \lambda_i \mathbb{1}_{A_i \cup B_{A_i}} \leq f + \varepsilon \mathbb{1}_E$ such that

$$\int_X (f + \varepsilon \mathbb{1}_E) dv \leq \sum_{i=1}^N \lambda_i v(A_i \cup B_i) + \delta, \tag{6}$$

where $A_i \subseteq E^c$ and $B_i \subseteq E$ for all $1 \leq i \leq N$. Since, $\sum_{i=1}^N \lambda_i \mathbb{1}_{A_i} \leq f$, $\sum_{i=1}^N \lambda_i \int_X \mathbb{1}_{A_i} dv \leq \int_X f dv$. Eq. (6) implies,

$$\int_X (f + \varepsilon \mathbb{1}_E) dv - \int_X f dv \leq \sum_{i=1}^N \lambda_i v(A_i \cup B_i) + \delta - \sum_{i=1}^N \lambda_i \hat{v}(A_i) \leq$$

$$\sum_{i=1}^N \lambda_i \varphi_{(E,\hat{v})}(B_i) + \delta \leq \varepsilon \int_E d\varphi_{(E,\hat{v})} + \delta.$$

Since δ is arbitrarily small, $\int_X (f + \varepsilon \mathbb{1}_E) dv - \int_X f dv \leq \varepsilon \int_E d\varphi_{(E,\hat{v})}$. As the right-hand side does not depend on f and since $\int_E d\varphi_{(E,\hat{v})} < \infty$, the proof is complete. □

The next example shows that the converse of Lemma 3 does not hold.

Example 5. *Consider the capacity over the Borel σ-algebra over $X = [0,1]$ defined by*

$$v(A) := \begin{cases} 1, & \{0\} \subsetneq A, \\ 0, & \textit{otherwise}. \end{cases}$$

Consider $E = [\frac{1}{2}, 1]$ and $f = \mathbb{1}_{\{0\}}$. The integral of f equals to 0, while $\int_X (f + \varepsilon \mathbb{1}_E) = 1$ for every ε. Finally, $\int_E d\varphi_{(E,\hat{v})} = 1$.

Lemma 4. *If for every $E \in \mathcal{F}$, $\lim_{\varepsilon \to 0} \psi_E(\varepsilon) = 0$, then v is loosely extendable.*

Proof. Fix some $E \in \mathcal{F}$. We show that if $\lim_{\varepsilon \to 0} \psi_E(\varepsilon) = 0$, then $\int_E d\varphi_{(E,\mu)} < \infty$ for every $\mu \in LsCore(v_{E^c})$. By Lemma 2 it implies that v is loosely extendable. If, on the contrary, $\int_E d\varphi_{(E,\mu)} = \infty$, then for every $c > 0$ there is $\sum_{i=1}^{N_c} \lambda_i \mathbb{1}_{A_i} \leq \mathbb{1}_E$ such that $\frac{1}{c} \leq \sum_{i=1}^{N_c} \lambda_i \varphi_{(E,\mu)}(A_i) \leq \sum_{i=1}^{N_c} \lambda_i \big[v(A_i \cup B_i) - \hat{v}(B_i)\big] + \varepsilon$, where $B_i \subseteq E^c$.

Denote $f = \sum_{i=1}^{N_c} \lambda_i \mathbb{1}_{B_i}$. We obtain $\frac{1}{c} \leq \int_X f + \mathbb{1}_E dv - \int_X f dv + \varepsilon$. Thus, $1 \leq \int_X cf + c\mathbb{1}_E dv - \int_X cf dv + c\varepsilon$. If ε is small enough so that $c\varepsilon < \frac{1}{2}$, then for every $c > 0$, we obtained a function cf such that $PD(cf) \subseteq E^c$ and $\frac{1}{2} \leq \int cf + c\mathbb{1}_E dv - \int cf dv$ which contradicts the assumption. □

We summarize Lemmas 3 and 4 in the following theorem.

Theorem 2. *$\int_E d\varphi_{(E,\hat{v})} < \infty$ for every $E \in \mathcal{F}$ implies $\lim_{\varepsilon \to 0} \psi_E(\varepsilon) = 0$ for every $E \in \mathcal{F}$, which implies that v is loosely extendable.*

Define $\widetilde{v}(f) = \inf_{\varepsilon > 0} \int_X (f + \varepsilon \mathbb{1}_{PD(f)^c}) dv$. Note that $\widetilde{v}(f) \leq \hat{v}(f)$ for every $f \in \mathcal{F}_\infty^+(X)$.

We conclude this section with a diagram summarizing the relations obtained between the various properties presented so far.

$$\begin{array}{ccc}
\int_E d\varphi_{(E,\hat{v})} < \infty, \ \forall E \in \mathcal{F} & & \\
\Downarrow & & \\
\lim_{\varepsilon \to 0} \psi_E(\varepsilon) = 0, \ \forall E \in \mathcal{F} & \Longrightarrow & v \text{ is loosely extendable} \\
\Downarrow & & \\
\widetilde{v}(f) = \int_X f dv, \ \forall f \in \mathcal{F}_\infty^+(X) & & \\
\Uparrow & & \\
\hat{v}(f) = \int_X f dv, \ \forall f \in \mathcal{F}_\infty^+(X) & \overset{\text{if } \int_X dv < \infty}{\Longleftrightarrow} & \int_X f dv = \inf_{\mu \in LsCore(v)} \int_X f d\mu, \ \ \forall f \in \mathcal{F}_\infty^+(X)
\end{array}$$

6. The integral and large cores

Sharkey [10] introduced the definition of large core in the case where X is finite. v has a large core if for every additive measure $\mu \in LsCore(\nu)$ there is $P \in Core(\nu)$ such that $\mu \geq P \geq \nu$. When X is finite, $LsCore(\nu)$ is always not empty. However, in the general case this should be explicitly assumed.

Definition 5. *A monotone set function ν has a* large core *iff $LsCore(\nu)$ is not empty and if for every additive measure $\mu \in LsCore(\nu)$ there is $P \in Core(\nu)$ such that $\mu \geq P \geq \nu$.*

Azrieli and Lehrer [1] proved that if X is finite, then a capacity v has a large core if and only if the integral w.r.t. v can be represented as the minimum of integrals w.r.t. capacities in the core of v. Moreover, v has a large core if and only if the integral w.r.t. v is additive w.r.t. constants.

Corollary 4. *If v is a loosely extendable capacity and has a large core, then Corollary 2 can be restated as follows:*

$$\int_X f dv = \inf_{P \in Core(v)} \int_X f dP$$

for every $f \in \mathcal{F}_\infty^+(X)$ such that $\inf\{f(x);\ x \in PD(f)\} > 0$.

Remark 3. *Example 4 shows that having a large core is not enough for representing an integral, that is, being loosely extendable is necessary in the large core case as well.*

Lemma 5. *Assume that v is a loosely extendable capacity with a large core. Then $\int_X (f+c) dv = \int_X f dv + c$ for every $c > 0$ and $f \in \mathcal{F}_\infty^+(X)$ such that* $\inf\{f(x); x \in PD(f)\} > 0$.

Proof. Always, $\int_X (f+c)dv \geq \int_X f dv + c$. On the other hand, given $\varepsilon > 0$ we can find some $P \in Core(\mathrm{v})$ such that

$$\int_X (f+c)dv \leq \int_X (f+c)dP = \int_X f dP + c \leq \int_X f dv + c + \varepsilon.$$

Thus, $\int_X (f+c)dv \leq \int_X f dv + c$. □

The next proposition shows the continuity of the integral w.r.t. capacities with large cores.

Proposition 3. *Let v be a loosely extendable capacity with a large core. Then the integral w.r.t. v is continuous over $\mathcal{F}^+_\infty(X)$. In particular, $\hat{v}(f) = \int_X f dv$ for all $f \in \mathcal{F}^+_\infty(X)$.*

Proof. Consider $f \in \mathcal{F}^+_\infty(X)$ and, as before, define $f_n = \mathbb{1}_{\{x;\ f(x)\geq \frac{1}{n}\}} f$ for every $n \in \mathbb{N}$. Clearly, $\lim_{n\to\infty} \int_X f_n dv \leq \int_X f dv$.

For the other implication, fix $\varepsilon > 0$. There exist $N \in \mathbb{N}$ such that $f \leq f_n + \frac{\varepsilon}{2}$ for any $n > N$, and there exist $P_n \in Core(v)$ such that $\int_X f_n P_n \leq \int_X f_n dv + \frac{\varepsilon}{2}$. We now have that, for any $n > N$,

$$\int_X f dv \leq \int_X f dP_n \leq \int_X f_n + \frac{\varepsilon}{2} dP_n = \int_X f_n dP_n + \frac{\varepsilon}{2} \leq \int_X f_n dv + \varepsilon.$$

Now, assume that $\{g_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^+_\infty(X)$ is such that $g_n \to f$ (in the norm). For every $k \in \mathbb{N}$, $f_k \in B_{\frac{1}{k}}(f)$, therefore, there exist $N \in \mathbb{N}$ such that $g_n \in B_{\frac{2}{k}}(f_k)$ for every $n > N$. Thus,

$$\int_X g_n dv \leq \int_X (f_k + \frac{2}{k}) dv = \int_X f_k dv + \frac{2}{k},$$

that is

$$\lim_{n\to\infty} \int_X g_n dv \leq \lim_{k\to\infty} \int_X f_k dv = \int_X f dv.$$

Conversely, define $g_{n_k} := \mathbb{1}_{\{x;\ g_n(x)\geq \frac{1}{k}\}} g_n$. Again, for every $k \in \mathbb{N}$, there exist $N \in \mathbb{N}$ such that $g_n \in B_{\frac{1}{k}}(f)$ for every $n > N$ that is, $g_{n_k} \in B_{\frac{2}{k}}(f)$, for every $n > N$. Thus,

$$\int_X g_n dv + \frac{2}{k} \geq \int_X g_{n_k} dv + \frac{2}{k} = \int_X (g_{n_k} + \frac{2}{k}) dv \geq \int_X f dv.$$

Therefore,

$$\lim_{n\to\infty} \int_X g_n dv \geq \int_X f dv.$$

Concluding that $\lim_{n\to\infty} \int_X g_n dv = \int_X f dv$. □

Theorem 1 implies the following.

Corollary 5. *Let v be a loosely extended capacity with a large core. Then,*

$$\int_X f dv = \inf_{P\in Core(v)} \int_X f dP$$

for every $f \in \mathcal{F}^+_\infty(X)$. Furthermore, $\int_X f + c dv = \int_X f dv + c$ for every $f \in \mathcal{F}^+_\infty(X)$ and $c > 0$.

7. Integral convergence theorems

7.1. **The notion of "almost everywhere".** The analysis of the integral's asymptotic behavior resorts to the notion of *almost everywhere* w.r.t. a non-additive capacity v.

Definition 6. *A* property q *over* X *is a function* $q : \mathcal{F} \to \mathcal{F}$ *that satisfies,* $q(A) = A \cap q(X)$ *for all* $A \in \mathcal{F}$.

For example, let $f, g : X \to \mathbb{R}_+$ be measurable functions. The function $e_{f,g}$, defined by $e_{f,g}(A) := \{x \in A;\ f(x) = g(x)\}$ for every $A \subseteq X$, is a property over X.

Definition 7 (Wang and Klir [11]). *A property* q *is satisfied* v*-almost everywhere iff* $v(q(X)^c) = 0$.

In the case where P is additive, if two functions f, g are P-almost surely equal, then their integrals coincide. The next example shows that in the non-additive case a stronger definition should be adopted in order to obtain integral equality.

Example 6. *Consider the following capacity over* $X = \{0, 1, 2\}$. *Define*

$$v(A) := \begin{cases} 1, & \{0\} \subsetneq A, \\ 0, & \textit{otherwise}. \end{cases}$$

Now define two functions,

$$f(x) = \begin{cases} 2, & x = 0, \\ 1, & x = 1, \\ 0, & x = 2, \end{cases} \text{ and } g(x) = \begin{cases} 2, & x = 0, \\ 1, & x = 1, \\ 2, & x = 2. \end{cases}$$

These functions coincide over a set of capacity 1 *and differ over a set of capacity* 0*, thus equal "almost everywhere" both according to the standard definition (the additive case) and to the definition of Wang and Klir* [11]*. Nevertheless*

$$\int_X f dv = 1 < 2 = \int_X g dv.$$

Whenever two functions f and g are equal P-almost surely, then for every $A \in \mathcal{F}$, $P(\{x \in A;\ f(x) = g(x)\}) = P(A)$. The example shows that in the non-additive case this is not necessarily so.

The next two lemmas examine the behavior of properties in two families of capacities.

Lemma 6. *Let v be a convex capacity and q a property, then $v(q(X)) = 1$ iff $v(q(A)) = v(A)$ for every $A \in \mathcal{F}$.*

Proof. Since $q(A) \subseteq A$, $v(q(A)) \leq v(A)$. Assume $v(q(X)) = 1$. Since v is convex we get,

$$v(A) + v(q(X)) \leq v(A \cup q(X)) + v(A \cap q(X)) = v(q(X)) + v(q(A)).$$

Thus, $v(A) \leq v(q(A))$ and we obtain $v(q(A)) = v(A)$.

On the other hand, if $v(q(A)) = v(A)$ for all $A \in \mathcal{F}$, in particular $v(q(X)) = v(X)$ and the result follows. □

Definition 8. *1. A capacity is said to be* totally balanced *iff $v(A) = \int_X 1\!\!1_A dv$ for all $A \in \mathcal{F}$.*
2. The integral w.r.t. a capacity v is said to be additive w.r.t. constants *iff $\int_X f + c dv = \int_X f dv + c$ for every measurable non-negative function f and $c > 0$.*

Lemma 7. *Assume that v is totally balanced and let q be a property such that $v(q(X)) = 1$. If the integral w.r.t. v is additive w.r.t. constants, then $v(q(A)) = v(A)$ for all $A \in \mathcal{F}$.*

Proof. Assume that $A \in \mathcal{F}$ is such that $v(q(A)) < v(A)$. Let $f = 1\!\!1_{A_q}$. Now

$$\int_X (1\!\!1_{q(A)} + 1) dv \geq v(A) + v(q(X)) > v(q(A)) + 1 = 1 + \int_X 1\!\!1_{q(A)} dv,$$

implying that the integral is not additive w.r.t. constants. □

Note that a convex capacity satisfies the hypothesis of Lemma 7.

In order to obtain integral convergence theorems we adopt the following definition of "almost everywhere":

Definition 9. *Property q is satisfied* v-a.e. *in X iff $v(q(A)) = v(A)$ for every $A \in \mathcal{F}$.*

Note that the new definition implies both the standard definition of "almost everywhere" and the definition by Wang and Klir [11].

The following lemma shows that if a property occurs almost everywhere w.r.t. a capacity v then it occurs almost everywhere w.r.t. the totally balanced cover $\hat{v}$. It is easy to verify that the converse does not always hold.

Lemma 8. *If property q occurs v-a.e. then it occurs $\hat{v}$-a.e.*

Proof. For $A \in \mathcal{F}$ such that $\hat{v}(A) < \infty$, for every $\varepsilon > 0$ there exist $\sum_{k=1}^{N} \lambda_k 1\!\!1_{A_k} \leq 1\!\!1_A$ such that $\hat{v}(A) = \int_X 1\!\!1_A dv \leq \sum_{k=1}^{N} \lambda_k v(A_k) + \varepsilon = \sum_{k=1}^{N} \lambda_k v(q(A_k)) + \varepsilon \leq \int_X 1\!\!1_{q(A)} dv + \varepsilon =$

$\hat{v}(q(A)) + \varepsilon$, therefore, $\hat{v}(A) \le \hat{v}(q(A))$. The other inequality is clear and we obtain the desired result. If $\hat{v}(A) = \infty$ than for every large L there exist $\sum_{k=1}^N \lambda_k \mathbb{1}_{A_k} \le \mathbb{1}_A$ such that $\sum_{k=1}^N \lambda_k v(A_k) > L$. From this point the proof is similar to the one above. $\square$

7.2. **Capacities that are continuous from below.**

Definition 10. *A monotone set function ν is* continuous from below *iff*

$$\lim_{n\to\infty} \nu(A_n) = \nu\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) \text{ whenever } A_1 \subseteq A_2 \subseteq \cdots .$$

Remark 4. *Let ν be an extended monotone set function. Assume that $\{f_n\}_{n\in\mathbb{N}}$ is an increasing sequence of non-negative measurable functions that converges ν-a.e. to a function f. That is, $\nu(\{x \in A;\ \lim f_n(x) = f(x)\}) = nu(A)$, for every $A \in \mathcal{F}$. If ν is continuous from below, then for every $A \in \mathcal{F}$ with $\nu(A) < \infty$, $\varepsilon' > 0$ and $\delta > 0$ there is $N \in \mathbb{N}$ such that for every $n > N$, $\nu(\{x \in A;\ f(x) - f_n(x) < \delta\}) > \nu(A) - \varepsilon'$.*

The following is the non-additive version of the Levi monotone convergence theorem.

Theorem 3 (Monotonic convergence 1)**.** *Let v be a capacity.* $\lim_{n\to\infty} \int_X f_n dv = \int_X f dv$ *for every increasing sequence of non-negative measurable functions $\{f_n\}_{n\in\mathbb{N}}$ converging v-a.e. to a function f iff $\hat{v}$ is continuous from below.*

Proof. Assume that v is such a capacity that $\hat{v}$ is continuous from below. Assume at first that $\int_X f dv < \infty$. Since $f_n \le f$, $\lim \int_X f_n dv \le \int_X f dv$. We will show that for every $\varepsilon > 0$, there exist $M \in \mathbb{N}$ such that for every $n \ge M$, $\int_X f_n d\hat{v} > \int_X f d\hat{v} - \varepsilon$, and by Lemma 1 we will have that $\int_X f_n dv \ge \int_X f dv$.

Fix $\varepsilon > 0$. There exist $\sum_{k=1}^N \lambda_k \mathbb{1}_{A_k} \le f$ such that

$$\int_X f d\hat{v} - \sum_{k=1}^K \lambda_k \hat{v}(A_k) < \varepsilon.$$

denote by $V := \max\{\hat{v}(A_k); 1 \le k \le N\}$.

By Lemma 8 we have that $\{f_n\}_{n\in\mathbb{N}}$ converges $\hat{v}$-a.e. to f. Applying Remark 4 to $\hat{v}$, $A = A_k$, $\varepsilon' = \frac{\varepsilon}{K\lambda_k}$ and $\delta = \frac{\varepsilon}{VK}$ $(k = 1, ..., K)$ one obtains an $N_k \in \mathbb{N}$ and a set $B_k \subseteq A_k$ that satisfy $\hat{v}(B_k) > \hat{v}(A_k) - \frac{\varepsilon}{K\lambda_k}$ and $f(x) - f_n(x) < \frac{\varepsilon}{K}$ for every $x \in B_k$ and every $n \ge N_k$. Set $M := \max\{N_1, ..., N_K\}$. Now, for every $n \ge M$ we get

$$\int_X f_n d\hat{v} > \sum_{k=1}^K \Big(\lambda_k - \frac{\varepsilon}{VK}\Big)\hat{v}(B_k) \ge \sum_{k=1}^K \lambda_k \hat{v}(B_k) - \varepsilon > \sum_{k=1}^K \lambda_k \Big(\hat{v}(A_k) - \frac{\varepsilon}{K\lambda_k}\Big) - \varepsilon > \int_X f d\hat{v} - 3\varepsilon.$$

Since ε is arbitrarily small, the result follows.

Now, if f is not integrable, that is $\int_X f dv = \infty$, given a large L, there exist $\sum_{k=1}^K \lambda_k 1\!\!1_{A_k} \leq f$ such that

$$\sum_{k=1}^{K} \lambda_k \hat{v}(A_k) > L.$$

The proof from this point is similar to the one above.

For the converse assume that $\hat{v}$ is not continuous from below, that is there exist a sequence $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$ increasing to A such that $\lim_n \hat{v}(A_n) < \hat{v}(A)$. Since $\hat{v}$ is totally balanced, $\int_X 1\!\!1_E d\hat{v} = \hat{v}(E)$ for every $E \in \mathcal{F}$, therefore $\lim_n \int_X 1\!\!1_{A_n} d\hat{v} < \int_X 1\!\!1_A d\hat{v}$. □

A conclusion from Theorem 3 is the non-additive version of the Fatou lemma.

Lemma 9 (Fatou). *$\int_X f_n dv \leq M$ for all $n \in \mathbb{N}$ implies $\int_X f dv \leq M$ for every sequence of non-negative measurable functions $\{f_n\}_{n\in\mathbb{N}}$ converging v-a.e. to a function f iff $\hat{v}$ is continuous from below.*

The proof is presented below as a part of Theorem 5's proof.

The integral's continuity from below, as a set function over $\mathcal{F}$, is an immediate consequence of the monotonic convergence theorem.

Corollary 6. *Let f be a non-negative measurable function, $A \in \mathcal{F}$ and $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ is increasing to A. Then $\lim_{n\to\infty} \int_{A_n} f dv = \int_A f dv$ iff $\hat{v}$ is continuous from below.*

7.3. **Large cores and convergence.**

Lemma 10. *If v is a capacity with a large core, then $\hat{v}$ is a capacity with a large core.*

Proof. $\hat{v}$ is clearly a capacity since $\hat{v}(X) = \int_X dv = \int_X 0 dv + 1 = 1$.

$\hat{v}(X) = 1$, and by Corollary 1 we have that $LsCore(\hat{v})$ is not empty. Now, fix $\mu \in LsCore(\hat{v})$, then $\mu \in LsCore(v)$. Since v has a large core, there exist some $P \in Core(v)$ such that $\mu \geq P \geq v$. Given an arbitrary $A \in \mathcal{F}$,

$$\hat{v}(A) = \int_X 1\!\!1_A dv \leq \int_X 1\!\!1_A dP = P(A).$$

Therefore, $P \in Core(\hat{v})$. □

The following is a second version of a monotonic convergence theorem.

Theorem 4 (Monotonic convergence 2)**.** *Assume that v is a loosely extendable capacity and has a large core. Then $\lim_{n\to\infty}\int_X f_n dv = \int_X f dv$ for every increasing sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}^+_\infty(X)$ converging to $f \in \mathcal{F}^+_\infty(X)$ over a set of capacity 1 iff $\hat{v}$ is continuous from below.*

Proof. Assume that $\hat{v}$ is continuous from below. By Lemma 10 the capacity $\hat{v}$ satisfies the conditions of Lemma 7 and so $\{f_n\}_{n\in\mathbb{N}}$ converges $\hat{v}$-a.e. to f. Applying Theorem 3 and Lemma 1 we obtain,

$$\lim_{n\to\infty}\int_X f_n dv = \lim_{n\to\infty}\int_X f_n d\hat{v} = \int_X f d\hat{v} = \int_X f dv.$$

The converse implication is immediate. □

Theorem 5 (Dominated convergence)**.** *Assume that v is a loosely extendable capacity having a large core. $\lim_{n\to\infty}\int_X f_n dv = \int_X f dv$ for every $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}^+_\infty(X)$ converging to f on a set of capacity 1 such that $f_n \le g$ for every $n \in \mathbb{N}$ where $g \in \mathcal{F}^+_\infty(X)$ iff $\hat{v}$ is continuous from below.*

Proof. Assume that $\hat{v}$ is continuous from below. Again, by Lemma 10 $\hat{v}$ satisfies the conditions of Lemma 7, thus $\{f_n\}_{n\in\mathbb{N}}$ converges $\hat{v}$-a.e. to f. $g \in \mathcal{F}^+_\infty(X)$, that is, g is integrable. $f \le g$ $\hat{v}$-a.e., thus f is integrable.

For every $n \in \mathbb{N}$ and $x \in X$, let

$$g_n(x) := \inf_{k\ge n} f_k(x).$$

For every $n \in \mathbb{N}$, g_n is measurable since $\{x;\ g_n(x) < c\} = \bigcup_{k\ge n}\{x;\ f_k(x) < c\}$. Moreover, $0 \le g_n \le f_n$, that is,

$$\int_X g_n d\hat{v} \le \int_X f_n d\hat{v}$$

for every $n \in \mathbb{N}$. Clearly, $\{g_n\}_{n\in\mathbb{N}}$ increases $\hat{v}$ a.e. to f. Applying Theorem 3, we obtain that

$$\int_X f d\hat{v} = \lim_{n\to\infty}\int_X g_n d\hat{v},$$

and from Lemma 1 we now have

$$\lim_{n\to\infty}\int_X f_n dv = \lim_{n\to\infty}\int_X f_n d\hat{v} \ge \lim_{n\to\infty}\int_X g_n d\hat{v} = \int_X f d\hat{v} = \int_X f dv.$$

Conversely, fix $\varepsilon > 0$. There exist an additive capacity $P \in Core(\hat{v})$ such that $\int_X f d\hat{v} \ge \int_X f dP - \varepsilon$. For every $n \in \mathbb{N}$, define $A_n := \{x \in X;\ |f(x) - f_n(x)| < \varepsilon\}$.

There exist $N \in \mathbb{N}$ such that $\int_{A_n^c} gdP < \varepsilon$ for all $n \geq N$. Now, for every $n \geq N$

$$\int_X fdv = \int_X fd\hat{v} \geq \int_X fdP - \varepsilon \geq \int_{A_n} fdP - \varepsilon \geq \int_{A_n} f_n dP - 2\varepsilon \geq$$

$$\geq \int_X f_n dP - 3\varepsilon \geq \int_X f_n d\hat{v} - 3\varepsilon = \int_X f_n dv - 3\varepsilon.$$

Since ε is arbitrarily small, we get that $\int_X fdv \geq \lim_{n\to\infty} \int_X f_n dv$, and the theorem is proved.

The other implications is obvious. □

8. Final comments

8.1. **Infinite additive measures.** Consider the case that a monotonic set function ν need not be finite, and define the loose core of ν to be all additive (not necessarily finite) measures greater than or equal to ν. In this case, given $f \in \mathcal{F}_\infty^+(X)$ such that $\inf\{f(x);\ x \in PD(f)\} > 0$, any additive measure in $LsCore(v_{PD(f)})$ could be trivially extended to an additive measure in $LsCore(v)$, resulting in

$$\int_X fdv = \inf_{\mu \in LsCore(v)} \int_X fd\mu.$$

8.2. **Dominated convergence without a large core.** We have proven in section 7.3 that every loosely extendable capacity with a large core satisfies the dominated convergence theorem. The following example shows that having a large core is not a necessary condition for a capacity to satisfy the dominated convergence theorem.

Example 7. *Recall the capacity v from Example 2. Although $LsCore(v)$ is empty, we will show that this capacity over $\mathbb{N}$ possesses the dominated convergence property.*

Notice, that for any function f, we have $\int_{\mathbb{N}} fdv = \sum_{i\in\mathbb{N}} v(\{i\})f(i)$. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ converges v-a.e. to f, and there exists some integrable g with $f_n \leq g$ for all integers n. In particular, $\{f_n\}_{n\in\mathbb{N}}$ converges to f pointwise.

Let $\varepsilon > 0$ and fix an integer K such that $\sum_{i>K} v(\{i\})g(i) < \frac{\varepsilon}{4}$. Let M be large enough so that for each $n > M$,

$$|f_n(i) - f(i)| < \frac{\varepsilon}{2\sum_{i\leq K} v(\{i\})} \text{ for all } i \leq K.$$

To see that $|\int_{\mathbb{N}} f_n dv - \int_{\mathbb{N}} f dv| < \varepsilon$ *for every* $n > M$*, notice that*

$$\left|\int_{\mathbb{N}} f_n dv - \int_{\mathbb{N}} f dv\right| = \left|\sum_{i\in\mathbb{N}} v(\{i\}) f_n(i) - \sum_{i\in\mathbb{N}} v(\{i\}) f(i)\right| \le \sum_{i\in\mathbb{N}} v(\{i\}) \left|f_n(i) - f(i)\right| =$$

$$\sum_{i\le K} v(\{i\}) \left|f_n(i) - f(i)\right| + \sum_{i>K} v(\{i\}) \left|f_n(i) - f(i)\right| \le \frac{\varepsilon}{2} + 2\frac{\varepsilon}{4} = \varepsilon.$$

It would be nice to find a necessary and sufficient condition for capacities which have such a convergence property.

8.3. **Open problems.** We leave open the question whether the converse of Theorem 2 is true. In other words, whether v is loosely extendable implies $\int_E d\varphi_{(E,\hat{v})} < \infty$ for every $E \in \mathcal{F}$.

Another question we leave open is whether, when v is loosely extendable, $\int_X f dv = \inf_{\mu\in LsCore(v)} \int_X f d\mu$ for every $f \in \mathcal{F}^+_\infty(X)$.

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