Econ Theory
DOI 10.1007/s00199-015-0880-0

RESEARCH ARTICLE

Joint games and compatibility

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Received: 16 July 2014 / Accepted: 30 April 2015

Abstract We introduce the concepts of joint games and compatibility. In a joint game, members of the grand coalition have the option to split and participate in different underlying games, thereby maximizing their total worths. In order to determine whether the grand coalition will remain intact, we introduce the notion of compatibility of these games. A set of games is compatible if the core of the joint game is non-empty. We find a necessary and sufficient condition for compatibility.

Keywords Joint game · Compatibility · Cooperative games · Core · Concave integral

JEL Classification C71

The authors thank the referees and the associate editor of *Economic Theory* for their valuable remarks that greatly improved the paper. We would also like to thank Szilvia Papai for providing in-depth critique of an earlier version of the paper. Lehrer acknowledges the support of the Israel Science Foundation, Grant No. 538/11 and BSF Grant 2010253.

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Published online: 09 May 2015

1 Introduction

In this paper, we introduce and study the notions of joint games and compatibility in the context of transferable utility cooperative games. A joint game is defined on the basis of a few underlying games. Each player in a joint game can take part in at least one of the underlying games. A coalition can then take advantage of this option and split itself into sub-coalitions, each participating in a different underlying game. The goal of the coalition is to find its optimal split, the one that maximizes the total worth of the sub-coalitions in their respective underlying game. Thus, the worth of a coalition in the joint game is determined by its optimal split and is typically greater than in each of the basis games. The aim of this paper was to formally introduce the notion of a joint game and to find when it has a non-empty core, which leads us to the notion of compatibility.

Consider, for instance, firms that are capable of producing the same product. Suppose that these firms operate in different countries, but their productivity and endowment change across countries. In this case, a consortium of firms can maximize its total output by properly splitting its members among the countries. The question arises as to whether this consortium of all the firms could be stable. To study this problem, we associate each country with a transferable utility cooperative game. In this game, the worth of any coalition is defined as the maximal production of the coalition if confined to produce in a particular country. This game is a market game and therefore has a non-empty core. However, on a larger scale, where firms can operate in different countries, the corresponding game is a joint game, which might have an empty core. In such a game, the grand coalition, the one that contains all the firms, typically benefits from the opportunity to operate in different countries. However, smaller coalitions may benefit as well, and sometimes on a larger scale than the grand coalition. This could be a source of instability.

Even though the joint game is introduced in a very natural way, its properties cannot be characterized very easily. It is not clear when the core of a joint game will be non-empty, even for very appealing cases when all the games involved are totally balanced or even when they are all convex. The main results of this paper concern the core of the joint game. Theorems 1 and 2 characterize the joint games which have a non-empty core. It turns out that the possibility of having sub-coalitions participating in different underlying games might improve the stability of the grand coalition. In such cases, the underlying games may have empty cores, while the joint game has a non-empty one. This happens when the opportunity to take part in different games increases the worth of the grand coalition to a level that could satisfy all coalitions in a way that would leave no group of players with an incentive to split from the grand coalition. However, this is not the most general case. It might also be the case that the underlying games have a non-empty core, while the core of the joint game is empty; this case is illustrated in the motivating example below that deals with market games (see Sect. 3). It turns out, however, that in this example, if the choices of the operating firms are reduced to only two countries, the core remains non-empty (see Theorem 3).

The study of the core of joint games is surprisingly related to a theory that has been developed in another context, that of decision making. Lehrer (2009) and later on Lehrer and Teper (2008) and Even and Lehrer (2014) introduced and studied the concave integral, which is used to compare different possible actions in a world with uncertainty. Whether or not a joint game has a non-empty core is determined by the concave integrals of the underlying games, when the latter are interpreted as non-additive probabilities. The Shapley–Bondareva theorem (Shapley 1967) characterizes when the core of a game is non-empty; in terms of the concave integral, it states that the core is non-empty precisely when the integral of the indicator of the grand coalition[1] does not exceed the worth of the grand coalition. The main result of this paper has the same spirit. It roughly states that the core of a joint game is non-empty if and only if the sum of the integrals (each is taken with respect to a different basis game) of functions that sum up to the indicator of the grand coalition does not exceed its worth.

Despite the popularity of the core and the well-established existence of multi-issue interactions between economic entities, only a few authors have studied the core of combined games or games with the possibility of membership in more than one coalitions. Bloch and de Clippel (2010) look at the core of combined games which are obtained by summing two coalitional games. They conclude that the set of all balanced transferable payoff games can be divided into equivalence classes where the core of the combination of two games is equal to the sum of the cores of the components if and only if the two games belong to the same class, for example, if both games are convex. Nax (2014) studies transferable utility cooperative games with multiple membership and considers economic environments featuring externalities and membership in multiple coalitions, and he proposes definitions of the core for this class of games. His definition of the core depends on what assumptions are made about how society reacts to coalitional deviations. He defines the core for a general conjecture and concludes that the core of a multiple membership game is the set of contracts that are feasible and un-blockable, given the conjecture.

To the best of our knowledge, there are not many papers in the literature of economic theory that discuss the idea of multi-issue interaction. However, in the literature of political economics, there are a few papers discussing this issue among which Conconi and Perroni (2002), Abrego et al. (2001), Horstmann et al. (2001) and Inderst (2000) can be named.

The paper is organized as follows: In Sect. 2, we introduce the joint game defined over K different cooperative games and state the results referring to the non-emptiness of the core; in Sect. 3, we introduce a motivating example of a joint game and the concept of compatibility using an example from market games. Section 4 deals with the compatibility of games including necessary and sufficient conditions for compatibility to hold. In Sect. 5, we have final comments, and proofs are given in the "Appendix" at the end.

2 Joint games

2.1 Games with the same grand coalition

Let v_1 and v_2 be the worths of two transferable utility cooperative games on a finite set of players N ($|N| = n$). For all $S \subseteq N$, $v_1(S)$ is the worth of S from participation

[1] The indicator of the grand coalition is the function defined over it and is equal 1.

in v_1 and $v_2(S)$ is the worth from participation in v_2. For any coalition, $S \subseteq N$, we suppose that the members of S can decide to participate in either v_1 or v_2, with the aim of participation being to maximize the worth of the coalition. In mathematical terms, let S_1 and S_2 be two exhaustive and mutually disjoint subsets of $S \subseteq N$ ($S_1 \cap S_2 = \emptyset$, $S_1 \cup S_2 = S$). The whole coalition S is looking for a partition $\{S_1^*, S_2^*\}$ such that

$$v_1\left(S_1^*\right) + v_2\left(S_2^*\right) = \max_{\substack{S_1 \cap S_2 = \emptyset \\ S_1 \cup S_2 = S}} \left[v_1(S_1) + v_2(S_2)\right]. \tag{2.1}$$

Here the coalition has the opportunity to collect a better value by assigning its members to participate in an appropriate game. We can generalize the same argument for K different games $v_1, \ldots, v_K$. Hence, we define a new worth as follows:

Definition 1 A joint cooperative game for $v_1, \ldots, v_K$ defined over N, is a cooperative game whose worth is defined as

$$v_1 \bullet \cdots \bullet v_K(S) = \max_{\substack{S_1, \ldots, S_K \\ \forall i \neq j,\ S_i \cap S_j = \emptyset, \\ S_1 \cup \cdots \cup S_K = S}} v_1(S_1) + \cdots + v_K(S_K). \tag{2.2}$$

Here we recall the definition of the core for coalitional games:

Definition 2 The *core* of a coalitional game with transferable payoffs $\langle N, v\rangle$ is the set of all payoff profiles, $(x_i)_{i \in N}$, such that $x(N) = v(N)$ and for any coalition S, $x(S) \geq v(S)$, where $x(S) = \sum_{i \in S} x_i$.

Example 1 Let $N = \{1, 2\}$ and define the following games: $v_1(1) = .9$, $v_1(2) = .2$, $v_1(N) = 1$, and $v_2(1) = .2$, $v_2(2) = .9$, $v_2(N) = 1$. The joint game, $v_1 \bullet v_2$, is additive: $v_1 \bullet v_2(1) = .9$, $v_1 \bullet v_2(2) = .9$, $v_1 \bullet v_2(N) = 1.8$. While $C(v_1) = C(v_2) = \emptyset$, the joint game has a non-empty core, $C(v_1 \bullet v_2) = \{(.9, .9)\}$. This simple example illustrates the typical case that in the joint game, the worths of all coalitions are larger than in each of the underlying games. In this case, the improvement in the worth of the grand coalition is high enough to satisfy all coalitions, making the core of the joint game non-empty.

Remark 1 It is worth mentioning that if we denote the set of all coalitional games on N by $\mathcal{G}(N)$, then $(\mathcal{G}(N), \bullet)$ has an Abelian semigroup structure. That is, $\bullet$ is a commutative, associative operator from $\mathcal{G}(N) \times \mathcal{G}(N)$ to $\mathcal{G}(N)$. The example shows that we cannot easily replace $\mathcal{G}(N)$ with a smaller set of games (e.g., the set of games with empty cores) and still retain the same algebraic structure.

2.2 Extension to games with different grand coalitions

One might argue that in real-world situations, some players can play in one place but not in another. Let $(v_1, N_1), \ldots, (v_K, N_K)$ be K cooperative games, where the grand coalition of v_i is N_i, $i = 1, \ldots, K$. Similar to the definition above, we introduce the joint game as follows.

Definition 3 A joint cooperative game for $(v_1, N_1), \ldots, (v_K, N_K)$ is a cooperative game whose grand coalition is $N = N_1 \cup \cdots \cup N_K$ and is defined as

$$v_1 \bullet \cdots \bullet v_K(S) = \max_{\substack{S_i \subseteq N_i,\ i=1,\dots,K \\ \forall i \neq j,\ S_i \cap S_j = \emptyset, \\ S_1 \cup \cdots \cup S_K = S}} v_1(S_1) + \cdots + v_K(S_K), \tag{2.3}$$

for every $S \subseteq N$.

3 A motivating example

In this section, we motivate the discussions in the paper, in particular the notions of joint games and compatibility, with an example from market games. Our example follows the same notation for market games as in Osborne and Rubinstein (1994). Consider a firm with n units, denoted by $1, \dots, n$, which can operate in K different countries $c_1, \dots, c_K$. In country c_i, unit j produces according to production function f_{ij} that uses m production factors and an endowment of $e_{ij} \in \mathbb{R}^m_+$. For a moment, let us focus on one country, say c_i. For any $S \subseteq \{1, \dots, n\}$, denote the aggregate endowment $\sum_{j \in S} e_{ij}$ by e^i_S. The optimal production can be regarded as a coalitional game: The worth of S in c_i is defined as,

$$v_i(S) = \max_{(x^j)_{j \in S} \in \mathcal{F}(e^i_S)} \sum_{j \in S} f_{ij}(x^j), \tag{3.1}$$

where $\mathcal{F}(e^i_S)$ is the feasibility set defined as

$$\mathcal{F}(e^i_S) = \left\{ \left(x^j\right)_{j \in S} \middle| x^j \geq 0\,,\ j \in S\,,\ \sum_{j \in S} x^j = e^i_S \right\}.$$

Now to be more specific, we consider three firms operating in three different countries. We assume that the production function of firm j in country c_i, i.e., f_{ij}, is a Cobb–Douglas function with two inputs, x and y: $f_{ij}(x, y) = A_{ij}\sqrt{xy}$, where A_{ij} is the total factor productivity (hence TFP). We assume that the aggregate endowment in each country is one unit of each input; for instance, in country c_1, the endowments of firms 1, 2 and 3 are (0, 0), (2/3, 2/3) and (1/3, 1/3), respectively. To make things even simpler, we further assume that it is not feasible for firm j to produce in country c_j, and therefore, $A_{jj} = 0$. The following table summarizes the endowments of each firm in each country and its TFP.[2]

Country		1			2			3	
Firm	1	2	3	1	2	3	1	2	3
TFP	0	1/3	1	1	0	1/3	1/3	1	0
Endowment	0	2/3	1/3	1/3	0	2/3	2/3	1/3	0

[2] In the third row of the table, **e** denotes the two-dimensional vector (e, e).

The game v_1 is given as follows,

$$v_1(1) = \max_{(x_1^1, x_2^1)=(0,0)} 0 = 0,$$

$$v_1(2) = \max_{(x_1^2, x_2^2)=(2/3,2/3)} (1/3)\sqrt{x_1^2 x_2^2} = 2/9,$$

$$v_1(3) = \max_{(x_1^3, x_2^3)=(1/3,1/3)} \sqrt{x_1^3 x_2^3} = 1/3,$$

$$v_1(2,3) = \max_{(x_1^2+x_1^3, x_2^2+x_2^3)=(1,1)} (1/3)\sqrt{x_1^2 x_2^2} + \sqrt{x_1^3 x_2^3} = 1,$$

$$v_1(1,3) = \max_{(x_1^1+x_1^3, x_2^1+x_2^3)=(1/3,1/3)} 0 + \sqrt{x_1^3 x_2^3} = 1/3,$$

$$v_1(1,2) = \max_{(x_1^1+x_1^2, x_2^1+x_2^2)=(2/3,2/3)} 0 + (1/3)\sqrt{x_1^2 x_2^2} = 2/9,$$

$$v_1(1,2,3) = \max_{(x_1^1+x_1^2+x_1^3, x_2^1+x_2^2+x_2^3)=(1,1)} 0 + (1/3)\sqrt{x_1^2 x_2^2} + \sqrt{x_1^3 x_2^3} = 1.$$

The following table summarizes the games v_1, v_2 and v_3 corresponding to the three countries,

Coalition	(1)	(2)	(3)	(2, 3)	(1, 3)	(1, 2)	(1, 2, 3)
v_1	0	2/9	1/3	1	1/3	2/9	1
v_2	1/3	0	2/9	2/9	1	1/3	1
v_3	2/9	1/3	0	1/3	2/9	1	1

We now turn to the joint games. We first consider the joint game $v_1 \bullet v_2$ and show how, for instance, $v_1 \bullet v_2(2,3)$ is computed.

$$v_1 \bullet v_2(2,3) = \max\{v_1(2) + v_2(3), v_1(3) + v_2(2), v_1(2,3), v_2(2,3)\} = 1.$$

The values of $v_1 \bullet v_2$ for different coalitions are given in the following table.

Coalition	(1)	(2)	(3)	(2, 3)	(1, 3)	(1, 2)	(1, 2, 3)
$v_1 \bullet v_2$	1/3	2/9	1/3	1	1	5/9	4/3

The cores of the games $v_1 \bullet v_2$, $v_2 \bullet v_3$ and $v_1 \bullet v_3$ are non-empty. For instance, $(3/9, 2/9, 7/9) \in C(v_1 \bullet v_2)$. However, the core of $v = v_1 \bullet v_2 \bullet v_3$ is empty. The table of v is the following:

Coalition	(1)	(2)	(3)	(2, 3)	(1, 3)	(1, 2)	(1, 2, 3)
v	1/3	1/3	1/3	1	1	1	4/3

Now let us take a closer look at this example. Observe that when the firms get access to a new market, the worth of each coalition cannot decrease. Therefore, by having access to a new market, firms' core allocations should be at least what they would receive without it.

A simple argument shows that $C(v_1 \bullet v_2) = \{(1/3, x, 1-x)|\ 2/9 \leq x \leq 1/3\}$. In particular, in any core allocation, firms 1 and 2 should receive at most $1/3+1/3 = 2/3$. Now assume that access to c_3 becomes available. Since $v_1 \bullet v_2 \bullet v_3(1, 2, 3) = v_1 \bullet v_2(1, 2, 3)$, the core $C(v_1 \bullet v_2 \bullet v_3)$ should be a subset of $C(v_1 \bullet v_2)$. Thus, in any core allocation of the former, firms 1 and 2 cannot receive more than $2/3$. However, with access to c_3, the worth of coalition $(1, 2)$ is 1, which implies that the core of $v_1 \bullet v_2 \bullet v_3$ is empty.

On the other hand, access to a new country can again change the situation. Suppose further that the firms have access to country 4 whose figures are described in the following table.

Country		4	
Firm	1	2	3
TFP	0	1/6	2
Endowment	0	5/6	1/6

The game v_4 and the joint game are given in the following table.

Coalition	(1)	(2)	(3)	(2, 3)	(1, 3)	(1, 2)	(1, 2, 3)
v_4	0	5/36	1/3	2	1/3	5/36	2
$v_1 \bullet v_2 \bullet v_3 \bullet v_4$	1/3	1/3	1/3	2	1	1	7/3

Observe that $(1/3, 1, 1) \in C(v_1 \bullet v_2 \bullet v_3 \bullet v_4)$. Thus, the core of the joint game $v_1 \bullet v_2 \bullet v_3 \bullet v_4$ is non-empty.

Remark 2 As one can see, even though the cores of $v_1 \bullet v_2$ and v_3 are non-empty, their joint game $v_1 \bullet v_2 \bullet v_3$ has an empty core, which is opposite to example 1.

4 Compatibility of games

4.1 Compatibility

In this section, we introduce the notion of compatibility. We say that games are compatible with each other if the core of the joint game is non-empty. This means that regardless of the stability of each individual game, the joint game is stable.

Definition 4 The games $(v_1, N_1), \dots, (v_K, N_K)$ are *compatible* if

$$C(v_1 \bullet \cdots \bullet v_K) \neq \emptyset.$$

In the following, we state a necessary and sufficient condition for non-emptiness of the core of a joint cooperative game, $v_1 \bullet \cdots \bullet v_K$. For this purpose, we introduce a concave integral for games (see Lehrer 2009). For two games v_1, v_2, by $v_1 \geq v_2$, we mean $v_1(S) \geq v_2(S)$, for all sets $S \subseteq N$. A game P is called *additive* if $P(S \cup T) = P(S) + P(T)$, for two disjoint sets $S, T \subseteq N$.

Definition 5 For a game v and a vector $X \geq 0$ in $\mathbb{R}^n$, the concave integral is defined as follows:

$$\int^{\text{cav}} X \mathrm{d}v = \min_{P \geq v} \int X \mathrm{d}P, \tag{4.1}$$

where the minimum is taken over all additive games, P, such that $P(S) \geq v(S), \forall S \subseteq N$.

Remark 3 (i) The concave integral is defined as the minimum over a set of additive games. This minimum (as opposed to infimum) is justified because the set of the additive games P such that $P \geq v$ can be restricted, without loss of generality, to a bounded set, say $\mathcal{P}_v$. In other words, the concave integral can be defined as an infimum over a compact set of additive games, and therefore, the infimum is attained.
(ii) The minimum $\min_{P \in \mathcal{P}_v} \int X \mathrm{d}P$ is obtained in an extreme point of $\mathcal{P}_v$. Moreover, the set $\mathcal{P}_v$ is defined by a finite number of linear inequalities, which makes $\mathcal{P}_v$ a polyhedron. As such, it has a finite number of extreme points. We conclude that the concave integral is the minimum of finitely many linear functions (the extreme points of $\mathcal{P}_v$) and is therefore continuous.

Remark 4 If v is additive, then $\int^{\text{cav}} X \mathrm{d}v = \int X \mathrm{d}v$, which is why we can equally use $\int^{\text{cav}} X \mathrm{d}v$ or $\int X \mathrm{d}v$ when working with additive games.

There is a dual approach to calculating $\int^{\text{cav}} X \mathrm{d}v$; the variable X can be decomposed as a linear combination of indicators. Clearly, X can be decomposed in many ways and among all the decompositions $X = \sum_{S \subseteq N} \alpha_S \cdot \mathbf{1}_S$, where $\alpha_S \geq 0$ for every $S \subseteq N$, the concave integral considers the one that maximizes $\sum_{S \subseteq N} \alpha_S v(S)$. Formally,[3]

$$\int^{\text{cav}} X \mathrm{d}v = \max \left\{ \sum_{S \subseteq N} \alpha_S v(S); \; \sum_{S \subseteq N} \alpha_S \cdot \mathbf{1}_S = X, \; \alpha_S \geq 0 \text{ for every } S \subseteq N \right\}. \tag{4.2}$$

Remark 5 Since our analysis is in finite spaces, one may wonder why we use the notation $\int$ in Definition 5 and Eq. (4.1). We have two reasons. First, we want to use standard notations used in the literature, such as the concave integral and the Choquet

[3] A similar formula has been used in Shapley and Shubik (1969) to show that every totally balanced game is a market game.

integral (Choquet 1955) (see below). Second, the integrals used here are indeed sums, but their domains change from one X to the other. That is, different X's might have different decompositions that attain the maximum in Eq. (4.2). These different domains are concisely captured by the integral notation.

The classical definition of the *totally balanced cover* of a game v is

$$B_v(T) := \max \left\{ \sum_{S\subseteq T} \alpha_S v(S);\ \ \sum_{S\subseteq T} \alpha_S \cdot \mathbf{1}_S = 1_T,\ \alpha_S \geq 0 \text{ for every } S \subseteq T \right\}$$
$$\text{for every } T \subseteq N.$$

Note that due to Eq. (4.2), the totally balanced cover of v can be written also as,

$$B_v(T) = \int^{\text{cav}} \mathbf{1}_T \mathrm{d}v \quad \text{for every} \quad T \subseteq N. \tag{4.3}$$

One can now use the notion of the concave integral to re-state the Shapley–Bondareva theorem (see Shapley 1967): For any game v,

$$C(v) \neq \emptyset \quad \text{if and only if} \quad B_v(N) \leq v(N). \tag{4.4}$$

Example 2 Consider v_1 in the previous example. There is an additive game which is the least of all additive games that satisfies $P(S) \geq v(S), \forall S \subseteq N$: $P(1) = 0.9$, $P(2) = 0.2$, $P(N) = 1.1$. Let $X = (3, 2)$. Then, $\int^{\text{cav}} X \mathrm{d}v_1 = 3 \cdot 0.9 + 2 \cdot 0.2 = 3.1$. As for the dual approach, the decomposition that maximizes the right-hand side of Eq. (4.2) with respect to v_1 is $X = 3 \cdot \mathbf{1}_{\{1\}} + 2 \cdot \mathbf{1}_{\{2\}}$. Indeed, $3v_1(1) + 2v_1(1) = 3.1 = \int^{\text{cav}} X \mathrm{d}v_1$. Note that X could be decomposed differently. For instance, $X = 2 \cdot \mathbf{1}_N + \mathbf{1}_{\{2\}}$. But then, $2v_1(N) + v_1(1) = 2 \cdot 1 + 0.9 = 2.9$ which is strictly smaller than $\int^{\text{cav}} X \mathrm{d}v_1$.

The following two theorems are the main results of this paper.

Theorem 1 *The games $v_1, \ldots, v_K$ defined over the same grand coalition N are compatible if and only if the following inequality holds*

$$\max_{\substack{f_1+\cdots+f_K=1\\ f_i:N\to\mathbb{R}_+}} \sum_{1\leq i\leq K} \int^{\text{cav}} f_i \mathrm{d}v_i \leq v_1 \bullet \cdots \bullet v_K(N). \tag{4.5}$$

Proof See Appendix. □

In order to characterize when games defined over different grand coalitions are compatible, we need some additional notation. Let $(v_1, N_1), \ldots, (v_K, N_K)$ be K cooperative games. Define, $N = N_1 \cup \cdots \cup N_K$ and

$$\mathcal{F}_i = \{f_i : N \to \mathbb{R}_+;\ f_i(\ell) = 0 \text{ whenever } \ell \notin N_i\}.$$

For $f_i \in \mathcal{F}_i$ denote by $f_{i|N_i} : N_i \to \mathbb{R}_+$ the restriction of f_i to N_i. That is, the function defined on N_i and coincides there with f_i. The next theorem generalizes Theorem 1 to games that might have different grand coalitions. Its proof hinges on that of Theorem 1.

Theorem 2 *The games* $(v_1, N_1), \ldots, (v_K, N_K)$ *are compatible if and only if the following inequality holds,*

$$\max_{\substack{f_1+\cdots+f_K=1 \\ f_i\in\mathcal{F}_i,\ i=1,\ldots,K}} \sum_{1\le i\le K} \int^{\mathrm{cav}} f_{i|N_i} \mathrm{d}v_i \le v_1 \bullet \cdots \bullet v_K(N). \tag{4.6}$$

Proof See Appendix. □

Definition 6 A cooperative game, v, is a *convex game* if for two subsets S_1, S_2,

$$v(S_1 \cap S_2) + v(S_1 \cup S_2) \ge v(S_1) + v(S_2). \tag{4.7}$$

Remark 6 The game v has a *large core* (Sharkey 1982) if for every $S \subseteq N$ and for every additive game Q that satisfies $v \le Q$, there is P in the core of v such that $P \le Q$. It is shown in Azrieli and Lehrer (2007) that v has a large core if and only if

$$\int^{\mathrm{cav}} X \mathrm{d}v = \min_{P\in C(v)} \int X \mathrm{d}P. \tag{4.8}$$

In other words, when v has a large core, the minimum in (4.1) can be taken over the core of v which is smaller than the set of all additive games that are greater than or equal to v. Furthermore, as noted in Azrieli and Lehrer (2007), any convex game has a large core.

A non-trivial application of Theorem 2 is the following theorem:

Theorem 3 *If* $(v_1, N_1), (v_2, N_2)$ *are convex games, then they are compatible.*

Proof See Appendix. □

The following example shows that more than two convex games might be non-compatible.

Example 3 Let $N = \{1, 2, 3\}$ and define three monotonic simple games. For each $i = 1, 2, 3$, the game v_i has one minimal winning coalition, $N\backslash\{i\}$. Since there is only one winning coalition, v_i is convex. Consider now $v = v_1 \bullet v_2 \bullet v_3$. It is also a simple game, and $v(S) = 1$ if and only if $|S| \ge 2$. It implies that the core of v is empty, rendering v_1, v_2 and v_3 non-compatible.

5 Final comments

5.1 Another sufficient condition for the non-emptiness of the joint game's core

In this section, we introduce another sufficient condition for the non-emptiness of the joint game's core.

Example 4 Let $N = \{1, 2\}$, v_1 be a game where $v_1(1) = 1$, $v_1(2) = .8$ and $v_1(N) = 2$. Also let $v_2(1) = .9$, $v_2(2) = .9$ and $v_2(N) = 2$. The cores of the two games have at least one member in common, and $v_1 \bullet v_2$ has a non-empty core. If we replace v_2 by v_2', where $v_2'(1) = .8$, $v_2'(2) = 1.1$ and leave $v_2'(N) = 2$, then the cores of v_1 and v_2' are disjoint, and the core of $v_1 \bullet v_2'$ is empty.

The following proposition states that, in this example, the linkage between the non-emptiness of the core of the joint game and the fact that the cores of the underlying games have at least one member in common is not coincidental.

Proposition 1 *Let $v_1, \ldots, v_K$ be a set of games on the same grand coalition N such that $v_1(N) = \cdots = v_K(N)$. Then, $v_1 \bullet \cdots \bullet v_K(N) = v_1(N)$ if and only if $C(v_1 \bullet \cdots \bullet v_K) = C(v_1) \cap \cdots \cap C(v_K)$.*

Proof See Appendix. □

This proposition implies in particular that when $v_1(N) = \cdots = v_K(N) = v_1 \bullet \cdots \bullet v_K(N)$, if there is a core allocation that is common to all games, then the core of the joint game is not empty.

An application of this theorem is for simple games. Let $v_1, \ldots, v_K$ be K simple games, each having a non-empty core. Let us denote the set of veto players (the intersection of all the winning coalitions) of game i by U_i. Then it is known that $C(v_i) = \{P \geq 0 | P(j) = 0\ ,\ \forall j \in U_i^c\}$. Therefore, $C(v_1) \cap \cdots \cap C(v_K) = \{P \geq 0 | P(j) = 0,\ j \in U_1^c \cup \cdots \cup U_K^c\}$. Therefore, by using Proposition 1, one can see that $v_1 \bullet \cdots \bullet v_K$ is a simple game with the veto players being $U_1 \cap \cdots \cap U_K$. On the other hand, there is no player who is a veto player in all games, if and only if the core of the joint game is empty.

5.2 Joint games and the least super-additive majorant

From a technical point of view, the definition of a joint game can be considered as a generalization of the notion 'least super-additive majorant' introduced in Shapley and Shubik (1969). If v is a game, then a least super-additive majorant is introduced as

$$\tilde{v}(S) = \max \sum_{\substack{S_i \cap S_j = \emptyset \\ S_1 \cup \cdots \cup S_n = S}} v(S_i). \tag{5.1}$$

The game $\tilde{v}$ is the smallest super-additive game that dominates v. This concept was first introduced in Shapley and Shubik (1969) and was used to study the core of super-additive games. This concept was not later studied because it was not found to be helpful. Here we revisit this concept and put the definition in a correct direction within the context of joint games.

Note that $\tilde{v} = \underbrace{v \bullet \cdots \bullet v}_{n \text{ times}}$. Now assume that in Eq. (5.1), one could arbitrarily choose games among the set $\{(v_1, N_1), \ldots, (v_K, N_K)\}$. The result would be,

$$\max \sum_{\substack{S_i \subseteq N_i \\ \forall i \neq j, S_i \cap S_j = \emptyset \\ S_1 \cup \cdots \cup S_n = S \\ v^i \in \{v_1, \ldots, v_K\}}} v^i(S_i). \tag{5.2}$$

This quantity is equal to $\tilde{v}_1 \bullet \cdots \bullet \tilde{v}_K(S)$. Thus, joint games are generalizations of least super-additive majorant games.

Given that this paper assumes that a coalition, S, can be split into different partitions, one may wonder how we explain the choice of (2.3) over the following alternative:

$$v_1 \bullet \cdots \bullet v_K(S) = \max_{\substack{S_i \subseteq N_i \\ \forall i \neq j,\ S_i \cap S_j = \emptyset, \\ S_1 \cup \cdots \cup S_K = S}} \tilde{v}_1(S_1) + \cdots + \tilde{v}_K(S_K). \tag{5.3}$$

Indeed, the definition in (5.3) can be accommodated by (2.3): It is the joint game for a set of the corresponding super-majorant games. We note that (5.3) cannot accommodate (2.3), meaning that (5.3) cannot be as general as (2.3). The definition we adopt [that given in (2.3)] would make better sense than that given in (5.3) in cases where a coalition S_i that decides to be in the environment of (v_i, N_i) cannot be further split: Its worth is $v_i(S_i)$ and not $\tilde{v}_i(S_i)$.

One can regard $\sim$ as an operator on $\mathcal{G}(N)$ (family of all games on N). As it was discussed earlier, $(\mathcal{G}(N), \bullet)$ has a semigroup structure. It is interesting that the operator $\sim$ is distributive on $(\mathcal{G}(N), \bullet)$ as it is shown in the following proposition

Proposition 2 *The operator* $\sim: \mathcal{G} \to \mathcal{G}$ *is distributive i.e., for any set Let* $v_1, \ldots, v_K$ *of games on the same grand coalition* N*, we have*

$$v_1 \widetilde{\bullet \cdots \bullet} v_K = \tilde{v}_1 \bullet \cdots \bullet \tilde{v}_K.$$

Proof See Appendix. □

A natural extension, which is related to our discussion in this section, is to extend the definition of compatibility by using c-cores. Let v be game. We introduce the game v^c as follows,

$$v^c(S) = \begin{cases} v(S) & S \neq N \\ \tilde{v}(N) & S = N \end{cases}. \tag{5.4}$$

The core of v^c is known as the *c-core* and is denoted by $C^c(v)$. This concept was first introduced in Guesnerie and Oddou (1979) and further studied in Sun et al. (2008).

Now we can introduce the notion of c-compatibility for K games, $v_1, \ldots, v_K$, on the same grand coalition N as:

$$C^c(v_1 \bullet \cdots \bullet v_K) \neq \emptyset.$$

The following result links c-compatibility to the compatibility of the least super-additive majorant,

Proposition 3 *Let $v_1, \ldots, v_K$ be a set of games with the same grand coalition N. Then, $v_1, \ldots, v_K$ are c-compatible if and only if $\tilde{v}_1, \ldots, \tilde{v}_K$ are compatible.*

Proof See Appendix. □

5.3 A concluding remark

In this paper, we have introduced the concept of a joint game and have developed testable conditions for determining whether or not its core is empty. In a joint game, it is typically better for the grand coalition to split into sub-coalitions and hence obtain a total value which is greater than the value of the grand coalition in the underlying games. We characterized when the core of the joint game is non-empty.

6 Appendix: Proofs

Proof of Theorem 1 We prove that the games $v_1, \ldots, v_K$ are compatible if and only if

$$\max_{\substack{f_1+\cdots+f_K=1 \\ f_i:N\to\mathbb{R}_+}} \sum_{1\le i\le K} \int^{\mathrm{cav}} f_i \mathrm{d}v_i \le v_1 \bullet \cdots \bullet v_K(N) \tag{6.1}$$

in six steps.

Step 1 Suppose that $u(X) = \inf_{P\in\mathcal{Q}} \int X \mathrm{d}P$, where $\mathcal{Q}$ is a comprehensive,[4] convex and closed set of additive games. We show that $u(X) = -\infty$ whenever one of the coordinates of X is negative. Indeed, let $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$ such that $x_{r_0} < 0$, for some $1 \le r_0 \le n$, and let $P = (p_1, \ldots, p_n) \in \mathcal{Q}$. For any $\lambda > 0$, define an additive game $P^\lambda = (p_1^\lambda, \ldots, p_n^\lambda)$ as $p_r^\lambda = p_r$ if $r \neq r_0$, and $p_{r_0}^\lambda = p_{r_0} + \lambda$. Since $\mathcal{Q}$ is comprehensive, $P^\lambda \in \mathcal{Q}$. Therefore,

$$u(X) = \min_{P\in\mathcal{Q}} \int X \mathrm{d}P \le \int X \mathrm{d}P^\lambda = \int X \mathrm{d}P + \lambda x_{r_0},$$

which tends to $-\infty$ when $\lambda \to \infty$.

Step 2 Suppose that $u_i(X) = \inf_{P_i\in\mathcal{Q}_i} \int X \mathrm{d}P_i$, for $i = 1, \ldots, K$, where $\mathcal{Q}_i$ is a comprehensive set of additive games. The convolution of $u_1, \ldots, u_K$, denoted $u_1 \star \cdots \star u_K$, is defined as

[4] $\mathcal{Q}$ is a comprehensive if $P \in \mathcal{Q}$ and $P' \ge P$, then $P' \in \mathcal{Q}$.

$$u_1 \star \cdots \star u_K(X) := \sup_{\left\{(X_l)_{l=1}^K \in (\mathbb{R}^n)^K \mid X_1+\cdots+X_K=X\right\}} u_1(X_1) + \cdots + u_K(X_K).$$

We claim that

$$u_1 \star \cdots \star u_K(X) = \inf_{P \in \cap_i \mathcal{Q}_i} \int X \mathrm{d}P.$$

In order to prove this claim, we use convex analysis. Consider a set of convex functions $f_1, \ldots, f_K$ from $\mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$, whose conjugates are defined as

$$f_i^*(P) = \sup_{X \in \mathbb{R}^n} \left\{ \int X \mathrm{d}P - f_i(X) \right\}, \quad i = 1, \ldots, K.$$

The infimal convolution of these K functions is

$$h(X) = \inf_{\left\{(X_l)_{l=1}^K \in (\mathbb{R}^n)^K \mid X_1+\cdots+X_K=X\right\}} \{f(X_1) + \cdots + f(X_K)\}.$$

We state the following version of Theorem 20, d) in Rockafellar (1973) to be used later. $\square$

Theorem 4 *Suppose that* $\exists \bar{X} \in \mathbb{R}^n$ *and* $M \in \mathbb{R}$ *such that the set*

$$\mathcal{K} := \left\{ (X_1, \ldots, X_K) \in (\mathbb{R}^n)^K \,\middle|\, \begin{array}{l} X_1 + \cdots + X_K = \bar{X}, \\ f_1(X_1) + \cdots + f_K(X_K) \leq M \end{array} \right\} \tag{6.2}$$

is non-empty and bounded. Then, $h^* = f_1^* + \cdots + f_K^*$.

Let $f_i = -u_i$. We verify that the condition of Theorem 4 is satisfied. Take $\bar{X} = \overrightarrow{0}$ and $M = 0$. First, for every $i = 1, \ldots, K$ $\mathcal{Q}_i$ is comprehensive and therefore, f_i is non-positive on $\mathbb{R}^n_+$. In particular, $f_i(\bar{X}) \leq 0$, implying that $\bar{X}^K := (\underbrace{\bar{X}, \ldots, \bar{X}}_{K \text{ times}}) \in \mathcal{K}$. Second, according to Step 1, $\mathcal{K} \subseteq (\mathbb{R}^n_+)^K$. This implies that $\bar{X}^K$ is the only point in $\mathcal{K}$, meaning that $\mathcal{K}$ is non-empty and bounded.

Having verified the condition of Theorem 4, we proceed to apply it to f_i. By the definition of the conjugate function, $f_i^*(Q) = \sup_{X \in \mathbb{R}^n} \left\{ \int X dQ + u_i(X) \right\}$. Since for all $\lambda > 0$, $u_i(\lambda X) = \lambda u_i(X)$, we have

$$\begin{aligned} f_i^*(Q) &= \sup_{X \in \mathbb{R}^n} \left\{ \textstyle\int X dQ + u_i(X) \right\} \\ &= \sup_{\lambda X \in \mathbb{R}^n} \left\{ \textstyle\int \lambda X dQ + u_i(\lambda X) \right\} \\ &= \lambda \sup_{X \in \mathbb{R}^n} \left\{ \textstyle\int X dQ + u_i(X) \right\} \\ &= \lambda f_i^*(Q). \end{aligned}$$

This implies that either $f_i^*(Q) = 0$ or $f_i^*(Q) = +\infty$.

We now elaborate on $(f_1^* + \cdots + f_K^*)(P)$. If P is such that for any i, $-P \in \mathcal{Q}_i$, then by $\int X \mathrm{d}P + u_i(X) \le \int X \mathrm{d}P - \int X \mathrm{d}P = 0$, we have $f_i^*(P) = 0$. On the other hand, if there exists i such that $-P \notin \mathcal{Q}_i$, then due to the assumption that $\mathcal{Q}_i$ is convex and closed, there is a unique $R \in \mathcal{Q}_i$ such that R is the closest point (in the Euclidean norm) to $-P$ in $\mathcal{Q}_i$. Let, $X := -P - R \neq 0$. The integral $\int X \mathrm{d}(-P) - \int X \mathrm{d}R'$ is bounded away from 0 for every $R' \in \mathcal{Q}_i$. Thus, $f_i^*(-P) = +\infty$. This implies that $f_1^* + \cdots + f_K^*$ is zero iff $-P \in \cap_{i=1}^K \mathcal{Q}_i$ and $+\infty$ otherwise. That is,

$$(f_1^* + \cdots + f_K^*)(P) = \begin{cases} 0, & \text{if } -P \in \bigcap_{i=1}^K \mathcal{Q}_i \\ +\infty, & \text{otherwise} \end{cases}. \tag{6.3}$$

Now define, $g(X) = -\inf_{P \in -\bigcap_{i=1}^K \mathcal{Q}_i} \int X \mathrm{d}P$. Since $\bigcap_{i=1}^K \mathcal{Q}_i$ is a comprehensive, convex and closed set, the same argument just used implies that g^* is equal to (6.3). Using Theorem 4, we conclude that $h^* = f_1^* + \cdots + f_K^* = g^*$. Since h and g are both lower semicontinuous functions, it implies that they are equal (see Rockafellar 1973). Thus, $h(X) = -\inf_{-P \in \bigcap_{i=1}^K \mathcal{Q}_i} \int X \mathrm{d}P$, which implies,

$$\begin{aligned} u_1 \star \cdots \star u_K(X) &= \sup_{\{(X_l)_{l=1}^K \in (\mathbb{R}^n)^K | X_1 + \cdots + X_K = X\}} u_1(X_1) + \cdots + u_K(X_K) \\ &= \inf_{\{(X_l)_{l=1}^K \in (\mathbb{R}^n)^K | X_1 + \cdots + X_K = X\}} \{f(X_1) + \cdots + f(X_K)\} \\ &= h(X) = -\inf_{-P \in \bigcap_{i=1}^K \mathcal{Q}_i} \int X \mathrm{d}P = \inf_{P \in \bigcap_{i=1}^K \mathcal{Q}_i} \int X \mathrm{d}P, \end{aligned}$$

which proves our claim.

Step 3 For a set of games v_i, $i = 1, \ldots, K$, and any additive game P, it is easy to see that

$$P(S) \ge v_1 \bullet \cdots \bullet v_K(S),\ \forall S \subseteq N \textbf{ iff } P(S) \ge v_i(S), \quad \forall S \subseteq N, \quad \forall i = 1, \ldots, K.$$

Step 4 Consider the games $v_1, \ldots, v_K$ and the functions $u_i(X) = \inf_{P \in \mathcal{Q}_i} \int X \mathrm{d}P$, where $\mathcal{Q}_i$ is the set of all additive games P such that $P \ge v_i$. $\mathcal{Q}_i$ is comprehensive, convex and closed, and therefore, Step 1 applies. By the definition of the $\star$ operator and Step 1 ($u_i(X_i) = -\infty$ for every $X_i \notin \mathbb{R}_+^n$), for $X \ge 0$, we therefore have,

$$\begin{aligned} \left(\int^{\mathrm{cav}} \cdot \, \mathrm{d}v_1 \star \cdots \star \int^{\mathrm{cav}} \cdot \, \mathrm{d}v_K \right)(X) = \max_{\substack{X_1 + \cdots + X_K = X \\ \forall i, X_i \ge 0}} & \int^{\mathrm{cav}} X_1 \mathrm{d}v_1 \\ & + \cdots + \int^{\mathrm{cav}} X_K \mathrm{d}v_K. \end{aligned}$$

Note that the maximum is replacing the supremum because all concave functions, by Remark 3, are continuous and the set $\left\{(X_i)_{i=1}^K | X_1 + \cdots + X_K = X, \forall i, X_i \ge 0\right\}$ is compact.

Step 5 For every $S \subseteq N$, the following equality holds,

$$B_{v_1 \bullet \cdots \bullet v_K}(S) = \max_{\substack{f_1+\cdots+f_K=\mathbf{1}_S \\ \forall i, f_i \geq 0}} \sum_{1 \leq i \leq K} \int^{\text{cav}} f_i \mathrm{d}v_i.$$

Indeed,

$$\begin{aligned}
\int^{\text{cav}} X d(v_1 \bullet \cdots \bullet v_K) &= \min_{P \geq v_1 \bullet \cdots \bullet v_K} \int^{\text{cav}} X \mathrm{d}P \\
&= \min_{\cap\{P \geq v_i\}} \int X \mathrm{d}P \\
&= \left(\int^{\text{cav}} \cdot \, \mathrm{d}v_1 \star \cdots \star \int^{\text{cav}} \cdot \, \mathrm{d}v_K\right)(X) \\
&= \max_{\substack{X_1+\cdots+X_K=X \\ \forall i, X_i \geq 0}} \int^{\text{cav}} X_1 \mathrm{d}v_1 + \cdots + \int^{\text{cav}} X_K \mathrm{d}v_K.
\end{aligned}$$

In the first equality, we use the definition of the concave integral; in the second equality, we use Step 3; in the third, we use Step 2; and in the fourth equality, we use Step 4. Using $X = \mathbf{1}_S$, we get the result.

Step 6 Given Step 5 and (4.4) we have,

$$\begin{aligned}
C(v_1 \bullet \cdots \bullet v_K) \neq \emptyset &\Leftrightarrow B_{v_1 \bullet \cdots \bullet v_K}(N) \leq v_1 \bullet \cdots \bullet v_K(N) \\
&\Leftrightarrow \max_{\substack{f_1+\cdots+f_K=1 \\ \forall i, f_i \geq 0}} \sum_{1 \leq i \leq K} \int^{\text{cav}} f_i \mathrm{d}v_i \leq v_1 \bullet \cdots \bullet v_K(N).
\end{aligned}$$

□

Proof of Theorem 2 Let $(v_1, N_1), \ldots, (v_K, N_K)$ be K games and recall that we have defined $N = N_1 \cup \cdots \cup N_K$. We show first that if $(v_1, N_1), \ldots, (v_K, N_K)$ are compatible then,

$$\max_{\substack{f_1+\cdots+f_K=1 \\ f_i \in \mathcal{F}_i,\ i=1,\ldots,K}} \sum_{1 \leq i \leq K} \int^{\text{cav}} f_{i|N_i} \mathrm{d}v_i \leq v_1 \bullet \cdots \bullet v_K(N).$$

Suppose that $f_1 + \cdots + f_K = 1$, where $f_i \in \mathcal{F}_i$, $i = 1, \ldots, K$ and that $\int^{\text{cav}} f_{i|N_i} \mathrm{d}v_i = \sum_{j=1}^{r_i} v_i(S_i^j)$, where $S_i^j \subseteq N_i$ and $\sum_{j=1}^{r_i} \mathbf{1}_{S_i^j} = f_i$. Then, $\sum_{1 \leq i \leq K} \int^{\text{cav}} f_{i|N_i} \mathrm{d}v_i = \sum_{1 \leq i \leq K} \sum_{j=1}^{r_i} v_i(S_i^j) \leq v_1 \bullet \cdots \bullet v_K(N)$. The inequality holds because $\sum_{1 \leq i \leq K} \sum_{j=1}^{r_i} \mathbf{1}_{S_i^j} = f_1 + \cdots + f_K = \mathbf{1}_N$.

As for the inverse direction, for any positive number, M, and $i = 1, \ldots, K$, define the game (v_i^M, N) as follows.

$$v_i^M(S) = v_i^M(S \cap N_i) - |S \setminus N_i| \cdot M.$$

The worth $v_i^M(S)$ coincides with $v_i(S)$ as long as $S \subseteq N_i$. Any player out of N_i is worth $-M$, which is also her contribution to any coalition in N_i. When M is sufficiently large, in order to obtain $v_1^M \bullet \cdots \bullet v_K^M(S)$, it is optimal to split S as $S = S_1 \cup \cdots \cup S_K$, where for any i, $S_i \subseteq N_i$. Since there are finitely many coalitions in N, for sufficiently large M, $v_1^M \bullet \cdots \bullet v_K^M(S) = v_1 \bullet \cdots \bullet v_K(S)$ for every $S \subseteq N$. It implies that $(v_1, N_1), \ldots, (v_K, N_K)$ are compatible whenever $(v_1^M, N_1), \ldots, (v_K^M, N_K)$ are.

We now use Theorem 1 and apply it to $(v_1^M, N_1), \ldots, (v_K^M, N_K)$. Theorem 1 states that these games are compatible if and only if,

$$\max_{\substack{f_1+\cdots+f_K=1 \\ f_i:N\to\mathbb{R}_+}} \sum_{1\le i\le K} \int^{\mathrm{cav}} f_i \mathrm{d}v_i^M \le v_1^M \bullet \cdots \bullet v_K^M(N). \tag{6.4}$$

We claim that for M large enough,

$$\max_{\substack{f_1+\cdots+f_K=1 \\ \forall i,\, f_i:N\to\mathbb{R}_+}} \sum_{1\le i\le K} \int^{\mathrm{cav}} f_i \mathrm{d}v_i^M = \max_{\substack{f_1+\cdots+f_K=1 \\ \forall i,\, f_i\in\mathcal{F}_i}} \sum_{1\le i\le K} \int^{\mathrm{cav}} f_i \mathrm{d}v_i^M. \tag{6.5}$$

We show first that for every $f_i : N \to \mathbb{R}_+$, $i = 1, \ldots, K$ that satisfy $f_1+\cdots+f_K = 1$, there are $f_i' \in \mathcal{F}_i$, $i = 1, \ldots, K$ that satisfy $f_1 + \cdots + f_K = 1$ and M sufficient large such that,

$$\sum_{1\le i\le K} \int^{\mathrm{cav}} f_i' \mathrm{d}v_i^M \ge \sum_{1\le i\le K} \int^{\mathrm{cav}} f_i \mathrm{d}v_i^M. \tag{6.6}$$

Suppose that $f_i : N \to \mathbb{R}_+$, $i = 1, \ldots, K$ satisfy $f_1+\cdots+f_K = 1$ and that there are j and $\ell \notin N_j$ with $f_j(\ell) > 0$. Denote by $f_i 1_{N_i}$ the function that coincides with f_i on N_i and is equal to 0 out of N_i. Note that $\int^{\mathrm{cav}} f_i \mathrm{d}v_i^M = \int^{\mathrm{cav}} f_i 1_{N_i} \mathrm{d}v_i - M \sum_{\ell\notin N_i} f_i(\ell)$. For every $\ell \in N$, there is an index $i(\ell)$ such that $\ell \in N_{i(\ell)}$. For every i define,

$$f_i' = f_i 1_{N_i} + \sum_{\ell;\ i(\ell)=i} \max_{\substack{j=1,\ldots,K \\ \text{s.t. } \ell\notin N_j}} f_j(\ell).$$

The function f_i' is 0 on $\ell \notin N_i$. Therefore, $f_i' \in \mathcal{F}_i$. Moreover, on $\ell \in N_i$, $f_i'(\ell) \ge f_i(\ell)$ with strict inequality when there is j such that $f_j(\ell) > 0$, $\ell \notin N_j$ and $i(\ell) = i$. In the latter case, $f_i'(\ell)$ is getting the total value of all $f_j(\ell)$ for which $\ell \notin N_j$. It is clear that $f_1' + \cdots + f_K' = 1$.

Regarding the left-hand side (LHS) of (6.6), note that since $f_i' \in \mathcal{F}_i$, $\int^{\mathrm{cav}} f_i' \mathrm{d}v_i^M = \int^{\mathrm{cav}} f'_{i|N_i} \mathrm{d}v_i$, meaning that it does not depend on M, and therefore, the LHS does not depend on M as well.

On the right-hand side (RHS), note that

$$\sum_{1\le i\le K} \int^{\mathrm{cav}} f_i \mathrm{d}v_i^M \le \sum_{i\ne j} \int^{\mathrm{cav}} f_i 1_{N_i} \mathrm{d}v_i + \int^{\mathrm{cav}} f_j 1_{N_j} \mathrm{d}v_j - f_j(\ell) M,$$

which tends to $-\infty$ as $M \to \infty$. We obtain that for M large enough, the LHS is strictly greater than the RHS. Note that the M we found depends on the functions $f_1, \ldots, f_K$ under discussion. However, due to the fact that the convex integral is continuous and that the set $\{(f_1, \ldots, f_K);\ \forall i, f_i, N \to \mathbb{R}_+, f_1 + \cdots + f_K = 1\}$ is compact, we conclude that there is one M such that Eq. (6.5) holds, as desired.

We obtain that there is M large enough such that [by Eqs. (6.4 and (6.5)], $v_1, \ldots, v_K$ are compatible if and only if,

$$\max_{\substack{f_1+\cdots+f_K=1 \\ \forall i, f_i \in \mathcal{F}_i}} \sum_{1 \le i \le K} \int^{\mathrm{cav}} f_i \mathrm{d}v_i^M \le v_1^M \bullet \cdots \bullet v_K^M(N). \tag{6.7}$$

However, for $M > 0$ when $f_i \in \mathcal{F}_i$, $\int^{\mathrm{cav}} f_i \mathrm{d}v_i^M = \int^{\mathrm{cav}} f_i \mathrm{d}v_i$. Moreover, when M is sufficiently large $v_1^M \bullet \cdots \bullet v_K^M(N) = v_1 \bullet \cdots \bullet v_K(N)$. This implies Eq. (4.6), and the proof of Theorem 2 is complete. □

Proof of Theorem 3 Let (v_1, N_1), (v_2, N_2) be convex games.

Step 1: We first assume that $N_1 = N_2$. Denote by $C(v_i)$ the core of v_i. Since v_i is convex, by Remark 6,

$$\int^{\mathrm{cav}} X \mathrm{d}v_i = \min_{P \in C(v_i)} \int^{\mathrm{cav}} X \mathrm{d}P, \quad X \ge 0.$$

We therefore have,

$$\begin{aligned}
\max_{\substack{f_1+f_2=1 \\ f_1, f_2 \ge 0}} \sum_i \int^{\mathrm{cav}} f_i \mathrm{d}v_i &= \max_{\substack{f_1+f_2=1 \\ f_1, f_2 \ge 0}} \sum_i \min_{P_i \in C(v_i)} \int^{\mathrm{cav}} f_i \mathrm{d}P_i \\
&= \max_{\substack{f_1+f_2=1 \\ f_1, f_2 \ge 0}} \min_{(P_i)_i \in C(v_1) \times C(v_2)} \sum_i \int^{\mathrm{cav}} f_i \mathrm{d}P_i \\
&= \min_{(P_i)_i \in C(v_1) \times C(v_2)} \max_{\substack{f_1+f_2=1 \\ f_1, f_2 \ge 0}} \sum_i \int^{\mathrm{cav}} f_i \mathrm{d}P_i,
\end{aligned} \tag{6.8}$$

where the last equality is obtained by using the minimax theorem, as explained in what follows. Consider the zero-sum game in which player 1's set of strategies is $\{f_1 + f_2 = 1, f_1, f_2 \ge 0\}$ and $C(v_1) \times C(v_2)$ is player 2's set of strategies. Both sets are compact. Finally, the payoff function $((P_i)_i, (f_i)_i) \mapsto \sum_i \int f_i \mathrm{d}P_i$ is bilinear. The minimax theorem allows us now to change the order of the minimum and the maximum in order to obtain the last equality.

Let (P_1^*, P_2^*) and (f_1^*, f_2^*) be the optimal strategies of the players. In particular, (f_1^*, f_2^*) solves the maximization problem in the LHS of Eq. (6.8).

For every $j \in N$, denote[5] $M(j) = \{i;\ P_i^*(j) \ge P_{-i}^*(j)\} = \mathrm{argmax}_{i=1,2} P_i^*(j)$. When $i \in M(j)$, $P_i^*(j)$ is greater than or equal to $P_{-i}^*(j)$. Since (f_1^*, f_2^*) solves

[5] $-i$ denotes $3 - i$.

$$\max_{\substack{f_1+f_2=1\\ f_1,f_2\geq 0}} \sum_i \int^{\mathrm{cav}} f_i \mathrm{d}P_i^*, \tag{6.9}$$

we infer that if $f_i^*(j) > 0$, then $i \in M(j)$. Indeed, if $i \notin M(j)$, then $P_{-i}^*(j) > P_i^*(j)$. Define now, $f_i'(j') = f_i^*(j')$, $f_{-i}'(j') = f_{-i}^*(j')$ for every $j' \neq j$, $f_i'(j) = 0$ and $f_{-i}'(j) = f_{-i}^*(j) + f_i^*(j)$. One obtains that f_1' and f_2' are nonnegative, $f_1' + f_2' = 1$ and,

$$\sum_i \int^{\mathrm{cav}} f_i^* \mathrm{d}P_i^* < \sum_i \int^{\mathrm{cav}} f_i' \mathrm{d}P_i^*,$$

a contradiction. Furthermore, any two nonnegative functions, f_1' and f_2', such that $f_1' + f_2' = 1$ and that satisfy the implication $[f_i'(j) > 0 \Rightarrow i \in M(j)]$ solve Eq. (6.9).

Claim There are two disjoint sets S_1 and S_2 such that $S_1 \cup S_2 = N$ (i.e., $\mathbf{1}_{S_1} + \mathbf{1}_{S_2} = 1$) and

(a) $j \in S_i$ implies $i \in M(j)$ for every $j \in N$ and $i = 1, 2$;
(b) $\int \mathbf{1}_{S_i} \mathrm{d}P_i^* = P_i^*(S_i) = v_i(S_i)$.

Before we prove the claim, we argue that (a) and (b) together would complete the proof. Indeed, by the previous paragraph, (a) would imply that $(\mathbf{1}_{S_1}, \mathbf{1}_{S_2})$ solves Eq. (6.9). (b) would imply that $\sum_i \int \mathbf{1}_{S_i} \mathrm{d}P_i^*(j) = \sum_i v_i(S_i)$. Together we would obtain

$$\max_{\substack{f_1+f_2=1\\ f_1,f_2\geq 0}} \sum_i \int^{\mathrm{cav}} f_i \mathrm{d}v_i = \sum_i \int^{\mathrm{cav}} f_i^* \mathrm{d}v_i = \sum_i \int^{\mathrm{cav}} f_i^* \mathrm{d}P_i^*$$
$$= \sum_i \int^{\mathrm{cav}} \mathbf{1}_{S_i} \mathrm{d}P_i^* = \sum_i v_i(S_i) \leq v_1 \bullet v_2(N).$$

Theorem 1 implies that v_1 and v_2 are compatible. □

Proof of the claim The game v_i is convex, and therefore, the concave integral takes a special form. Let π_i be a permutation over N such that $f_i^*(\pi_i(1)) \leq f_i^*(\pi_i(2)) \leq \cdots \leq f_i^*(\pi_i(n))$. Define $A_i(j) = \{j' \in N;\ f_i^*(j') \geq f_i^*(\pi_i(j))\}$. It is clear that $A_i(n) \subseteq A_i(n-1) \subseteq \cdots \subseteq A_i(1)$. Note that f_i^* attains its maximum on the set $A_i(n)$ and $A_i(1) = N$. By Lovász (1983),[6]

$$\int^{\mathrm{cav}} f_i^* \mathrm{d}v_i = \sum_{j=1}^{n} \Big(f_i^*(\pi_i(j)) - f_i^*(\pi_i(j-1))\Big) v_i(A_i(j)),$$

where $f_i^*(\pi_i(0)) = 0$.

Recall that $f_1^* + f_2^* = 1$.

Case 1: $\max f_1^* < 1$. Then, $\min f_2^* > 0$ implies $2 \in M(j), \forall j$, and therefore, $A_2(1) = N$. In this case, set $S_1 = \emptyset$ and $S_2 = N$. Properties (a) and (b) are satisfied, and the proof of the claim is complete.

[6] This is also known as the Choquet integral (Choquet 1955).

Case 2: max $f_2^* < 1$. This is similar to the previous case.
Case 3: max $f_1^* =$ max $f_2^* = 1$. Denote by j^* the smallest index such that f_2^* is strictly positive on $A_2(j^*)$ (it exists because otherwise, $f_2^* = 0$ contradicting max $f_2^* = 1$). Note that the complement of $A_2(j^*)$ is the set where $f_2^* = 0$, which is precisely where $f_1^* = 1$, namely $A_1(n)$. Define, $S_1 = A_1(n)$ and $S_2 = A_2(j^*)$. The sets S_1 and S_2 are complements of one another. Moreover, (a) is satisfied (because $j \in S_i$ implies that $f_i^*(j)$ is positive and therefore in $M(i)$).

We now show that (b) is satisfied. As (P_1^*, P_2^*) and (f_1^*, f_2^*) are the optimal strategies, we have

$$\begin{aligned}\sum_{i=1}^{2}\int^{\mathrm{cav}} f_i^* \mathrm{d}v_i &= \sum_{i=1}^{2}\int f_i^* \mathrm{d}P_i^* \\ &= \sum_{i=1}^{2}\sum_{j=1}^{n}\Big(f_i^*(\pi_i(j)) - f_i^*(\pi_i(j-1))\Big)P_i^*(A_i(j)) \\ &\geq \sum_{i=1}^{2}\sum_{j=1}^{n}\Big(f_i^*(\pi_i(j)) - f_i^*(\pi_i(j-1))\Big)v_i(A_i(j)) \\ &= \sum_{i=1}^{2}\int^{\mathrm{cav}} f_i^* \mathrm{d}v_i,\end{aligned}$$

where the inequality is due to the fact that P_i^* is in the core of v_i. We obtain that this inequality is actually an equality, and therefore, $P_i^*(A_i(j)) = v_i(A_i(j))$ for every j. In particular, it holds for S_i, $i = 1, 2$, and hence (b). This shows the claim and the proof of Theorem 3 in case of identical grand coalitions.

Step 2 We now lift the restriction that $N_1 = N_2$ and allow different grand coalitions, N_1 and N_2. Denote $N = N_1 \cup N_2$ and use the same technique we employed in the proof of Theorem 2: For $i = 1, 2$ and a positive M, define v_i^M over N. Recall that in v_i^M the worth of $\ell \notin N_i$ is $-M$, which is also her contribution to any coalition she does not belong to. Thus, v_i^M is also convex. Furthermore, from the proof of Theorem 2, we know that when M is sufficiently large, v_1^M and v_2^M are compatible if and only if v_1 and v_2.

Using Step 1, v_1^M and v_2^M are compatible as convex games. Thus, when M is large enough, v_1 and v_2 are compatible, as desired. □

Proof of Proposition 1 Denote $v = v_1 \bullet \cdots \bullet v_K$. Suppose first that $v(N) = v_1(N)$. We show that $C(v) = C(v_1) \cap \cdots \cap C(v_K)$. Assume that $P \in C(v_1) \cap \cdots \cap C(v_K)$. Let $S \subseteq N$ and let $S_1, \ldots, S_K$ be a partition of S to pairwise disjoint sets such that $v(S) = \sum_i v_i(S_i)$. One obtains, $P(S) = \sum_i P(S_i) \geq \sum_i v_i(S_i) = v(S)$. Furthermore, $P(N) = v_1(N) = v(N)$ and therefore, $P \in C(v)$. Now assume that $P \in C(v)$. For every coalition $S, v(S) \geq v_i(S)$, implying $P(S) \geq v(S) \geq v_i(S)$ for every $i = 1, \ldots, K$. Since $P(N) = v(N) = v_i(N)$, $P \in C(v_i)$ for every $i = 1, \ldots, K$.

As for the inverse direction, $C(v) = C(v_1) \cap \cdots \cap C(v_K)$ readily implies that $v(N) = v_1(N)$, which completes the proof. □

Proof of Proposition 2 We prove the theorem by induction on the number of games. First assume that $K = 2$. We have

$$\begin{aligned}
\tilde{v}_1 \bullet \tilde{v}_2(S) &= \max_{\substack{S_1 \cap S_2 = \emptyset \\ S_1 \cup S_2 = S}} \left[\tilde{v}_1(S_1) + \tilde{v}_2(S_2)\right] \\
&= \max_{\substack{S_1 \cap S_2 = \emptyset \\ S_1 \cup S_2 = S}} \left[\max_{\substack{S_1^i \cap S_1^j = \emptyset \\ S_1^1 \cup \cdots \cup S_1^n = S_1}} \left\{ v_1\left(S_1^1\right) + \cdots + v_1\left(S_1^n\right) \right\} \right. \\
&\quad \left. + \max_{\substack{S_2^i \cap S_2^j = \emptyset \\ S_2^1 \cup \cdots \cup S_2^n = S_2}} \left\{ v_2\left(S_2^1\right) + \cdots + v_2\left(S_2^n\right) \right\} \right] \\
&= \max_{\substack{S_1 \cap S_2 = \emptyset \\ S_1 \cup S_2 = S}} \max_{\substack{S_1^i \cap S_1^j = \emptyset \\ S_1^1 \cup \cdots \cup S_1^n = S_1}} \max_{\substack{S_2^i \cap S_2^j = \emptyset \\ S_2^1 \cup \cdots \cup S_2^n = S_2}} \left[v_1\left(S_1^1\right) + \cdots + v_1\left(S_1^n\right) \right. \\
&\quad \left. + v_2\left(S_2^1\right) + \cdots + v_2\left(S_2^n\right) \right] \\
&= \max_{\substack{T_i \cap T_j = \emptyset \\ T_1 \cup \cdots \cup T_n = S}} \left[\max_{\substack{S_1^1 \cap S_2^1 = \emptyset \\ S_1^1 \cup S_2^1 =: T_1}} \left\{ v_1\left(S_1^1\right) + v_2\left(S_2^1\right) \right\} \right. \\
&\quad \left. + \cdots + \max_{\substack{S_1^n \cap S_2^n = \emptyset \\ S_1^n \cup S_2^n =: T_n}} \left\{ v_1\left(S_1^n\right) + v_2\left(S_2^n\right) \right\} \right] \\
&= \max_{\substack{T_i \cap T_j = \emptyset \\ T_1 \cup \cdots \cup T_n = S}} \left[v_1 \bullet v_2(T_1) + \cdots + v_1 \bullet v_2(T_n) \right] \\
&= \widetilde{v_1 \bullet v_2}(S).
\end{aligned}$$

Now assume the induction hypothesis: The statement of the theorem holds for $2, \ldots, K-1$ games. Let $v := v_1 \bullet \cdots \bullet v_{K-1}$. By using the induction hypothesis and by the semigroup structure of $(\mathcal{G}(N), \bullet)$, we have,

$$\begin{aligned}
\widetilde{v_1 \bullet \cdots \bullet v_K} &= \widetilde{v \bullet v_K} \\
&= \tilde{v} \bullet \tilde{v}_K \\
&= (\tilde{v}_1 \bullet \cdots \bullet \tilde{v}_{K-1}) \bullet \tilde{v}_K \\
&= \tilde{v}_1 \bullet \cdots \bullet \tilde{v}_K.
\end{aligned}$$

□

Proof of Proposition 3 We need the following three lemmas.

Lemma 1 *For any game* v*,*

$$\{P\text{linear}|P \geq v\} = \{P\text{linear}|P \geq \tilde{v}\} = \{P\text{linear}|P \geq v^c\}.$$

Proof This is an immediate result of the linearity of P and the definition of $\tilde{v}$ and v^c. $\square$

An immediate consequence of this lemma is the following one,

Lemma 2 *For any game* v*,*

$$B_v = B_{\tilde{v}} = B_{v^c}.$$

The third lemma is,

Lemma 3 *For any game* v*,*

$$C^c(v) \neq \emptyset \quad \text{iff} \quad C(\tilde{v}) \neq \emptyset.$$

Proof By (4.4), $C^c(v) \neq \emptyset$ is equivalent to $B_{v^c}(N) \leq v^c(N)$. Since $v^c(N) = \tilde{v}(N)$, by Lemma 2, $B_{v^c}(N) \leq v^c(N)$ is equivalent to $B_{\tilde{v}}(N) \leq \tilde{v}(N)$, which is, by (4.4), equivalent to $C(\tilde{v}) \neq \emptyset$. $\square$

Now we complete the proof of the proposition. We have,

$$\begin{aligned}
v_1, \ldots, v_K \text{ are c-compatible } &\Leftrightarrow C^c(v_1 \bullet \cdots \bullet v_K) \neq \emptyset \\
&\Leftrightarrow C(v_1 \widetilde{\bullet \cdots \bullet} v_K) \neq \emptyset \\
&\Leftrightarrow B_{\widetilde{v_1 \bullet \cdots \bullet v_K}}(N) \leq v_1 \widetilde{\bullet \cdots \bullet} v_K(N) \\
&\Leftrightarrow B_{\tilde{v}_1 \bullet \cdots \bullet \tilde{v}_K}(N) \leq v_1 \widetilde{\bullet \cdots \bullet} v_K(N) \\
&\Leftrightarrow B_{\tilde{v}_1 \bullet \cdots \bullet \tilde{v}_K}(N) \leq \tilde{v}_1, \ldots, \tilde{v}_K(N) \\
&\Leftrightarrow C(\tilde{v}_1 \bullet \cdots \bullet \tilde{v}_K) \neq \emptyset \\
&\Leftrightarrow \tilde{v}_1, \ldots, \tilde{v}_K \text{ are compatible.}
\end{aligned}$$

The first equivalence is the definition of compatibility. The second one is due to Lemma 3. The third equivalence is by (4.4), while the fourth and the fifth are due to Proposition 2. The last two equivalences result from (4.4) and the definition of compatibility, respectively. $\square$

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