JUSTIFIABLE PREFERENCES

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ABSTRACT. A new equilibrium concept of non-cooperative games has emerged from the learning literature: conjectural equilibrium. In this equilibrium, every player obtains a partial information about the strategies used by others and plays a best response to one strategy consistent with this information. In other words, in conjectural equilibrium it is sufficient that there exists a consistent belief that justifies a player's choice. Motivated by this notion, we characterize preference relations over acts that can be represented by a utility function and a set of priors over states, such that an act f is preferred to act g if there is a prior under which the expected utility induced by f is higher than that induced by g. These kind of preferences are called *justifiable preferences.*

We further introduce a generalized model of ambiguity that involves a collection of multiple priors, namely *multiple multiple priors.* We combine the models of Knightian and justifiable preferences into one: act f is preferred to act g if according to at least one multiple prior in the collection, f is unanimously preferred to g. We finally axiomatize justifiable preferences that are generated by partially–specified probabilities.

Keywords: Justifiable preferences, multiple priors, multiple multiple priors, partially–specified probabilities.

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1. Introduction

An extension of Nash equilibrium emerged from the learning literature: *conjectural equilibrium* (Battigalli [2]).[1] In such an equilibrium each player receives a partial information regarding the action profile played by others. This information does not reveal precisely what other players play. Rather, it provides a player with a set of possible strategies that the others might play. This set contains all the strategy profiles that are consistent with a player's knowledge about others strategies. In equilibrium, each player plays a best response to one of these consistent–with–information strategies. In other words, a strategy profile constitutes a conjectural equilibrium, if every player's strategy is justified by a belief consistent with his information.

Motivated by the notion of justifiability in conjectural equilibrium, we take an axiomatic approach in an Anscombe–Aumann setting [1], and characterize *justifiable preferences*. More formally, we characterize a binary relation $\succeq$ over acts, such that there exist a vN–M utility function u and a closed and convex set of probability distributions P over the state space, where $f \succeq g$ if and only if there exists $p \in P$ such that, with respect to p, the expected value of $u(f)$ is at least as high as that of $u(g)$.

Justifiable preferences need not be transitive. In fact, it is transitive if and only if the collection of priors consists of a single prior. In this case it admits an expected–utility representation. Though transitivity is considered by many economists as the "cornerstone" of normative decision theory, non–transitive literature is abundant, and for the past fifty years there has been a growing body of evidence, empirical and theoretical,[2] that intransitive preferences can reflect underlying rational choices. One of the first who dealt with non–transitive preferences was May [20]. To quote May,

> *Theories of choice may be built to describe behavior as it is or as it "ought to be". In the belief that the former should precede the latter, this paper is concerned solely with descriptive theory and, in particular, with the intransitivity of preferences.*

To illustrate the definition of justifiable preferences consider an example of a firm that delegates responsibility to its employees. Every employee is competent and trustworthy

[1]Closely related solution concepts are Fudenberg and Levine's [8] self–confirming equilibrium and Kalai and Lehrer's [12, 13] subjective equilibrium.

[2]See the related literature Section 6.1.

in the eyes of the management, and considers only the benefit of the firm. All employees share common sources of relevant information, which usually provide just a partial picture. However, each agent typically has his own background, education, knowledge, guts feeling, intuition and instincts. It is quite plausible in such a situation that different agents would justify their decisions by different assessments about the real state, all are consistent with their information. It might therefore happen that several agents would choose f over g, and others would make the opposite choice. This can also occur when choosing between g and h. Furthermore, it is possible that every agent would choose h over f. To an outsider who observes the conduct of the firm (and not the agents), it seems that the firm is indifferent between f and g, and also between g and h. However, it seems that h is strictly preferred to f. In this case the firm's observable preferences appear intransitive and perhaps irrational. However, when confronted, these preferences are justifiable: management can justify every particular decision made, and back up its competent employees.

Our model joins a vast literature of non–expected utility attempting to explain behavioral evidence, which violate expected–utility maximization. The most prominent illustration of such a behavior is provided by Ellsberg [5], which shows that partially informed decision makers typically do not adopt a unique prior that rationalizes their choices, and therefore, do not adhere to expected–utility theory (Savage [21] and Anscombe–Aumann [1]). This observation raises several questions like how decision makers perceive uncertainty? and, what might be a decision maker's attitude towards uncertainty? The literature offers several alternatives to these issues.

One way to model uncertainty is by a convex closed set of probability distributions, the multiple priors model. This set reflects the ambiguity of the decision maker generated by the partial information he obtained. The maxmin expected–utility model (Gilboa and Schmeidler [10]) suggests that a decision maker is ambiguity averse: she considers the worst case scenario and chooses the alternative that maximizes the minimal expectation. More formally, she prefers f to g if the minimal expectation of f over all possible priors is greater than the minimal expectation of g.

The maxmin approach is also applied to non–cooperative normal–form games. In beliefs equilibrium (see Lo [17]) each player is a maxmin decision maker who believes that the other players play strategies in a certain set (the beliefs set). Furthermore, this set is consistent in the sense that it contains the strategies actually being played. In both, beliefs and conjectural equilibrium players play against ambiguous strategies of others. While in beliefs equilibrium players are maxmin decision makers, in conjectural equilibrium players are allowed to respond to any strategy profile in their beliefs set. In other words, conjectural equilibrium does not restrict a player to any particular attitude to ambiguity: it allows any justifiable decision.

Following the characterization of justifiable preferences, we proceed with the second contribution of this paper: extension of the classical ambiguity model of multiple priors, where the decision maker's ambiguity is reflected by a single set of priors. As noted above one possible source of ambiguity (see also Lehrer [16]) is the partial information a decision maker obtains about the actual distribution over states. Quite often, and especially when the decision problems considered have a significant importance, different and parallel agencies are established to collect information. Based on the information it gathered, each agency provides its own assessments. Since information is typically incomplete, an assessment amounts to, possibly, multiple priors that settle with the information available. Thus, the decision maker is provided with a collection of sets of probability distributions. This is the *multiple multiple priors* that reflects a hyper ambiguity: not only that there are several priors, there are several sets of priors.

We axiomatize a decision maker whose preferences are determined by multiple multiple priors in the following manner. An act f is preferred to g if with respect to at least one set of probability distributions, f dominates g in the Knightian sense (Bewley [3]). In the context, previously discussed, of many parallel agencies that are in charge of collecting information and providing assessments, f is preferred to g if according to at least one agency, f is unanimously preferred to g.

The last part of the paper is devoted to justifiable preferences induced by partially–specified probabilities (see Lehrer [16]). Here, multiple priors emerge from concrete partial information about the real distribution governing the state space. This partial information induces a vN–M representation of the preference over a set of acts that includes the set of constant acts. There are typically many distributions that can serve as the prior of this vN–M representation. The multiple priors that generates the justifiable preferences consists of all these distributions.

2. Characterization of justifiable preferences

Consider a decision making model in an Anscombe–Aumann [1] setting. Let X be a non–empty finite set of *outcomes*, and let $Y = \Delta(X)$ be the set of all *lotteries*,[3] that is probability distributions over X. Let S be a finite non–empty set of *states of nature*. Now, consider the collection $L = Y^S$ of all functions from states of nature to lotteries. Such functions are referred to as *acts*. Endow this set with the product topology, where the topology on Y is the relative topology inherited from $[0,1]^X$. We denote by L_c the collection of all constant acts. Abusing notations, for an act $f \in L$ and a state $s \in S$, we denote by $f(s)$ the constant act that assigns the lottery $f(s)$ to every state of nature.

Mixtures (convex combinations) of lotteries and acts are performed pointwise. In particular, if $f, g \in L$ and $\alpha \in [0,1]$, then $(\alpha f + (1-\alpha)g)(s) = \alpha f(s) + (1-\alpha)f(s)$ for every $s \in S$.

The primitive of such a decision model is a binary relation $\succeq$ over L, which represents the preferences of a decision maker (DM) over all acts. $\succ$ is the asymmetric part of the relation, that is $f \succ g$ if $f \succeq g$ but it is not true that $g \succeq f$. $\sim$ is the symmetric part, that is $f \sim g$ if $f \succeq g$ and $g \succeq f$. The binary relation $\succeq$ is *reflexive* if $f \sim f$ for every act f. $\succeq$ is *complete* over $K \subseteq L$ if for every $f, g \in K$, either $f \succeq g$ or $g \succeq f$. $\succeq$ is complete if it is complete over L. It is *transitive* over $K \subseteq L$ if for $f, g, h \in K$, $f \succeq g$ and $g \succeq h$ imply $f \succeq h$. Lastly, the relation is *non–trivial* if there are two acts f and g such that $f \succ g$.

Following is a list of assumptions (axioms) about a binary preference relation $\succeq$ over acts.

A1 *Relation.* $\succeq$ is reflexive, complete and non–trivial.

A1 is a structural assumption. Completeness is assumed while transitivity is not required.

For two acts $f, g \in L$, we denote $f \succeq^S g$ and $f \succ^S g$, if respectively, $f(s) \succeq g(s)$ for every $s \in S$, and $f(s) \succ g(s)$ for every $s \in S$

A2 *Unambiguous transitivity.* (i) $f \succeq g$ and $g \succeq^S h$ imply $f \succeq h$; and (ii) $f \succeq g$ and $h \succeq^S f$ imply $h \succeq g$.

A2 combines two preferences. The first, $f \succeq g$, suggests that the DM managed to decide between f and g in spite of the ambiguity. This is the original preference relation.

[3]Given a finite set A, $\Delta(A)$ denotes the collection of all probability distributions over A.

The second, $g \succeq^S h$, reflects a domination of g over h beyond uncertainty, and takes into consideration only the DM's tastes. **A2** requires transitivity in the following sense: if $f \succeq g$, and g is beyond any doubt as good as h, then $f \succeq h$.

A3 *Strict monotonicity.* $f \succ^S g$ implies that $f \succ g$.

A3 is a strict monotonicity assumption. It states that a strictly preferred lottery in every state of nature yields a strictly preferred act. This is not the common monotonicity assumption, however, it is necessary for our representation, and the relation between the two will be discussed later.

A4 *Continuity.* For any act f the sets $\{g : g \succeq f\}$ and $\{g : f \succeq g\}$ are closed.

A5 *Independence.* $f \succeq g$ if and only if $\alpha h + (1-\alpha)f \succeq \alpha h + (1-\alpha)g$ for every $h \in L$ and $\alpha \in [0,1]$.

A5 is the classical independence assumption.

A6 *Favorable mixing.* If $g \succ f$, and $h \in L$, $\alpha \in [0,1]$ are such that $\alpha f + (1-\alpha)h \succeq g$, then $\lambda f + (1-\lambda)h \succeq g$ for every $\lambda \leq \alpha$.

Consider the case where an act g is strictly preferred to f, and that the preference is reversed once f is mixed with h with probability α. This suggests that h is better than g and that the weight of α on h is sufficient for reversing the order. **A6** states that if f is mixed with h with a probability greater than α, the DM would still reverse her preference.

When transitivity over all acts is assumed then **A6** is implied by **A5**. Once transitivity is relaxed and only **A2** is assumed, **A6** is not implied by **A5**.

The following lemma is immediate and will be useful in obtaining the representation of justifiable preferences. The proof is omitted.

Lemma 1. *Assume that $\succeq$ satisfies **A2**. Then,*
(1) $\succeq$ is transitive over L_c; and
(2) $\succeq$ satisfies monotonicity*, that is, $f \succeq^S g$ implies that $f \succeq g$.*

A function $u : Y \to \mathbb{R}$, also referred to as a *utility function*, is *affine* if for every $q \in Y$ it satisfies $u(q) = \sum_{x \in X} q(x)u(x)$. Given such a utility function and an act $f \in L$, we denote by $u(f) = (u(f(s)))_{s \in S}$.

We are now ready to present our characterization.

Theorem 1. *$\succeq$ is a binary relation over L. Then the following are equivalent:*
*(1) $\succeq$ satisfies **A1**–**A6**.*
(2) There exist a non–constant affine utility function $u : Y \to \mathbb{R}$, and a non–empty, closed and convex set P of probability distributions over S, such that for every two acts f and g,

$$f \succeq g \quad \Leftrightarrow \quad \exists p \in P \text{ such that } p \cdot u(f) \geq p \cdot u(g).^4$$

Moreover, P is unique and u is unique up to a positive linear transformation.

Remark 1. *Theorem 1 shows that strict monotonicity of $\succeq$ is necessary for the set P to consist of probability distributions. Consider the following example of a preference $\succeq$ over utility vectors in $\mathbb{R}^2$, where the strict monotonicity assumption **A3** is dropped. Let P be the convex–hull of $\{(-1,2),(0,1)\}$, and define $\succeq$ by $x \succeq y$ if there exists $p \in P$ such that $p \cdot (x-y) \geq 0$. Although P consists of signed probability distributions, $\succeq$ satisfies monotonicity.*

*Denote by Q the l_1^S unit ball, that is, $Q = \{q \in \mathbb{R}^S : \sum_{s_i \in S} |q(s_i)| \leq 1\}$. Following the proof of Theorem 1, dropping the strict monotonicity **A3**, Assumptions $1,2,4,5$ and 6 yield a representation as in Theorem 1, where $P \subseteq Q$ is closed, convex and contains a non–negative element of Q but not 0.*

Remark 2. *Let $\succeq$ satisfy **A1**–**A6**. Then, $\succeq$ admits an expected–utility representation if and only if P is a singleton.*

Before turning to the proof of Theorem 1 we need to establish an auxiliary result regarding the geometric nature of our assumptions. Let N be a finite non–empty set, and let N also denote the number of elements in N. The collection of all functions $x : N \to \mathbb{R}$ can by identified with the linear space $\mathbb{R}^N$, where multiplication of such functions by scalars and addition of functions are performed pointwise. For $x, y \in \mathbb{R}^N$, $x \geq y$ if $x(i) \geq y(i)$ for every $1 \leq i \leq N$, and $x > y$ if $x(i) > y(i)$ for every $1 \leq i \leq N$.

Consider a binary relation $\succeq^*$ over a closed and convex $C \subseteq \mathbb{R}^N$. The following are some possible properties for such a relation:
(i). For every $x, y \in C$, either $x \succeq^* y$ or $y \succeq^* x$, and $x \sim^* x$
(ii). $x > y$ implies $x \succ^* y$.
(iii). The sets $\{x \in C : x \succeq^* y\}$ are closed.

[4]Given vectors $x, y \in \mathbb{R}^S$, the inner–product of x and y is denoted by $x \cdot y$. The inner–product of x and a probability distribution p over S is the expectation of x with respect to p.

(iv). $x \succeq^* y$ if and only if $\alpha z + (1-\alpha)x \succeq^* \alpha z + (1-\alpha)y$ for $z \in C$ and $\alpha \in [0,1]$.
(v). If $y \succ^* x$, and $z \in C$, $\alpha \in [0,1]$ are such that $\alpha x + (1-\alpha)z \succeq^* y$, then $\lambda x + (1-\lambda)z \succeq^* y$ for every $\lambda \leq \alpha$.

Proposition 1. *Let $\succeq^*$ be a binary relation over $[0,1]^N$. Then the following are equivalent:*
(1) $\succeq^*$ satisfies (i)-(v).
(2) There exists a unique closed and convex set P of probability distributions over N, such that

$$x \succeq^* y \quad \Leftrightarrow \quad \exists p \in P \text{ such that } p \cdot x \geq p \cdot y.$$

Proof. It is easy to see that (2) implies (1). For the converse assume that $\succeq^*$ satisfies (i)–(v).

First we extend $\succeq^*$ to a complete binary relation over $\mathbb{R}^N$ as follows. We start with $\mathbb{R}^N_+$, the set of all nonnegative vectors is $\mathbb{R}^N$. Let $x, y \in \mathbb{R}^N_+$ and define $x \succeq^* y$ if and only if $\lambda x \succeq^* \lambda y$ for sufficiently small positive λ. To see that this is well defined, apply (iv) where $z = 0$. We have obtained that $\succeq^*$ is (positively) homogeneous over $\mathbb{R}^N_+$, formally

$$x \succeq^* y \;\Leftrightarrow\; \lambda x \succeq^* \lambda y, \quad \text{for every } \lambda > 0.$$

Now consider $x, y \in \mathbb{R}^N$. We say that $x \succeq^* y$ if and only if there are $x', y' \in \mathbb{R}^N_+$ such that $x' \succeq^* y'$ and $x' - y' = x - y$. Since $\succeq^*$ is by now defined over all $\mathbb{R}^N_+$, it is complete over $\mathbb{R}^N$. It remains to show that $\succeq^*$ is well defined. That is, for every distinct $x, y, x', y' \in \mathbb{R}^N_+$, if $x \succeq^* y$ and $x' - y' = x - y = z$ then $x' \succeq^* y'$. Define, $w = \min(x, y)$ coordinate–wise. Then $x - w, y - w \in \mathbb{R}^N_+$, $(x - w) - (y - w) = x - y$ and $x > x - w$. $x \succeq^* y$ implies by homogeneity that $\frac{1}{2}w + \frac{1}{2}(x - w) \succeq^* \frac{1}{2}w + \frac{1}{2}(y - w)$, and again by (iv) that $(x - w) \succeq^* (y - w)$.

Define $w' = \min(x', y')$. Note that $x' - w' = x - w = \max(z, 0)$ coordinate–wise, thus $x' - (x - w) = w'$ is in $\mathbb{R}^N_+$. The previous argument applied to x', y' and w' (that play the roles of x, y and w) shows that $x' \succeq^* y'$. We conclude that $x \succeq^* y$ if and only if $x - y \succeq^* 0$ for every $x, y \in \mathbb{R}^N$.

Next we show that the set $K = \{x \in \mathbb{R}^N : x \succeq^* 0\}$ is a positive homogeneous closed cone, and that its complement K^c is convex. By homogeneity K is a positive homogeneous, and by (iii) it is closed. To see that K^c is convex, assume to the contrary that there exist $x, y, z \in \mathbb{R}^N$ and $\alpha \in (0,1)$ such that $0 \succ^* y$, $0 \succ^* z$ and $\alpha y + (1-\alpha)z \succeq^*$

0. However, by (v), $0 \succ^* y$ and $\alpha y + (1-\alpha)z \succeq^* 0$ imply $z \succeq^* 0$ (with $\lambda = 0$), which contradicts $0 \succ^* z$.

As a closed cone whose complement is convex, K can be written as a union of hyperspaces H_i of $\mathbb{R}^N$. For every H_i there is a unique vector $p_i \in \mathbb{R}^N$ such that $p_i \cdot h \geq 0$ for every $h \in H_i$ and the sum of coordinates of p_i is 1. Denote P the convex–hull of $\{p_i\}_i$. We have that $x \in K$ if and only if there exits $p \in P$ such that $p \cdot x \geq 0$.

By (ii), K^c contains the negative orthant[5]. Thus, P consists of probability distributions only. Otherwise there would exist $p \in P$ with $p(s) < 0$ for some $s \in S$, which would imply that there exists $x < 0$ with $p \cdot x > 0$. This implies that K and the negative orthant are not disjoint. A contradiction.

We therefore obtained that under the assumptions stated above, $x \succeq^* y$ if and only if $x - y \succeq^* 0$ if and only if there exists a probability distribution $p \in P$ such that $p \cdot (x - y) \geq 0$. □

We now turn to the proof of Theorem 1.

Proof of Theorem 1. The necessity of **A1**–**A6** is obvious. We now prove the converse.

Since **A1**, **A2**, **A4** and **A5** are satisfied, the hypothesis of the von Neumann–Morgenstern theorem hold. The theorem assures the existence, and uniqueness up to a positive linear transformation, of an affine function $u : Y \to \mathbb{R}$, which represents the preferences restricted to L_c. By **A1** the function u is non–constant. Moreover, u can be normalized so that the minimal utility is 0 and the maximal utility is 1.

The existence of such utility function u induces a preference relation over $[0,1]^S$. For $f, g \in L$, $u(f) \succeq^* u(g)$ if and only $f \succeq g$. $\succeq^*$ is well defined due to assumption **A2** of $\succeq$. Furthermore, $\succeq^*$ satisfies properties (i)–(v).

Now, applying Proposition 1 completes the proof. □

3. Decision models with multiple priors

3.1. **Generalizations of the maxmin model.** The maxmin model has been extended in several directions. Ghirardato et al. [9] axiomatize a model termed α–maxmin, differentiating ambiguity attitude from ambiguity. In this model, a DM valuates an act partially by maxmin and partially by maxmax.[6] The variational preferences model,

[5]By negative orthant of $\mathbb{R}^N$ we mean the collection of all $x < 0$.

[6]The maxmax preference order induced by a set of priors P and a utility function u is defined by: $f \succeq g \quad \Leftrightarrow \quad \max_{p \in P} p \cdot u(f) \geq \max_{p \in P} p \cdot u(g)$.

introduced by Maccheroni et al. [18], suggests that using a particular prior involves a cost, and that agents maximize the net worst–case utility that takes into account also the cost involved. As the maxmin model, variational preferences are ambiguity averse. Recently, Cerreia et al. [4] established a representation, resorting to multiple priors, for general uncertainty averse preferences. Smooth preferences, presented and axiomatized by Klibanoff et al. [15], suggest that the decision maker's ambiguity is reflected by multiple priors, however there exists an additional subjective distribution over the set of priors, suggesting the probability that a specific prior is the "correct" one.

3.2. **Bewley's Knightian model.** Bewley [3] introduces a different approach towards uncertainty in the multiple priors model, and axiomatizes Knightian preferences. Under Knightian preferences the decision maker prefers f to g if f dominates g in the sense that according to all priors, the expected–utility induced by f is greater than that induced by g. Formally, for every two acts f and g,

$$f \succ' g \quad \Leftrightarrow \quad \forall p \in P,\ p \cdot u(f) > p \cdot u(g),$$

where P is a given convex closed set of probability distribution over S. In other words, f is preferred over g if all possible priors unanimously agree that indeed f is better than g. The obvious shortcoming of the Knightian preference order is that it is incomplete.[7]

In the case where the set of priors consists of more than one prior, there are many (i.e., continuum) ways to extend the Knightian preferences to a complete binary relation. An obvious way, which we call a *Bayesian extension*, is to adopt one of the many possible priors, say p, and to declare that f is preferred over g if $p \cdot u(f) \geq p \cdot u(g)$. Following one particular prior, like in the Bayesian extension, has many desirable properties such as transitivity, ambiguity aversion and time consistency when temporal aspects are involved. Gilboa et al. [11] complete the Knightian preferences to maxmin preferences.

Another way to extend the incomplete Knightian preferences is as follows. For every two non–comparable acts f, g define $f \succeq' g$. In words, whenever preferences are indecisive about the comparison of f and g, the definition of $\succeq'$ implies that both, $f \succeq' g$ and $g \succeq' f$. This extension is complete and is precisely the justifiable preferences presented above.

[7]In fact, such preferences are complete if and only if P consists of a singleton, which implies expected utility preferences.

3.3. Knightian and Justifiable preferences. To see the dual roles of the Knightian and the justifiable preferences, one needs to notice that Knightian preferences satisfy

$$f \succ' g \;\Leftrightarrow\; \min_{p\in P} p\cdot(u(f)-u(g)) > 0, \tag{1}$$

whereas justifiable preferences admit the form

$$f \succeq g \;\Leftrightarrow\; \max_{p\in P} p\cdot(u(f)-u(g)) \geq 0. \tag{2}$$

A binary relation over acts is formally defined as a subset of pairs contained in the product of the set of acts with itself. Eq. (1) implies that the (irreflexive) Knightian preferences are the intersection of all the irreflexive Bayesian preferences induced by the priors in P. Moreover, all the irreflexive Bayesian preferences are open and convex (again, as sets of pairs) and so are the Knightian preferences.

On the other hand, Eq. (2) implies that any Bayesian preference order is a subset of the justifiable preferences. Moreover, the justifiable preferences are the union of all the Bayesian extensions.

The reader is referred to subsection 6.2 below for a further elaboration on this point.

4. Multiple multiple priors: a model of ambiguity

The models mentioned so far refer to a decision maker facing one set of priors. This section is devoted to a new model of ambiguity, the *multiple multiple priors* model, and to a representation of a particular decision method that is based on it. This ambiguity model and the resulting preferences encompass the Knightian and justifiable preferences.

4.1. Multiple multiple priors. Recall the example of the firm given in the introduction. The firm's employees are partially informed regarding nature's true state. They share common sources of information, such as those provided by the firm, however, each of them might obtain private information (depending on business acquaintances and past experiences for example) different than that obtained by the others. Also, it is natural that, due to uncertainty, different agents would end up with different information, some overlapping but some also contradicting each other. For example, each agent ends up with the true expectation of a collection of random variables, or the true probability of a partial collection of events. The list (of random variables or events) obtained by one agent need not be exactly the same as that obtained by the others. Typically information is incomplete, and each agent's ambiguity amounts to a set of priors, each of which is consistent with the information obtained. Therefore, the firm is provided with a collection of sets of priors, and has to base its decisions according to it.

4.2. **Knightian and justifiable preferences combined.** Next we characterize a model which extends both Knightian and justifiable preferences. For this we need to weaken the completeness of $\succeq$ as stated by **A1**.

A1'. $\succeq$ is reflexive, complete over L_c and non–trivial.

Definition 1. *A set $\mathcal{P}$ is a* collection of multiple priors *if each $P \in \mathcal{P}$ is a set of priors. It is* minimal *if it contains no two sets of priors that one contains the other.*

Theorem 2. *For a binary relation $\succeq$ over L the following are equivalent:*
*(1) $\succeq$ satisfies **A1'** and **A2–A5**.*
(2) There exist a minimal collection $\mathcal{P}$ of closed and convex sets of probability distributions over S with $\bigcup_{P\in\mathcal{P}} P$ being closed, and a non–constant affine function $u : Y \to \mathbb{R}$, such that for every two acts f and g,

$$f \succeq g \quad \Leftrightarrow \quad \max_{P\in\mathcal{P}} \min_{p\in P} \{p \cdot (u(f) - u(g))\} \geq 0.$$

Moreover, u is unique up to a positive linear transformation.

Consider the example of the firm discussed in the previous Section 4.1. Theorem 2 suggests that, subject to the assumptions described, the firms' decisions will be based on a few sets of multiple priors, obtained by its agents, in the following sense. [8] An act f is preferred to g if there exists at least one agency that prefers f to g in the Knightian sense. That is, there exists an agent for whom the expected–utility of f is greater than that of g according to every prior consistent with his information.

For a numerical illustration of the model consider the following example. Suppose that an urn contains 30 red balls, and additional 60 balls that are either black or white. A ball is randomly drawn from the urn and the decision maker is given the choice between lottery **R** of receiving \$100 if a red ball is drawn, and lottery **B** of receiving \$100 if a black ball is drawn. Assume that two different sources of information provide the decision maker with two sets of possible distributions over $\{r, b, w\}$, $P_b = \{p \in \Delta(r, b, w) : \ p(r) = \frac{1}{3}, p(b) \geq \frac{1}{2}\}$ and $P_w = \{p \in \Delta(r, b, w) : \ p(r) = \frac{1}{3}, p(w) \geq \frac{1}{6}\}$. With respect to any $p \in P_b$, the expected utility of **B** is greater than that of **R**. However, there are two priors in P_w such

[8]We follow the "observed preferences" interpretation given in the example.

that with respect to one the expected utility of **B** is greater than that of **R**, while the expected utility of **R** with respect to the other is greater than that of **B**. The model presented in Theorem 2 suggests that the decision maker would strictly prefer **B** to **R**.

It is clear by the assumptions of Theorem 2 that this model is an extension of Knightian and justifiable preferences. In terms of representation, when $\mathcal{P}$ consists of a single closed and convex set of probability distributions over S, one obtains Knightian preferences. And when $\mathcal{P}$ consists only of singletons, the resulting preferences are justifiable.

As in the Proof of Theorem 1, we first examine the geometric implications of our assumptions. Recall the binary relation $\succeq^*$ over a closed and convex set $C \subseteq \mathbb{R}^N$ discussed in the Section 2. We describe two additional properties for such a relation. The first is a weakening property (i).
(i'). For every $x \in C$, $x \sim^* x$.

Given (i'), as opposed to (i), $\succeq^*$ need not be complete. However, property (ii) implies that if $x, y \in C$ are two constants, then wither $x \succ^* y$ or $y \succ^* x$ (thus, $\succeq^*$ is complete over the constants in C).

(vi). $x \succeq^* y$ and $z \geq x$ implies that $z \succeq^* y$.

Proposition 2. *Let $\succeq^*$ be a binary relation over $[0,1]^N$. Then the following are equivalent:*
(1) $\succeq^*$ satisfies (i') and (ii)-(vi).
(2) There exists a minimal collection $\mathcal{P}$ of closed and convex sets of probability distributions over N with $\bigcup_{P \in \mathcal{P}} P$ being closed, such that

$$x \succeq^* y \quad \Leftrightarrow \quad \max_{P \in \mathcal{P}} \min_{p \in P} p \cdot (x - y) \geq 0.$$

Proof. First, as in the proof of Proposition 1, the relation $\succeq^*$ is extended to $\mathbb{R}^N$, to obtain an homogeneous binary relation (typically intransitive and incomplete). Furthermore, it satisfies $x \succeq^* y$ if and only if $x - y \succeq^* 0$ for every $x, y \in \mathbb{R}^N$.

Let $K = \{x : x \succeq^* 0\}$. K is, as obtained in the proof Proposition 1, a positive homogeneous closed cone, satisfying $\mathbb{R}^N_+ \subseteq K$. Next we show that K can be split into a union: $K = \bigcup_{j \in J} K_j$, where for every $j \in J$, K_j is a positive homogeneous closed and convex cone that contains $\mathbb{R}^N_+$.

Let $x \in K \setminus \mathbb{R}^N_+$, $y \in \mathbb{R}^N_+$ and $\alpha \in (0,1)$. Then, $\alpha x + (1-\alpha)y \geq \alpha x \succeq^* 0$. By (vi), $\alpha x + (1-\alpha)y \in K$. Thus, $K_x = \text{conv}\{\{\lambda \text{x} : \lambda > 0\} \cup \mathbb{R}^{\text{N}}_+\}$ is a closed and convex cone contained in K, and $K = \bigcup_{x \in K} K_x$.

As in Bewley [3] (see his proof of Theorem 1, p. 105), for every $x \in K$ there exists closed and convex set P_x of probability distributions over N, such that $y \in K_x$ if and only if for every $p \in P_x$, $p \cdot y \geq 0$. Denote by $\mathcal{P}$ the collection of all P_x. We obtained that $y \in K$ if and only if there exists $P \in \mathcal{P}$ such that for every $p \in P$, $p \cdot y \geq 0$. This implies that

$$y \in K \quad \Leftrightarrow \quad \max_{P \in \mathcal{P}} \min_{p \in P} p \cdot y \geq 0 \tag{3}$$

The set P_x is convex and therefore $p \in P_x$ if and only if there exists $y \in K_x$ such that $p \cdot y = 0$ and $y \neq 0$. To show that $\bigcup_{P \in \mathcal{P}} P$ is closed assume that a sequence $\{p^k\}_k \subseteq \bigcup_{P \in \mathcal{P}} P$ converges to p. For every p^k there is $y^k \in K$ such that $p^k \cdot y^k = 0$. As K is a cone, the vector y^k can be normalized to be with norm 1, and since K is closed, $\{y^k\}_k$ has a converging subsequence to a point $x \neq 0$ in K. Thus, $p \cdot x = 0$ and therefore $p \in P_x$. In particular, the limit of $\{p^k\}_k$ is in $\bigcup_{P \in \mathcal{P}} P$, and $\bigcup_{P \in \mathcal{P}} P$ is closed, as desired.

Consider the set of all collections $\mathcal{P}$ of closed and convex sets of distributions that satisfy Eq. (3) and $\bigcup_{P \in \mathcal{P}} P$ is closed. This set is partially ordered by inclusion. A minimum in this order is a minimal collection and it satisfies the properties stated in Proposition 2 (2). In particular, $\min_{p \in P} p \cdot x$ for every $P \in \mathcal{P}$, and $\max_{P \in \mathcal{P}} \min_{p \in P} p \cdot x$ are well defined for every $x \in \mathbb{R}^N$. □

We are now ready to prove Theorem 2.

Proof of Theorem 2. As in the proof of Theorem 1, **A1'** along with the other assumptions ensure that the preference relation restricted to L_c satisfies the hypothesis of the von Neumann–Morgenstern theorem. This assures the existence and uniqueness up to a positive linear transformation of a non–constant, normalized affine function $u : Y \to \mathbb{R}$, which represents this relation.

The binary relation $\succeq^*$ given by $u(f) \succeq^* u(g)$ if and only $f \succeq g$, for every $f, g \in L$, is well defined due to **A2**. Furthermore, $\succeq^*$ satisfies properties (i') and (ii)-(vi). Applying Proposition 2 completes the proof. □

5. Justifiable preferences and partially–specified probabilities

5.1. **Partially–specified probabilities.** In [16], Lehrer suggests a different perception to uncertainty than that of multiple priors, non–additive prior (Schmeidler [22]), and others appearing in the literature. This alternative is information based. The decision maker obtains a *partially–specified probability (PSP)*, in particular, either the true probability of some, but maybe not all, events, or the true expectation of a partial list of random variables. The decision maker then evaluates the alternatives, according to her attitude to uncertainty, utilizing only the PSP and completely ignores unavailable information.

5.2. **Justifiable preferences and PSP.** In this subsection we characterize justifiable preferences that are generated by PSP.

Definition 2. *An act f is* primitive *if for any $\alpha \in [0,1)$:*
(i) for any constant act c such that $f \succeq c$, the inequality $h \succeq \alpha f + (1-\alpha)g$ implies $h \succeq \alpha c + (1-\alpha)g$ and $\alpha c + (1-\alpha)g \succeq h$ implies $\alpha f + (1-\alpha)g \succeq h$; and
(ii) for any constant act c such that $c \succeq f$, the inequality $\alpha f + (1-\alpha)g \succeq h$ implies $\alpha c + (1-\alpha)g \succeq h$ and $h \succeq \alpha c + (1-\alpha)g$ implies $h \succeq \alpha f + (1-\alpha)g$.

Primitive acts are essentially those that keep transitivity in a broad sense. A primitive act f satisfies simple transitivity when constants are involved. That is, $h \succeq f$ and $f \succeq c$ with c being constant imply $h \succeq c$. However, primitive act satisfy a broader sense of transitivity. Assume, for instance, that $f \succeq c$ with c being constant and that $h \succeq \alpha f + (1-\alpha)g$, then c can replace f to produce $h \succeq \alpha c + (1-\alpha)g$. This is transitivity and mixing combined

Remark 3. *Note that the definition immediately implies that whenever f is primitive, and c, d are constants such that $c \succeq f \succeq d$, then $c \succeq d$.*

A7 *Primitives determine preferences.* For any two acts g_1, g_2 the following are equivalent:
(i) $g_1 \succeq g_2$.
(ii) If for every two primitive acts f_1, f_2, $\alpha g_2 + (1-\alpha)f_1 \succeq^S \alpha g_1 + (1-\alpha)f_2$ for every $\alpha \in (0,1)$, then $f_1 \succeq f_2$.
According to **A7** the primitive acts are those which the preference relation should be founded on. It suggests that the fact that $\alpha g_2 + (1-\alpha)f_1$ dominates $\alpha g_1 + (1-\alpha)f_2$ for every $\alpha \in (0,1)$ cannot co–exist with $f_2 \succ f_1$, attests that the domination of $\alpha g_2 + (1-$

$\alpha)f_1$ over $\alpha g_1 + (1-\alpha)f_2$ stems from $f_1 \succeq f_2$ and not from $g_2 \succ g_1$. **A7** states that if the primitive acts do not provide a strong evidence that $g_2 \succ g_1$, by default $g_1 \succeq g_2$.

Theorem 3. *For a binary relation $\succeq$ over L the following are equivalent:*
*(1) $\succeq$ satisfies **A1**-**A5** and **A7**.*
(2) There exist a collection of acts F, a non–constant affine function $u : Y \to \mathbb{R}$ and a prior distribution p over S, such that for every two acts g_1 and g_2, $g_1 \succeq g_2$ if and only if

$$(4)\quad \min\{p\cdot(u(f_1)-u(f_2)) : \gamma[u(f_1)-u(f_2)] \geq u(g_1)-u(g_2),\ f_1, f_2 \in F,\ \gamma > 0\} \geq 0.$$

Furthermore, u is unique up to a positive linear transformation.
(3) There exist a collection of acts F, a non–constant affine function $u : Y \to \mathbb{R}$ and a prior distribution p over S, such that for every two acts g_1 and g_2,

$$g_1 \succeq g_2 \quad \Leftrightarrow \quad \exists q \in P \text{ such that } q\cdot u(g_1) \geq q \cdot u(g_2),$$

where $P = \{q :\ q\cdot u(f) = p\cdot u(f) \text{ for every } f \in F\}$. Furthermore, u is unique up to a positive linear transformation.

Theorem 3 (3) is the PSP version of justifiable preferences. The DM is informed of the expectations, with respect to p, of all the variables $u(f)$, $f \in F$. The set of priors P that consists of all the distributions q that agree with p on F induces a justifiable preference relation.

In order to prove Theorem 3, we need to establish some preliminary results regarding the collection of all primitive acts, and how they relate to one another with respect to $\succeq$.

Lemma 2. ***A2** and **A5** guarantee that every constant act is primitive.*

Proof. Suppose that f, c are constants and $f \succeq c$. By **A5**, $\alpha f+(1-\alpha)g \succeq^S \alpha c+(1-\alpha)g$ for every act g and $\alpha \in [0,1]$. Let $h \succeq \alpha f + (1-\alpha)g$. Then, by **A2**, $h \succeq \alpha c + (1-\alpha)g$. All other implications required in Definition 2 are shown in a similar fashion. □

Lemma 3. ***A1**, **A2**, **A4** and **A5** guarantee that for every primitive act f there is a unique[9] constant act c such that $c \sim f$.*

[9]Here we mean unique up to the equivalence class. That is, if c, c' are constants such that $c \sim f$ and $c' \sim f$, then by Remark 3 $c \sim c'$.

Proof. Consider the collection L_c of all constant acts endowed with the relative topology. Completeness of $\succeq$ implies that any constant is a member of at least one the following closed sets of constants, $\{c \in L_c : c \succeq f\}$ and $\{c \in L_c : f \succeq c\}$. Remark 3 guarantees that the intersection of these sets contains at most one act, and the convexity of the set of constants (which implies connectedness) implies that the intersection contains exactly one act. □

For any primitive act f denote by c_f the constant act that satisfies $c_f \sim f$.

Lemma 4. ***A1**, **A2**, **A4** and **A5** guarantee that the set of primitive acts is convex.*

Proof. Let f_1, f_2 be primitive acts and suppose that $\gamma f_1 + (1-\gamma)f_2 \succeq c$. Let $h \succeq \alpha(\gamma f_1 + (1-\gamma)f_2) + (1-\alpha)g$. We need to show that $h \succeq \alpha c + (1-\alpha)g$. Since f_1, f_2 are primitive acts, $h \succeq \alpha(\gamma c_{f_1} + (1-\gamma)c_{f_2}) + (1-\alpha)g$ and $\gamma c_{f_1} + (1-\gamma)c_{f_2} \succeq c$. Thus, Assumption 2 implies $h \succeq \alpha c + (1-\alpha)g$. All other implications required in Definition 2 are shown in a similar fashion. □

Lemma 5. ***A1**, **A2**, **A4** and **A5** guarantee that $\succeq$ is transitive over the set of primitive acts.*

Proof. Assume f_1, f_2, f_3 are primitive acts that satisfy $f_1 \succeq f_2 \succeq f_3$. Then, $c_{f_1} \succeq f_1 \succeq f_2$ which implies $c_{f_1} \succeq f_2$. On the other hand $c_{f_1} \succeq f_2 \succeq c_{f_2}$ which implies $c_{f_1} \succeq c_{f_2}$. In the same manner $c_{f_2} \succeq c_{f_3}$. By Lemma 1 we have that $c_{f_1} \succeq c_{f_3}$. Now, $f_1 \succeq c_{f_1} \succeq c_{f_3}$ implies $f_1 \succeq c_{f_3}$, and $f_1 \succeq c_{f_3} \succeq f_3$ implies $f_1 \succeq f_3$. □

We are now ready to prove Theorem 3.

Proof of Theorem 3. Let $F \subseteq Y$ be the set of all primitive acts. This set is convex, includes all constant acts, and $\succeq$ is transitive over F. Therefore, there exists a utility function $u : Y \to \mathbb{R}$ and a prior distribution p over S such that for every two primitives f_1 and f_2,

$$f_1 \succeq f_2 \;\Leftrightarrow\; p \cdot u(f_1) \geq p \cdot u(f_2). \tag{5}$$

Now, define a binary relation $\succeq^*$ over all acts as follows: for every two acts g_1 and g_2, $g_1 \succeq^* g_2$ if and only if Eq. (4) holds. Let us show that $\succeq^*$ and $\succeq$ coincide.

Suppose that $g_1 \succeq^* g_2$ and assume to the contrary that $g_2 \succ g_1$. **A7** implies that there exist two primitive acts f_1, f_2 such that $\alpha g_2 + (1-\alpha)f_1 \succeq^S \alpha g_1 + (1-\alpha)f_2$ for every $\alpha \in (0,1)$, and $f_2 \succ f_1$. By Eq. (5) we obtain $p \cdot u(f_2) > p \cdot u(f_1)$. However,

$\alpha g_2 + (1-\alpha) f_1 \succeq^S \alpha g_1 + (1-\alpha) f_2$ implies that $\frac{1-\alpha}{\alpha}[u(f_1) - u(f_2)] \geq u(g_1) - u(g_2)$. Thus, the minimum in Eq. (4) is less than 0, which contradicts the assumption that $g_1 \succeq^* g_2$.

Conversely, assume that $g_1 \succeq g_2$. By **A7** the minimum in Eq. (4) is at least 0, which implies that $g_1 \succeq^* g_2$.

The equivalence of (2) and (3) is a routine application of a separation theorem and is therefore omitted. □

6. Comments

6.1. **Intransitivity and related literature.** May [20] showed that transitivity is not necessary for axiomatic characterizations of preferences, and that intransitivity is a natural result when choosing among alternatives that have several attributes with conflicting criteria. He also showed several natural examples in which transitivity is violated and how experiments may be designed to create such violations.

Tversky [24] created an experimental situation in which individuals revealed consistent patterns of intransitive choices. Tversky was also one of the first to propose a nontransitive preference representing functional, which generalizes transitive preferences.

Following Tversky, many representations of preferences that accommodate intransitivity have been suggested and axiomatized (see Fishburn [6] for a survey on nontransitive preferences in decision theory). The basic form for most representations is a functional J over couples of acts into $\mathbb{R}$, such that $f \succeq g \iff J(f,g) \geq 0$.

Finally, there is an abundance of theoretical results that deal with agents whose preferences are intransitive. Excellent example are Sonnenschein [8] and Kim and Richter [5], who prove equilibrium existence in pure exchange economies where preferences need not be transitive. Also, there is a body of literature in social choice, cooperative decision making and team theory (for example, Fishburn [7] and Marschak and Radner [19]) in which it is customary to abandon transitivity.

6.2. **Extensions of Knightian preferences.** In Subsection 3.2 we mentioned that all Bayesian completions of the Knightian preferences are contained (as subsets) in the justifiable preferences. The same is true for the maxmin extension. It turns out that any complete preferences contained in the justifiable preferences contain the Knightian preferences.

Two issues arise from this observation. The first can be phrased as a question: for given Knightian preferences, what are the continuous and complete extensions that are subsets of the corresponding justifiable preferences? Equivalently, what are the continuous and complete preference relations contained in given justifiable preferences?

The second issue concerns an axiomatization. Beyond the Bayesian and the maxmin preferences one can think of other preference relations, such as the maxmax, that are subsets of justifiable preferences and satisfy completeness and continuity. It would be interesting to axiomatize the continuous and complete preference relations that are subsets of a justifiable preferences (w.l.o.g. the set of priors could be the set of all probability distributions over S). And in the case of such preferences, what in terms of the relation itself is the unique minimal and convex set of priors that generates the corresponding justifiable preferences.

6.3. **General justifiability.** A general notion of justifiability is hidden in the unambiguous transitivity assumption **A2**. Theorem 1 states that a weakened transitivity axiom together with those that yield expected–utility maximization imply an order induced by taking the maximum over a collection of preferences, all admit an expected–utility representation. That is, the decision maker is looking for one out of several expected–utility maximization preferences to justify his choice. Theorem 2 presents a similar result for Knightian preferences. Bewley's axioms [3], with **A2** replacing transitivity, yield a relation induced by taking the maximum over a collection of Knightian preferences. In other words, the decision maker justifies every choice with one Knightian witness out of several others.

Let $\mathcal{A}$ be a set of axioms that includes transitivity. A preference relation is *$\mathcal{A}$–justifiable* if it is the maximum over a collection of preference orders that satisfy $\mathcal{A}$. In light of the observations above, a natural question arises as to what should the set $\mathcal{A}$ be in order to obtain $\mathcal{A}$–justifiable preferences from replacing the transitivity axiom in $\mathcal{A}$ by **A2**?

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