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Lower Equilibrium Payoffs in Two-Player Repeated Games with Non-Observable Actions

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Abstract: We characterize, by the one-shot game terms, the set of lower equilibrium payoffs of the undiscounted repeated game with non-observable actions.

1 Introduction

The classical theory of repeated games deals with standard information, i.e., after each stage of the game the players get information about the actions (of each one of the players) that took place in that stage. [L1] deals with the case in which each player is informed of the equivalence class of the action of each of the other players in the previous stage. Here we refer to the general case in which the actions are non-observable and the information the players get is a function of the actions.

We characterize the Nash lower equilibrium payoffs in undiscounted two-player repeated games by the one-shot game terms. Two sets, C_1 and C_2, of pairs of strategies are defined. C_1 is the set of all the pairs (p_1, p_2), where p_j is a mixed strategy of player j ($j = 1, 2$), which have the following property: Among all those strategies p which satisfy both that p induces the same distribution on the signals of player 2 as p_1 does, and that p does not decrease the possibility to distinguish between actions of player 2, p_1 is the best response against $p_2 \cdot C_2$ is defined in a similar way. By playing $(p_1, p_2) \in C_1$ many times repeatedly, player 2 can detect a deviation of player 1.

The set of the lower equilibrium payoffs is proved to be the payoffs which are both individually rational and included in the intersection of the convex hulls of the payoffs sets associated with C_1 and with C_2.

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2 Definitions and Notations

Definition 2.1: A two-players repeated game G^* with non-observable actions is defined by:

1. Finite sets Σ_1, Σ_2, called *action-sets.*

2. Functions $l_1, l_2; l_i : \Sigma_1 \times \Sigma_2 \to L_i$, $i = 1, 2,$. l_i is called the *information-function* and L_i is called the *signals set* of player i, $i = 1, 2$. l_1 and l_2 satisfy:
   (i) $l_1(s, t) \neq l_1(s', t')$ when $s \neq s'$ for all $t, t' \in \Sigma_2$.
   (ii) $l_2(s, t) \neq l_2(s' t')$ when $t \neq t'$ for all $s, s' \in \Sigma_1$.

3. Functions $h_1, h_2; h_i : \Sigma_1 \times \Sigma_2 \to R$, $i = 1, 2$, called *payoff-functions.*

Notation 2.2: Denote the range of h_i by X_i, $i = 1, 2$.

The sets of pure strategies of a player in the repeated game, denoted by F_i, are defined as follows.

Definition 2.3

$$F_i = \{(f_i^1, f_i^2, f_i^3, \ldots); \text{for each } n \in \mathbb{N},\ f_i^n : L_i^{n-1} \to \Sigma_i\}$$

for $i = 1, 2$, where L_i^0 is any single-element set.

Intuitively, when player i chooses the pure strategy f_i, $i = 1, 2$, the game is played as follows. At the first stage, player i plays f_i^1, gets his payoff $h_i(f_1^1, f_2^1)$, and the signal $l_i(f_1^1, f_2^1)$. At the second stage, player i acts $f_i^2(l_i(f_1^1, f_2^1))$, gets his payoff $h_i(f_1^2(l_1(f_1^1, f_2^1)), f_2^2(l_2(f_1^1, f_2^1)))$ and the signal $l_i(f_1^2(l_1(f_1^1, f_2^1)), f_2^2(l_2(f_1^1, f_2^1)))$, and so forth.

A mixed strategy of player i is a probability measure μ_i on F_i.

Notation 2.4: The set of all the mixed strategies of player i is denoted by $\Delta(F_i)$, $i = 1, 2$.

For each pair of pure strategies $f = (f_1, f_2) \in F_1 \times F_2$ there is a correspondent string of signals $(s_1^n(f), s_2^n(f))_{n=1}^{\infty} \in (L_1 \times L_2)^{\mathbb{N}}$.

The correspondence is defined as follows:

$s_i^0(f)$ is the element of L_i^0

$$\vdots$$

$$s_i^n(f) = l_i(f_1^n(s_1^1(f), s_1^2(f), \ldots, s_1^{n-1}(f)), f_2^n(s_2^1(f), s_2^2(f), \ldots, s_2^{n-1}(f)))$$

$$\vdots$$

There is also a correspondent string of payoffs:

$$(x_1^n(f), x_2^n(f))_{n=1}^{\infty} \in (X_1 \times X_2)^{\mathbb{N}}.$$

This correspondence is defined as follows:

$$x_i^1(f) = h_i(f_1^1, f_2^1)$$

$$\vdots$$

$$x_i^n(f) = h_i(f_1^n(s_1^1(f), \ldots, s_1^{n-1}(f)), f_2^n(s_2^1(f), \ldots, s_2^{n-1}(f))).$$

$$\vdots$$

Let $\mu = (\mu_1, \mu_2) \in \Delta(F_1) \times \Delta(F_2)$. By the correspondences introduced above, two measures are induced: μ_X on $(X_1 \times X_2)^{\mathbb{N}}$, and μ_L on $(L_1 \times L_2)^{\mathbb{N}}$.

Definition 2.5: A *behavior strategy* of player i, $i = 1, 2$, in G^* is a sequence $f_i = (f_i^1, f_i^2, \ldots)$ of functions

$$f_i^n : L_i^{n-1} \to \Delta(\Sigma_i), \quad n = 1, 2, \ldots$$

A pair (f_1, f_2) of behavior strategies induces measure on $F_1 \times F_2$, and thus on $(X_1 \times X_2)^{\mathbb{N}}$ and on $(L_1 \times L_2)^{\mathbb{N}}$.

Remark 2.6: A repeated game with non-observable actions is a game with perfect recall, and thus, by Kuhn's theorem ([A1], [K]), we are allowed to concentrate in behavior strategies whenever it is convenient.

Definition 2.7: Let $\mu = (\mu_1, \mu_2) \in \Delta(F_1) \times \Delta(F_2)$ and $n \in \mathbb{N}$,

$$H_i^n(\mu_1, \mu_2) = \mathrm{Exp}_\mu \left(\frac{1}{n} \sum_{k=1}^{n} x_i^k(f) \right), \quad i = 1, 2.$$

$H_i^n(\mu_1, \mu_2)$ is the expectation of the average-payoff of player i at the n first stages of the repeated game, when μ_1 is the strategy played by player 1, and μ_2 is that played by player 2.

Definition 2.8:

(1) $H_1^*(\mu_1, \mu_2) = \lim_n H_1^n(\mu_1, \mu_2)$ if it exists.

$H_2^*(\mu_1, \mu_2) = \lim_n H_2^n(\mu_1, \mu_2)$ if it exists.

(2) $H^*(\mu_1, \mu_2) = (H_1^*(\mu_1, \mu_2), H_2^*(\mu_1, \mu_2))$ if both H_1^* and H_2^* are defined.

Definition 2.9: $(\mu_1, \mu_2) \in \Delta(F_2) \times \Delta(F_2)$ is a *lower-equilibrium* if:

(i) $H^*(\mu_1, \mu_2)$ is defined.

(ii) For every $\bar{\mu}_1 \in \Delta(F_1)$, $\liminf_n H_1^n(\bar{\mu}_1, \mu_2) \leqslant H_1^*(\mu_1, \mu_2)$, and for every $\bar{\mu}_2 \in \Delta(F_2)$, $\liminf_n H_2^n(\mu_1, \bar{\mu}_2) \leqslant H_2^*(\mu_1, \mu_2)$.

Notation 2.10: LEP = $\{H^*(\mu_1, \mu_2) | (\mu_1, \mu_2)$ is a lower-equilibrium$\}$.

Notation 2.11: If Σ is a set and $s \in \Sigma$, then δ_s will denote the Dirac-measure on s, and will be the measure corresponding to s in the set of the probability measures over $\Sigma : \Delta(\Sigma)$.

Sometimes we will refer to δ_s as s.

Remark 2.12: The functions $h = (h_1, h_2)$ and $l = (l_1, l_2)$ can be extended to $\Delta(\Sigma_1) \times \Delta(\Sigma_2)$ in a natural way, such that h_i and l_i will be ranged to R and to $\Delta(L_i)$ respectively ($i = 1, 2$).

Notation 2.13

(1) $d_1 = \underset{q \in \Delta(\Sigma_2)}{\text{Min}} \; \underset{p \in \Delta(\Sigma_1)}{\text{Max}} \; h_1(p, q)$.

(2) $\tau_1 \in \Delta(\Sigma_1)$ is a strategy which satisfies $d_1 = \underset{q \in \Delta(\Sigma_2)}{\text{Min}} \; h_1(\tau_1, q)$.

(3) $\sigma_2 \in \Delta(\Sigma_2)$ is a strategy which satisfies $d_1 = \underset{p \in \Delta(\Sigma_1)}{\text{Max}} \; h_1(p, \sigma_2)$.

(4) d_2, τ_2 and σ_1 are defined in a similar way.

(5) $\text{IR} = \{(a, b) \in \mathbb{R}^2 \,|\, a \geqslant d_1 \text{ and } b \geqslant d_2\}$. IR is the set of all individually rational payoffs.

3. The Main Theorem

The characterization of the set of lower equilibrium payoffs is done mainly by a partial order defined on $\Delta(\Sigma_i)$. We will give the following definitions for strategies of player 1. One can apply similar definitions for player 2.

Definition 3.1

(1) Let $s, s' \in \Sigma_1$. s is equivalent to s' ($s \sim s'$) if for every $t \in \Sigma_2$ $l_2(s, t) = l_2(s', t)$.

(2) Let $s \in \Sigma_1$. The set $[s] = \{s' \in \Sigma_1 \,|\, s' \sim x\}$ is the *equivalent class* of s.

(3) Let $p, p' \in \Delta(\Sigma_1)$. p is *equivalent* to p' if for every $t \in \Sigma_2$

$$l_2(p', t) = l_2(p, t) \quad \text{(in the sense of Remark 2.12)}$$

In words, $p' \sim p$ if the distributions over the signals of player 2 are the same under p as under p', for any action t.

Definition 3.2

(1) Let $s, s' \in \Sigma_1$. s' is greater than $s (s' \succ s)$ if $s' \sim s$ and if for every $t, t' \in \Sigma_2$

$$l_1(s, t) \neq l_1(s, t') \quad \text{implies} \quad l_1(s', t) \neq l_1(s', t')$$

(2) Let $p, p' \in \Delta(\Sigma_1)$. p' is *greater* than p $(p' \succ p)$ if $p' \sim p$ and if there are two random variables X, X' ranged to Σ_1, with distributions p and p', respectively, and finally $X' \succ X$.

In words, p' is greater than p, in the sense of the partial order $\succ$, if $p' \sim p$ and if by playing p' the player can distinguish between two actions of his opponent with a greater probability than he can do so by playing p.

We could define the relation $\succ$ in another way: $p' \succ p$ if $p' \sim p$ and if there are nonnegative constants $\beta_{s',s}$ such that $p_s = \Sigma_{s'}\beta_{s',s}$, $p'_{s'} = \Sigma_s \beta_{s',s}$ and if $\beta_{s',s} > 0$ then $s' \succ s$.

Definition 3.3

(1) $C_1 = \{(p, q) \in \Delta(\Sigma_1) \times \Delta(\Sigma_2) \mid h_1(p, q) = \operatorname{Max}_{p' \succ p} h_1(p', q)\}$

(2) $C_2 = \{(p, q) \in \Delta(\Sigma_1) \times \Delta(\Sigma_2) \mid h_2(p, q) = \operatorname{Max}_{q' \succ q} h_2(p, q')\}$,

i.e., C_i is the set of pairs of the one-shot game mixed strategies, in which player i cannot profit by any deviation without being discovered by player 3-i, or without decreasing his potential of getting information. Intuitively, if $(p, q) \in C_1$ is played repeatedly many times, then player 1 can profit only by a detectable deviation.

Definition 3.4

(1) $D_1 = \{(p, q) \in \Delta(\Sigma_1) \times \Delta(\Sigma_2) \mid h_1(p, q) = \operatorname{Max}_{p' \sim p} h_1(p', q)\}$

(2) $D_2 = \{(p, q) \in \Delta(\Sigma_1) \times \Delta(\Sigma_2) \mid h_2(p, q) = \operatorname{Max}_{q' \sim q} h_2(p, q')\}$.

Here the element of decreasing the potential to get information is dropped. D_1 and D_2 will play a role whenever at least one of the players has a trivial information function, namely, whenever one player cannot get any information about his opponent's actions. This player, on one hand, cannot lose the possibility of getting information because he has no such possibility, and in the other hand he cannot recognize that his opponent had decreased his possibility of getting information.

Definition 3.5

(1) Player 1 has *trivial information* if for any $s \in \Sigma_1$ and $t, t' \in \Sigma_2, l_1(s, t) = l_1(s, t')$, and a similar definition for player 2.

(2) A game G^* is a *game with trivial information* if at least one player has a trivial information, and otherwise it is a *game with non-trivial information.*

Main Theorem: In a two-players repeated game with non-observable actions the following hold:

(i) If the game is a game with non-trivial information, then

$$\text{LEP} = \text{conv}\, h(C_1) \cap \text{conv}\, h(C_2) \cap \text{IR},$$

(ii) If the game is a game with trivial information, then

$$\text{LEP} = \text{conv}\, h(D_1) \cap \text{conv}\, h(D_2) \cap \text{IR},$$

where, for all $E \subseteq \Delta(\Sigma_1) \times \Delta(\Sigma_2)$, $h(E) = \{h(p, q) | (p, q) \in E\}$.

Example 3.6: Standard information.

A game with standard information is a game where $l_i(s, t) = (s, t)$ for all $(s, t) \in \Sigma_1 \times \Sigma_2$. In such a game, $C_i = D_i = \Delta(\Sigma_1) \times \Delta(\Sigma_2)$, $i = 1, 2$, and therefore LEP $= h(\Delta(\Sigma_1) \times \Delta(\Sigma_2)) \cap$ IR. This, in fact, is a part of the content of the folk theorem.

Example 3.7: Repeated prisoner's dilemma with non-observable actions:

	L	R
T	1,1	4,0
B	0,4	3,3

payoffs

	L	R
T	n,n	n,n'
B	n',n	c,c

signals

In this game a player gets a signal c (for cooperation) only when both players act the cooperative actions. Here $T \not\to B$ and $L \not\to R$, thus LEP is again all the individually rational and feasible payoffs.

Example 3.8: Trivial information for both players.

Let $l_i(s_1, s_2) = s_i$, $i = 1, 2$. Here,

$$D_i = \{(p_1, p_2) | p_i \text{ is the best response against } p_{3-i}\}.$$

Note that $D_1 \cap D_2$ is the set of all Nash equilibria in the one-shot game. In this example we have

$$h(\text{conv}\,(D_1 \cap D_2)) \subset \text{LEP} = \text{conv}\, h(D_1) \cap \text{conv}\, h(D_2) \cap \text{IR}.$$

$$= \text{conv}\, h(D_1) \cap \text{conv}\, h(D_2).$$

Example 3.9: The repeated game of

	L	M	R
U	2,2	0,3	0,0
B	1,1	3,0	0,0
D	0,0	0,0	0,0

payoffs

	L	M	R
U	a,x	a,x'	a,x"
B	a',x	a',x'	a',x"
D	a",y	a",x'	b,x"

signals

$L \sim M$ but $M \not\succ L$, because $l_2(U, L) = x \neq y = l_2(D, L)$ but $l_2(U, M) = x' = l_2(D, M)$. Therefore $h(U, L) = (2,2) \in h(C_2)$. Obviously $(2,2) \in h(C_1)$. $U \succ B$ and $B \succ U$. Therefore (U, M), $(B, L) \notin C_1$ $(h_1(U, L) > h_1(B, L)$, and $h_1(B, M) > h_1(U, M))$. $L \succ M$, therefore $(B, M) \notin C_2$. Also the Nash equilibrium $((1/2, 1/2, 0), (3/4, 1/4, 0))$ of the one-shot game is in $C_1 \cap C_2$. The payoff associated with this equilibrium is $\left(1\frac{1}{2}, 1\frac{1}{2}\right)$, Since $(d_1, d_2) = (0,0)$, we get

$$\text{LEP} = \text{conv}\left\{(0,0), (2,2), \left(1\frac{1}{2}, 1\frac{1}{2}\right)\right\} = \text{conv}\ \{(0,0), (2,2)\}.$$

Example 3.10: If we would change the former example so that $l_2(U, M) = z$ then U would not be equivalent to B any more and thus $(U, M) \in C_1 \cap C_2$ and

$$\text{LEP} = \text{conv}\ \{(0,0), (2,2), (0,3)\}.$$

In examples 3.7, 3.9 and 3.10, $h(C_1 \cap C_2) = h(C_1) \cap h(C_2)$. However, in the following example the situation is different:

Example 3.11: The repeated game of

	L	M	R
U	3,0	2,2	0,0
B	2,2	$2\frac{1}{2}$,3	0,0
D	0,0	0,0	0,0

payoffs

	L	M	R
U	a,x	a,x'	a,x"
B	a,x	a,x'	c,x"
D	a",x	a",z	a",x"

signals

$M \succ L$ so (B, L), $(U, L) \notin C_2$. $B \succ U$ then $(U, M) \notin C_1$. However, $(U, M) \in C_2$ and since $U \not\succ B$, $(B, L) \in C_1$. Hence,

$$(2,2) \in \text{conv}\ h(C_1) \cap \text{conv}\ h(C_2) \setminus \text{conv}\ h(C_1 \cap C_2).$$

$(2^1/_2, 3)$ is also included in $h(C_1) \cap h(C_2)$. Thus,

$$\text{LEP} = \text{conv}\ \{(0,0)(2,2), (2^1/_2, 3)\}.$$

Lemma 3.12: $h(C_1)$ and $h(C_2)$ are closed sets.

Proof: We will prove that C_1 is a closed set. Let $\{(p_n, q_n)\}_{n=1}^{\infty} \subset C_1$ be a sequence that converge to (p, q). If $(p, q) \notin C_1$ then there is $p' \succ p$ and $\epsilon > 0$ s.t. $h_1(p', q) > h_1(p, q) + \epsilon$. In particular there is a set function $\phi(s)$ and constants $(\beta_{s',s})_{s' \in \phi(s)}$ such that $\delta_{s'} \succ \delta_s$ for $s' \in \phi(s)$, $p = \Sigma\, \alpha_s \delta_s$, $\alpha_s = \sum_{s' \in \phi(s)} \beta_{s',s}$ and $p' = \sum_{s \in \Sigma_1} \sum_{s' \in \phi(s)} \beta_{s',s} \delta_{s'}$.

Denote $p_n = \sum_{s \in \Sigma_1} \alpha_s^n \delta_s$. Let $(\beta_{s',s}^n)_{s' \in \phi(s)}$ be a vector in the set $\{(\bar\beta_{s',s})_{s' \in \phi(s), s \in \Sigma_1} \mid \bar\beta_{s',s} \geqslant 0, \sum_{s' \in \phi(s)} \bar\beta_{s',s} = \alpha_s^n$ for all $s\}$ which achieves the minimum distance (with respect to the maximum norm) from the vector $(\beta_{s',s})_{s' \in \phi(s)}$. Define $p'_n = \sum_{s \in \Sigma_1} \sum_{s' \in \phi(s)} \beta_{s',s}^n \delta_{s'}$. Obviously $p'_n \succ p_n$ and $p'_n \to p'$. By the continuity of h, whenever n is big enough, $h_1(p'_n, q_n) > h_1(p_n, q_n) + \epsilon/2$, a contradition to the fact that $(p_n, q_n) \in C_1$.

Q.E.D.

Lemma 3.13: Let L be a straight line in $\mathbb{R}^2$ s.t. conv $h(C_1) \subseteq L^+$ (the open half of the plan). Then there is an $\alpha > 0$ such that for any $h(p_1, p_2) \in L^- = (L^+)^c$, there is $p'_1 \succ p_1$ s.t. $h_1(p'_1, p_2) > h_1(p_1, p_2) + \alpha$.

Proof: Assume to the contrary that for any n there is (p_n, q_n) s.t. $h(p_n, q_n) \in L^-$, but for all $p' \succ p_n$, $h_1(p', q_n) \leqslant h_1(p_n, q_n) + 1/n$.

We can assume that (p_n, q_n) tends to (p, q). L^- is closed. Therefore, $h(p, q) \in L^-$ and $(p, q) \notin C_1$. By a similar argument to that of the previous lemma, if $p' \succ p$ and $h_1(p', q) > h_1(p, q) + \epsilon$ for a certain $\epsilon > 0$, then one can find $p'_n \succ p_n$ so that $h_1(p'_n, q_n) > h_1(p_n, q_n) + \alpha/2$ for any sufficiently large n, which contradicts the assumption.

Q.E.D.

Denote[2] for any $\gamma > 0$ $L_\gamma^- = \{x \in L^- \mid \text{dist}(x, L) \geqslant \gamma\}$,

Lemma 3.14: Let L be as in the preceding lemma. Then there is a $\delta > 0$ such that if

$$\sum_{k=1}^{l} \alpha_k h(p_k, q_k) \in L_\gamma^- \quad \text{where } \sum_{1}^{l} \alpha_k = 1 \text{ and } \alpha_k \geqslant 0, \quad k = 1, \ldots, l$$

[2] If $x \in \mathbb{R}^n$ and $A \subsetneq \mathbb{R}^n$ is a closed set then $\text{dist}(x, A) = \min_{y \in A} \|x - y\|_\infty$.

then

$$\sum_{h(p_k, q_k) \in L^-} \alpha_k > \delta .$$

Proof: Clear. Q.E.D.

Lemma 3.15

(i) If $(p, q) \notin C_1$ then there is $p' \succ p$ s.t. $(p', q) \in C_1$, $p = \sum_{s \in \Sigma_1} \alpha_s \delta_s$ and $p' = \sum_{s \in \Sigma_1} \alpha_s \delta_{\phi(s)}$, where $\delta_{\phi(s)} \succ \delta_s$.

(ii) If $(p, q) \notin C_2$ then there is $q' \succ q$ s.t. $(p, q') \in C_2$, $q = \sum_{s \in \Sigma_2} \alpha_s \delta_s$ and $q' = \sum_{s \in \Sigma_2} \alpha_s \delta_{\phi(s)}$, where $\delta_{\phi(s)} \succ \delta_s$.

Proof: We will prove (i).

Let $p'' \succ p$ be the strategy which achieves Max $\{h_1(\hat{p}, q) | \hat{p} \succ p\}$. In particular there are set function $\bar{\phi}(s)$ and constants $(\beta_{s',s})_{s' \in \bar{\phi}(s), s \in \Sigma_1}$ which satisfy $\sum_{s' \in \phi(s)} \beta_{s',s} = \alpha_s$ and $\delta_{s'} \succ \delta_s$ for all $s' \in \bar{\phi}(s)$. Let $\phi(s) \in \bar{\phi}(s)$ be one of the actions in $\phi(s)$ which achieves Max $\{h_1(\delta_{s'}, q) | s' \in \bar{\phi}(s)\}$. Define $p' = \sum \alpha_s \delta_{\phi(s)}$. By definition, $h_1(p', q) \geqslant h_1(p'', q)$, and $p' \succ p$. Now if $(p', p) \notin C_1$ then there is $\bar{p} \succ p'$ s.t. $h_1(\bar{p}, q) > h_1(p', q)$. The partial order $\succ$ is transitive, so $\bar{p} \succ p$ and we have got $h_1(\bar{p}, q) > h_1(p', q) \geqslant h_1(p'', q)$, in contradiction to the choice of p''. Q.E.D.

Lemma 3.16: Let $0 < \epsilon < 1$ and $v, r', r \in \Delta(\Sigma_i)$ so that $r \succ r'$. Then

$$(1 - \epsilon)v + \epsilon r \succ (1 - \epsilon)v + \epsilon r'.$$

Proof: Clear. Q.E.D.

Lemma 3.17: Let $(p, q) \in \Delta(\Sigma_1) \times \Delta(\Sigma_2)$. Then,

(1) If $(p, q) \in C_1$ and $p = \sum_{a \in \Sigma_1} \alpha_a \delta_a$ ($\alpha_a \geqslant 0$, $\sum_{a \in \Sigma_1} \alpha_a = 1$), then $\alpha_a > 0$ implies that $(\delta_a, q) \in C_1$.

(2) If $(p, q) \in C_2$ and $q = \sum_{b \in \Sigma_2} \beta_b \delta_b$ ($\beta_b \geqslant 0$, $\sum_{b \in \Sigma_2} \beta_b = 1$), then $\beta_b > 0$ implies that $(p, \delta_b) \in C_2$.

Proof: We will prove (1) and by a similar argument one can prove (2).

If the conditions of (1) hold but there is one action $a \in \Sigma_1$ s.t. $\alpha_a > 0$ and $(\delta_a, q) \notin C_1$, then there is a strategy $r \in \Delta(\Sigma_1)$ s.t. $r \succ a$ and $h_1(r, q) > h_1(\delta_a, q)$. Define $p' = \sum_{b \neq a} \alpha_b \delta_b + \alpha_a r$. $p' \succ p$ by the preceding lemma. Furthermore,

$$h_1(p', q) = \sum_{b \neq a} \alpha_b \cdot h_1(\delta_b, q) + \alpha_a h_1(r, q)$$

$$= h_1(p, q) - \alpha_a(h_1(r, q) - h_1(\delta_a, q)) > h_1(p, q).$$

This is in contradiction with $(p, q) \in C_1$. Q.E.D.

4 Proof of the Main Theorem

The proof is divided into four steps; the first three steps deal with the non-trivial information.

Step 1: LEP $\subseteq$ IR.

Step 2: LEP $\subseteq$ conv $h(C_1) \cap$ conv $h(C_2) \cap$ IR.

Step 3: conv $h(C_1) \cap$ conv $h(C_2) \cap$ IR $\subseteq$ LEP.

The fourth step deals with the trivial information:

Step 4: LEP = conv $h(D_1) \cap$ conv $h(D_2) \cap$ IR.

At steps 1 and 2 we will concentrate only in behavior-strategy. (Recall Definition 2.5 and Remark 2.6.)

Step 1: LEP $\subseteq$ IR.

Let (f_1, f_2) be a pair of behavior-strategies. If $H_1^*(f_1, f_2) < d_1$, then by deviating to the behavior-strategy $g_1 = (g_1^1, g_1^2, \ldots)$ where, for each n, g_1^n is defined to be τ_1, player 1 can increase his expected payoff, i.e., liminf $\text{Exp}_{(g_1, f_2)}\left(\frac{1}{n}\sum_{k=1}^{n} x_1^k\right) \geqslant d_1$. We have got that (f_1, f_2) is not a lower-equilibrium strategy.

Step 2: LEP $\subseteq$ conv $h(C_1) \cap$ conv $h(C_2) \cap$ IR.

Assume that $H^*(f_1, f_2) \in$ IR $\setminus$ (conv $h(C_1) \cap$ conv $h(C_2)$). Therefore, without loss of generality it can be assumed that $H^*(f_1, f_2) \notin$ conv $h(C_1)$. According to Lemma 3.12, conv $h(C_1)$ is a closed set. Hence, there is a separation line L, that divides $\mathbb{R}^2$ into two parts: L^- (the close one) and L^+. $H^*(f_1, f_2) \in L^-_{2\gamma}$ and conv $h(C_1) \subseteq L^+$. (Recall the notation of L^-_γ before Lemma 3.14.)

In order to define the behavior-strategy $\bar{f}_1$ by which player 1 can increase his expected payoff, we first have to prove a few lemmata.

Definition 4.1: Let V, U be finite sets, and P a probability measure on $V \times U$. If there are non-negative constants $\{x_v\}_{v \in V}$, $\{y_u\}_{u \in U}$, and a $\{0, 1\}$-valued function $\phi(v, u)$ s.t. $P(v, u) = \phi(v, u) \cdot x_v \cdot y_u$, then P is $(\{x_v\}_{v \in V}, \{y_u\}_{u \in U}, \phi)$-*semi-independent* or simply *semi-independent.*

Lemma 4.2: Let A, B and $\bar{B}$ be finite sets, μ is a $(\{x_a\}_{a \in A}, \{y_b\}_{b \in B}, \phi)$-semi-independent probability on $A \times B$, and σ is a $(\{x_a\}_{a \in A}, \{z_{\bar{b}}\}_{\bar{b} \in \bar{B}}, \bar{\phi})$-semi-independent probability on $A \times \bar{B}$. Also let

$$g : B \to \Delta^u \quad \text{and} \quad \psi : A \times B \to A \times \bar{B}.$$

Suppose that the following three conditions hold:[3]

(1) If $(a, b) \in \text{supp}(\mu)$ then $\psi_1(a, b) = a$, where $\psi = (\psi_1, \psi_2)$,

(2) $\mu(\psi^{-1}(a, \bar{b})) = \sigma(a, \bar{b})$,

(3) $\psi(a, b) = (a, \bar{b})$, $\psi(a', b') = (a', \bar{b})$ and $(a, b), (a', b') \in \text{supp}(\mu)$ imply that (a, b'), $(a', b) \in \text{supp}(\mu)$ and $\psi_2(a, b') = \psi_2(a', b) = \bar{b}$.

Then, there is a function $\bar{g} : \bar{B} \to \Delta^u$ s.t.

$$E_\mu(g|a) = E_\sigma(\bar{g}\,|a) \qquad \text{for every } a \in A.$$

Proof: Denote by μ_1, μ_2 the marginal probabilities of μ on A and B respectively, and by σ_1, σ_2 the marginal probabilities of σ on A and $\bar{B}$ respectively. By (1), (2) we get for every $a \in A$, $\mu_1(a) = \sigma_1(a)$.

---

3 supp (μ) and supp (σ) are the supports of μ and σ in $A \times B$ and in $A \times \bar{B}$ respectively.

By (3), if $(a, \bar{b}), (a', \bar{b}) \in \text{supp}(\sigma)$ and if $\psi^{-1}(a, \bar{b}) \cap \text{supp}(\mu) = \{a\} \times B_1, \psi^{-1}(a', \bar{b}) \cap \text{supp}(\mu) = \{a'\} \times B_2$, then, $B_1 = B_2$.

Thus, we can define by $B(\bar{b})$ the projection of $\psi^{-1}(a, \bar{b}) \cap \text{supp}(\mu)$ to B, for some $(a, \bar{b}) \in \text{supp}(\sigma)$. $B(\bar{b})$ is well defined.

Let $(a, \bar{b}) \in \text{supp}(\sigma)$. Writing $\phi(a, b)$ as ϕ_{ab} we get,

$$\sigma(a\ \bar{b}) = x_a z_{\bar{b}} = \mu(\psi^{-1}(a, \bar{b})) = \sum_{b \in B(\bar{b})} x_a y_b \phi_{ab} = \sum_{b \in B(\bar{b})} x_a y_b$$

Hence,

$$\sum_{b \in B(\bar{b})} y_b = z_{\bar{b}}.$$

Define for every $\bar{b} \in \bar{B}$

$$\bar{g}(\bar{b}) = \frac{\sum_{b \in B(\bar{b})} y_b \cdot g(b)}{z_b}.$$

Now,

$$E_\mu(g|a) = \sum_{b \in B} \frac{\mu(a, b) \cdot g(b)}{\mu_1(a)} = \sum_{b \in B} \frac{\phi_{ab} \cdot x_a \cdot y_b \cdot g(b)}{\mu_1(a)}$$

$$= \sum_{(a, b) \in \text{supp}(\mu)} \frac{x_a \cdot y_b \cdot g(b)}{\mu_1(a)}$$

$$= \sum_{\bar{b} \in \bar{B}} \bar{\phi}_{a\bar{b}} \sum_{\substack{(a, b) \in \psi^{-1}(a, \bar{b}) \\ (a, b) \in \text{supp}(\mu)}} \frac{x_a \cdot y_b \cdot g(b)}{\mu_1(a)}$$

$$= \sum_{\bar{b} \in \bar{B}} \frac{x_a \bar{\phi}_{a\bar{b}}}{\mu_1(a)} \sum_{\substack{(a, b) \in \psi^{-1}(a, \bar{b}) \\ (a, b) \in \text{supp}(\mu)}} y_b \cdot g(b)$$

$$= \sum_{\bar{b} \in \bar{B}} \frac{x_a \bar{\phi}_{a\bar{b}}}{\mu_1(a)} \sum_{b \in B(\bar{b})} y_b \cdot g(b)$$

$$= \sum_{\bar{b} \in \bar{B}} \frac{x_a}{\mu_1(a)} \bar{\phi}_{a\bar{b}} \cdot z_{\bar{b}} \cdot g(\bar{b})$$

$$= \sum_{\bar{b} \in \bar{B}} \frac{\sigma(a, \bar{b})}{\sigma_1(a)} \bar{g}(\bar{b}) = E_\sigma(\bar{g}|a). \qquad \text{Q.E.D.}$$

Lemma 4.3: Let (e_1, e_2) be a pair of behavior strategies $e_i = (e_i^1, e_i^2, \ldots), i = 1, 2$, and $n \in \mathbb{N}$. If μ_n is the probability induced by $((e_1^k), (e_2^k))_{k=1}^n$ on $L_1^n \times L_2^n$, then μ_n is semi-independent.

Proof: Through induction on n.

For $n = 1$, define $\phi_1 : L_1 \times L_2 \to \{0, 1\}$.

If $(a, b) \in L_1 \times L_2$, then

$$\phi_1(a, b) = \begin{cases} 1 & \text{if there are } u \in \Sigma_1 \text{ and } v \in \Sigma_2 \text{ s.t. } l_1(u, v) = a \text{ and } l_2(u, v) = b \\ 0 & \text{otherwise} \end{cases}$$

and

$$\text{prob}_{(e_1, e_2)}(a, b) = e_1^1(u) \cdot e_2^1(v) \cdot \phi_1(a, b),$$

where

$$(l_1(u, v), l_2(u, v)) = (a, b)$$

and

$$e_i^1 = (e_i^1(1), e_i^1(2), \ldots, e_i^1(|\Sigma_i|)) \in \Delta(\Sigma_i), \quad i = 1, 2.$$

Furthermore, since a player knows his actions, the same u is good for every $b' \in L_2$. I.e. for every $b' \in L_2$, there is $v' \in \Sigma_2$ s.t.

$$\text{prob}_{(e_1, e_2)}(a, b') = e_1^1(u) \cdot e_2^1(v') \cdot \phi_1(a, b')$$

and the same v is good for all $a' \in L_1$. That concludes the proof of $n = 1$.

By assuming that $\text{prob}_{(f_1,f_2)}(\cdot)$ reduced to $(L_1 \times L_2)^n$ is $(\{x_a\}_{a\in L_1^n}, \{y_b\}_{b\in L_2^n}, \phi_n)$-semi-independent, we will prove that $\text{prob}_{(f_1,f_2)}(\cdot)$ reduced to $(L_1 \times L_2)^{n+1}$ is semi-independent.

Let there be $a' \in L_1^{n+1}$ and $b' \in L_2^{n+1}$. Denote the first n coordinates of a' by a and its last coordinate by α, and the first n coordinates of b' by b, and its last one by β.

$$\text{prob}_{(e_1,e_2)}(a', b') = x_a \cdot y_b \cdot \phi_n(a, b) \cdot e_1^n(a)(u) \cdot e_2^n(b)(v) \cdot \phi_1(\alpha, \beta),$$

where

$$e_1^n(a) = (e_1^n(a)(1), e_1^n(a)(2), \ldots, e_1^n(a)(|\Sigma_1|)) \in \Delta(\Sigma_1)$$

$$e_2^n(a) = (e_2^n(a)(1), e_2^n(a)(2), \ldots, e_2^n(a)(|\Sigma_2|)) \in \Delta(\Sigma_2)$$

and

$$l_1(u, v) = \alpha, \quad l_2(u, v) = \beta.$$

Furthermore, the constant $x_a \cdot e_1^n(a)(u)$ holds for every $b' \in L_2^{n+1}$, and the constant $y_b \cdot e_2^n(b)(v)$ holds for every $a' \in L_1^{n+1}$. Set $\phi_{n+1}(a', b') = \phi_n(a, b) \cdot \phi_1(\alpha, \beta)$. This concludes the proof of the inductive step. Q.E.D.

$\bar{f}_1$ will be defined in the following way. To begin with, a sequence of behavior strategies of player $1: g_1, g_2, \ldots$, will be defined. This sequence will satisfy the following properties:

(P1) $(g_n^1, \ldots, g_n^n) = (g_{n+1}^1, \ldots, g_{n+1}^n), \quad n = 1, 2, \ldots.$

In words, g_{n+1} coincides with g_n on the first n functions.

(P2) There is a constant $\alpha > 0$ and an integer N s.t. if $n > N$, then

$$H_1^n(g_n, f_2) > H_1^n(f_1, f_2) + \alpha$$

(recall Definition 2.7).

After this, $\bar{f}_1$ will be defined by

$$\bar{f}_1^n = g_n^n, \quad n = 1, 2, \ldots.$$

In Lemma 4.4 it will be proved that if $\{g_n\}_{n=1}^{\infty}$ is a sequence as here described, then $\bar{f}_1$ is a "good" deviating strategy for player 1.

Lemma 4.4: If there is a sequence $\{g_n\}_{n=1}^{\infty}$ of player 1's strategies in the repeated game that have properties (P1) and (P2), then, provided that $\bar{f}_1$ is defined to be the diagonal, i.e., $\bar{f}_1^n = g_n^n (n = 1, 2, \ldots)$, we have

$$\liminf_n H_1^n(\bar{f}_1, f_2) > H_1^*(f_1, f_2).$$

Proof: By (P2), there are $\alpha > 0$ and an integer N s.t. if $n > N$, then

$$H_1^n(g_n, f_2) > H^*(f_1, f_2) + \alpha.$$

The desired inequality holds because by (P1) and the definition of $\bar{f}$, $H_1^n(\bar{f}_1, f_2) = H_1^n(g_n, f_2)$. Q.E.D.

Define now the sequence $\{g_n\}_{n=1}^{\infty}$ by induction. $g_0 = f_1$. Assume that $g_1, \ldots, g_{n-1}$ were defined to be behavior-strategies of player 1 which satisfy (P1). Namely, g_{i+1} coincides with g_i on the first i functions, $1 \leqslant i \leqslant n-1$. Assume, furthermore, that these behavior-strategies satisfy the following properties:

For any integers $1 \leqslant i \leqslant n-1, i < m$ and $w \in L_2^{m-1}$

$$\sum_{v \in L_1^{m-1}} \text{prob}_{(f_1, f_2)}(v|w) \cdot f_1^m(v) = \sum_{v \in L_1^{m-1}} \text{prob}_{(g_i, f_2)}(v|w) \cdot g_i^m(v). \tag{4.1}$$

and for all $1 \leqslant i \leqslant n-1$

$$E_{(g_i, f_2)}(x_1^i, x_2^i) \in \text{conv}\, h(C_1). \tag{4.2}$$

In words, in player 2's point of view, player 1 plays the same strategy, no matter if he follows the strategy f_1 or the strategy g_i. g_n will be defined as follows:

$$g_n^i = g_{n-1}^i, \quad i = 1, \ldots, n-1. \tag{4.3}$$

Denote for every $a \in L_1^{n-1}$,

$$k_n(a) = \sum_{b \in L_2^{n-1}} \text{prob}_{(g_n, f_2)}(b \mid a) \cdot f_2(b). \tag{4.4}$$

Let $a \in L_1^{n-1}$. If $(g_{n-1}^n(a), k_n(a)) \in C_1$, then define $g_n^n(a) = g_{n-1}^n(a)$. However, if $(g_{n-1}^n(a), k_n(a)) \notin C_1$, then there is a strategy $p(a) \in \Delta(\Sigma_1)$, s.t. $p(a) \succ g_{n-1}^n(a)$, and $(p(a), k_n(a)) \in C_1$. So, define:

$$g_n^n(a) = p(a). \tag{4.5}$$

At this point we have for each $a \in L_1^{n-1}$

$$(g_n^n(a), k_n(a)) \in C_1 \quad \text{and} \quad g_n^i = g_{n-1}^i \quad \text{for } i \leqslant n-1.$$

In order to define g_n^k for $k > n$ in such a way that it will satisfy (4.1) for $i = n$, we have to use Lemma 4.2.

Denote $L_2^n = A$, $L_1^n = B = \bar{B}$.

$$g = g_{n-1}.$$

$$\mu = \text{prob}_{(g_{n-1}^i, f_2^i)_{i=1}^n}(\cdot),$$

i.e., μ is the probability induced by $(g_{n-1}^i, f_2^i)_{i=1}^n$ on $A \times B$.

$$\sigma = \text{prob}_{(g_n^i, f_2^i)_{i=1}^n}(\cdot).$$

According to Lemma 4.3, μ and σ are semi-independent and, by the proof of Lemma 4.3, we learn that the constants of μ and of σ on A are the same ones. There remains to define $\psi : A \times B \to A \times \bar{B}$. Fix $u \in L_1^{n-1}$. Denote for the moment $g_{n-1}^n(u) = (\alpha_1, \ldots, \alpha_{|\Sigma_1|})$. Since $g_n^n(u) = p(u) \succ g_{n-1}^n(u)$, by Lemma 3.15, $p(u)$ can be chosen to be:

$$p(u) = \sum_{s \in \Sigma_1} \alpha_s \delta_{\phi(s)}, \quad \text{where } \phi(s) \succ s.$$

If $u \in L_1^{n-1}$, $v \in L_2^{n-1}$, $s \in \Sigma_1$ and $t \in \Sigma_2$, let u and $l_1(s, t)$ be joined to become a string of signals in $\in L_1^n$, and let v and $l_2(s, t)$ be joined to become a string of signals in L_2^n; denote them by $(u, l_1(s, t))$, and by $(v, l_2(s, t))$ respectively.

Define for every $s \in \Sigma_1$, $t \in \Sigma_2$ and $(u, v) \in L_1^{n-1} \times L_2^{n-1}$

$$\psi((v, l_2(s, t)), (u, l_1(s, t))) = ((v, l_2(\phi(s), t)), (u, l_1(\phi(s), t))).$$

On all the remaining points of $A \times B$, ψ can be defined arbitrarily. $\phi(s) \succ s$, in particular $\phi(s) \sim s$; therefore,

$$l_2(\phi(s), t) = l_2(s, t) \quad \text{and} \quad (v, l_2(\phi(s), t)) = (v, l_2(s, t)).$$

So, ψ satisfies (1) of Lemma 4.2.

In order to prove that ψ satisfies (3), assume that

$$\psi_2((v, l_2(s, t)), (u, l_1(s, t))) = \psi_2((v', l_2(s', t')), (u', l_1(s', t')));$$

then, $u = u'$ and $l_1(\phi(s), t)) = l_1(\phi(s'), t')$. Because player 1 knows his actions $\phi(s) = \phi(s')$ and because $\phi(s) \succ s$ and $\phi(s') \succ s'$, we get $l_1(s, t) = l_1(s, t')$ and $l_1(s', t) = l_1(s', t')$. Furthermore $\phi(s) \sim s$ and $\phi(s') \sim s'$. Thus, $s \sim s'$, in particular $l_2(s, t) = l_2(s', t)$ and $l_2(s', t') = l_2(s, t')$. If $\mu((v, l_2(s, t)), (u, (l_1(s, t))) > 0$ and $\mu(v', l_2(s', t')), (u', l_1(s', t'))) > 0$, then $g_{n-1}^{n-1}(u)(s)$, $g_{n-1}^{n-1}(u')(s')$, $f_2^{n-1}(v)(t)$ and $f_2^{n-1}(v')(t')$ are all positive numbers. We have got that

$$((v', l_2(s', t')), (u, l_1(s, t))) = ((v, l_2(s, t')), (u, l_1(s, t')))$$

and

$$((v, l_2(s, t)), (u, l_1(s', t'))) = ((v, l_2(s', t)), (u, l_1(s', t)))$$

are in supp (μ). The other conclusion required in (3) follows immediatedly from the definition of ψ on the points of this form.

The proof that ψ satisfies (2) is derived from the definition of g_n^n.

Apply, now, Lemma 4.2 to get $\bar{g}$. Define g_n^{n+1} to be $\bar{g}$. We have got

$$E_{(g_n^i, f_2^i)_1^n}(g_n^{n+1} \mid w) = E_{(g_{n-1}^i, f_2^i)_1^n}(g_{n-1}^{n+1} \mid w)$$

for every $w \in L_2^n$. Moreover, by the definition of g_n^n

$$\text{prob}_{(g_n^i, f_2^i)_1^n}(w) = \text{prob}_{(g_{n-1}^i, f_2^i)_1^n}(w),$$

thus,

$$E_{(g_n^i, f_2^i)_1^{n+1}}(x_i^{n+1}) = E_{(g_{n-1}^i, f_2^i)_1^{n+1}}(x_i^{n+1}), \quad i = 1, 2,$$

i.e., the expected payoff for both players in the $n + 1$ stage is the same, whether g_{n-1}^{n+1} or g_n^{n+1} is played by player 1.

By applying Lemma 4.2 repeatedly, we will define g_n^l for all $l > n + 1$, and get the strategy g_n.

(4.1) for $i = n$ is given by the following: if $n \leqslant m$ and $w \in L_2^{n-1}$, then by the definition of $(g_n^j)_{j=1}^m$ and by adding (4.1) for $i < n$, we get:

$$E_{(g_n^j, f_2^j)_{j=1}^{m-1}}(g_n^m \mid w) = E_{(g_{n-1}^j, f_2^j)_{j=1}^{m-1}}(g_{n-1}^m \mid w) = E_{(f_1, f_2)}(f_1^m \mid w).$$

We have got $g_1, \ldots, g_n$ which satisfy (4.1) and (4.2) for $1 \leqslant i \leqslant n$. Continue inductively this way in order to get the sequence $g_1, g_2, \ldots$. It remains to prove that this sequence has (P1) and (P2). (P1) results immediately from the definition (see (4.3)). In order to prove that the sequence $g_1, g_2, \ldots$ has (P2) we need some notions and lemmata:

Definition 4.5: Let M be a set of integers. The *lower density of M,* denoted by LD(M), is $\liminf_t \# M \cap \{1, \ldots, t\}/t$.

Lemma 4.6: If $H^*(f_1, f_2) \notin \text{conv}\, C_1$, then the set $M = \{n \in \mathbb{N} \mid E(x_1^n, x_2^n) \in L_\gamma^-\}$ has a positive lower density, namely, LD(M) $= \eta > 0$.

Proof: Clear.

Let $n \in M$. Because of (4.1),

$$E_{(g_{n-1}, f_2)}(x_1^n, x_2^n) = E_{(f_1, f_2)}(x_1^n, x_2^n) \in L_\gamma^- \tag{4.6}$$

By Lemma 3.14 there is a $\delta > 0$ such that

$$\text{prob}_{(g_{n-1},f_2)}\{a \in L_1^{n-1} \mid h(g_{n-1}^n(a), k_n(a)) \in L^-\} > \delta \tag{4.7}$$

(recall (4.4)).

By Lemma 3.13 and by (4.5), there is $\alpha > 0$ such that if $h(g_{n-1}^n(a), k_n(a)) \in L^-$, then

$$h_1(g_n^n(a), k_n(a)) > h_1(g_{n-1}^n(a), k_n(a)) + \alpha. \tag{4.8}$$

(4.6), (4.7) and (4.8) give that

$$E_{(g_n,f_2)}(x_1^n) > E_{(g_{n-1},f_2)}(x_1^n) + \delta \cdot \alpha = E_{(f_1,f_2)}(x_1^n) + \delta \cdot \alpha. \tag{4.9}$$

Because the sequence $g_1, g_2, \ldots$ has (P1), by (4.9) and according to Lemma 4.6, if n is big enough, then

$$H_1^n(g_n, f_2) > H_1^*(f_1, f_2) + \delta \cdot \alpha \cdot \eta/2.$$

This means that the sequence $g_1, g_2, \ldots$ has also (P2), and the proof of this step is finished.

Step 3: conv $h(C_1) \cap$ conv $h(C_2) \cap$ IR $\subseteq$ LEP.

We will show that for every $(\alpha_1, \alpha_2) \in h(\text{conv } C_1) \cap h(\text{conv } C_2) \cap \text{IR}$, there is a lower equilibrium strategy $f = (f_1, f_2)$ s.t. $H^*(f) = (\alpha_1, \alpha_2)$.

Let $(\alpha_1, \alpha_2) \in h(\text{conv } C_1) \cap h(\text{conv } C_2)$. By the Caratheodory Theorem, for each $i \in \{1, 2\}$ there are 3 pairs of mixed strategies $\{(p_{i,l}, q_{i,l})\}_{l=1}^3 \subseteq C_i$ and three positive constants γ_i^l, $l = 1, 2, 3$, with total sum 1 so that

$$\sum_{l=1}^{3} \gamma_i^l \cdot h(p_{i,l}, q_{i,l}) = (\alpha_1, \alpha_2).$$

Furthermore, by Lemma 3.17, $p_{1,l}$ is a pure strategy of player 1 and $q_{2,l}$ is a pure strategy of player 2, $l = 1, 2, 3$.

In order to define f we need the following notation:

Notation 4.7: Let $\epsilon > 0$ and $x = (x_1, \ldots, x_n) \in \Delta^n$, the simplex of dimension $n-1$.

x^ϵ is the point in $\{(y_1, \ldots, y_n) \in \Delta \mid y_i \geqslant \epsilon, 1 \leqslant i \leqslant n\}$,

which achieves the minimum distance from x with respect to the maximum norm.

For every $x \in \Delta^n$ and $\epsilon > 0$

$$\|x - x^\epsilon\|_\infty \leqslant (n-1)\epsilon.$$

Divide $\mathbb{N}$ into an infinite number of sets $M_1, M_2, B_1, B_2, B_3, B_4, \ldots$ as follows:

(1) $B_1 = \{1\} \quad 2 \in M_1$

$B_2 = \{3\} \quad 4 \in M_2.$

(2) If B_{2k} has been defined, then let $b_{2k} = \text{Max } B_{2k}$

$b_{2k} + 1 \in M_2.$

$B_{2k+1} = \{b_{2k} + 2, b_{2k} + 3, \ldots, b_{2k} \cdot 2(k+1)\}$

and let

$b_{2k+1} = \text{Max } B_{2k+1}.$

$b_{2k+1} + 1 \in M_1$

$B_{2k+2} = \{b_{2k+1} + 2, \ldots, b_{2k+1} \cdot 2(k+1)\}$

and so forth.

Remark 4.8

(1) For any $l \in \mathbb{N}$, $\#B_l / \# \bigcup_{k-1}^{l-1} B_k \geqslant l$, and

(2) M_1 and M_2 are infinite and

$$\limsup \#M_i \cap \{1, \ldots, t\}/t = 0, \quad i = 1, 2.$$

In the sequel, $B_1, B_2, \ldots$ will be called blocks. All the blocks with odd indices will be devoted to player 1 and all the others to player 2, in the sense that in blocks with odd indices, player 2 (by playing a modification of strategies in C_1) checks player 1 while in the remaining blocks player 1 (by playing a modification of strategies in C_2) checks player 2.

The payoffs at stages of $M_1 \cup M_2$ will have no influence on the payoff's average because of the zero density.

In addition to the information player i gets during the play in block B_k, he also gets information about the block B_k during the stages of M_i. By these data player i will be able to check if his opponent has deviated in block B_k or not. The additional information received in M_i is needed because the information received in "real-time" is not sufficient to detect all possible deviations. The information collected in "real-time" is available for a discovery of deviations to strategies which are *non-equivalent* to the strategy that should have been played. The information collected not in "real-time", namely in M_1 or in M_2, is required for a discovery of deviations to strategies which are *not greater* (in the sense of $\succ$) than the strategy that should have been played.

How player i can get information about what was going on at stage t long after stage t has passed? Both players have non-trivial information, therefore player 1 has three actions $v_1, s_1, s_2 \in \Sigma_1$ and player 2 has three actions $v_2, t_1, t_2 \in \Sigma_2$ such that

$$l_1(v_1, t_1) \neq l_1(v_1, t_2) \quad \text{and} \quad l_2(s_1, v_2) \neq l_2(s_2, v_2). \tag{4.11}$$

Since L_1 and L_2 are finite, by a finite number of "Yes-No" questions, player i can identify the signal player 3-i got at any former stage.

In a precise way:

Let $L_1 = \{x_1, \ldots, x_{|L_1|}\}$ and $L_2 = \{y_1, \ldots, y_{|L_2|}\}$, and denote the question "Did you get the signal s at stage t?" by $\psi^t(s)$.

To each stage t in a block with an even index we will correlate $|L_2| - 1$ stages in M_2, say the stages of the set $R_2(t)$, and for each stage t in a block with an odd index we will correlate $|L_1| - 1$ stages in M_1, say the stages of $R_1(t)$. Now, at the j-th stage of $R_2(t)$ player 2 has to answer the question $\psi^t(y_j)$ i.e. to act t_1 for "Yes" and t_2 for "No", and player 1 has to play v_1 in order to get the answer. If player 1 gets the signal $l_1(v_1, t_1)$, he understands that the answer to question $\psi^t(y_j)$ is "Yes" and he understands "No" otherwise (see (4.11)). The procedure is similar to stages in M_1 with exchanged roles.

Player 2 has to answer honestly because in the stages of even index blocks he plays pure strategies, and therefore player 1 (knowing his own actions) knows what signals player 2 should have received. Hence, he knows on which action player 2 has to report "Yes" and on which "No".

The strategy f will be defined as follows: Divide the block B_k into three parts B_k^1, B_k^2 and B_k^3, in such a way that for any segment S in B_k of length k and for any $1 \leqslant l \leqslant 3$:

$$|\#B_k^l \cap S/k - \gamma_1^l| < 2/k \quad \text{if } k \text{ is odd} \quad \text{and}$$

$$|\#B_k^l \cap S/k - \gamma_2^l| < 2/k \quad \text{if } k \text{ is even.} \tag{4.12}$$

If $t \in B_k^l$ and k is odd, then player 1 has to play $p_{1,l}$ and player 2 has to play $q_{1,l}^{1/k}$ (see Notation 4.7), unless player 2 has come to the conclusion that player 1 had deviated some time before B_k had started. In this case player 2 will play σ_2, by which the punishment is executed, forever.

Alternatively, if $t \in B_k$ and k is even, then player 1 has to play $p_{2,l}^{1/k}$ and player 2 has to play $q_{2,l}$ unless player 1 comes to the conclusion that player 2 had deviated sometime in the past, before B_k. In this case player 1 will punish his opponent forever by playing σ_1.

How does a player decide whether or not his opponent has deviated? In blocks with odd indices, player 1 plays only pure strategies, therefore when player 2 is acting some $a \in \Sigma_2$, he is expected to get some signal with probability 1. If he does not get it, he knows that player 1 has deviated. Furthermore, he knows what signal (in L_1) player 1 should have got and thus on what signal player 1 should have reported (in the corresponding stages of M_2). If the signal reported does not fit the expected one, then player 2 comes to the conclusion that player 1 had deviated.

Player 1 checks player 2 in a similar way.

Lemma 4.9: $H^*(f) = (\alpha_1, \alpha_2)$

Proof: Let $t \in \mathbb{N}$. Denote by v_i^t the expected payoff of player i at stage t, i.e., $v_i^t = E(x_i^t)$, $1 = 1, 2$. Player 2 checks player 1 in blocks with odd indices. In addition, in these blocks, player 1 plays only pure strategies. Therefore, the probability that player 2 will punish player 1 because he found a deviation in block B_k (although, actually, player 1 did not deviate at all) is zero. Similarly, the probability that player 2 is being punished although he did not deviate is zero. Let $n = \text{Max}\,(|\Sigma_1|, |\Sigma_2|)$. For every odd k, $1 \leqslant l \leqslant 3$, and $t \in B_k^l$, we have

$$\|(v_1^t, v_2^t) - h(p_{1,l}, q_{1,l}^{1/k})\|_\infty \leqslant (n-1)W/k, \tag{4.13a}$$

where $W = 2$ Max $\{\|h(s,r)\|_\infty \mid (s,r) \in \Sigma_1 \times \Sigma_2\}$.

For every even k and $1 \leqslant l \leqslant 3$, if $t \in B_k^l$, then

$$\|(v_1^t, v_2^t) - h(p_{2,l}^{1/k}, q_{2,l})\|_\infty \leqslant (n-1)W/k. \tag{4.13b}$$

Because of (4.12), (4.13a) and (4.13b), we have

$$\|(1/\#B_k) \sum_{t \in B_k} (v_1^t, v_2^t) - (\alpha_1, \alpha_2)\| \leqslant (n-1)W/k + 2W/k.$$

By Remark 4.8(1),

$$\|\frac{1}{b_k} \sum_{t=1}^{b_k} (v_1^t, v_2^t) - (\alpha_1, \alpha_2)\|_\infty \leqslant (n-1)W/k + 2W/k + W/k + kW/b_k. \tag{4.14}$$

The term W/k appears because $\#B_k / \sum_{k'<k} \#B_{k'} < 1/k$, and kW/b_k appears because $\#((M_1 \cap M_2) \cap \{1, \ldots, b_k\}) \leqslant k$. The right hand term of (4.14) tends to zero. Since B_k^1, B_k^2, B_k^3 are distributed homogeneously in B_k (in the sense of (4.12)), the average of the expected payoffs at a stage in the middle of B_k is not far from (α_1, α_2). In a precise way, let $T \in B_k$. By (4.12), (4.13a) and (4.13b),

$$\|1/(T - (b_{k-1} + 1)) \sum_{t=b_{k-1}+2}^{T} (v_1^t, v_2^t) - (\alpha_1, \alpha_2)\|_\infty$$

$$\leqslant [(k-1)W/(T - (b_{k-1} + 1))] + [(n-1)W/k] + 2W/k. \tag{4.15}$$

The first term of the right hand of (4.15) appears because the evaluation of the expected average is done on segments of length k and there are at most $k-1$ stages that are not contained in such a segment.

$$\|\frac{1}{T} \sum_{t=1}^{T} (v_1^t, v_2^t) - (\alpha_1, \alpha_2)\|_\infty = \|\frac{1}{T} \sum_{t=1}^{b_{k-1}} (v_1^t, v_2^t) + \frac{1}{T}(\sum_{t \in B_k} (v_1^t, v_2^t)) - (\alpha_1, \alpha_2)\|_\infty$$

$$\leqslant \|\frac{1}{T} \sum_{t=1}^{b_{k-1}} (v_1^t, v_2^t) - (\alpha_1, \alpha_2)\|_\infty + \|\frac{1}{T} \sum_{\substack{t \in B_k \\ t \leqslant T}} (v_1^t, v_2^t) - (\alpha_1, \alpha_2)\|_\infty$$

By (4.14) and (4.15), this is less or equal to

$$(b_{k-1}/T)[(n-1)W/(k-1)+3W/(k-1)+(k-1)W/b_{k-1}]$$
$$+[(T-(b_{k-1}+1))/T][(k-1)W/(T-(b_{k-1}+1))+(n-1)W/k+2W/k] \underset{k\to\infty}{\to} 0.$$

This concludes the proof. Q.E.D.

In order to prove that f is a lower equilibrium strategy we need the following probabilistic proposition.

Proposition 4.10: Let $\{\beta_n\}$ be a decreasing sequence of positive reals such that $\sum_{n=1}^{\infty} n\beta_n < \infty$, and let $\{A_n\}$ be a sequence of events which satisfies $A_n^c \underset{\beta_n}{\subseteq} A_m$ for all $n < m$. Then,

$$\text{prob}\,(A_n) \underset{n\to\infty}{\to} 1,$$

where A_n^c is the complement of A_n, and $A \underset{\epsilon}{\subseteq} B$ if prob $(A \setminus B) \leqslant \epsilon \cdot \text{prob}\,(A)$.

Proof: Assume in the contrary that there is $1/2 > \delta > 0$ and a subsequence $\{A_{n_k}\}$ s.t. prob $(A_{n_k}) < 1-\delta$ for all k. Let l be an integer such that the following holds:

$$l > 1/\delta^2 \quad \text{and} \quad \sum_{k=l}^{\infty} k\beta_k < 1.$$

It is known that $A_{n_k}^c \underset{\beta_{n_k}}{\subseteq} A_{n_{k'}}$ for every $k < k'$. So,

$$\text{prob}\left(\bigcup_{k=l}^{2l} A_{n_k}^c\right) = \text{prob}\,(A_{n_l}^c) + \text{prob}\left(\bigcup_{k=l+1}^{2l} A_{n_k}^c\right)$$
$$-\text{prob}\left(\left(\bigcup_{k=l+1}^{2l} A_{n_k}^c\right) \cap A_{n_l}^c\right)$$
$$\geqslant (1 - l\beta_l)\delta + \text{prob}\left(\bigcup_{k=l+1}^{2l} A_{n_k}^c\right)$$

Continue this way inductively and get:

$$\operatorname{prob}\left(\bigcup_{k=l}^{2l} A_{n_k}^c\right) \geqslant \sum_{k=l}^{2l} (1-(2l-k)\beta_k)\delta > l\delta - \delta \sum_{k=l}^{\infty} k\beta_k > (l-1)\delta > 1,$$

a contradiction. Q.E.D.

Lemma 4.11: f is a lower equilibrium strategy.

Proof: Let g_2 be a strategy of player 2. We will show that

$$\liminf_t H_2^t(f_1, g_2) \leqslant \alpha_2.$$

By a similar argument one can show that

$$\liminf_t H_1^t(g_1, f_2) \leqslant \alpha_1 \quad \text{for every } g_1.$$

Both arguments will give the desired proof. Denote by μ the probability measure induced by (f_1, g_2) on $F_1 \times F_2$ (see Definition 2.3). Fix an $\eta > 0$. We will define a sequence $\{A_n\}_{n=1}^{\infty}$ of events inductively.

A_n will be the event in which the average of the random variables $\{x_2^t\}_{t=1}^{b_{l_n}}$ is less than $\alpha_2 + \eta$, where b_{l_n} is the end stage of the block B_{l_n} which has an even index and starts after all the questions about the block $B_{l_{n-1}}$ have already been asked. In a precise way: A_2 is the event

$$\left\{(1/b_2) \sum_{t=1}^{b_2} x_2^t \leqslant \alpha_2 + \eta\right\}.$$

If A_{n-1} is defined, let B_{l_n} be the first block with an even index which satisfies

$$\operatorname{Min} B_{l_n} > \operatorname*{Max}_{t \in B_{l_{n-1}}} R_2(t)$$

(recall the definition of $R_2(t)$ at the beginning of this step), and let A_n be the event

$$\left\{(1/b_{l_n}) \sum_{t=1}^{b_{l_n}} x_2^t \leqslant \alpha_2 + \eta\right\}.$$

Claim: If $\mu(A_n) \to 1$, then $\liminf E\left(\frac{x_2^1 + \ldots + x_2^T}{T}\right) \leqslant \alpha_2 + \eta$.

Proof of Claim: The random variables $\{x_2^t\}$ are uniformly bounded. The proof is, therefore, clear.

According to Proposition 4.10, and by the preceding claim, it is enough to prove that for every n and $n < n'$, $A_n^c \underset{\beta_n}{\subseteq} A_n$, for some sequence $\{\beta_n\}$ which satisfies $\sum_{n=1}^{\infty} n\beta_n < \infty$.

We will show that whenever n is big enough,

$$\text{prob}\,(A_{n'}|A_n^c) \geqslant 1 - \bar{c}(l_n^{-10} + l_{n'}^{-10})$$

for some constant $\bar{c}$, provided that prob $(A_n^c) > 0$, and thus

$$A_n^c \underset{\bar{c}\cdot(l_n^{-10}+l_{n'}^{-10})}{\subseteq} A_{n'}.$$

Since $l_n < l_{n'}$, we can define $\beta_n = 2\bar{c}l_n^{-10}$ and get $\sum_{n=1}^{\infty} \beta_n \cdot n < \sum_{n=1}^{\infty} \beta_n l_n < \infty$. Fix an n, and assume that A_n^c is given from this moment on. The event $A_{n'}^c$ $(n < n')$ is included in the union of two events. The first one is that player 1 did not discern any deviation in block B_{l_n} and the second is that player 1 did discern a deviation in block B_{l_n} and from that moment on he takes measures in order to punish player 2 (this he does also in block $B_{l_{n'}}$), but after all this happened, $A_{n'}^c$ did, all the same, occur.

For evaluating the probabilities of these events we need Lemma 5.5 of [L1]:

Proposition 4.12: Let $Y_1, \ldots, Y_n$ be a sequence of identically distributed Bernoulli random variables with parameter p, and let $R_1, \ldots, R_n$ be a sequence of Bernoulli random variables such that for each $1 \leqslant l \leqslant n$, Y_l is independent of $R_1, \ldots, R_l, Y_1, \ldots, Y_{l-1}$, then

$$\text{prob}\left\{\left|\frac{R_1Y_1 + \ldots + R_nY_n}{n} - p\cdot\frac{R_1 + \ldots + R_n}{n}\right| > \epsilon\right\} < \frac{1}{n\epsilon^2}$$

for every $\epsilon > 0$.

The event A_n^c is included in the event $\{(1/\#B_{l_n}) \sum_{t \in B_{l_n}} x_2^t > \alpha_2 + \eta/2\}$ whenever n is big enough because $\#B_{l_n} / \sum_{l<l_n} \#B_l \geqslant l_n \underset{n\to\infty}{\to} \infty$. Fix $j \in \{1, 2, 3\}$, and for the moment let $l = l_n$.

Define for every $s \in \Sigma_1$, $r \in \Sigma_2$ and $t \in B_l^j$,

$Y_t(s) = 1$ if player 1 acted s at stage t, and 0 otherwise,

and

$R_t(r) = 1$ if player 2 acted r at stage t, and 0 otherwise;

finally define

$$w_l^j(s, r) = \frac{1}{\#B_l^j} \sum_{t \in B_l^j} R_t(r) \cdot Y_t(s)$$

and

$$u_l^j(r) = \frac{1}{\#B_l^j} \sum_{t \in B_l^j} R_t(r).$$

By this definitions we have

$$(1/\#B_l^j) \sum_{t \in B_l^j} x_2^t = (1/\#B_l^j) \sum_{t \in B_l^j} \sum_{\substack{r \in \Sigma_2 \\ s \in \Sigma_1}} R_t(r) \cdot Y_t(s) \cdot h_2(s, r)$$

$$= \sum_{s \in \Sigma_1} \sum_{r \in \Sigma_2} h(s, r) \cdot w_l^j(s, r).$$

According to Proposition 4.12, with probability of at least $1 - (l^2/\#B_l^j)$, the last term is less or equal to

$$\sum_{r \in \Sigma_2} \sum_{s \in \Sigma_1} [p_{2,j}^{1/l}(s) \cdot u_l^j(r) \cdot h_2(s, r) + W/l]$$

$$= \sum_{r \in \Sigma_2} [h_2(p_{2,j}^{1/l}, r) \cdot u_l^j(r) + W \cdot |\Sigma_1|/l]$$

$$= h_2(p_{2,j}^{1/l}, u_l^j) + W \cdot |\Sigma_1| \cdot |\Sigma_2|/l,$$

where $u_l^j = (u_l^j(r))_{r \in \Sigma_2}$.

If $(1/\#B_l) \sum_{t\in B_l} x_2^t > \alpha_2 + \eta/2$, then for some $j \in \{1,2,3\}$

$$h_2(p_{2,j}^{1/l}, u_l^j) + W \cdot |\Sigma_1| \cdot |\Sigma_2|/l > \alpha_{2,j} + \eta/4,$$

where $\alpha_{2,j} = h_2(p_{2,j}, q_{2,j})$. Whenever $l = l_n$ is big enough, we have

$$h_2(p_{2,j}^{1/l}, u_l^j) > \alpha_{2,j} + \eta/8.$$

Since h_2 is continuous,

$$h_2(p_{2,j}, u_l^j) > \alpha_{2,j} + \eta/10.$$

However, $(p_{2,j}, q_{2,j}) \in C_2$, therefore $u_l^j \not\succ q_{2,j}$, and furthermore, there is a certain $r_0 \in \Sigma_2$ s.t. $r_0 \not\succ q_{2,j}$ and $u_l^j(r_0) > 2/l$.

Two cases:

(i) $r_0 \sim q_{2,j}$. In this case there is some $s \in \Sigma_1$ such that $l_1(s, r_0) \neq l_1(s, q_{2,j})$ and according to Proposition 4.12,

$$\text{prob}\,\{|(1/\#B_l^j) \sum_{t\in B_l^j} Y_t(s) \cdot R_t(r_0) - p_{2,j}^{1/l}(s) \cdot u_l^j(r_0)| > 1/l^3\} < l^6/\#B_l^j. \tag{4.16}$$

In particular with probability of at least $1 - l^6/\#B_l^j$,

$$(1/\#B_l^j) \sum_{t\in B_l^j} Y_t(s) \cdot R_t(r_0) \neq 0.$$

Say $Y_{t_0}(s) \cdot R_{t_0}(r_0) = 1$. In other words, at stage t_0 player 1 acted s and player 2 acted r_0. However, player 1 had expected to get the signal $l_1(s, q_{2,j})$ but he got $l_1(s, r_0)$ which is different. Therefore, player 1 comes to the conclusion that player 2 has deviated and thus he punishes player 2 (by playing σ_1) from block B_{l_n+1} on forever.

(ii) $r_0 \sim q_{2,j}$ but $r_0 \not\succ q_{2,j}$. This means that there are $s_1, s_2 \in \Sigma_1$ such that

$$l_2(s_1, q_{2,j}) \neq l_2(s_2, q_{2,j}) \quad \text{but} \quad l_2(s_1, r_0) = l_2(s_2, r_0).$$

Since $p_{2,j}^{1/l}(s) \geqslant 1/l$ for all $s \in \Sigma_1$, by (4.16) with probability of at least $1 - 2l^6/\#B_l^j$ there holds:

$$\sum_{t \in B_l^j} Y_t(s) R_t(r_0) > \#B_l^j/2l^2.$$

Now, player 2 has to report (at the stages of M_2) about his signals. In particular he has to report whether $l_2(s_1, r_0)$ or $l_2(s_2, r_0)$ were the signals at those stages whereby $Y_t(s_1)R_t(r_0) + Y_t(s_2)R_t(r_0) = 1$. But player 2 does not know this difference, because by acting r_0 he cannot distinguish between s_1 and s_2. The probability to guess correctly (without *any* mistake) in which stages s_1 was carried out by player 1 and in which stages s_2 was carried out is less than $2^{-l^{50}}$ (because $\#B_l^j/2l^2 > l^{50}$ whenever $l = l_n$ is big enough).

To recapitulate, the probability of the first event (i.e. that player 1, given that A_n^c had occurred, did not discern a deviation) is less than

$$l^2/\#B_l^j + l^6/\#B_l^j + 2l^6/\#B_l^j + 2^{-l^{50}} \leqslant c' l^{-10}$$

for some constant c', whenever $l = l_n$ is big enough.

We come now to the evaluation of the probability of the second event (i.e., that player 1 played so as to punish player 2, but it so happened that the average payoff of player 2 at block $B_{l_{n'}}$ is greater than $\alpha_2 + \eta/2$). Denote $l = l_{n'}$ and define $Y_t(s)$ and $R_t(r)$ for all $s \in \Sigma_1$ and $r \in \Sigma_2$ as above.

By a calculation similar to the former one, we can get the following: With probability of at most $|\Sigma_1| \cdot |\Sigma_2| \cdot l^2/\#B_l \leqslant c'' \cdot l_{n'}^{-10}$ there are $s \in \Sigma_1$ and $r \in \Sigma_2$ that satisfy:

$$\left|(1/\#B_l) \sum_{t \in B_l} Y_t(s) \cdot R_t(r) - (\sigma_1(s)/\#B_l) \sum_{t \in B_l} R_t(r)\right| > 1/l,$$

and therefore with probability of at least $1 - c'' l_{n'}^{-10}$,

$$(1/\#B_l) \sum_{t \in B_l} x_2^t \leqslant d_2 + |\Sigma_1| \cdot |\Sigma_2| \cdot W/l < \alpha_2 + \eta/2$$

whenever $l = l_{n'}$ is big enough (so that $|\Sigma_1| \cdot |\Sigma_2| \cdot W/l < \eta/2$).

Summary: Let $\bar{c} = \text{Max}\,(c', c'')$.

We obtain that

$$\text{prob}\,(A_{n'} \mid A_n^c) \geqslant 1 - \bar{c}(l_n^{-10} + l_{n'}^{-10})$$

as desired. The proof of Lemma 4.11 is finished. Q.E.D.

Step 4: Trivial information.

Step 1 does not depend on the information, therefore LEP $\subseteq$ IR.

Let player 1 be the player with trivial information. By the definitions, $C_1 = D_1$. The proof of step 2 provides that LEP $\subseteq$ conv $h(D_1)$. LEP $\subseteq$ conv $h(D_2)$, because otherwise let $f = (f_1, f_2)$ be a lower equilibrium strategy with $H^*(f) \notin$ conv $h(D_2)$. Since the information of player 1 is trivial the actions of player 1 do not depend on the previous actions of player 2. Therefore a deviation g_2 of player 2 can be defined as follows: for every $m \in \mathbb{N}$ and $w \in L_2^{m-1}$ let $g_2^m(w)$ be the strategy $q(w)$ which is the best response against $\sum_{u \in L_1^{m-1}} \text{prob}_{f_1}(u) f_1(u)$. The proof that $g_2 = (g_2^1, g_2^2, \ldots)$ is a "good" deviation is similar to the proof appearing at Step 2.

The opposite direction of the inclusion, namely that conv $h(D_1) \cap$ conv $h(D_2) \cap$ IR $\subseteq$ LEP is proved in a way similar to that in Step 3, except for the element of asking questions during the game, which is dropped here.

5 Concluding Remarks

5.1 We required in Definition 2.1.2(i) that a player will be informed about his own actions. By Dalkey's Theorem [D], any mixed (or behavior) strategies in which a player can rely on his own previous actions has an equivalent *mixed* strategy in which a player does not rely on his actions. Therefore, we could drop that requirement and get the same results.

5.2 We could define the notion of upper equilibrium by exchanging liminf with limsup (in Definition 2.9), or instead define an equilibrium by any Banach limit. The question of characterization the set of all the payoffs associated with upper (Banach) equilibria in the general case is still open. In [L2], which relies on this paper, a characterization of these sets in the case of observable payoffs is given. Another case in which we have a full characterization is the case of semi-standard information in which a player is informed about the class that includes his opponent's action (see [L1]).

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