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A NOTE ON THE MONOTONICITY OF V_n

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We give an example of a repeated game with incomplete information, lack of information on one side, and non-standard information, in which the value V_3, of the three-fold repeated game, is greater than V_2.

Let K be a finite set of states of nature. Two players, denoted PI and PII, play a zero-sum repeated game, in which the payoff matrix depends on the actual state k. Nature chooses a state k according to a probability distribution P over K. PI knows k, but PII does not know it. However, PII knows the distribution P. Thus PII has a lack of information.

It is well known that if the information is standard (after each stage the players are informed of the actions that took place in that stage), then the sequence of the V_n (the value of the n-fold repeated game) is monotonic non-increasing. The intuition is that the player with the lack of information can learn more about the true state, if the game continues for a longer span of time. By using this knowledge he can ensure himself a better payoff in the longer game. The proof of this is based on a recursive formula; V_n is expressed in terms of the $(n-1)$-fold repeated game. The recursive formula follows from the standard information.

The question is whether the sequence V_n is always monotonic, even when the information is not standard. In such games there are two information functions, t_1 and t_2. After a stage in which each player i played a_i, player j gets the signal $t_j(a_1, a_2)$. The conjecture that the sequence is monotonic was supported by the above intuition.

We will show in the next example that $V_2 < V_3$. Thus the conjecture is answered negatively. Note that always $V_1 \geqslant V_2$. Denote by Σ_i the set of actions of player i.

Example. Let $K = \{1, 2, 3\}$, $P = (1/3, 1/3, 1/3)$, $\Sigma_1 = \{X, Y\}$ and $\Sigma_2 = \{A, B, C, R, S\}$. See table 1. T means that the information is trivial (the player knows only his own actions), and D means that the information is discrete (the player knows also his opponent's action).

Denote by G_n the n-fold repeated game. In G_1, PII can ensure 4 by playing R. Thus $V_1 \leqslant 4$. In G_2, PII can ensure 3 by playing the following strategy: at the first stage, play R or S with probability 1/2 each. At the second stage, play

A if he has observed one of the signals b or d,
B if he has observed one of the signals c or f,
C if he has observed one of the signals a or e.

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Table 1

$k=1$						$k=2$					$k=3$				
	A	B	C	R	S	A	B	C	R	S	A	B	C	R	S
Payoffs															
X	100	4	0	4	4	0	100	4	4	4	4	0	100	4	4
Y	100	0	4	4	4	4	100	0	4	4	0	4	100	4	4
Signals of PI															
X	D	D	D	T	T	D	D	D	T	T	D	D	D	T	T
Y	D	D	D	T	T	D	D	D	T	T	D	D	D	T	T
Signals of PII															
X	T	T	T	f	e	T	T	T	a	d	T	T	T	b	c
Y	T	T	T	a	c	T	T	T	b	e	T	T	T	f	d

At the first stage he obviously ensures 4. In order to explain the second stage, let us assume for the moment that $k=1$, and that PI has played X at the first stage. Because of his trivial signals (whenever PII is acting R or S), PI does not know whether PII got the signal f or e. He knows only that PII got each one of these signals with probability 1/2. Thus, he expects PII to play B or C with probability 1/2 each. PII ensures by this the payoff 2. The same explanation holds for all other possibilities. Thus $V_2 \leqslant (4+2)/2 = 3$.

In G_3, PI can ensure at least 10/3 by the following strategy, σ. At the first two stages, play (1/2, 1/2) – each one of $\{X, Y\}$ with probability 1/2. At the third stage, if your information at the previous stages was trivial, play again (1/2, 1/2); otherwise the first non-trivial signal was either A or B or C, and then play as follows:

when $k=1$: (1/2, 1/2) or X or Y, for A or B or C, respectively;
when $k=2$: X or (1/2, 1/2) or Y, for A or B or C, respectively;
when $k=3$: X or Y or (1/2, 1/2), for A or B or C, respectively.

It is enough to show that this strategy gives PI at least the payoff 10/3 versus any pure strategy of PII. Fix a pure strategy of PI. If at the first stage neither R nor S are played, then the payoff 100 gets a probability 1/3, and thus the payoff at G_3 is greater than 100/9. So we can assume that R or S are played at the first stage, and the payoff then is 4.

If at the second stage PII plays R or S, then the payoff is 4, and because the payoff 2 is ensured by PI at the third stage, the payoff in G_3 is at least 10/3. Hence, A or B or C are played at the second stage by PII. To avoid the payoff 100, PII must play according to the strategy of G_2 that was described above. In this case the payoff at the second stage is 2. However, by playing A, B or C, PII does not get any information. Furthermore, he reveals the information he has. It gives PI the opportunity to play his best response against the expected action of PII at the third stage. One can check that the response described by the strategy σ ensures PI the payoff 4 at the third stage, and thus the payoff 10/3 in G_3. We conclude that $V_3 \geqslant 10/3 > 2 \geqslant V_2$.

This example raises another question: Is there an integer N (depending on the game) from which on the sequence V_n is non-increasing?

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