Subjective multi-prior probability: A representation of a partial likelihood relation*

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Abstract

This paper deals with an incomplete relation over events. Such a relation naturally arises when likelihood estimations are required within environments that involve ambiguity, and in situations which engage multiple assessments and disagreement among individuals' beliefs. The main result characterizes binary relations over events, interpreted as likelihood relations, that can be represented by a unanimity rule applied to a set of prior probabilities. According to this representation an event is at least as likely as another if and only if there is a consensus among all the priors that this is indeed the case. A key axiom employed is a cancellation condition, which is a simple extension of similar conditions that appear in the literature.

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1 Introduction

1.1 Motivation and Background

Estimating the odds and comparing the likelihood of various events are essential parts of processes carried out by many organizations. For instance, the US' intelligence community produces National Intelligence Estimates, in which the likelihood of various events is assessed. Question such as, 'Is it more likely that democracy will prevail in Libya, or that a military regime will be established?', and alike, seem natural to ask. These questions and many others call for likelihood comparisons of different events. Other examples in which probabilistic estimates are used include forecasts published by central banks, that address issues such as the odds of inflation or recession, assessments of market trends supplied by committees of experts, estimated likelihood of natural events, such as global warming, that are based on individual opinions of scientists and on many experiments, and so forth. In all these instances, statements of the type 'event A is more likely than event B' seem fundamental to the respective context.

Typically, assessments of the kind given above rely on 'objective' data, such as reports of military movements, temperature measurements and alike. Such assessments are usually intended to be as closely based on the data as possible, in order to be considered 'objective' themselves. Frequently, though, the events examined involve some degree of ambiguity. Knowledge or available information might be insufficient to determine which of two events under consideration is more likely. For example, due to insufficient or conflicting data about customers' preferences, a committee of experts might be unable to determine which of the smart phones and the tablet PCs markets will grow more rapidly in the next few years. A likelihood relation in such situations might therefore leave the comparison between some pairs of events unspecified. A question arises, as to what kind of probabilistic representation, if any, may describe such likelihood relations, and reflect indetermination with regard to some pairs of events.

The Bayesian approach suggests that an individual's likelihood relation over events can be represented by a prior probability measure. This probability is subjective in nature, as it emerges from subjective likelihood comparisons. Ramsey (1931), de Finetti (1931,1937), Savage (1954), and others who followed their footsteps, introduced conditions on a binary relation over events that characterize when it may be represented by a prior probability. However, situations in which some events are incomparable cannot be captured by a single prior probability, as a single probability determines likelihood order between any two events.

Arguments that question the validity of the completeness assumption of preference relations, are by now well-known. von Neumann and Morgenstern (1944) already doubted that an individual can always decide among all alternatives. Aumann (1962) stated that 'of all the axioms of utility theory, the completeness axiom is perhaps the most questionable', and doubted completeness on descriptive as well as normative grounds.

In later models, involving alternatives with unknown probabilities, completeness was challenged based on ambiguity considerations. The leading rationale was that when the decision situation is unclear (due, for instance, to lack of information) an individual might be unable, or unwilling, to make decisions among some alternatives. In most of these models, ambiguity was reflected in a difficulty to assess probabilities and choose one prior probability to describe the decision maker's belief. As a result, belief was represented by a set of prior probabilities. Those models that are closely related to the current work are further discussed in subsection 1.4 below.

For the same reasons of ambiguity, an individual may decline to judge among some pairs of events and prefer to consider them incomparable. It is therefore reasonable to assume that a likelihood relation over events is incomplete.

1.2 Subjective multi-prior probability

This paper proposes a characterization of a subjective probability, which differs from the aforementioned classical works. Here, completeness is not assumed. The paper formulates conditions on a binary relation over events, that are necessary and sufficient for the relation to be represented by a *set* of prior probabilities. This set of priors should be considered subjective, as it emerges from subjective likelihood comparisons.

The nature of the representing set of priors is that one event is considered at least as likely as another, precisely when all thee priors in this set agree that this is the case. When the representing set contains at least two non-identical priors, there always exist two events over which there is no consensus among the priors. Hence, the likelihood order induced by two different priors or more is necessarily incomplete.

Formally, let $\succsim$ denote a binary relation over events, where $A \succsim B$ for events A and B is interpreted as 'A is at least as likely as B'. The main theorem of the paper introduces necessary and sufficient conditions ('axioms') on $\succsim$ that guarantee the existence of a set of prior probabilities, $\mathcal{P}$, for which,

$$A \succsim B \quad \Leftrightarrow \quad \mu(A) \geq \mu(B) \text{ for all } \mu \in \mathcal{P}. \tag{1}$$

When this happens it is said that the 'at least as likely as' relation has a *subjective multi-prior probability* representation.

The representing set of prior probabilities is not necessarily unique. This is the reason why reference is made to the maximal, w.r.t. inclusion, representing set of priors. In the case of a finite state set, the set of priors is not restricted in any sense. However, when the state space is infinite, the priors should agree on any null-set, or more formally, to satisfy a kind of uniform absolute continuity condition. Still, even in this case, the priors need not agree on the probability of any event rather than null and universal events.

The main result of the paper is separated into two cases: when the state space is finite and when it is infinite. While for the finite case, de Finetti's three basic assumptions and GFC suffice to characterize those relations that are representable by a multi-prior probability, the infinite case requires an additional assumption regarding the richness of the state space, akin to the Archimedean assumption of Savage.

The use of a set of prior probabilities to describe an individual's belief elicits likelihood judgements that are more robust to ambiguity than a description by a single probability measure. For suppose that an individual is uncertain as to the prior probability to choose. Choice of a single prior probability is sensitive to uncertainty in the sense that two different probabilities might exhibit reversal of likelihood order between some pair of events. On the other hand, representing the uncertainty through a set of prior probabilities leaves such conflicting comparisons undetermined, reflecting the individual's lack of knowledge regarding the 'right' prior to choose.

The technique of employing sets of probabilities to produce robust models is used extensively in Bayesian Analysis. Robust Bayesian Analysis replaces a single Bayesian prior with a set of priors so as to make the statistical model less sensitive to the prior chosen. A well known example of this approach is the ε-contamination class (see, for instance, Berger 1994). The axiomatization presented here may therefore lend normative ground to these robust, multi-prior Bayesian techniques[1].

Two additional results, somewhat complementary to the elicitation of a subjective multi-prior probability, are contained in the paper. One compares the ambiguity perceived by two individuals, holding subjective multi-prior probability beliefs, by means of inclusion of priors sets. One individual is considered to perceive more ambiguity than another if he or she is less decisive, that is, leaves more comparisons undetermined. In the representation, the maximal (w.r.t. set inclusion) set of probabilities of the first individual contains that of the second individual. The other result proposes a scheme to complete a partial likelihood relation, that obtains a subjective multi-prior probability. This scheme suggests aversion to ambiguity.

1.3 Axiomatization

Some notations are needed in order to facilitate the following discussion. Let S denote a nonempty state-space, with a typical element s. An event is a subset of S which is a member of an algebra Σ over S. For an event E, $\mathbf{1}_E$ denotes the indicator function of E.[2] A binary relation $\succsim$ is defined over Σ, with $\succ$ denoting its asymmetric part. A probability measure P *agrees* with $\succsim$ if it represents it, in the sense that $A \succsim B \quad \Leftrightarrow \quad P(A) \geq P(B)$. A probability measure P *almost agrees* with $\succsim$ if the former equivalence is relaxed to $A \succsim B \quad \Rightarrow \quad P(A) \geq P(B)$. In other words, P almost agrees with $\succsim$ if it cannot be the case that $A \succsim B$, and at the same time, $P(B) > P(A)$. It is possible, however, to have $P(A) \geq P(B)$ yet $\neg(A \succsim B)$.

De Finetti introduced four basic postulates that must be satisfied by an 'at least as likely as' relation. These postulates define an entity known since as a *qualitative probability.* The four postulates are:

Complete Order:	The relation $\succsim$ is complete and transitive.
Cancellation:	For any three events, A, B and C, such that $A \cap C = B \cap C = \emptyset$, $A \succsim B \quad \Leftrightarrow \quad A \cup C \succsim B \cup C$.
Positivity:	For every event A, $A \succsim \emptyset$.
Non Triviality:	$S \succ \emptyset$.

de Finetti assumed that the 'at least as likely as' relation is complete, and, in addition, satisfying three basic assumptions: Transitivity; Positivity, which states that any event is at least as likely as the empty event; and Non-triviality, which states that 'all' (the universal event) is strictly more likely than 'nothing' (the empty event).

Beyond these three assumptions, de Finetti introduced another, more substantial postulate, Cancellation. Cancellation implies a form of separability over events in the following sense. Any

[1]We thank Tzachi Gilboa for bringing this example to our attention

[2]That is, $\mathbf{1}_E$ is the function that attains the value 1 on E and 0 otherwise.

event has its own likelihood weight, which is unrelated to other disjoint events. In other words, any event has the same marginal contribution, no matter what other disjoint events it is annexed to[3].

The four de Finetti's assumptions are necessary for the relation to have an agreeing probability. de Finetti posed the question whether these postulates are also sufficient to guarantee existence of an agreeing probability, or even of an almost agreeing probability.

The question was answered negatively by Kraft et al.(1959). They introduced a counter example with a relation over a finite state space, that satisfies all the above conditions and yet has no almost agreeing probability. Kraft et al.(1959) suggested a strengthening of the Cancellation condition, and showed that with a finite state space, the strengthened condition together with the above basic conditions imply that the relation has an almost agreeing probability. Later on, other conditions that derive agreeing or almost agreeing probabilities were proposed by Scott (1964), Kranz et al. (1971) and Narens (1974). These conditions stated that for two sequences of events, $(A_i)_{i=1}^n$ (the A-sequence) and $(B_i)_{i=1}^n$ (the B-sequence),

$$\begin{aligned}&\text{If} \quad \sum_{i=1}^{n} \mathbf{1}_{A_i}(s) = \sum_{i=1}^{n} \mathbf{1}_{B_i}(s) \ \text{ for all } s \in S, \\ &\text{and} \quad A_i \succsim B_i \quad \text{for } i = 1, \ldots, n-1, \\ &\text{then} \ \ B_n \succsim A_n.\end{aligned} \tag{2}$$

The idea behind Finite Cancellation is similar in essence to that lying at the basis of de Finetti's Cancellation condition. Like Cancellation, Finite Cancellation is based on the assumption that each state has always the same marginal contribution of likelihood, no matter to which other states it is added. However, Finite Cancellation takes this rationale a step further.

The equality, that appears in the axiom, between the two sums of indicators, means that each state appears the same number of times in each of the sequences. Following the rationale explained above, the weight of each state does not depend on the order and manner in which it appears in each sequence. Hence, the equality between the two sums suggests that it cannot be that the A-sequence has an overall likelihood weight greater than that of the B-sequence. In order to explain this point we turn to an analogy from accounting.

Consider a simple double-entry booking procedure with credit and debit that express likelihood weights. In each of the $n-1$ first lines of the ledger there are pairs of events with $A_i \succsim B_i$: A_i is recorded on the credit side, while B_i on the debit side. Another pair of events A_n and B_n is now considered. Suppose that A_n is recorded on the credit column and B_n on the debit column. This reflects the fact that A_n is at least as likely as B_n. It is argued that this ranking contradicts the rationale behind Cancellation, of invariant marginal contribution.

When A_n is ranked more likely than B_n, each event in the A-sequence is as likely as its counterpart in the B-sequence. Therefore, the credit balance outweighs the debit balance. Such an assertion contradicts the conclusion that the two sequences should have the same overall likelihood weight, unless each pair consists of equally likely events. It implies, in particular, that

[3]This assumption is violated, for instance, when states are evaluated through a nonadditive probability v. Under such an evaluation, the marginal contribution of event C when added to event A, $v(A \cup C) - v(A)$, is not necessarily the same as its marginal contribution when added to event B, $v(B \cup C) - v(B)$.

if A_n and B_n are comparable, it must be that $B_n \succsim A_n$. Finite Cancellation explicitly states that A_n and B_n are comparable, and that B_n is at least as likely as A_n.[4]

This paper follows the path of studies described above and introduces an axiom termed *Generalized Finite Cancelation* (GFC). This axiom postulates that for two sequences of events, $(A_i)_{i=1}^n$ and $(B_i)_{i=1}^n$, and an integer $k \in \mathbb{N}$,

$$\begin{aligned}&\text{If} \quad \sum_{i=1}^{n-1} \mathbf{1}_{A_i}(s) + k\mathbf{1}_{A_n}(s) = \sum_{i=1}^{n-1} \mathbf{1}_{B_i}(s) + k\mathbf{1}_{B_n}(s) \quad \text{for all } s \in S,\\ &\text{and} \quad A_i \succsim B_i \quad \text{for } i = 1, \ldots, n-1,\\ &\text{then} \quad B_n \succsim A_n.\end{aligned}$$

GFC strengthens Finite Cancellation. Similarly to Finite Cancellation, it is concerned with sequences of events with identical accumulation of indicators, which again should have the same overall likelihood weight. However, the conclusion in GFC applies to multiple repetitions of the last pair of events, A_n and B_n. As in Finite Cancellation, if the first $n-1$ events in one sequence are judged to weigh at least as much as the first $n-1$ events in the other sequence, then the last pair of events, repeated more than once in the aggregation, should balance the account. Thus, B_n is regarded at least as likely as A_n.

This generalization of Finite Cancellation is required since the relation in this paper is not assumed to be complete. For a complete relation, GFC is implied by Finite Cancellation: $\neg(B_n \succsim A_n)$ translates to $A_n \succ B_n$, and under the assumptions of the axiom, a contradiction to Finite Cancellation is inflicted. When the relation is incomplete, such a contradiction may not arise, thus GFC specifically requires that $B_n \succsim A_n$ be concluded.

As this work is concerned with incomplete relations, the role of GFC is two fold. The first is to preserve consistency of the 'at least as likely as' relation, as explained above. The second is to allow for extensions of the relation to yet undecided pairs of events, on the basis of others. Suppose that a pair of yet-undecided events can play the part of A_n and B_n for some sequences of events $(A_i)_{i=1}^n$ and $(B_i)_{i=1}^n$. GFC then explicitly prescribes a completion rule that imposes the principle of each state having its own marginal contribution, unrelated to other states.

Any relation $\succsim$ with a subjective multi-prior probability representation as in (1) satisfies GFC. In the opposite direction, Theorems 1 and 2, the main results of the paper, show that GFC, along with basic conditions (and when S is infinite an additional richness assumption), imply that the relation $\succsim$ admits a subjective multi-prior representation.

An analogue axiom to GFC, formulated on mappings from states to outcomes, appeared in Blume, Easley and Halpern (2009) (under the name 'extended statewise cancellation'). In that paper, the axiom was applied in a different framework, and was used to obtain a representation that contains a *subjective* state space. Hence, the result of Blume et al. cannot be employed to characterize a likelihood relation over a given, primitive state space.

[4] For further details on Cancellation axioms and almost agreeing probabilities see Fishburn (1986) for a thorough survey, and Wakker (1981) for a discussion and related results.

1.4 Related literature

Incomplete Expected Utility models over objective lotteries (on a set of prizes) began with Aumann (1962). Dubra, Maccheroni and Ok (2004) gave a full characterization of an incomplete preference relation over lotteries, which obtains a unanimity representation over a set of utility functions. The path of dropping completeness for the sake of ambiguity, as suggested in this paper, was taken in the literature by several authors. The papers which are most related to the current work are Bewley (2002), Ghirardato et al (2003), and Nehring (2009). These papers are discussed next.

Bewley (2002) axiomatized preference among alternatives that are mappings from events to a rich set of consequences. Bewley worked in an environment with exogenous probabilities, as in Anscombe and Aumann (1963; as rephrased by Fishburn 1970), and showed that dropping completeness and maintaining the rest of Anscombe-Aumann axioms yields a multi-prior expected utility representation. That is, for every pair of alternatives f and g,

$$f \succsim g \quad \Leftrightarrow \quad \mathbb{E}_\mu(u(f)) \geq \mathbb{E}_\mu(u(g)) \ \text{ for every } \mu \in \mathcal{P}, \tag{3}$$

for a unique convex and closed set of priors $\mathcal{P}$ and a vN-M utility function u ($\mathbb{E}_\mu$ denotes the expectation operator w.r.t. the probability measure μ). In other words, under quite 'standard' set of axioms, which is known to imply an expected utility representation with a unique probability measure (as Anscombe and Aumann show), giving up the completeness assumption leads to an expected utility representation with a set of probability measures.

Prior to Bewley, Giron and Rios (1980) phrased similar axioms over alternatives which map an abstract state space to the real line. Giron and Rios took the alternatives to be bounded, and assumed that they consist of a convex set (they mentioned randomization as justification for convexity). Giron and Rios showed that their set of axioms imply a multi-prior expected utility representation as in (3).

Both models, that by Giron and Rios and the one by Bewley, if applied with only two consequences, may be used to extract beliefs without any implication to risk attitude. However, they both assume convexity, which amounts to the fact that events may be mixed using exogenous, objective probabilities. Hence, their setup is less general than the purely subjective setup of Savage.

Ghirardato et al (2003) axiomatized multi-prior expected utility preferences in an environment without exogenous probabilities. Instead, they assumed that the set of consequences is connected, and that there exists an event on which all probabilities agree. The structure and axioms they employed do not permit to apply the model for two consequences alone. It necessitates, in order to obtain the mixtures used in their axioms, to specify preference on all binary alternatives contingent on the agreed-upon event (that is, on all alternatives that yield one consequence on this event and another outside of it). Thus, the model cannot be used to identify 'pure' belief.

Nehring (2009) presents a model based on an incomplete 'at least as likely as' relation over events. Nehring formulates assumptions on the relation, and proves a multi-prior representation result as in (1). In Nehring's work, there is a unique convex and closed set of priors that represents the relation. The model is placed in a general setup, where events belong to an algebra over a state space. No assumptions are a-priori made on this space, and in particular no objective probabilities are assumed. Nonetheless, one of the axioms Nehring applies to characterize the multi-prior representation is *Equidivisibility*, which hinges upon an explicit assumption that any event can be divided into two equally-likely events. Nehring's assumption is quite a strong one. It states that any event can be split into two events in a way that *all* the priors involved agree that they are equally probable. Formally, the set of priors in Nehring's representation theorem is explicitly assumed to have the property that for any event A, there exists an event $B \subset A$ such that all priors μ agree that $\mu(B) = \mu(A)/2$. It implies, in particular, that all the priors agree on a rich algebra of events: the one generated by dividing the entire space into 2^n equally likely events, for any integer n. This assumption restricts the plurality of the representing set of priors. Moreover, it does not work in the finite case.

The setup employed in this paper is as general as that of Savage[5]. Neither exogenous probabilities nor any manner of mixture is assumed or required. As the assumptions apply to events alone, they are free of any risk attitude.

1.5 Outline of the paper

Section 2 describes the essentials of the subjective multi-prior probability model. It details the setup and assumptions, and then formulates representation theorems, separated to the cases of a finite and infinite spaces. Section 3 presents some examples, and Section 4 elaborates on three issues: a relative notion of 'more ambiguous than'; an embedding of ambiguity attitude that yields a complete relation; and minimal sets of priors. All proofs appear in the last section.

2 The subjective multi-prior probability model

2.1 Setup and assumptions

Let S be a nonempty set, Σ an algebra over S, and $\succsim$ a binary relation over Σ. A statement $A \succsim B$ it to be interpreted as 'A is at least as likely as B. For an event $E \in \Sigma$, $\mathbf{1}_E$ denotes the indicator function of E. In any place where a partition over S is mentioned, it is to be understood that all atoms of the partition belong to Σ.

The following assumptions ('axioms') are employed to derive a subjective multi-prior probability belief representation.

P1. Reflexivity:
For all $A \in \Sigma$, $A \succsim A$.

P2. Positivity:
For all $A \in \Sigma$, $A \succsim \emptyset$.

[5]In fact the setup is even somewhat less restrictive than that of Savage, in that it requires Σ to be an algebra and not a σ-algebra.

P3. Non Triviality:
$\neg(\emptyset \succsim S)$.

The first three assumptions are standard. Positivity and Non Triviality are two of de Finetti's suggested attributes. Since Completeness is not supposed, Reflexivity is added in order to identify the relation as a weak one. Transitivity is implied by the other axioms, hence it is not written explicitly.

The next assumption is the central one in the derivation of the main result of the paper.

P4. Generalized Finite Cancellation:
Let $(A_i)_{i=1}^n$ and $(B_i)_{i=1}^n$ be two sequences of events from Σ, and $k \in \mathbb{N}$ an integer. Then

$$\begin{aligned}
&\text{If} \quad \sum_{i=1}^{n-1} \mathbf{1}_{A_i}(s) + k\mathbf{1}_{A_n}(s) = \sum_{i=1}^{n-1} \mathbf{1}_{B_i}(s) + k\mathbf{1}_{B_n}(s) \quad \text{for all } s \in S, \\
&\text{and} \quad A_i \succsim B_i \quad \text{for } i = 1, \ldots, n-1 \ , \\
&\text{then} \quad B_n \succsim A_n \quad .
\end{aligned}$$

Generalized Finite Cancellation is based on the basic assumption that every state has the same marginal contribution of likelihood, no matter to which other states it is added. Generalized Finite Cancellation applies this underlying assumption to sequences of events that have exactly the same aggregation for each and every state, that is, in cases where each state appears the same number of times in each of the sequences. As each state should add the same likelihood weight to each of these aggregations, the axiom requires that likelihood judgements would be balanced among the two sequences.

Basic Cancellation (de Finetti's condition) is obtained from Generalized Finite Cancellation by letting $A_1 = A$, $B_1 = B$, $A_2 = B \cup C$ and $B_2 = A \cup C$, for events A, B and C such that $(A \cup B) \cap C = \emptyset$. The indicators sum is identical,

$$\mathbf{1}_{A_1} + \mathbf{1}_{A_2} = \mathbf{1}_A + \mathbf{1}_{B\cup C} = \mathbf{1}_A + \mathbf{1}_B + \mathbf{1}_C = \mathbf{1}_{A\cup C} + \mathbf{1}_B = \mathbf{1}_{B_1} + \mathbf{1}_{B_2},$$

and hence $A \succsim B$ implies $A \cup C \succsim B \cup C$ and vice versa. Transitivity is obtained by letting $A_1 = B_3 = A$, $A_2 = B_1 = B$, and $A_3 = B_2 = C$. The following remark sums several implications of Generalized Finite Cancellation.

Remark 1. Generalized Finite Cancellation implies that $\succsim$ satisfies Transitivity, and together with Positivity yields that $A \succsim B$ whenever $A \supset B$. It also results in additivity: $A \succsim B \;\Leftrightarrow\; A \cup E \succsim B \cup E$, whenever $A \cap E = B \cap E = \emptyset$. In particular, $A \succsim B \;\Leftrightarrow\; B^c \succsim A^c$,[6] hence $S \succsim B$ for all events B.

[6]This is true since $A = (A \cap B^c) \cup (A \cap B) \succsim (B \cap A^c) \cup (A \cap B) = B \;\Leftrightarrow\; A \cap B^c \succsim B \cap A^c \;\Leftrightarrow\; B^c = (A \cap B^c) \cup (A^c \cap B^c) \succsim (B \cap A^c) \cup (A^c \cap B^c) = A^c$.

2.2 Subjective multi-prior probability representation

2.2.1 The case of a finite space S

The next theorem states that when S is finite, assumptions **P1-P4** are sufficient to obtain a subjective multi-prior probability representation of $\succsim$. For simplicity, Σ is assumed to be the collection of all subsets of S (that is, $\Sigma = 2^S$).

Theorem 1. *Suppose that S is finite, and let $\succsim$ be a binary relation over events in S. Then statements (i) and (ii) below are equivalent:*

(i) $\succsim$ satisfies axioms P1 through P4.

(ii) There exists a nonempty set $\mathcal{P}$ of additive probability measures over events in S, such that for every $A, B \subseteq S$,

$$A \succsim B \quad \Leftrightarrow \quad \mu(A) \geq \mu(B) \;\; \text{for every } \mu \in \mathcal{P}.$$

The set of prior probabilities needs not be unique. However, the union of all representing sets is itself a representing set, and is maximal w.r.t. inclusion by its definition. When analyzing judgements made under ambiguity, the maximal set w.r.t. inclusion seems to be a natural choice to express belief, as it takes into consideration all priors that may be relevant to the case at hand. In fact, the maximal w.r.t. inclusion set includes all the probability measures that almost agree with the relation.

Suppose, for instance, that after identifying a subjective multi-prior probability, an individual is interested also in the extreme probabilities of each event. The maximal set yields the largest range of probabilities for each event, without casting unnecessary limits. On that account, the notion of a maximal w.r.t. inclusion set is claimed to be a satisfactory notion of unique belief.

When S is finite, then even if completeness is assumed, there is no one, unique probability measure that represents the relation. That is to say, even if $\succsim$ is complete, the maximal representing set is not a singleton (see Example 1 below).

2.2.2 The case of an infinite space S

To obtain representation when S is infinite, an additional richness assumption is required. Without this assumption, for an infinite S, it is possible to obtain a set $\mathcal{P}$ of probability measures that only *almost agrees* with $\succsim$, in the sense that for all events A and B, $A \succsim B \Rightarrow \mu(A) \geq \mu(B)$ for every $\mu \in \mathcal{P}$, but not necessarily the other way around. To obtain the full 'if and only if' representation, a Non Atomicity axiom is added. The assumption requires a definition of strong preference for its formulation (the definition originates in Nehring, 2009).

Definition 1. *For two events $A, B \in \Sigma$, the notation $A \succ\succ B$ states that there exists a finite partition $\{G_1, \ldots, G_r\}$ of S, such that $A \setminus G_i \succsim B \cup G_j$ for all i, j.*

Strong preference, in the representation (thus when all the axioms are assumed to hold), whenever the set $\mathcal{P}$ is the maximal w.r.t. inclusion set[7], is equivalent to the condition that there exists $\delta > 0$ for which $\mu(A) - \mu(B) > \delta > 0$ for every $\mu \in \mathcal{P}$.

P5. Non Atomicity:
If $\neg(A \succsim B)$ then there exists a finite partition of A^c, $\{A'_1, \ldots, A'_m\}$, such that for all i, $A'_i \succ\succ \emptyset$ and $\neg(A \cup A'_i \succsim B)$.

Remark 2. As $\neg(A \succsim B) \Leftrightarrow \neg(B^c \succsim A^c)$, Non Atomicity can equivalently be phrased as:
If $\neg(A \succsim B)$ then there exists a finite partition of B, $\{B_1, \ldots, B_m\}$, such that for all i, $B_i \succ\succ \emptyset$ and $\neg(A \succsim B \setminus B_i)$.

Non Atomicity is the incomplete-relation version of Savage's richness assumption P6. Adding Completeness makes **P5** (along with the definition of strict preference) identical to Savage's P6, as negation of preference simply reduces to strict preference in the other direction. The setup used here is somewhat weaker than that in Savage, as Σ is assumed to be an algebra and not necessarily a σ-algebra. Still, adding Completeness yields a unique probability that represents the 'at least as likely as' relation $\succsim$ (see Kopylov (2007) for this result for an even more general structure of Σ)[8]. The difference from Savage's theorem is that the derived probability need not be convex-ranged, only *locally dense*:

Definition 2. A probability measure μ over Σ is *locally dense*, if for every event B, the set $\{\mu(A) \mid A \subset B\}$ is dense in $[0, \mu(B)]$.

In a similar fashion, Non Atomicity implies that each probability measure in the subjective multi-prior probability set is locally dense.

Definition 3. A set $\mathcal{P}$ of probability measures is said to be *uniformly absolutely continuous,* if:

(a) For any event B, $\mu(B) > 0 \Leftrightarrow \mu'(B) > 0$ for every pair of probabilities $\mu, \mu' \in \mathcal{P}$.

(b) For every $\varepsilon > 0$, there exists a finite partition $\{G_1, \ldots, G_r\}$ of S, such that for all j, $\mu(G_j) < \varepsilon$ for all $\mu \in \mathcal{P}$.

The following is a representation result of the subjective multi-prior probability model for the case of an infinite set S .

Theorem 2. *Let* $\succsim$ *be a binary relation over* Σ. *Then statements (i) and (ii) below are equivalent:*

(i) $\succsim$ *satisfies axioms* ***P1-P5***.

[7] or any other compact set

[8] Generalized Finite Cancellation obviously implies additivity (de Finetti's Cancellation), thus when adding completeness to axioms P2 through P5, Savage's result follows.

(ii) There exists a nonempty, uniformly absolutely continuous set $\mathcal{P}$ of additive probability measures over Σ, such that for every $A, B \in \Sigma$,

$$A \succsim B \quad \Leftrightarrow \quad \mu(A) \geq \mu(B) \text{ for every } \mu \in \mathcal{P}. \tag{4}$$

Corollary 1. *Assume that $\succsim$ admits a subjective multi-prior probability representation as in (ii) of Theorem 2. Then all the probability measures in the representing set are locally dense.*

It is a known fact that existence of fine partitions, as depicted in the definition of uniform absolute continuity, imply local denseness (see, for instance, Lehrer and Schmeidler, 2005). Hence, if $\succsim$ admits a subjective multi-prior probability representation as in (ii) of Theorem 2, then all the probability measures in the representing set are locally dense.

Observation 1. *The union of all sets of probabilities that represent $\succsim$ as in (ii) of Theorem 2 is itself a representing set of $\succsim$. Thus, the union of all sets that represent $\succsim$ satisfies (ii) of Theorem 2. By its definition, every other representing set is contained in it, namely it is the maximal w.r.t. inclusion representing set of $\succsim$.*

Note that for general sets of probability measures, the fact that all sets in the union are uniformly absolutely continuous does not imply that the union itself is uniformly absolutely continuous (as the union may be over an infinite number of sets). However, if the sets represent $\succsim$, then (i) of Theorem 2 guarantees uniform absolute continuity of the union. As already explained in the introduction, the set of probabilities which represents $\succsim$ need not be unique. Nevertheless, the maximal representing set, which is the union of all representing sets, is unique by its definition.

A few words on the reason for lack of uniqueness, even when Non Atomicity is assumed, are in order. The difficulty in obtaining uniqueness is that it is impossible, under the assumptions made, to produce an 'objective measuring rod' for probabilities. The probabilities in the priors set need not agree on any event which probability is strictly between zero and one. Moreover, there even need not be events with probability that is ε-close to some value $0 < p < 1$, according to all measures (see Example 2 in section 3). On top of that, the domain to which the relation applies is scarce. Translating events, to which the relation applies, to vectors in $\mathbb{R}^S$, only indicator vectors are subject to comparison, and preference calls among vectors with values other than zero and one are meaningless. The problem is not only technical, for there are examples with two distinct (convex and closed) representing sets of priors. An example may be found in Nehring (2009; Example 1).

3 Examples

Example 1. Let $S = \{H, T\}$, $\Sigma = 2^S$. If it is the case that $H \succsim T$, then any probability measure of the sort $(H : p \ ; \ T : 1 - p)$, for $0.5 \leq p \leq 1$, represents the relation. In that case, even though the relation is complete, there is no unique representing probability measure, or unique representing probability measures set.

In the following examples, λ denotes the Lebesgue measure.

Example 2. Let $S = [0,1)$ and denote by Σ the algebra generated by all intervals $[a,b)$ contained in $[0,1)$. Define measures π_1 and π_2 through their densities:

$$f_1(s) \quad = \quad \begin{cases} \frac{1}{2} & s \in [0,0.5) \\ \frac{3}{2} & s \in [0.5,1) \end{cases} \qquad , \qquad f_2(s) \quad = \quad \begin{cases} \frac{3}{2} & s \in [0,0.5) \\ \frac{1}{2} & s \in [0.5,1) \end{cases}$$

In other words, π_1 distributes uniform weight of $\frac{1}{4}$ on $[0,0.5)$ and uniform weight of $\frac{3}{4}$ on $[0.5,1)$, and π_2 distributes uniform weight of $\frac{3}{4}$ on $[0,0.5)$ and uniform weight of $\frac{1}{4}$ on $[0.5,1)$. Let $\mathcal{P}$ denote the convex set generated by π_1 and π_2. A set $\mathcal{P}$ of this form may express an individual's ambiguity as to the way the probabilistic weight is divided between the lower and upper halves of a considered range, with the constraint that at least $\frac{1}{4}$ of the weight is placed on each half, and with otherwise uniform belief.

For every pair of measures $\pi, \pi' \in \mathcal{P}$, $\pi(A) = 0 \Leftrightarrow \pi'(A) = 0$, and for $\varepsilon > 0$, $\lambda(E) < \frac{2}{3}\varepsilon$ guarantees $\pi_i(E) < \varepsilon$ for $i = 1, 2$, thus for all measures in $\mathcal{P}$. Therefore, fine partitions as required exist, and $\mathcal{P}$ is uniformly absolutely continuous. The subjective multi-prior probability induced by $\mathcal{P}$ is:
$A \succsim B$, if and only if,

$$\begin{aligned} \frac{1}{2}\,[\ \lambda(A) - \lambda(B)\ ] \quad &\geq \quad \lambda(B \cap [0.5,1)) - \lambda(A \cap [0.5,1)) \\ &\text{and} \\ \frac{1}{2}\,[\ \lambda(A) - \lambda(B)\ ] \quad &\geq \quad \lambda(B \cap [0,0.5)) - \lambda(A \cap [0,0.5))\ . \end{aligned}$$

In this example, a necessary condition to obtain $A \succsim B$ is $\lambda(A) \geq \lambda(B)$, but this condition is not sufficient. It should also be that on each half separately, event B does not have a 'large-enough edge'. For instance, $\neg$ ($[0,0.5) \succsim [0.8,1)$), but $[0,0.45) \cup [0.95,1) \succsim [0.8,1)$. Events A and B are considered equally likely if they have the same Lebesgue measure, divided in the same manner between $[0,0.5)$ and $[0.5,1)$.

Example 3. Let $S = [0,1)$ and Σ the algebra generated by all intervals $[a,b)$ contained in $[0,1)$. For $A \in \Sigma$ such that $\lambda(A) = \frac{1}{2}$, let π_A to be the probability measure defined by the density:

$$f_A(s) = \begin{cases} \frac{1}{2} & s \in A \\ \frac{3}{2} & s \notin A \end{cases}$$

Let $\mathcal{P}$ be the convex and closed set generated by all probability measures π_A. All measures in the set are mutually absolutely continuous with λ. For $\varepsilon > 0$, letting $\{E_1, \ldots, E_n\}$ be a partition with $\lambda(E_i) < \frac{2}{3}\varepsilon$ obtains $\pi_A(E_i) < \varepsilon$ for all measures π_A, hence for all measures in $\mathcal{P}$. The set $\mathcal{P}$ is therefore uniformly absolutely continuous.

The resulting subjective multi-prior probability representation satisfies assumptions **P1-P5**. Note that for any event $B \in \Sigma$ for which $0 < \lambda(B) < 1$,

$$\begin{aligned} \max_{\pi \in \mathcal{P}} \pi(B) \quad &= \quad \frac{3}{2}\min(\lambda(B), \frac{1}{2}) + \frac{1}{2}\max(0, \lambda(B) - \frac{1}{2}) \\ &> \quad \frac{1}{2}\min(\lambda(B), \frac{1}{2}) + \frac{3}{2}\max(0, \lambda(B) - \frac{1}{2}) = \min_{\pi \in \mathcal{P}} \pi(B). \end{aligned}$$

That is, the measures in $\mathcal{P}$ do not agree on any event which is non-null and non-universal. This example points to the problem of producing an 'objective measuring rod'. One cannot hope to be able to produce events which are unambiguous, in the sense that all probabilities in the set assign those events the same probability. Moreover, there are even no events with probability ε-close to a fixed value $0 < p < 1$. Note also that for events A and B, it cannot be that all measures in $\mathcal{P}$ agree that A and B have the same probability, therefore events cannot be partitioned into equally-likely events.

4 Extensions and comments

4.1 A relative notion of 'more ambiguous than'

Next, a relative notion of 'more ambiguous than' is presented. One subjective multi-prior probability is considered to be more ambiguous than another, if it is less decisive, and leaves more comparisons open. As can be expected, the 'more ambiguous than' relation is characterized by inclusion of the $\succsim$-maximal representing sets of priors. Being more ambiguous means having a larger, in the sense of set inclusion, set of priors.

Proposition 1. *Let $\succsim_1$ and $\succsim_2$ be two binary relations over Σ, satisfying P1-P5, with $\mathcal{P}_1$ and $\mathcal{P}_2$ the maximal representing sets (unions of all representing sets) of $\succsim_1$ and $\succsim_2$, respectively. Then the following statements are equivalent:*

(i) $\succsim_1 \subseteq \succsim_2$.

(ii) $\mathcal{P}_2 \subseteq \mathcal{P}_1$.

A result similar to this one is presented in Ghirardato, Maccheroni and Marinacci (2004). The setup in Ghirardato et al. contains alternatives which are mappings from a state space to a set of consequences. In that paper, a definition of 'reveals more ambiguity than' is given, that amounts to inclusion of one relation in another, where both relations admit a multiple-priors expected utility representation. Ghirardato et al. show that such inclusion of the relation implies reversed inclusion of the priors sets [9].

4.2 Ambiguity Attitude

A subjective multi-prior probability may be viewed as representing the unambiguous judgements of an individual. According to this view, some comparisons between events are left open due to ambiguity, and those that are determined are the 'unambiguous part' of the individual's likelihood judgements.

In the representation, a set of prior probabilities represents the scenarios the individual considers possible. An event is judged at least as likely as another, if this is the case under every possible scenario. The set of priors thus represents all the ambiguity that the individual perceives in the situation, and the unanimity rule characterizes those comparisons that are not affected by this ambiguity.

[9] In order to obtain the other direction in Ghirardato et al., an additional assumption of identical utility for both relations is required.

After ambiguity itself is identified, attitudes towards it may be added. One obvious possibility is completion using one probability measure from the multi-prior probability set. Another possibility, in a setup with an infinite space S, is suggested next. It exhibits aversion to ambiguity, and describes an individual which judges events by their lowest possible probability. A similar treatment, of completing an incomplete relation, is suggested by Gilboa et al. (2010) and Ghirardato, Maccheroni and Marinacci (2004). In these models, a set of priors is obtained as a description of the ambiguity perceived by an individual. The relations discussed in those papers apply to alternatives which map states to lotteries over an abstract set of consequences, and appraisal of acts involves a procedure of expected utility. Here, the relation applies to events alone, but the treatment of ambiguity separately from ambiguity attitude is similar in spirit.

In order to obtain an ambiguity averse completion, the framework of Sarin and Wakker (1992) is adopted. In that framework, an exogenous sub-sigma-algebra is taken as part of the model primitives, and is meant to represent a (rich) set of unambiguous events. An assumption of *Ambiguity Aversion* is further used. The aversion to ambiguity is expressed by supposing that the individual ranks unambiguous events strictly above other events, whenever this does not contradict consistency with the incomplete, 'original' relation.

Let $\mathcal{A}$ denote a sigma-algebra contained in Σ, and $\succsim$ a binary relation over Σ. Events in $\mathcal{A}$ should be thought of as *unambiguous*. For a complete relation $\succsim'$, the relations $\succ'$ and $\sim'$ denote its asymmetric and symmetric components, respectively.

First, a weak order assumption is required for the relation $\succsim'$.

P1'. Weak Order:
For any two events A and B, either $A \succsim' B$ or $B \succsim' A$. (Completeness)
For any three events A, B and C, if $A \succsim' B$ and $B \succsim' C$ then $A \succsim' C$. (Transitivity)

Next, a richness assumption, Savage's postulate P6 confined to unambiguous events, is stated.

P6-UA. Unambiguous Non Atomicity.:
If $C \succ' B$ for events $B, C \in \Sigma$, then there exists a finite partition of S, $\{A_1, \ldots, A_m\}$ with $A_i \in \mathcal{A}$ for all i, such that $C \succ' B \cup A_i$ for all i.

Last, two linking assumptions are added. The first assumption, *Consistency*, states that $\succsim'$ is a completion of $\succsim$, so $\succsim'$ is consistent with the preferences exhibited by $\succsim$. The second, *Ambiguity Aversion*, subscribes a completion rule that favors events with known probabilities. This assumption imitates Epstein's (1999) definition of uncertainty aversion for alternatives, mapping events to an abstract set of consequences. It uses as a benchmark for uncertainty neutrality the relation $\succsim$, and suggests to complete it by preferring unambiguous events. The completion in fact identifies an *'unambiguous equivalent'* for every event, much in the same manner as is done with certainty equivalents. As noted above, similar development may be found in Gilboa et al. (2010; the relevant axiom is termed there *Default to Certainty*, or *Caution* in its weaker version).

P7. Consistency:
For any two events A and B, $A \succsim' B$ whenever $A \succsim B$.

P8. Ambiguity Aversion:
Let $E \in \Sigma$ and $A \in \mathcal{A}$ be events. If $\neg(E \succsim A)$ then $A \succ' E$.

Proposition 2. *Let* $\succsim$ *and* $\succsim'$ *be binary relations over* Σ*, and let* $\mathcal{A}$ *be a* σ*-algebra contained in* Σ*. Then statements (i) and (ii) below are equivalent.*

(i) The following conditions are satisfied:

(a) $\succsim$ *satisfies axioms P1 through P5. On* $\mathcal{A}$, $\succsim$ *is complete.*

(b) $\succsim'$ *satisfies P6-UA.*

(c) $\succsim$ *and* $\succsim'$ *satisfy Consistency (P7) and Ambiguity Aversion (P8).*

(ii) There exists a nonempty, uniformly absolutely continuous set of probabilities $\mathcal{P}$*, such that for any pair of events* A *and* B*,*

$$A \succsim B \quad \Leftrightarrow \quad \mu(A) \geq \mu(B) \text{ for every } \mu \in \mathcal{P} \tag{5}$$

$$A \succsim' B \quad \Leftrightarrow \quad \min_{\mu \in \mathcal{P}} \mu(A) \geq \min_{\mu \in \mathcal{P}} \mu(B) \tag{6}$$

Furthermore, the probabilities in $\mathcal{P}$ *agree on* $\mathcal{A}$*, and their common part on* $\mathcal{A}$ *is unique and convex-ranged there.*

The theorem suggests how to complete a subjective multi-prior probability, in a manner that expresses ambiguity aversion. An ambiguity seeking rule can analogously be described through the use of maximum instead of minimum in (6). As in Theorem 2, the set of probability measures $\mathcal{P}$ is not necessarily unique, yet a maximal (w.r.t. inclusion) set may be identified, which represents the relation (see Observation 1 above).

Note that the set of unambiguous events was taken as a primitive of the model, and not derived from the relation. Epstein and Zhang (2001) derived a set of unambiguous events endogenously from preference. They then formulated conditions that obtain probabilistic sophistication over acts that are measurable with respect to the derived set of unambiguous events. The method of identifying endogenously a set that can play the part of unambiguous events was not applied here, as no adequate manner to define unambiguous events was yet found. In the setup considered here, with completeness not assumed, the definition of Epstein and Zhang yields that all the events in Σ are unambiguous.

4.3 Minimal set of priors

Nehring (2009) presents an example with two sets of priors $\mathcal{P}_1 \subsetneqq \mathcal{P}_2$ that represent the same likelihood order. The priors in $\mathcal{P}_2$ that are not in $\mathcal{P}_1$ do not restrict the order induced by $\mathcal{P}_1$ in the sense that when all $p \in \mathcal{P}_1$ agree that $p(A) \geq p(B)$, every $q \in \mathcal{P}_2 \setminus \mathcal{P}_1$ concurs. In this example the intersection of the two representing sets is also a representing set: it is $\mathcal{P}_1$. The question arises whether the intersection of every two representing sets is alwys a representing one.

The following example answers this question on the negative.

Example 4. Suppose that $S = \{0,1\}^{\mathbb{N}}$, the set of all sequences consisting of 0's and 1's. The algebra Σ is the one generated by events of the type $C(i,a) = (x_1, x_2, ...) \in S;\ x_1 = a$, where $a = 0, 1$. The probability measure, denoted $(p_1, p_2, ...)$, is the one induced by an infinite sequence of independent tosses of coins, where the probability of getting 1 in the i-th toss is p_i ($p_i \in (0,1)$) and the probability of getting 0 is $1 - p_i$. For simplicity a coin that assigns probability p to the outcome 1 and probability $1 - p$ to the outcome 0 is referred to as p-coin.

We define two disjoint sets of priors. Let $\mathcal{P}_1$ be the convex hull of the probability measures of the type $(p_1, ..., p_n, \frac{1}{2}, \frac{1}{2}, ...)$, where $p_1, ..., p_n \in \{\frac{1}{3}, \frac{1}{2}\}$. In other words, any distribution in $\mathcal{P}_1$ is such that the coins from time $n+1$ and on are $\frac{1}{2}$-coins, while up to time n the coins could be any combination of $\frac{1}{2}$-coins and $\frac{1}{3}$-coins.

The set $\mathcal{P}_2$ is defined in a way similar to that of $\mathcal{P}_1$ with the difference that coins from time $n+1$ and on are $\frac{1}{3}$-coins. Note that the distributions in $\mathcal{P}_1$ are generated by only finitely many $\frac{1}{3}$-coins and infinitely many $\frac{1}{2}$-coins. On the other hand, the distribution in $\mathcal{P}_2$ are generated by only finitely many $\frac{1}{2}$-coins and infinitely many $\frac{1}{3}$-coins. Thus, $\mathcal{P}_1$ and $\mathcal{P}_2$ are disjoint.

We argue that $\mathcal{P}_1$ and $\mathcal{P}_2$ induce the same likelihood relation over Σ. Let A, B be two events in Σ. There is time, say n, such that A and B are defined by conditions imposed only on the first n coordinates. Thus, the probability of A and B assigned by a particular distribution depends only on its behavior on the n first coordinates. However, for every distribution $\mu \in \mathcal{P}_1$ there is $\nu \in \mathcal{P}_2$ that behaves on the first n coordinates like μ. In particular, if $\mu(A) \geq \mu(B)$ (or $\mu(A) > \mu(B)$), then $\nu(A) \geq \nu(B)$ (or $\nu(A) > \nu(B)$). This implies that if $\nu(A) \geq \nu(B)$ for every $\nu \in \mathcal{P}_2$, then $\mu(A) \geq \mu(B)$ for every $\mu \in \mathcal{P}_1$. For a similar reason the inverse of the previous statement is also true. We conclude that $\mathcal{P}_1$ and $\mathcal{P}_2$ induce the same likelihood relation over Σ.

This example shows that two representing sets can be disjoint. In particular, the intersection of two representing sets is not representing the same relation.

A careful look at this example reveals that the sets $\mathcal{P}_1$ and $\mathcal{P}_2$ are not closed (w.r.t. the weak* topology). For instance, the probability $(\frac{1}{3}, \frac{1}{3}, ...)$ is in the closure of $\mathcal{P}_1$. The reason is that $(\frac{1}{3}, \frac{1}{3}, ...)$ is a cumulative point of the following sequence of probability measures in $\mathcal{P}_1$: $(\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}...)$, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}...)$, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}...)$,... .

It turns out that the closure of one set contains the other and therefore, the closures of $\mathcal{P}_1$ and of $\mathcal{P}_2$ coincide. This conclusion leaves us with two unanswered questions: whether the intersection of two closed (w.r.t. the weak* topology) representing sets of priors can be disjoint, and whether the intersection itself is a representing set for the same relation.

5 Proofs

Let $B_0(S, \Sigma)$ denote the vector space generated by linear combinations of indicator functions $\mathbf{1}_A$ for $A \in \Sigma$, endowed with the supremum norm. Define a subset $D_{\succsim}$ of $B_0(S, \Sigma)$,

$$D_{\succsim} = closure\{ \sum_{i=1}^{n} \alpha_i \ [\ \mathbf{1}_{A_i} - \mathbf{1}_{B_i}\ ] \mid A_i \succsim B_i,\ \alpha_i \geq 0,\ n \in \mathbb{N} \ \} .$$

That is, $D_{\succsim}$ is the closed convex cone generated by indicator differences $\mathbf{1}_A - \mathbf{1}_B$, for $A \succsim B$. By Reflexivity, Positivity and Nontriviality, $D_{\succsim}$ has vertex at zero, $D_{\succsim}$ is not the entire space $B_0(S, \Sigma)$, and it contains every nonnegative vector $\psi \in B_0(S, \Sigma)$.

The next claims show that under assumptions P1 through P4, preference is preserved under convex combinations (the proof is joint for the finite and infinite cases). In some of the claims, conclusions are more easily understood considering the following alternative formulation of Generalized Finite Cancellation:

Let A and B be two events, and $(A_i)_{i=1}^n$ and $(B_i)_{i=1}^n$ two sequences of events from Σ, that satisfy,

$$A_i \succsim B_i \text{ for all } i, \text{ and for some } k \in \mathbb{N},$$
$$\sum_{i=1}^{n} [\mathbf{1}_{A_i}(s) - \mathbf{1}_{B_i}(s)] = k[\mathbf{1}_A(s) - \mathbf{1}_B(s)] \text{ for all } s \in S.$$

Then $A \succsim B$.

Claim 1. *Suppose that*[10] $r_i \in \mathbb{Q}_{++}$, *and* $A_i \succsim B_i$ *for* $i = 1, \ldots, n$. *If* $\mathbf{1}_A - \mathbf{1}_B = \sum_{i=1}^n r_i(\mathbf{1}_{A_i} - \mathbf{1}_{B_i})$, *then* $A \succsim B$.

Proof. Let k denote the common denominator of $r_1, \ldots, r_n$, and write $r_i = \frac{m_i}{k}$ for $m_i \in \mathbb{N}$ and $i = 1, \ldots, n$. It follows that $k(\mathbf{1}_A - \mathbf{1}_B) = \sum_{i=1}^n m_i(\mathbf{1}_{A_i} - \mathbf{1}_{B_i})$ for all $s \in S$. By GFC applied to sequences $(A_i)_{i=1}^N$ and $(B_i)_{i=1}^N$, where each A_i and each B_i repeats m_i times ($N = m_1 + \ldots + m_n$), it follows that $A \succsim B$. ■

Claim 2. *Suppose there are* $\alpha_i > 0, i = 1, \ldots, n,$ *such that* $\mathbf{1}_A - \mathbf{1}_B = \sum_{i=1}^n \alpha_i(\mathbf{1}_{A_i} - \mathbf{1}_{B_i})$. *Then there are* $r_i \in \mathbb{Q}_{++}, i = 1, \ldots, n,$ *such that* $\mathbf{1}_A - \mathbf{1}_B = \sum_{i=1}^n r_i(\mathbf{1}_{A_i} - \mathbf{1}_{B_i})$.

Proof. Consider the partition induced by $A_1, \ldots, A_n, B_1, \ldots, B_n, A, B$, and denote it by $\mathcal{A}$, with atoms denoted by a. The assumed indicators identity for all $s \in S$ translates to the following finite system of linear equations, with the variables $\alpha_1, \ldots, \alpha_n$.

$$\sum_{i=1}^{n} \delta_i(a)\alpha_i = \delta(a), \quad a \in \mathcal{A},$$
$$\delta_i(a) = \mathbf{1}_{A_i}(s) - \mathbf{1}_{B_i}(s), \ s \in a, \quad \delta(a) = \mathbf{1}_A(s) - \mathbf{1}_B(s), \ s \in a.$$

Since all coefficients in the above equations, $\delta(a)$ and $\delta_i(a)$, are integers, then by denseness of the rational numbers in the reals it follows that if a nonnegative solution to this system exists, then there also exists a nonnegative *rational* solution. Thus, if indeed $\mathbf{1}_A - \mathbf{1}_B = \sum_{i=1}^n \alpha_i(\mathbf{1}_{A_i} - \mathbf{1}_{B_i})$ for some $\alpha_i > 0$, then there is a solution $r_i \in \mathbb{Q}_{++}$. ■

Conclusion 1. *Suppose that* $A_i \succsim B_i$ *for* $i = 1, \ldots, n$. *If there are* $\alpha_i > 0$, $i = 1, \ldots, n,$ *such that* $\mathbf{1}_A - \mathbf{1}_B = \sum_{i=1}^n \alpha_i(\mathbf{1}_{A_i} - \mathbf{1}_{B_i})$, *then* $A \succsim B$.

[10] $\mathbb{Q}_{++}$ is the set of strictly positive rational numbers.

In order to show that $D_{\succsim}$ contains exactly those indicator differences which correspond to preference, it should further be proved that preference is preserved under the closure operation. This is done separately for the finite and infinite cases.

5.1 Proof of Theorem 1

First it is proved that under assumptions **P1-P4**, the multiple-priors representation follows (direction (i)⇒(ii)).

When S is finite, there are finitely many pairs of events. Therefore $D_{\succsim}$ is generated by a finite number of vectors. In fact,

$$D_{\succsim} = \sum_{(A,B):A\succsim B} \alpha_{(A,B)}(\mathbf{1}_A - \mathbf{1}_B), \quad \alpha_{(A,B)} \geq 0 \ ,$$

hence $D_{\succsim}$ is closed, and by Conclusion 1, $\mathbf{1}_A - \mathbf{1}_B \in D_{\succsim}$ if and only if $A \succsim B$. Define[11] $\mathcal{V} = \{v \in \mathbb{R}^S \mid v \cdot \varphi \geq 0 \text{ for all } \varphi \in D_{\succsim}\}$. The set $\mathcal{V}$ is a closed convex cone, and contains the zero function.

Claim 3.

$$\varphi \in D_{\succsim} \quad \Leftrightarrow \quad v \cdot \varphi \geq 0 \ \textit{ for every } v \in \mathcal{V}.$$

Proof. By definition of $\mathcal{V}$, if $\varphi \in D_{\succsim}$ then $v \cdot \varphi \geq 0$ for every $v \in \mathcal{V}$. Now suppose that $\psi \notin D_{\succsim}$. Since $D_{\succsim}$ is a closed convex cone then by a standard separation theorem (see Dunford and Schwartz, Corollay V.2.12) there exists a nonzero vector separating $D_{\succsim}$ and ψ. Since $0 \in D_{\succsim}$ and $a\psi \notin D_{\succsim}$ for all $a > 0$, there exists $w \in \mathbb{R}^S$ such that $w \cdot \varphi \geq 0 > w \cdot \psi$, for every $\varphi \in D_{\succsim}$. It follows that $w \in \mathcal{V}$, and the proof is completed. ■

Conclusion 2.

$$A \succsim B \quad \Leftrightarrow \quad v \cdot (\mathbf{1}_A - \mathbf{1}_B) \geq 0 \ \textit{ for every } v \in \mathcal{V}.$$

For every event $A \in \Sigma$ and vector $v \in \mathcal{V}$, denote $v(A) = v \cdot \mathbf{1}_A = \sum_{s \in A} v(s)$. According to the Non-triviality assumption, $\mathcal{V} \neq \{0\}$. By Positivity, $\mathbf{1}_A \in D_{\succsim}$ for all $A \in \Sigma$, therefore $v(A) \geq 0$ for every $v \in \mathcal{V}$. It follows that the set $\mathcal{P} = \{\pi = v/v(S) \mid v \in \mathcal{V} \setminus \{0\}\}$ is a nonempty set of additive probability measures over Σ, such that:

$$A \succsim B \quad \Leftrightarrow \quad \pi(A) \geq \pi(B) \text{ for every } \pi \in \mathcal{P}.$$

By its definition, $\mathcal{P}$ is the maximal set w.r.t. inclusion that represents the relation.

The other direction, from the representation to the axioms, is trivially implied from properties of probability measures (GFC directly follows by taking expectation on both sides).

[11] For $x = (x_1, ..., x_{|S|}), y = (y_1, ..., y_{|S|}) \in \mathbb{R}^S$, $x \cdot y$ denotes the inner product of x and y. That is, $x \cdot y = \sum_{i=1}^{|S|} x_i y_i$.

5.2 Proof of Theorem 2

5.2.1 Proof of the direction (i)⇒(ii)

Claim 4. *If* $A \succ\succ B$ *, then* $\mathbf{1}_A - \mathbf{1}_B$ *is an interior point of* $D_{\succsim}$*.*

Proof. By definition, $A \succ\succ B$ implies that there exists a partition $\{A_1, \ldots, A_k\}$ of A and a partition $\{B'_1, \ldots, B'_l\}$ of B^c, such that for all i, j, $A \backslash A_i \succsim B \cup B_j$. First observe that it cannot be that $A = \emptyset$, for it would imply, on the one hand, that $A = \emptyset \succsim B$, by definition of strong preference, and on the other hand, by Generalized Finite Cancellation, $\emptyset \succsim B^c \;\Leftrightarrow\; B \succsim S$, contradicting Non Triviality. Similarly $B = S$ is impossible. Hence, $k, l \geq 1$. Using the definition of strong preference, monotonicity of $\succsim$ w.r.t set inclusion and the structure of $D_{\succsim}$ obtains:

$$\begin{aligned}
&\mathbf{1}_A - \mathbf{1}_B - \mathbf{1}_{A_i} \in D_{\succsim}\ ,\ i = 1, \ldots, k, \quad \text{and} \\
&\mathbf{1}_A - \mathbf{1}_B - \mathbf{1}_{B_j} \in D_{\succsim}\ ,\ j = 1, \ldots, l, \quad \text{therefore} \\
&(k+l)(\mathbf{1}_A - \mathbf{1}_B) - \mathbf{1}_A - \mathbf{1}_{B^c} = (k+l-1)(\mathbf{1}_A - \mathbf{1}_B) - \mathbf{1}_S \in D_{\succsim} \quad \Rightarrow \\
&(\mathbf{1}_A - \mathbf{1}_B) - \frac{1}{k+l-1}\mathbf{1}_S \in D_{\succsim}.
\end{aligned}$$

It is next shown that there exists a neighborhood of $\mathbf{1}_A - \mathbf{1}_B$ in $D_{\succsim}$. Let $\varepsilon < \frac{1}{2(k+l-1)}$ and let $\varphi \in B_0(S, \Sigma)$ be such that $\|\mathbf{1}_A - \mathbf{1}_B - \varphi\| < \varepsilon$. For all $s \in S$, $\varphi(s) > \mathbf{1}_A(s) - \mathbf{1}_B(s) - \frac{1}{2(k+l-1)}$, therefore φ dominates $\mathbf{1}_A - \mathbf{1}_B - \frac{1}{k+l-1}\mathbf{1}_S$. It follows that $\varphi = \mathbf{1}_A - \mathbf{1}_B - \frac{1}{k+l-1}\mathbf{1}_S + \psi$ for $\psi \in D_{\succsim}$ (since ψ is nonnegative), hence $\varphi \in D_{\succsim}$ and $\mathbf{1}_A - \mathbf{1}_B$ is an internal point of $D_{\succsim}$. ■

Claim 5. *If* $\mathbf{1}_A - \mathbf{1}_B$ *is on the boundary of* $D_{\succsim}$*, then* $A \succsim B$*.*

Proof. Suppose on the contrary that for some events A and B,
$\mathbf{1}_A - \mathbf{1}_B$ is on the boundary of $D_{\succsim}$, yet $\neg(A \succsim B)$. As $\mathbf{1}_A - \mathbf{1}_B$ is on the boundary of $D_{\succsim}$, there exists $\varepsilon' > 0$ and $\varphi \in D_{\succsim}$ such that $\mathbf{1}_A - \mathbf{1}_B + \delta\varphi \in D_{\succsim}$ for every $0 < \delta < \varepsilon'$.

On the other hand, employing Non Atomicity, there exists an event $F \subseteq A^c$ such that $F \succ\succ \emptyset$ and $\mathbf{1}_A - \mathbf{1}_B + \mathbf{1}_F \notin D_{\succsim}$. The previous claim entails that $\mathbf{1}_F$ is an internal point of $D_{\succsim}$, hence there exists $\varepsilon_\varphi > 0$ such that $\mathbf{1}_F + \delta\varphi \in D_{\succsim}$ for all $|\delta| < \varepsilon_\varphi$. Let $0 < \delta < \min(\varepsilon_\varphi, \varepsilon')$, then $\mathbf{1}_A - \mathbf{1}_B + \delta\varphi + \mathbf{1}_F - \delta\varphi = \mathbf{1}_A - \mathbf{1}_B + \mathbf{1}_F$ is in $D_{\succsim}$, since it is a sum of two vectors in $D_{\succsim}$. Contradiction. ■

Conclusion 3.

$$A \succsim B \quad \Leftrightarrow \quad \mathbf{1}_A - \mathbf{1}_B \in D_{\succsim}.$$

Proof. The set $D_{\succsim}$ contains, by its definition, all indicator differences $\mathbf{1}_{A'} - \mathbf{1}_{B'}$ for $A' \succsim B'$ (thus also the zero vector), and their positive linear combinations. However, by the previous claims, if $\mathbf{1}_A - \mathbf{1}_B$ may be represented as a positive linear combination of indicator differences $\mathbf{1}_{A'} - \mathbf{1}_{B'}$ for which $A' \succsim B'$, or if $\mathbf{1}_A - \mathbf{1}_B$ is on the boundary of $D_{\succsim}$, then $A \succsim B$. That is, every indicators difference $\mathbf{1}_A - \mathbf{1}_B$ in the closed convex cone generated by indicator differences indicating preference also satisfies $A \succsim B$. ■

Denote by $B(S,\Sigma)$ the space of all Σ-measurable, bounded real functions over S, endowed with the supremum norm. Denote by $ba(\Sigma)$ the space of all bounded, additive functions from Σ to $\mathbb{R}$, endowed with the total variation norm. The space $ba(\Sigma)$ is isometrically isomorphic to the conjugate space of $B(S,\Sigma)$. Since $B_0(S,\Sigma)$ is dense in $B(S,\Sigma)$, $ba(\Sigma)$ is also isometrically isomorphic to the conjugate space of $B_0(S,\Sigma)$.

Consider an additional topology on $ba(\Sigma)$. For $\varphi \in B_0(S,\Sigma)$ and $m \in ba(S,\Sigma)$, let $\varphi(m) = \int_S \varphi dm$. Every φ defines a linear functional over $ba(S,\Sigma)$, and $B_0(S,\Sigma)$ is a total space of functionals on $ba(S,\Sigma)$.[12] The $B_0(S,\Sigma)$ topology of $ba(S,\Sigma)$, by its definition, makes a locally convex linear topological space, and the linear functionals on $ba(S,\Sigma)$ which are continuous in this topology are exactly the functionals defined by $\varphi \in B_0(S,\Sigma)$. Event-wise convergence of a bounded generalized sequence μ_α in $ba(S,\Sigma)$ to μ is identical to its convergence to μ in the following topologies: the $B_0(S,\Sigma)$ topology, the $B(S,\Sigma)$ topology, and the weak* topology (see Maccheroni and Marinacci, 2001). Hence, the notion closedness of bounded subsets of $ba(S,\Sigma)$ is identical in all three topologies.

Let

$$\mathcal{M} = \{m \in ba(\Sigma) \mid \int_S \varphi dm \geq 0 \text{ for all } \varphi \in D_{\succsim} \}. \tag{7}$$

The set $\mathcal{M}$ is a convex cone, and contains the zero function. For a generalized sequence $\{m_\tau\}$ in $\mathcal{M}$, which converges to m in the $B_0(S,\Sigma)$ topology, $m_\tau(\xi) \to m(\xi)$ for every $\xi \in B_0(S,\Sigma)$. Therefore, having $m_\tau(\varphi) \geq 0$ for every $\varphi \in D_{\succsim}$ and every τ, yields that $m \in \mathcal{M}$. The set $\mathcal{M}$ is thus closed in the $B_0(S,\Sigma)$ topology.[13]

Claim 6.

$$\varphi \in D_{\succsim} \quad \Leftrightarrow \quad \int_S \varphi dm \geq 0 \textit{ for every } m \in \mathcal{M}.$$

Proof. According to the definition of $\mathcal{M}$, it follows that if $\varphi \in D_{\succsim}$ then $\int_S \varphi dm \geq 0$ for every $m \in \mathcal{M}$. Now suppose that $\psi \notin D_{\succsim}$. Since $D_{\succsim}$ is a closed convex cone, and $B_0(S,\Sigma)$, endowed with the supnorm, is locally convex, then by a Separation Theorem (see Dunford and Schwartz (1957), Corollary V.2.12) there exists a non-zero, continuous linear functional separating $D_{\succsim}$ and ψ. Hence, since $0 \in D_{\succsim}$ and $a\psi \notin D_{\succsim}$ for all $a > 0$, there exists $m' \in ba(\Sigma)$ such that $\int_S \varphi dm' \geq 0 > \int_S \psi dm'$, for every $\varphi \in D_{\succsim}$. It follows that $m' \in \mathcal{M}$, and the proof is completed. ■

Conclusion 4.

$$A \succsim B \quad \Leftrightarrow \quad \int_S (\mathbf{1}_A - \mathbf{1}_B) dm \geq 0 \quad \textit{for every } m \in \mathcal{M}.$$

Proof. Follows from Conclusion 3 and the previous claim. ■

[12] That is, $\varphi(m) = 0$ for every $\varphi \in B_0(S,\Sigma)$ implies that $m = 0$.

[13] This part of the proof is very similar to a proof found in 'Ambiguity from the differential viewpoint', a previous version of Ghirardato, Maccheroni and Marinacci (2004).

According to the Nontriviality assumption, $\mathcal{M} \neq \{0\}$. By Positivity, $\mathbf{1}_A \in D_{\succsim}$ for all $A \in \Sigma$, therefore $\int_S \mathbf{1}_A dm \geq 0$ for every $m \in \mathcal{M}$. It follows that the set $\mathcal{P} = \{\pi = m/m(S) \mid m \in \mathcal{M} \setminus \{0\}\}$ is a nonempty, $B_0(S,\Sigma)$-closed and convex set of additive probability measures over Σ, such that:

$$A \succsim B \quad \Leftrightarrow \quad \pi(A) \geq \pi(B) \text{ for every } \pi \in \mathcal{P}.$$

Observation 2. The set $\mathcal{P}$ is bounded (in the total variation norm), hence it is compact in the $B(S,\Sigma)$ topology, thus in the $B_0(S,\Sigma)$ topology (see Corollary V.4.3 in Dunford and Schwartz). Boundedness of $\mathcal{P}$ also implies that it is weak* closed.

The set $\mathcal{P}$, by its definition, is maximal w.r.t. inclusion (any $\pi' \notin \mathcal{P}$ yields $\int_S \varphi d\pi' < 0$ for some $\varphi \in D_{\succsim}$, hence $\pi'(A) < \pi'(B)$ for some pair of events that satisfy $A \succsim B$), $B_0(S,\Sigma)$-closed and convex.

Claim 7.

$$A \succ\succ B \quad \Rightarrow \quad \pi(A) - \pi(B) > \delta > 0 \textit{ for every } \pi \in \mathcal{P}.$$

Proof. By definition of strong preference, $A \succ\succ B$ if and only if there exists a partition $\{G_1, \ldots, G_r\}$ of S, such that $A \setminus G_i \succsim B \cup G_j$ for all i, j. It means that there are partitions $\{A_1, \ldots, A_k\}$ of A, and $\{B'_1, \ldots, B'_l\}$ of B^c, such that $\pi(A) - \pi(A_i) \geq \pi(B) + \pi(B'_j)$ for all $\pi \in \mathcal{P}$ and all i, j. It cannot be that $\pi(A) - \pi(A_i) = 0$ or $\pi(B) + \pi(B'_j) = 1$, since $\pi(B \cup B'_j) > 0$ for some j and $\pi(A \setminus A_i) < 1$ for some i. Hence, $k \geq 2$ and $l \geq 2$, and, for all $\pi \in \mathcal{P}$,

$$\begin{aligned} &\pi(A) - \pi(B) \geq \pi(A_i) + \pi(B'_j), \quad \text{for all } i, j \quad \Rightarrow \\ &(k + l - 1)(\pi(A) - \pi(B)) \geq 1 \end{aligned}$$

and the proof is completed with $\delta = 1/(k + l)$, for instance. ■

In particular, it follows that $\pi(A) > \pi(B)$ for every $\pi \in \mathcal{P}$, whenever $A \succ\succ B$.

Lemma 1. *The probability measures in $\mathcal{P}$ are uniformly absolutely continuous.*

Proof. Let B be an event, and suppose that $\mu'(B) > 0$ for some $\mu' \in \mathcal{P}$. The inequality implies that $\neg(B^c \succsim S)$, therefore by Non Atomicity and claim 7 there exists an event $E \subseteq B$ that satisfies $\mu(E) > 0$ for every $\mu \in \mathcal{P}$. It follows that $\mu(B) > 0$ for every $\mu \in \mathcal{P}$, hence $\mu(B) > 0 \Leftrightarrow \mu'(B) > 0$, for every event B and any $\mu, \mu' \in \mathcal{P}$.

Non Atomicity further results that $\neg(B^c \cup E \succsim S)$, which implies that $\mu(B \setminus E) > 0$ for some, hence for all, $\mu \in \mathcal{P}$. Therefore, if $\mu(B) > 0$ there exists $E \subset B$ with $\mu(B) > \mu(E) > 0$. All probability measures in $\mathcal{P}$ are thus non atomic.

According to the above arguments, there exists an event F_1 such that
$0 < \mu(F_1) < 1$ for all $\mu \in \mathcal{P}$. As this implies $\neg(F_1^c \succsim S)$, it follows from Non Atomicity that there exists a partition of F_1, $\{E_1, \ldots, E_m\}$, such that:

$$E_i \succ\succ \emptyset \quad \text{and} \quad \neg(\emptyset \succsim F_1 \setminus E_i) \text{ for } i = 1, \ldots, m.$$

For a fixed i, the preference $E_i \succ\succ \emptyset$ entails that there exists a partition of S, $\{G_1, \ldots, G_{r_i}\}$, that satisfies $E_i \succsim G_j$, $j = 1, \ldots, r_i$. Taking the refinement of the partitions for each i, there exists a partition $\{G_1, \ldots, G_r\}$ such that $E_i \succsim G_j$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, r$. It follows that for each i, j, $\mu(G_j) \leq \mu(E_i)$ for every $\mu \in \mathcal{P}$. As $\emptyset \succsim \emptyset$, the partition $\{E_1, \ldots, E_m\}$ must consist of at least two atoms. Hence, for each j,

$$\mu(G_j) \leq \frac{1}{m} \sum_{i=1}^{m} \mu(E_i) \leq \frac{1}{2}\mu(F_1) < \frac{1}{2}, \text{ for all } \mu \in \mathcal{P}.$$

Let $F_2 = G_j$ for G_j such that $\mu(G_j) > 0$ (there must exist such j since the G_j's partition S). Again $\neg(\emptyset \succsim F_2)$, and by Non Atomicity there exists a partition of F_2, $\{E'_1, \ldots, E'_l\}$, such that:

$$E'_i \succ\succ \emptyset \ \text{ and } \ \neg(\emptyset \succsim F_2 \setminus E'_i) \ \text{ for } i = 1, \ldots, l.$$

As in the previous step, it follows that $l \geq 2$ and there exists a partition $\{G'_1, \ldots, G'_k\}$ with $\mu(G'_j) \leq \mu(E'_i)$ for all indices i, j and all probabilities $\mu \in \mathcal{P}$. Thus, for all j,

$$\mu(G'_j) \leq \frac{1}{l} \sum_{i=1}^{l} \mu(E'_i) \leq \frac{1}{2}\mu(F_2) < \frac{1}{4}, \text{ for all } \mu \in \mathcal{P}.$$

In the same manner, for all $n \in \mathbb{N}$ there exists a partition $\{G_1, \ldots, G_r\}$ such that for all j, $\mu(G_j) < \frac{1}{2^n}$ for all $\mu \in \mathcal{P}$. It follows that the probabilities in $\mathcal{P}$ are uniformly absolutely continuous. ■

The proof of the direction $(i) \Rightarrow (ii)$ is in fact completed. One more corollary is added, that will prove useful in the sequel. This is the opposite direction to Claim 7.

Corollary 2. *For any events A and B, if there exists $\delta > 0$ such that $\mu(A) - \mu(B) > \delta$ for every $\mu \in \mathcal{P}$, then $A \succ\succ B$.*

Proof. Let A and B be events such that $\mu(A) - \mu(B) > \delta > 0$. By uniform absolute continuity, there exists a partition $\{G_1, \ldots, G_r\}$ of S such that for all j, $\mu(G_j) < \delta/2$ for every $\mu \in \mathcal{P}$. It is so obtained that

$$\mu(A \setminus G_i) > \mu(A) - \delta/2 > \mu(B) + \delta/2 > \mu(B \cup G_j), \text{ for all } i, j.$$

■

Conclusion 5.

$$A \succ\succ B \quad \Leftrightarrow \quad \mu(A) - \mu(B) > \delta > 0 \ , \ \textit{for every } \mu \in \mathcal{P}.$$

5.2.2 Proof of the direction (ii)⇒(i)

Suppose that for every $A, B \in \Sigma$, $A \succsim B$ if and only if $\pi(A) \succsim \pi(B)$ for every probability measure π in a $\succsim$-maximal set $\mathcal{P}$, and that $\mathcal{P}$ is uniformly absolutely continuous, and all probabilities in the set are non atomic. Assumptions P1 through P5 are shown to hold.

P1 Reflexivity and P2 Positivity.
For every $A \in \Sigma$ and every $\pi \in \mathcal{P}$, $\pi(A) \geq \pi(A)$ and $\pi(A) \geq 0$, hence $A \succsim A$ and $A \succsim \emptyset$.

P3 Nontriviality. The $\succsim$-maximal set $\mathcal{P}$ is nonempty, thus $\pi(B) > \pi(A)$ for some $A, B \in \Sigma$ and $\pi \in \mathcal{P}$, implying $\neg(A \succsim B)$.

P4 Generalized Finite Cancellation.
Let $(A_i)_{i=1}^n$ and $(B_i)_{i=1}^n$ be two collections of events in Σ, such that $A_i \succsim B_i$ for all i, and $\sum_{i=1}^n (\mathbf{1}_{A_i}(s) - \mathbf{1}_{B_i}(s)) \leq k\,(\mathbf{1}_A(s) - \mathbf{1}_B(s))$ for all $s \in S$, for some $k \in \mathbb{N}$ and events $A, B \in \Sigma$. Then for every π in $\mathcal{P}$, $k\mathbb{E}_\pi\,(\mathbf{1}_A - \mathbf{1}_B) \geq \sum_{i=1}^n \mathbb{E}_\pi(\mathbf{1}_{A_i} - \mathbf{1}_{B_i}) \geq 0$. It follows that $\pi(A) \geq \pi(B)$ for every $\pi \in \mathcal{P}$, hence $A \succsim B$.

Claim 8. *If* $\mu(F) > 0$ *for an event* F *and probability measure* $\mu \in \mathcal{P}$, *then* $F \succ\succ \emptyset$.

Proof. Suppose that F is an event with $\mu(F) > 0$. By uniform absolute continuity of $\mathcal{P}$, there exists a partition $\{E_1, \ldots, E_m\}$ of S, such that $\mu(F) > \mu(E_i)$ for all i and all $\mu \in \mathcal{P}$. In the same manner, for any $F \setminus E_i$ there exists a partition $\{G_1^i, \ldots, G_{k_i}^i\}$ such that $\mu(F \setminus E_i) > \mu(G_j^i)$ for all j and all $\mu \in \mathcal{P}$. Let $\{G_1, \ldots, G_r\}$ be the refinement of $\{E_1, \ldots, E_m\}$ and the partitions for each i, then $\mu(F \setminus G_i) > \mu(G_j)$ for all i, j and all $\mu \in \mathcal{P}$. By definition, $F \succ\succ \emptyset$. ■

P5 Non Atomicity.
Suppose that $\neg(A \succsim B)$. By the representation assumption, $\mu'(B) > \mu'(A)$ for some $\mu' \in \mathcal{P}$. Note that necessarily $\mu'(A^c) > 0$. It is required to show that there exists a partition $\{A'_1, \ldots, A'_k\}$ of A^c such that for all i, $A'_i \succ\succ \emptyset$ and $\neg(A \cup A'_i \succsim B)$.

Uniform absolute continuity of the set $\mathcal{P}$ implies that there exists a partition $\{G_1, \ldots, G_r\}$ of S, such that for all j, $\mu(G_j) < \mu'(B) - \mu'(A)$ for all μ, thus specifically for μ'. The partition $\{G_1, \ldots, G_r\}$ induces a partition $\{A'_1, \ldots, A'_k\}$ of A^c such that $\mu'(A'_i) > 0$ and $\mu'(A \cup A'_i) < \mu'(B)$. By the representation and the previous claim, $A'_i \succ\succ \emptyset$ and $\neg(A \cup A'_i \succsim B)$ for all i.

5.3 Proof of Proposition 1

If $\mathcal{P}_2 \subseteq \mathcal{P}_1$ then obviously $\pi(A) \geq \pi(B)$ for every $\pi \in \mathcal{P}_2$, whenever $\pi(A) \geq \pi(B)$ for every $\pi \in \mathcal{P}_1$. In the other direction, let $\succsim_1$ and $\succsim_2$ be two binary relations over Σ with a subjective multi-prior probability representation. Denote by $\mathcal{P}_1$ the $\succsim_1$-maximal set and by $\mathcal{P}_2$ the $\succsim_2$-maximal set. Assume that for every pair of events A and B, $A \succsim_1 B$ implies $A \succsim_2 B$. By the representation, it follows that $\pi(A) \geq \pi(B)$ for every $\pi \in \mathcal{P}_1$ implies $\pi(A) \geq \pi(B)$ for every $\pi \in \mathcal{P}_2$. The proof that $\mathcal{P}_2 \subseteq \mathcal{P}_1$ is given separately for the finite and the infinite cases.

5.3.1 Proof of (i)⇒(ii) when S is finite

Recall the construction of $\mathcal{V}$ and the derived $\succsim$-maximal set of probabilities from the proof of Theorem 1. Let $\mathcal{V}_1 = \{a\pi \mid a \geq 0, \pi \in \mathcal{P}_1\}$, $\mathcal{V}_2 = \{a\pi \mid a \geq 0, \pi \in \mathcal{P}_2\}$ be the closed convex cones associated with $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. Suppose on the contrary that there exists $\pi' \in \mathcal{P}_2 \setminus \mathcal{P}_1$. As $\pi' \notin \mathcal{P}_1$, then also $a\pi' \notin \mathcal{V}_1$ for every $a \geq 0$, and since $\mathcal{V}_1$ is a cone, a separation theorem yields that there exists a non-zero $\varphi \in \mathbb{R}^S$ for which $\varphi \cdot v \geq 0 > \varphi \cdot$, for every $v \in \mathcal{V}_1$.

Let $D_{\succsim_1}$, $D_{\succsim_2}$ denote the closed convex cones generated by indicator differences $\mathbf{1}_A - \mathbf{1}_B$ for $A \succsim B$ (see the definition of $D_{\succsim}$ in the proof of Theorem 1). Employing Claim 3, the separating φ satisfies $\varphi \in D_{\succsim_1} \setminus D_{\succsim_2}$. That is, $\varphi = \sum_{i=1}^n \alpha_i(\mathbf{1}_{A_i} - \mathbf{1}_{B_i})$ for $A_i \succsim_1 B_i$, but $\sum_{i=1}^n \alpha_i(\mathbf{1}_{A_i} - \mathbf{1}_{B_i}) \notin D_{\succsim_2}$. It follows that there exist events A, B such that $A \succsim_1 B$ while not $A \succsim_2 B$. Contradiction.

5.3.2 Proof of (i)⇒(ii) when S is infinite

For $\varphi \in B_0(S, \Sigma)$ and $m \in ba(S, \Sigma)$, let $\varphi(m) = \int_S \varphi dm$. Every φ defines a linear functional over $ba(S, \Sigma)$, and $B_0(S, \Sigma)$ is a total space of functionals on $ba(S, \Sigma)$.[14] Consider the $B_0(S, \Sigma)$ topology of $ba(S, \Sigma)$. By its definition, in this topology $ba(S, \Sigma)$ is a locally convex linear topological space, and the linear functionals on $ba(S, \Sigma)$ which are continuous in the $B_0(S, \Sigma)$ topology are exactly the functionals defined by $\varphi \in B_0(S, \Sigma)$.

Recall the construction of $\mathcal{M}$ and the derived $\succsim$-maximal set of probabilities from the proof of Theorem 2. Let $\mathcal{M}_1 = \{a\pi \mid a \geq 0, \pi \in \mathcal{P}_1\}$, $\mathcal{M}_2 = \{a\pi \mid a \geq 0, \pi \in \mathcal{P}_2\}$ be the closed convex cones associated with $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. Suppose on the contrary that there exists $\pi' \in \mathcal{P}_2 \setminus \mathcal{P}_1$. As $\pi' \notin \mathcal{P}_1$, then also $a\pi' \notin \mathcal{M}_1$ for every $a \geq 0$, and a Separation Theorem, employed for the $B_0(S, \Sigma)$ topology of $ba(S, \Sigma)$, yields that there exists a non-zero $\varphi \in B_0(S, \Sigma)$, separating $\mathcal{M}_1$ and π'. Since $\mathcal{M}_1$ is a cone, there exists φ for which $\int_S \varphi dm \geq 0 > \int_S \varphi d\pi'$, for every $m \in \mathcal{M}_1$.

Let $D_{\succsim_1}$, $D_{\succsim_2}$ denote the closed convex cones generated by indicator differences $\mathbf{1}_A - \mathbf{1}_B$ for $A \succsim B$ (see Claim 5, the definition of $D_{\succsim}$ in the proof of Theorem 2 and Observation 1). Employing Claim 6, the separating φ satisfies $\varphi \in D_{\succsim_1} \setminus D_{\succsim_2}$. That is, $\varphi = \sum_{i=1}^n \alpha_i(\mathbf{1}_{A_i} - \mathbf{1}_{B_i})$ for $A_i \succsim_1 B_i$, but $\sum_{i=1}^n \alpha_i(\mathbf{1}_{A_i} - \mathbf{1}_{B_i}) \notin D_{\succsim_2}$. It follows that there exist events A, B such that $A \succsim_1 B$ while not $A \succsim_2 B$. Contradiction.

5.4 Proof of Proposition 2

Assume that (i) of the proposition holds. By Theorem 2, for any two events A and B, $A \succsim B$ if and only if $\mu(A) \geq \mu(B)$ for every probability measure μ in a uniformly absolutely continuous set $\mathcal{P}$. According to Savage (1954), $\succsim$ on $\mathcal{A}$ is represented by a unique probability, hence all probabilities in $\mathcal{P}$ coincide on $\mathcal{A}$. Savage's theorem also implies that the common part on $\mathcal{A}$ is convex-ranged. Denote the restriction of $\mathcal{P}$ to $\mathcal{A}$ by π. Note that since $\succsim$ on $\mathcal{A}$ is complete, $\succsim$ and $\succsim'$ coincide there. Note that according to Observation 2 $\mathcal{P}$ is compact, thus it attains its infimum (see Lemma I.5.10 in Dunford and Schwartz (1957)).

Let $E \in \Sigma$ and $A \in \mathcal{A}$ be events, and suppose that $\pi(A) > \min_{\mu \in \mathcal{P}} \mu(E)$. Then $\pi(A) > \mu'(E)$ for some $\mu' \in \mathcal{P}$, and as $\pi(A) = \mu'(A)$, then by the representation of $\succsim$ it follows that $\neg(E \succsim A)$. Employing Ambiguity Aversion yields the preference $A \succ' E$. In the other direction,

[14]That is, $\varphi(m) = 0$ for every $\varphi \in B_0(S, \Sigma)$ implies that $m = 0$.

if $\min_{\mu\in\mathcal{P}} \mu(E) \geq \pi(A)$, then $\mu(E) \geq \mu(A)$ for all $\mu \in \mathcal{P}$, therefore $E \succsim A$ and by Consistency also $E \succsim' A$. Thus, summing the two implications,

$$E \succsim' A \quad \Leftrightarrow \quad \min_{\mu\in\mathcal{P}} \mu(E) \geq \pi(A). \tag{8}$$

Recall that π over $\mathcal{A}$ is convex-ranged. Hence, for any $E \in \Sigma$ there exists $A \in \mathcal{A}$ with $\pi(A) = \min_{\mu\in\mathcal{P}} \mu(E)$. It is next shown that $E \sim' A$ must hold. First, by (8), $E \succsim' A$. Suppose on the contrary that $E \succ' A$. By Unambiguous Non Atomicity, there exists an unambiguous partition of S, $\{A_1, \ldots, A_m\}$ with $A_i \in \mathcal{A}$ for all i, such that $E \succ' A \cup A_i$. According to the above arguments, $\min_{\mu\in\mathcal{P}} \mu(E) \geq \pi(A \cup A_i)$ for all i, hence $\min_{\mu\in\mathcal{P}} \mu(E) > \pi(A)$. Contradiction. Therefore $E \sim' A$.

Let E and F be events, and A and B events in $\mathcal{A}$, such that $\pi(A) = \min_{\mu\in\mathcal{P}} \mu(E)$ and $\pi(B) = \min_{\mu\in\mathcal{P}} \mu(F)$. Then, employing Transitivity (P1'),

$$E \succsim' F \quad \Leftrightarrow \quad A \succsim' B \quad \Leftrightarrow \quad A \succsim B \quad \Leftrightarrow \quad \min_{\mu\in\mathcal{P}} \mu(E) \geq \min_{\mu\in\mathcal{P}} \mu(F) \tag{9}$$

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