ONE-SHOT PUBLIC MEDIATED TALK

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Abstract

We show that any correlation device with rational coefficients can be generated by a mechanism, where each player sends a private message to a mediator who in turn makes a public deterministic announcement. It is then shown that the mechanism can be adapted also to situations with differential information, where the correlation device itself depends on the players' private messages that may vary with their realized types. All the mechanisms suggested are immunized against individual deviations. Therefore, by using them, players can implement any correlated or communication equilibrium.

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1. Introduction

In a mediated talk (see Lehrer (1994)) players are allowed to communicate through a mediator. Each one of them transmits a private message to the mediator. The latter, in turn, produces a public announcement which depends (deterministically) on the individual private messages. Players are allowed to communicate for a long time. After the conversation ends, each player takes an action relying on the private message as well as on the public announcement.

The motivation of this research is twofold. First, the mediated talk is a mechanism that can be designed to enable players to improve payoffs without violating incentive compatibility constraints. Lehrer (1994) shows that in a complete information game, unbounded (without time limit) mediated talk can generate any correlated equilibrium distribution (Aumann, 1974). Here we simplify the mechanism to cover the case of a one-shot communication phase and we generalize it to incomplete information games.

The second motivation is the examination of the extent to which existing mediating mechanisms can be correlation devices between players. Mediated talk exists everywhere. Any voting procedure involves private votes and public results; citizens cast their votes privately and the election results then become public. The public outcome is certainly a function of all private messages and it is therefore a mediated talk. When citizens, or committee members, take actions based on their own vote and on the publicly known outcome, their actions are necessarily correlated by the means of the mediated talk. Obviously, the primary use of an election is not as a mediating mechanism, but it has an inevitable consequence: it provides players with private and interrelated information.

The existing mediated talks are most often one-shot mechanisms. For instance, citizens or committee members cast their votes and the outcome is announced. This is also the case with tax returns which remain private and a resulting tax policy that becomes public. Thus, in order to examine the power of existing mediated talks, one should focus on one-shot mechanisms. Roughly speaking, what we show here is that, by one-shot mechanisms, everything can be generated. Therefore, we have gained no extra predicting power (had it been otherwise, namely, the case where some correlations are impossible, we would be able to predict that these correlations cannot be induced.)

We deal here with one-shot mediated talk and show that any correlated distribution (with rational numbers probabilities) can be produced in a way that is immunized against unilateral deviations. As an application, we show that by adding a mediated talk phase to any game, any correlated equilibrium distribution of this original game can be obtained as an equilibrium of the extension.

Another result, perhaps the most important one, is the application to communication equilibrium (Forges, 1986). In a communication extension of a game, each player sends some private message (input) to the mediator. In turn, the mediator chooses randomly private signals (outputs), one for each player. Then the players take actions based on their own input and the private output they received. We show that every communication equilibrium distribution can be generated using a mediated talk . The mediated talk mechanism takes advantage of the initial private communication phase and enables the mediator to make only one public announcement rather than many private ones. Furthermore the mediator's announcement is deterministic rather than random, as in a general communication device.

Finally we introduce a universal mechanism. In Lehrer (1994) any correlated distribution requires its particular mechanism. Here, to the contrary, we introduce a universal mechanism that can be adapted to any specific correlated equilibrium.

The main idea of the construction of a mediated talk is to use a finite collection of jointly controlled lotteries. All of them but one remain latent. The active device is selected by the profile of private inputs. It produces some public output while none of the players is told which device is employed. Then, each individual uses his private input for decoding the public announcement.

2. Correlation Device and Mediated Talk

Inspired by Aumann (1974, 1987), we introduce a (finite) correlation device (or information structure) for n agents as a list of n random variables $Y_i,\ i = 1, \dots, n$, defined on the same probability space $(\Omega, \mathcal{A}, \mathcal{P})$ and ranged to finite output sets $A_i,\ i = 1, \dots, n$. One may think of the probability space as the state space. If ω in Ω is the state, the information of agent i is $Y_i(\omega)$. The knowledge of each agent i is represented by the (finite) partition generated by Y_i. The output of an agent (e.g., a player in a game, computer component) is a function of the information available to him. In simple words, one has the following:

Definition 1. An *correlation device* is a distribution Q over a product set $A = \times_{i=1}^{n} A_i$. An element $a \in A$ is chosen with probability $Q(a)$ and agent i is informed of the component a_i.

Definition 2. A *public mediated talk* is defined by (finite) private message sets, $S_i,\ i = 1, \dots, n$, and an *announcement map* f from $S = \times_{i=1}^{n} S_i$ to some finite set X (the set of public announcements). Each player i chooses a private message s_i to be sent to a mediator who makes the public announcement $x = f(s_1, \dots, s_n)$.

The goal of the paper is to mimic the information structure by a mechanism where agents choose (independently) a private message and interpret the resulting public announcement accordingly. Formally we have the following:

Definition 3. A *public mediated talk mechanism* consists of:

1) independent random variables σ_i (called *mixed messages*) which take values in S_i;

2) a public mediated talk $(S_1, ..., S_n, f, X)$;

3) *decoding maps* θ_i from $S_i \times X$ to some set B_i.

The map θ_i allows agent i to interpret the public announcement x according to his private message s_i. (The maps θ_i are usually called *strategies* in a game theoretical context.)

Given $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $\theta = (\theta_1, \ldots, \theta_n)$, we denote by $P_{\sigma,\theta}$ the distribution induced on $B = \times_{i=1}^n B_i$ by σ, f and θ. Explicitly

$$P_{\sigma,\theta}(b) = \sum_{\substack{s;\\ \theta_i(s_i,f(s))=b_i, i=1,\ldots,n}} \prod_{i=1}^n \sigma_i(s_i).$$

Definition 4. A public mediated talk mechanism, $M = (f,\ \sigma,\ \theta)$ is *adapted* to a correlation device Q on A, if $B_i = A_i, \forall i$.

M simulates Q if in addition it satisfies the following:

(1) $$P_{\sigma,\theta}(a \mid s_i) = Q(a) \quad \text{for every } a \in A, \quad s_i \in S_i, \quad \text{and } i = 1, \ldots, n,$$

(2) $$P_{\sigma,\theta}\big(a_{-i} \mid s_i, x\big) = Q\big(a_{-i} \mid \theta_i(s_i, x)\big),$$

for every $a_{-i} \in A_{-i}$, every $s_i \in S_i$ and $x \in X$ having positive probability under $P_{\sigma,\theta}$ and every $i = 1, \ldots, n$.

Remark 1. Notice that for any random variable τ_i with range S_i, and for every $s_i \in S_i$,

$$P_{\tau_i,\sigma_{-i},\theta}\big(\cdot \,|s_i\big) = P_{\sigma,\theta}\big(\cdot \mid s_i\big) \quad \text{and}$$
$$P_{\tau_i,\sigma_{-i},\theta}\big(\cdot \mid s_i, x\big) = P_{\sigma,\theta}\big(\cdot \mid s_i, x\big) \ .$$

Therefore, any unilateral deviation does not affect the distribution over A given s_i, neither does it affects the distribution over A_{-i} given (s_i, x). Moreover, by conditions (1) and (2) both these distributions coincide with the corresponding distributions defined by Q.

Now we are ready to state the first result of the paper.

Theorem 1. *Given any correlation device with rational values, there exists a public mediated talk mechanism that simulates it.*

Example 1. Consider the following 2×2 game:

	ℓ	r
t	7,7	3,8
b	8,3	0,0

Figure 1: The Payoff Matrix

where $A_1 = \{t, b\}$ and $A_2 = \{\ell, r\}$, and the following correlated equilibrium distribution:

$Q =$

	ℓ	r
t	1/2	1/4
b	1/4	0

Figure 2: The Correlation Distribution

The payoff associated with this correlated equilibrium cannot be sustained by any Nash equilibrium nor by any combination of Nash equilibria. Thus, the players might want to resort to some external mediating device that will generate the (canonical) correlation device Q (see e.g., Mertens, Sorin and Zamir (1994, Ch. II, §3)). They can do it obeying the following procedure. Each player selects privately a number in $\{1, \ldots, 4\}$ with probability 1/4 each and then transmits it to a machine which produces a deterministic public announcement according to the following matrix:

		1	2	3	4
	1	a	a	a	b
	2	a	a	b	a
Player I	3	a	b	b	b
	4	b	a	b	b

Figure 3: The Signaling Matrix

The machine publicly announces x if players I and II selections were i, j, respectively, and if the (i, j) cell of the matrix is x, $x = a, b$. In other words, $S_i = \{1, \ldots, 4\}$ and σ_i assigns each symbol a probability of 1/4. After receiving the public announcement, the players play the following strategies.

The private selected signal was...	...and the public announcement is...	...then play
1	a	t
1	b	b
2	a	t
2	b	b
3	a	b
3	b	t
4	a	b
4	b	t

Figure 4: The Strategy of Player I (θ_1)

The private selected signal was...	...and the public announcement is...	...then play
1	a	ℓ
1	b	r
2	a	ℓ
2	b	r
3	a	r
3	b	ℓ
4	a	r
4	b	ℓ

Figure 5: The Strategy of Player II (θ_2)

One can check that if the players play the strategies just defined, then indeed Q is generated. Moreover, given these strategies, and the uniform selection of player II, all the rows of the signaling matrix (Figure 3) are equivalent in the sense that all induce the same distribution over joint actions. The same observation holds for player II. Therefore, no player has an incentive to deviate neither in the communication phase nor in the play phase.

To see that this example satisfies (1) and (2) of the theorem, notice that, given θ_1 and θ_2, the conditional distribution, given any s_i, induced by σ_1 and σ_2 over A is Q. Moreover, given s_i and x, the probability of any a_{-i} is exactly $Q\big(a_{-i} \mid a_i\big)$, where $a_i = \theta_i\big(s_i, x_i\big)$. For instance, suppose that $s_1 = 1$ and $x = c$. Here, $\theta_1(1, c) = t$, $Q(\ell|t) = 2/3$, and $Q(r|t) = 1/3$. Indeed, given $s_1 = 1$ and $x = c$, the probability that player 2 will play ℓ is $2/3$, while the probability of r being played is $1/3$.

Example 2. In simulating Q we employed public mediation that used only two symbols, a and b. In order to generate the distribution $Q' = \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 0 \end{pmatrix}$ over the set of joint actions we must use three symbols. One way to do it is to use the following signaling matrix:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} d & d & b \\ d & c & c \\ b & c & b \end{pmatrix} \end{array}$$

Figure 6

Here each player chooses one of the numbers $1, 2$ and 3 with equal probability. The strategies that induce Q' are easy to construct.

3. The Mediated Talk Extension of a Game

Let G be an n-player game. We will extend the game G to a new game G^* by adding a pre-play communication phase. In this phase player i selects (possibly randomly) a message s_i from a finite set S_i. Then a deterministic mediator publicly announces $f(s_1, \ldots, s_n)$. In the play phase each player chooses an action which may depend on the message s_i and on the announcement $f(s_1, \ldots, s_n)$. G^* is called a *mediated talk extension* of G.

Obviously, any profile of individual strategies in such an extension induces a correlated distribution in G. We are concerned here with the inverse question – whether any correlated equilibrium distribution of G can be generated by a Nash equilibrium of a mediated talk extension of G. We answer this question in the affirmative.

Corollary 1. *Let C be a correlated equilibrium distribution of G with rational entries. Then there exists a mediated talk extension of G having a Nash equilibrium that induces the distribution C.*

Remark 2. The mechanism described here defines only Nash equilibrium of the extended game and not a strong equilibrium. Thus, it is immunized only against unilateral deviations.

4. Proof of Theorem 1

We will first show the proof in the two player case. It extends easily to the n player case as indicated later. Suppose that the distribution Q over A can be written as $Q = \left(c_{ij}/d\right)_{0 \le i \le n-1, 0 \le j \le m-1}$, where all c_{ij} and d are integers. The signaling matrix to be constructed is of the size $dn \times dm$. Actually, it will be described as a $n \times m$ matrix where each cell is a $d \times d$ matrix.

Let $a_1, \ldots, a_y$ be a string of y symbols. The *latin square* corresponding to this string is the following matrix

$$\begin{pmatrix} a_1 & \cdots & a_{y-1}a_y \\ a_2 & \cdots & a_y a_1 \\ a_3 & \cdots & a_1 a_2 \\ . & \cdots & . \\ . & \cdots & . \\ . & \cdots & . \\ a_y a_1 & \cdots & a_{y-1} \end{pmatrix}$$

where the lines are successive shifts of the same string. For a vector $b_{0,0}, b_{0,1}, \ldots, b_{n-1,m-1}$ of nm symbols we denote by $K\left(b_{0,0}, b_{0,1}, \ldots, b_{n-1,m-1}\right)$ the latin square corresponding to the string which consists of $c_{0,0}$ times $b_{0,0}$ and then $c_{0,1}$ times $b_{0,1}$, and so forth. Thus, $K\left(b_{0,0}, \ldots, b_{n-1,m-1}\right)$ is a $d \times d$ matrix (because $\sum c_{ij} = d$).

Example 3. As in Example 1, let $Q = \begin{pmatrix} 2/4 & 1/4 \\ 1/4 & 0 \end{pmatrix}$. In this case

$$K(a,b,c,d) = \begin{pmatrix} a & a & b & c \\ a & b & c & a \\ b & c & a & a \\ c & a & a & b \end{pmatrix}$$

This is so because $c_{0,0} = 2, c_{0,1} = c_{1,0}$ and $c_{1,1} = 0$. Therefore, in every row and column, a appears twice, b and c appear once, and d does not appear at all.//

Now fix $n \times m$ different symbols $\left(b_{ij}\right)_{0 \le i \le n-1, 0 \le j \le m-1}$. In what follows, for any integers x and y, $x(n)$ and $y(m)$ will stand for the numbers x modulo n and y modulo m, respectively. Before we get to the announcement map defined by the signaling matrix we need one more convention. When $\left(b_{ij}\right)_{i,j}$ is referred to a string, rather than a matrix, the string is defined in a natural way: the first row first then the second row and so forth.

The signaling matrix consists of an $n \times m$ grand matrix where for every $0 \leq k \leq n-1$ and $0 \leq \ell \leq m-1$ in the (k, ℓ) cell, there stands the matrix

$$K\left(b_{i+k(n),j+\ell(m)}\right)_{0 \leq i \leq n-1, 0 \leq j \leq m-1} .$$

Now let $S_1 = \{0, \ldots, n-1\} \times \{1, \ldots, d\}$ and $S_2 = \{0, \ldots, m-1\} \times \{1, \ldots, d\}$. For every $(k, d_1) \in S_1$ and $(\ell, d_2) \in S_2$ define $f((k, d_1), (\ell, d_2))$ to be the $\left(d_1, d_2\right)$ entry of the matrix standing in the cell (k, ℓ) of the grand matrix.

Example 3 (Continued). With the distribution of Example 1, the grand matrix is 2×2 (the size of the original distribution) consisting of cells which are 4×4 matrices. Let $b_{00}, b_{01}, b_{10}, b_{11}$ be four different symbols. According to the construction, in the $(0, 0)$ cell of the grand matrix stands the matrix

$$K\left(b_{ij}\right) = K(b_{00}, b_{01}, b_{10}, b_{11}) = \begin{pmatrix} b_{00} & b_{00} & b_{01} & b_{10} \\ b_{00} & b_{01} & b_{10} & b_{00} \\ b_{01} & b_{10} & b_{00} & b_{00} \\ b_{10} & b_{00} & b_{00} & b_{01} \end{pmatrix}$$

Recall that $K(\cdot)$ is a latin square where b_{ij} is replicated c_{ij} times in each row and column. In the $(0, 1)$ cell of the grand matrix stands the matrix

$$K\left(b_{i,j+1(2)}\right)_{i,j} = K\left(b_{01}, b_{00}, b_{11}, b_{10}\right) = \begin{pmatrix} b_{01} & b_{01} & b_{00} & b_{11} \\ b_{01} & b_{00} & b_{11} & b_{01} \\ b_{00} & b_{11} & b_{01} & b_{01} \\ b_{11} & b_{01} & b_{01} & b_{00} \end{pmatrix}$$

To facilitate the reading set $b_{00} = x, b_{01} = y, b_{10} = w, b_{11} = z$. The signaling matrix is therefore

$$\begin{pmatrix} x & x & y & w & y & y & x & z \\ x & y & w & x & y & x & z & y \\ y & w & x & x & x & z & y & y \\ w & x & x & y & z & y & y & x \\ \\ w & w & z & x & z & z & w & y \\ w & z & x & w & z & w & y & z \\ z & x & w & w & w & y & z & z \\ x & w & w & z & y & z & z & w \end{pmatrix}$$

For instance, if $d_1 = 2, d_2 = 3, k = 1$ and $\ell = 0$, then f equals x. One can see that any symbol appears in any row and column only once or exactly three times. Moreover, if, for instance, x appears only once in a certain column, then it appears two more times in its row. Such an x, that appears once in its column will later be associated with the right signal in A_2 of player II. Since it appears only once player II knows what player I is going to do; player I will play top because according to the distribution Q the probability of top given right is 1. Furthermore, when player I is prescribed to play top he should assign probability 2/3 on left and 1/3 on right (these are the conditional probabilities), and therefore in the same row there appear two more x's which correspond to the left column.//

Now that S_1, S_2 and f are defined, in order to complete the description of the mediated talk mechanism it remains to define σ_i and θ_i, $i = 1, 2$. σ_i is uniform over S_i and θ_i is defined as follows. If the public announcement is b_{ij} then θ_1 is $a_1 = (i - k)(n)$ and θ_2 is $a_2 = (j - \ell)(m)$, where k and ℓ are the respective messages sent. Notice that θ_1 does not depend on the second index of the public announcement and θ_2 does not depend on the first.

We first show that if each player i follows the decoding map θ_i just described, then the distribution over A given any s_i is exactly Q. For any cell (k, ℓ) the corresponding matrix is $K\big(b_{i+k(n),j+\ell(m)}\big)$, where in each row there are c_{ij} times the symbol $b_{i+k(n),j+\ell(m)}$. Moreover, each symbol in any row is assigned the same probability. In the case where $b_{i+k(n),j+\ell(m)}$ is the public announcement then by the above decoding map, player I's output is $a_1 = ((i+k)(n)-k)(n) = i$ and player II's output is $a_2 = ((j+\ell)(m)-\ell)(m) = j$. Therefore, the joint output (i, j) is prescribed c_{ij} times out of a total of d. In other words, the joint output (i, j) is prescribed with probability c_{ij}/d. Since this is true for any cell, (i, j) is assigned the probability c_{ij}/d for any row (namely, for any s_1). This shows (1).

Next we show (2). Namely, for every message s_1 and public announcement x, we show that the probability of $a_{-1} \in A_{-1}$ is $Q\big(a_{-1}|a_1\big)$, where $a_1 = \theta_1(s_1, x)$. Suppose that indeed player II abides by σ_2. Thus the choice of player II is uniformly distributed over the columns of the signaling matrix. Fix an arbitrary (k, d_1). We now take a look at the d_1 row of the matrices that stand in the cells $(k, 0), (k, 1), \ldots, (k, m = 1)$ of the grand matrix. In the $(k, 0)$ matrix there are c_{ij} times $b_{i+k(n),j(m)}$; in the $(k, 1)$ matrix there are c_{ij} times $b_{i+k(n),j+1(m)}$, and so on. Thus, the symbol $b_{i+k(n),j}$ appears $\sum_\ell c_{i,j-\ell(m)}$ times, out of which (recall σ_2) $c_{i,j-\ell(m)}$ times are associated with the action of choosing $j - \ell(m)$ column. Rearranging the parameters, we obtain that the symbol $b_{i+k(n),j+\ell(m)}$ appears $\sum_r c_{ir}$ times. So all the symbols with the first index $i+k(n)$ appear $\sum_r c_{ir}$ times. Moreover, out of these $\sum_r c_{ir}$ appearances c_{ij} are associated with the j-th column. For each one of these symbols θ_1 obtains the value i (i.e., $\theta_1(s_1, x) = i$, where $s_1 = (k, d_1)$ and $x = b_{i+k(n),j+\ell(m)}$). Therefore, given (s_1, x), the probability of (i, j) being prescribed by (θ_1, θ_2) is $c_{ij}/\sum_r c_{ir}$, which is $Q(j|i)$. Since the same argument holds for player II, it proves (2).

The proof given is for the 2-player case. For the sake of completeness, we provide the adaptation needed for the n-player case. Let Q be a distribution over A, where $Q(a)$ is rational for all $a \in A$. Let $Q(a) = c(a)/d$, where $c(a)$ is an integer. Let $A_i = \{0, \ldots, m_i - 1\}$. The $d \times \cdots \times d$ (n times) latin cube consisting of the symbols $1, \ldots, d$, $L(k_1, \ldots, k_n)$, is a $d \times \cdots \times d$ matrix whose $i_1, \ldots, i_n$ entry equals $i_1 + \cdots + i_n + k_1 + \cdots + k_n$ (mod d), where $k_i = 0, \ldots, m_i - 1$. The grand matrix consists of $m_1 \times \cdots \times m_n$ latin cubes, where the $k_1, \ldots, k_n$ cube is $L(k_1, \ldots, k_n)$. Let b be a function $b : \{1, \ldots, d\} \to A$ s.t. the number of ℓ's s.t. $b(\ell) = a$ is $c(a)$. We denote b_i the projection of to A_i. Thus, $b_i(\ell)$ is an output of player i.

We define $S_i = \{0, \ldots, m_i - 1\} \times \{1, \ldots, d\}$. Let ℓ be the $(d_1, \ldots, d_n)$ entry of the cube $L(k_1, \ldots, k_n)$. Define $f((k_1, d_1), \ldots, (k_n, d_n))$ to be $b(\ell)$.

As in the 2-player case, σ_i is uniform over S_i. As for θ_i, assume that the public announcement is $(a_1, \ldots, a_n)$, then the decoded signal of player i, given the message sent (k_i, d_i), is $(a_i - k_i)$ (mod m_i).//

Proof of the Corollary. Let C be a correlated equilibrium distribution in G. Theorem 1 states that there exists a mediated talk mechanism which induces the distribution C over A (this is a consequence of (1)). In order to show that this mediated talk mechanism defines a Nash equilibrium of the extension we show that no player can gain by adopting another mixed message $\overline{\sigma}_i$ or by adopting another decoding map $\overline{\theta}_i$, or both. By (2), given σ_{-i} and θ, for every s_i and x which satisfy $\theta_i(s_i, x) = a_i$, one has $P(a_{-i}|s_i, x) = Q(a_{-i}|a_i)$. Since C is a (canonical) correlated equilibrium, a_i is a best response against $Q(a_{-i}|a_i)$. Therefore, given σ_i, σ_{-i} and θ_{-i}, θ_i (which prescribes a_i) is a best response. By Remark 1, any alternative τ_i does not change properties (1) and (2) and therefore whatever the alternative:

1. the probability of playing a is the one assigned by C for any $a \in A$, and
2. whenever a_i is prescribed it is indeed an optimal response.

We have proved that (σ_i, θ_i) is a best response to $(\sigma_{-i}, \theta_{-i})$ and therefore it is a Nash equilibrium in the extended game G^*.//

Remark 3. In the construction of the signaling matrix we introduce the grand matrix which consists of the submatrices $K(\cdot)$. The first components of the private messages (sent by the players to the mediator) select the specific $K(\cdot)$ that becomes active. The second components (d_1 and d_2) determine the public announcement from the $K(\cdot)$ already chosen.

One may consider all the submatrices K as jointly controlled devices. The active device is jointly chosen by the players through k and l. Then, the players jointly control the lottery using d_1 and d_2, without knowing which one is active.

Remark 4. In the proof we use decoding maps θ_i that do not depend on the particular Q under consideration. As a matter of fact the same decoding map is good for every Q.

5. From Correlated to Communication Devices.

Forges (1986) introduced the concept of communication equilibrium. Before the mediator correlates between the players, he receives some information from them. For instance, in a game where players have differential information (e.g., their own types), the correlation applied may depend on the data sent by the players. Thus the outcome may (partially) reveal their private information.

Example 4. Suppose that player I may be of two types: 1 and 2, which are equally likely. Player I knows his type while player II knows only the prior distribution over player I's type: $(1/2, 1/2)$. Let the payoffs be

		ℓ	r	Probability
type 1:	t	6,6	3,8	1/2
	b	7,3	0,0	

		ℓ	r	Probability
type 2:	t	0,0	7,3	1/2
	b	3,8	6,6	

Figure 7

Consider now the following mediation. If player I tells the mediator that he is of type 1, the mediator chooses one of the joint actions (t, ℓ), (t, r) and (b, ℓ) with probability $1/2, 1/4$ and $1/4$ respectively. However, if player I reports that he is of the second type, then the mediator chooses each of (r, b), (r, t) and (ℓ, b) with probability $1/2, 1/4$ and $1/4$ respectively. Whatever the choice of the mediator, he informs player I of the row chosen and player II of the column chosen.

Notice that, once player II receives some information from the mediator, his prior over player I's type changes. For instance, if ℓ is sent, then the posterior ascribes type 1 the probability $3/4$ (as opposed to the prior $1/2$).

The distribution induced on any matrix is not a correlated equilibrium distribution. Nevertheless, due to differential information, given that players play according to the announcement of the mediator, the procedure induces an equilibrium; player I has the incentive to reveal his true type and to stick to the mediator's announcement and moreover, player II also has no incentives to deviate.

This conclusion depends strongly on the specific posteriors. Therefore, any simulating mechanism should always generate the same posteriors as the simulated device.

In order to generate this communication equilibrium by a mediated talk we adopt the matrix of Example 1 and define two signaling matrices, one for each type:

$$\begin{pmatrix} x & x & y & w & y & y & x & z \\ x & y & w & x & y & x & z & y \\ y & w & x & x & x & z & y & y \\ w & x & x & y & z & y & y & x \\ & & & & & & & \\ w & w & z & x & z & z & w & y \\ w & z & x & w & z & w & y & z \\ z & x & w & w & w & y & z & z \\ x & w & w & z & y & z & z & w \end{pmatrix}$$

for type 1, and

$$\begin{pmatrix} y & w & z & z & x & z & w & w \\ w & z & z & y & z & w & w & x \\ z & z & y & w & w & w & x & z \\ z & y & w & z & w & x & z & w \\ & & & & & & & \\ z & x & y & y & w & y & x & x \\ x & y & y & z & y & x & x & w \\ y & y & z & x & x & x & w & y \\ y & z & x & y & x & w & y & x \end{pmatrix}$$

for type 2. The private messages of player II are in $\{1,2\}\times\{1,2,3,4\}$, while player I has also to inform the mediator of his type. If the type is 1, then the active signaling matrix is the first. Otherwise, it is the second matrix. One can confirm that the posteriors of player II are either $(3/4, 1/4)$ or $(1/4, 3/4)$, as needed.

Notice that, if instead of $\begin{pmatrix} 0 & 1/4 \\ 1/4 & 1/2 \end{pmatrix}$ the distribution on the second type's matrix would have been $\begin{pmatrix} 0 & 1/3 \\ 1/3 & 1/3 \end{pmatrix}$, for instance, then the corresponding matrix would have been of size 6×6. To have a dimension for player II independent of player I's type we replicate each matrix to get two matrices of size 24×24. //

We introduce now the formal communication model and the corresponding extension of mediated talk.

Definition 5. A *communication device* for n agents is a map Q from a product set $T = \times_{i=1}^{n} T_i$ to distributions over a product set $A = \times_{i=1}^{n} A_i$.

The sets T_i are the input sets and for each profile $t \in T$ of inputs the communication device Q selects a profile $a \in A$ according to the distribution $Q(t)$. Finally the component a_i is announced to agent i.

For any distribution D over T, D and Q induce a distribution $D \otimes Q$ over $T \times A$ as follows: $D \otimes Q(t, a) = D(t)Q(t)(a)$. Moreover, for any $\widetilde{t}_i \in T_i$ one defines the distribution $D \otimes Q(.; \widetilde{t}_i)$ over $T \times A$ by: $D \otimes Q(t, a; \widetilde{t}_i) = D(t)Q(t_{-i}, \widetilde{t}_i)(a)$. In both cases, $D \otimes Q(.|t_i)$ and $D \otimes Q(.; \widetilde{t}_i|t_i)$ denote the conditional probabilities given that the i-th component chosen according to D is t_i.

In a framework of incomplete information games, one possible interpretation is that T_i is the set of agent i's types. The profile $t = (t_1, \ldots, t_n)$ is selected according to the publicly known distribution D. Each agent sends privately to the communication device $\widetilde{t}_i$ which may or may not be equal to t_i. The distribution Q on the signals of the agents depends upon the profile announced. Thus $D \otimes Q(.; \widetilde{t}_i|t_i)$ is the distribution computed by player i on $T_{-i} \times A$ if being of type t_i, he announces $\widetilde{t}_i$, while all other agents announce their realized types. Let $\widetilde{T}_i$ be a copy of T_i and $\widetilde{T} = \times_i \widetilde{T}_i$.

Definition 6. A mediated talk mechanism *adapted* to the communication device Q is a triple $M = (\sigma, f, \theta)$ where :

1) σ_i is a *message map* from T_i to distributions on $S_i \times \widetilde{T}_i$, $i = 1, \ldots, n$,

2) f is an *announcement map* from $S \times \widetilde{T}$ to X,

3) θ_i is a *decoding map* from $T_i \times S_i \times \widetilde{T}_i \times X$ to A_i, $i = 1, \ldots, n$.

As in Definition 2, the S_i are finite sets of messages and X is the finite set of public announcements. Given any distribution D on T one defines a distribution $D \otimes M$ on $T \times A$ as follows:

$$D \otimes M(t, a) = D(t) \left(\sum_{\substack{\widetilde{t}, s, x \\ f(s, \widetilde{t}) = x, \\ \theta_i(t_i, s_i, \widetilde{t}_i, x) = a_i, i = 1, \ldots, n}} \prod_{i=1}^{n} \sigma_i(t_i)(s_i, \widetilde{t}_i) \right)$$

and given $\widetilde{t}_i \in \widetilde{T}_i$:

$$D \otimes M(t, a; \widetilde{t}_i) = D(t) \left(\sum_{\substack{\widetilde{t}_{-i}, s, x \\ f(s, \widetilde{t}) = x, \\ \theta_j(t_j, s_j, \widetilde{t}_j, x) = a_j, j = 1, \ldots, n}} \sigma_i(t_i)(s_i|\widetilde{t}_i) \prod_{\substack{j=1 \\ j \neq i}}^{n} \sigma_j(t_j)(s_j, \widetilde{t}_j) \right).$$

The interpretation of these two probabilities is similar to the interpretation of $D \otimes Q(t, a)$ and $D \otimes Q(t, a; \widetilde{t}_i)$.

Definition 7. A mediated talk mechanism M *simulates* the communication device Q if M is adapted to Q and in addition, for every distribution D on T, one has:

$$D \otimes Q = D \otimes M \tag{3}$$

$$D \otimes Q(.;\widetilde{t}_i|t_i, a_i) = D \otimes M(.;\widetilde{t}_i|s_i, t_i, x) \quad \text{on} \quad A_{-i} \times T_{-i} \tag{4}$$

whenever $\theta_i(t_i, s_i, \widetilde{t}_i, x) = a_i$ and $(t_i, s_i, \widetilde{t}_i, x)$ has positive probability under D and M.

In words, (3) says that the communication device and the mediated talk mechanism induce the same distribution over $T \times A$ for any "entrance distribution" D.

(4) means that if all agents $j, j \neq i$, are following their strategies σ_j, given their types t_j, then agent i of type t_i would have, with the communication device and the mediated talk mechanism, the same conditional probabilities on the unknown parameters in $T_{-i} \times A_{-i}$. This is true whatever being his revealed type $\widetilde{t}_i$, his private information $(t_i, s_i, \widetilde{t}_i, x)$ and the value of his decoding map $a_i = \theta_i(t_i, s_i, \widetilde{t}_i, x)$.

We are now ready to state our second main result.

Theorem 2. *For any communication device with rational values there exists a mediated talk mechanism that simulates it.*

Proof: Let d be the common denominator for all $Q(t)$, $t \in T$. We use the construction of Theorem 1 and adapt it to the private information setup. Two modifications are needed. First, in addition to the private signal chosen in Theorem 1, here each player sends privately a type. Thus, $S_i = T_i \times \{0, \ldots, m_i - 1\} \times \{1, \ldots, d\}$ and $\sigma_i(t_i)$ is uniform over $\{t_i\} \times \{0, \ldots, m_i - 1\} \times \{1, \ldots, d\}$. The other change in the mediated talk mechanism is the following. Let f_t be the announcement map constructed in Theorem 1 for the correlation device $D(t)$, using the common denominator d. Here $f(t, s)$ is defined as $f_t(s)$, where t is the types profile (privately) sent and s the profile of messages in S. Finally we define $\theta_i(t_i, s_i, \widetilde{t}_i, x)$ as in the complete information case hence its value is $\theta_i(s_i, x)$. This is well defined due to Remark 4.

In other words, the mediated talk mechanism consists of blocks, where the t-th block is a mediated talk mechanism associated with the (complete information) announcement map f_t corresponding to $Q(t)$. Notice that no matter what $Q(t)$ is, as long as d is common to all, $\sigma(t_i)$ is always uniform over $\{0, \ldots, m_i - 1\} \times \{1, \ldots, d\}$, hence therefore consistent with the construction in Section 4. Moreover, the decoding strategy $\theta_i(t_i)$ is always the same.

If t is the profile of types and the agents follow σ, in particular they all tell the true t_i, then the active announcement map will be f_t and therefore, using the results of section 4, the induced distribution on A will be $Q(t)$, hence (3).

As for (4), if agent i of type t_i announces $\widetilde{t}_i$, the active announcement map will be $f_{t_{-i},\widetilde{t}_i}$ with probability $D(t_{-i}|t_i)$. Since θ_{-i} does not depend on t_{-i}, this is also the marginal distribution $D \otimes M(t_{-i}; \widetilde{t}_i|s_i, t_i, x)$ over T_{-i}, for every $(t_i, s_i, \widetilde{t}_i, x)$. Moreover, given σ_{-i} and θ_{-i} the distribution induced by $f_{t_{-i},\widetilde{t}_i}$ on A is $Q(t_{-i}, \widetilde{t}_i)$. Now, by construction, if $\theta_i(s_i, x) = \theta_i(s'_i, y) = a_i$, the probabilities of x and y given a_i are the same, hence the updating of player i depends only on a_i. Thus, the conditional probability on $T_{-i}\times A_{-i}$ given $(t_i, s_i, \widetilde{t}_i, x)$ depends in fact upon $t_i, \widetilde{t}_i$ and a_i. Furthermore, since as indicated above, the induced distribution is $Q(t_{-i}, \widetilde{t}_i)$, the conditional distribution $D \otimes M(a_{-i}; \widetilde{t}_i|s_i, t_i, x)$ over A_{-i} is equal to $Q(t_{-i}, \widetilde{t}_i)(a_{-i}|a_i)$.

This indeed implies that both the marginal on T_{-i} given $(t_i, \widetilde{t}_i)$ and the conditional on A_{-i} given $(t_{-i}, \widetilde{t}_i, a_i)$ coincide in both mechanisms, hence the result. //

We now consider an n player game G with incomplete information. Let T_i be the set of player i's types and D be the initial probability on $T = \times T_i$. The action space of player i is A_i and his payoff function is a real map g defined on $T\times A$, where $A = \times A_i$. We extend the game G to a new game G_C by adding a communication device C as follows: after the selection of the types according to D, the device is used and then players choose actions in G. A Nash equilibrium of G_C is by definition a *communication equilibrium* of G. Let Q be the distribution induced on $T\times A$ by some communication equilibrium of G. The revelation principle (see Myerson(1991, §6.3) or Mertens, Sorin and Zamir (1994, §II.3.c)) states that Q is a *canonical communication equilibrium*. Namely if the communication device Q is used in G, an equilibrium is obtained when each player sends his type to the mediator and play in the game the action privately announced through Q.

Theorem 2 implies the following corollary.

Corollary 2. *Let Q be a communication equilibrium distribution of an incomplete information game G. Assume Q has rational values. Then there exists a mediated talk extension of G that has a Nash equilibrium which induces the distribution Q over the product set of types and actions.*

Proof: Given Q, we consider the mediated talk mechanism defined in Theorem 2. A potential deviation of player i in the mediated talk extension of G is of the form $(\widetilde{t}_i, b_i)$. By condition (4) above, the corresponding payoff for player i would be the same as the payoff he would get in G_Q by using $(\widetilde{t}_i, b_i)$. Since Q is a canonical communication equilibrium, there is no profitable deviation. Property (3) achieves now the proof.//

6. Final Comments

The fact that players can, by using independent randomizations, generate some correlated device and moreover, in a way immunized against deviations, dates back to 1968 when Aumann and Maschler introduced the jointly controlled lottery (see Aumann and Maschler (1995) and Mertens, Sorin and Zamir (1994), §II.3, for extensions).

In the framework of a finite game where the set of correlated or communication equilibrium distributions has finitely many extreme points, one can introduce a universal multi-stage mediated talk mechanism that can generate any equilibrium. This can be done by associating a mediated talk mechanism to each of the extreme points and adding a jointly controlled lottery. Any equilibrium induces a distribution over the extreme points. The jointly controlled lottery will be used sequentially to choose an extreme point according to this distribution and the selected mediated talk mechanism will be then employed (see Forges (1990) and Mertens, Sorin and Zamir (1994)).

Recall that as soon as one deals with a mechanism that allows for private outputs at some stage one, can assume that all future outputs are public. In fact, these subsequent outputs can be encoded in several ways specific to each player by using some codes sent previously as private outputs to them (see Forges (1990) and Mertens, Sorin and Zamir (1994)).

The main contribution of this paper is to show that one can get rid of the private outputs in case where players can send private inputs (which is the basis of communication devices). Moreover, we provide an explicit construction taking care simultaneously of the randomness and of the private information aspects of correlated or communication devices.

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