Relative Utility

Galit Ashkenazi and Ehud Lehrer

School of Mathematical Sciences
Tel Aviv University
Tel Aviv, 69978
Israel

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Abstract

The primitive of the model is a partial order that indicates which of two agents is more prosperous. This partial order depends on numerical attributes that contain relevant data about the agents, such as initial endowments, needs, etc. We give natural sufficient conditions which ensure that this partial order can be represented linearly. A linear representation implies the existence of an agent's utility function that depends on his/her own situation as well as on that of others.

[1]The first version was entitle "Well-Being Indices".

1 Introduction

Individuals tend to compare themselves to others. Furthermore, they have a sense of how well they do compared to their peer group. A certain salary is considered high in one profession while considered low in another profession (that may require just as much training). This is so, simply because a salary is compared with other salaries within the same profession. In this paper we deal with the issue of this "relativeness". The notion of relative utility is defined and axiomatized.

Let a society consist of n agents. A vector $x \in \mathbb{R}^k$ represents the society's configuration. This vector contains relevant data about the agents involved. It may contain agents' salaries, needs, education, the resources of the institutions the agents belong to, these institutions' needs, etc.

The primitive of the model is a partial order $\succsim_x$ called the prosperity partial order, defined on the set of agents. The prosperity partial order depends on the configuration x. $i \succsim_x j$ is interpreted as "agent i is more prosperous than agent j". This partial order represents the relative situation of the agents with respect to their mates and may be either subjective, as in the case when one compares her salary with the salaries of her peer group, or objective, as in the following example. A foundation (like the NSF), receives individual and group applications. These applications provide information about the needs and endowments of the applicants under consideration. All this information makes up the society configuration. By considering this society configuration, the foundation ranks individuals and allocates funds accordingly. The foundation in this case is external to the group of applicants and, therefore, the induced partial order may be considered objective.

The main goal of the paper is to find natural conditions that ensure that there exist vectors γ_i, one for each agent, such that $i \succsim_x j$ if and only if $\langle x, \gamma_i \rangle \geq \langle x, \gamma_j \rangle$. The vectors γ_i are induced by the prosperity partial order and they enable one to compare the utilities of different agents. When $\langle x, \gamma_i \rangle \geq \langle x, \gamma_j \rangle$, the relative utility of agent i is greater than that of agent j. If such vectors exist, we say that $\succsim$ admits linear representation and $\langle x, \gamma_i \rangle$ is called the relative utility of agent i under the configuration x.

The vector γ_i represents the relative utility of agent i, which in turn takes into consideration the endowments of all other agents. In this respect, the utility of agent i is relative to the situations of other agents. (However, this utility has meaning only when comparing the prosperity of different agents induced by one configuration. The relative utility does not allow comparison of the prosperity of one agent induced by different configurations.)

It turns out that the following axioms guarantee linear representation.

Axiom-CONV. For any two agents i, j, two society configurations x_1, x_2, and two positive numbers, α_1 and α_2, if $i \succ_{x_1} j$ (resp. $i \succsim_{x_1} j$) and $i \succsim_{x_2} j$, then $i \succ_{\alpha_1 x_1 + \alpha_2 x_2} j$ (resp. $i \succsim_{\alpha_1 x_1 + \alpha_2 x_2} j$).

Axiom-OPEN. For every x and every two agents i, j, if $i \succ_x j$, then there exists a neighborhood of x, H, such that for every $x' \in H$, $i \succ_{x'} j$.

Axiom-ORDER. For every society configuration c, $\succ_x$ is an order (defined on the set of agents).

Axiom-RECIPROCITY. If there is a configuration x, where $i \succ_x j$, then there is a configuration x' such that $i \succ_{x'} j$.

We say that a configuration x' is *insignificant* for a group of agents G, if adding x' to any configuration x does not change the partial order $\succ_x$ over G (i.e., $i \succ_{x+x'} j$ if and only if $i \succ_x j$ for every i and j in G).

Axiom-No TRIUMVIRATE. If there are at least four agents, then there is no set of three agents G such that when x' is insignificant for G, it is also insignificant for all agents.

We show that if the partial order $\succsim$ satisfies **CONV, OPEN, ORDER, RECIPROCITY**, and **No TRIUMVIRATE**, then $\succsim$ admits linear representation.

2 Relative utility

2.1 Prosperity partial orders

Let $N = \{1, ..., n\}$ be the set of agents, and let the vector $x = (x_1, ..., x_k) \in \mathbb{R}^k$ contain relevant data about the agents. This vector may contain data about individual and institutional endowments, individual and group needs, etc. Such a vector will be called a *society configuration*, or *configuration* in short. The set of all possible configurations is called the *configuration set* and is denoted by X.

We assume that for any vector $x \in X$, there exists a partial order $\succ_x$ defined on the set of agents N. This partial order will be referred to as a prosperity partial order, or simply, prosperity order. We interpret $i \succ_x j$ as i being more prosperous than j when x is the society configuration.

Our goal is to transform the prosperity order to interpersonal comparable utility functions.

2.2 A representation of a prosperity order

Definition 1 A function $d : X \longrightarrow \mathbb{R}^N$ is a *representation* of $\succ_x$ when $i \succ_x j$ if and only if $d(x)_i > d(x)_j$ for every i and j.

It is clear that the order $\succ_x$ has a representation if and only if for every $x \in x$, $\succ_x$ is a complete order.

Definition 2 When $d : X \longrightarrow \mathbb{R}^N$ represents $\succ_x$, $d(x)_i$ is the *relative utility* of agent i at x.

Here the relative utility of agent i depends not only on his or her endowments and needs, but also on those of other agents. Moreover, $d(x)_i$ and $d(x)_j$ are comparable in the sense that when the utility of i, $d(x)_i$, is greater than that of j, $d(x)_j$, agent i is indeed considered more prosperous than agent j.

Note that $d(\cdot)_i$ is not a utility function, and does not provide information whether i is better off at the configuration x or at the configuration y. The vector $d(x)$ provides a ranking of the agents according to the prosperity order.

Definition 3 $d : X \longrightarrow \mathbb{R}^N$ is a *linear representation* if for every i there are vectors γ_i, $i = 1, ..., n$, in $\mathbb{R}^k$ such that $d(x)_i = \langle \gamma_i, x \rangle$.

In words, $d : X \longrightarrow \mathbb{R}^N$ is a linear representation if for each agent i there is a vector γ_i, which represents the relative weights (negative or positive) of the various components of the configuration, such that the dot product of x and γ_i is equal to $d(x)_i$.

Our goal is to identify a set of some natural properties of the partial order $\succ_x$ that characterize those having a linear representation.

3 Axiomatization

In all this discussion we confine ourselves to two types of configuration sets, $\mathbb{R}^k$ and $\mathbb{R}^k_+$, of the non-negative vectors. We introduce a few axioms that characterize the prosperity orders that admit a linear representation. It turns out that the axiomatization changes with the configuration set X. We start with the case where there is no restriction over this set.

3.1 Full domain

In this subsection we provide a characterization of the prosperity partial order that admits linear representation when a partial order $\succ_x$ is defined for every $x \in \mathbb{R}^k$, that is, when $X = \mathbb{R}^k$.

Axiom-CONV. For any two agents $i, j \in N$, two configurations x_1, x_2, and two positive numbers, α_1 and α_2, if $i \succ_{x_1} j$ (resp. $i \succsim_{x_1} j$) and $i \succsim_{x_2} j$, then $i \succ_{[\alpha_1 x_1 + \alpha_2 x_2]} j$ (resp. $i \succsim_{[\alpha_1 x_1 + \alpha_2 x_2]} j$).

The axiom **CONV** states that if with respect to the vector x_1, agent i is considered more prosperous than agent j and with respect to the vector x_2 agent i is considered more prosperous than or equally prosperous to agent j, then in any positive linear combination of the two vectors, agent i is considered more prosperous than agent j.

Define R_{ij} (resp. W_{ij}) to be the set of vectors c for which agent i is more prosperous than agent j (resp. more prosperous than or equally prosperous to agent j). **CONV** implies that both R_{ij} and W_{ij} are convex cones. Furthermore, in any neighborhood of a point in W_{ij} there is a point in R_{ij}.

Axiom-OPEN. For every x and every two agents i, j, if $i \succ_x j$, then there exists a neighborhood of x, H, such that for every $x' \in H$, $i \succ_{x'} j$.

The axiom **OPEN** states that if with respect to x, agent i is more prosperous than agent j, then so too is the situation when x is slightly perturbed. **OPEN** implies that R_{ij} is open. Let ℓ be an integer. Let $\mathbf{0}$ be a configuration with all zero components.

ℓ **No Cycles Condition-**ℓ**NCC**. If there is an $\ell \times \ell$ matrix $\left(r_{ij}\right)_{i,j \in N}$ such that for every $i, j \in N$, $r_{ij} \in W_{ij} \cup \{\mathbf{0}\}$, and for every $i \in N$, $\sum_j r_{ij} = \sum_j r_{ji}$, then $r_{ij} \notin R_{ij}$ for every i and j.

Note that if for every $i, j \in N$, $(0,0) \notin R_{ij}$, then 2**NCC** is implied by the fact that the sets R_{ij} are induced by a partial order (since, $R_{ij} \bigcap W_{ji} = \emptyset$).

It turns out that the axioms **CONV** and **OPEN** together with the n**NCC** are sufficient to characterize the prosperity orders that admit a linear representation, as stated in the following theorem.

Theorem 1 *A prosperity partial order has a linear representation if and only if it satisfies the axioms **CONV** and **OPEN** together with the n**NCC**.*

Theorem 1 indicates in particular that n**NCC**, along with **CONV** and **OPEN**, implies that the prosperity partial order can be extended to an order. That is, any vector x actually induces an order on the set of agents. The n**NCC** is strongly related to cycles and we elaborate on this in the sequel (Chapters 4 and 7).

The no-cycle condition does not have a natural intuition and we therefore suggest an alternative axiomatization. (The no-cycle condition will play a major role in any case.)

Axiom-ORDER. For every configuration x, $\succ_x$ is an order.

A prosperity partial order that satisfies **ORDER** is called a prosperity order.

Lemma 1 3**NCC** *implies* **ORDER**.

Proof. Suppose to the contrary that there exist i_1, i_2, i_3 such that $r \in W_{i_1 i_2} \cap W_{i_2 i_3} \cap W_{i_3 i_1}$, with at least one r_{ij} in its respective R_{ij}. Set $r_{i_1 i_2} = r_{i_2 i_3} = r_{i_3 i_1} = r$. All other r_{ij}'s are set to zero. Thus, $\sum_j r_{i_t j} = r_{i_t i_{t+1}} = r = r_{i_{t+1} i_t} = \sum_j r_{j i_t}$ (where $3+1$ is understood here to be 1). This is a contradiction to 3**NCC**. ∎

As will be shown in Proposition 1 below, in the case of a full domain, **ORDER** implies 3**NCC**.

Axiom-RECIPROCITY. If R_{ij} is not empty, then neither is R_{ji}.

RECIPROCITY states that if there is a vector x where the agent i is more prosperous than agent j, then there is another vector where agent j is more prosperous than agent i.

We say that a vector x' is *insignificant* for a group of agents L if adding x' to any vector x does not change the partial order $\succ_x$ over L (i.e., $i \succ_{x+x'} j$ if and only if $i \succ_x j$ for every i and j in L).

Axiom-No TRIUMVIRATE. If there are at least four agents, then there is no set of three agents L such that when x' is insignificant for L, it is also insignificant for all other agents.

When a difference of x', added to a vector x, does not change the partial order induced on L for any x, we say that x' is insignificant for L. The axiom **No TRIUMVIRATE** states that if there are at least four agents, then there is no three-agent set L such that whenever x' is insignificant for L, a difference of x' also does not affect the partial order over all agents. That is, the partial orders $\succ_{x+x'}$ and $\succ_x$ coincide.

Theorem 2 *When* **No TRIUMVIRATE** *is satisfied, then a prosperity partial order has a linear representation if and only if it satisfies the axioms* ***ORDER***, ***CONV***, ***OPEN*** *and* ***RECIPROCITY***.

Notice that up to this point we dealt with partial orders induced by any point in a Euclidean space ($\mathbb{R}^k$ in this case). If the partial order is induced only by points in a subset of a Euclidean space (for instance, only by points in the positive orthant), then **RECIPROCITY** is not sufficient. This case is discussed in the next subsection.

3.2 Restricted domain

Here we discuss the case where the vectors x are in X_+. It turns out that Theorem 1 is valid here. That is, a prosperity partial order has a linear representation if and only if it satisfies the axioms **CONV** and **OPEN** together with the n**NCC**. However, Theorem 2 is not true.

Axiom-DIFF. If $i \succ_x j$, $j \succsim_{(x+d)} i$, $j \succ_{x'} m$, $m \succsim_{(x'+d)} j$ and $m \succ_{x''} i$, then $i \not\succsim_{(x''+d)} m$.

The intuition of **DIFF** is as follows. Suppose that the difference d transforms agent i from being more prosperous than agent j in x to being less prosperous than j in $x+d$. Thus, d is responsible for worsening the situation of i compared to that of j. The same happens to j compared to m. That is, j is more prosperous than m in x', but turns out to be less prosperous than m in $x'+d$. **DIFF** requires that the same difference d cannot change the situation of m compared to that of i in the same manner. That is, if m is more prosperous than i in x'', then it is impossible that m will be less prosperous than i in $x'' + d$.

Note that in the case of a full domain, **CONV** and **ORDER** imply **DIFF**. This is so because there is an order $\succ_{-d}$ induced by the vector $-d$. If **DIFF** is violated, $i \succ_{(x''+d)} m$. By **CONV**, this implies that $i \succ_{-d} j$. Otherwise, $j \succ_{(x+d)+(-d)} i$, which contradicts $i \succ_x j$. Similarly, $j \succ_{-d} m$ and $m \succ_{-d} i$. We obtained that the order $\succ_{-d}$ is cyclic, which is a contradiction.

Axiom-HOPE. For every $i, j \in N$, R_{ij} is not empty.

HOPE ensures that there is at least one vector c where agent i is more prosperous than agent j.

Theorem 3 *Suppose that a prosperity partial order satisfies **HOPE** and **No TRIUMVIRATE**. It has a linear representation if and only if it satisfies the axioms **CONV**, **OPEN** and **DIFF**.*

Theorems 2 and 3 deal with cases where the **No TRIUMVIRATE** axiom is satisfied. In the case of three agents only this axiom imposes no further restriction and linear representation is ensured by the other axioms. The case where **No TRIUMVIRATE** is not satisfied, which is the degenerate case, needs in addition a requirement that is equivalent to **NCC**. We refer to the degenerate case in the next chapter and prefer not to discuss it in the game theoretical context.

The proofs need some duality results which are presented in the following section.

4 Linear representation

In this section we present a general duality result that may also be applicable in different contexts. We therefore prefer to make it independent of the current discussion. For this reason, we repeat some conditions and notation that were mentioned before.

The analysis will be divided into two cases. In the first, every point in a Euclidean space induces a partial order over a finite set. In the second case, only points in the positive orthant induce a partial order. The next subsection, though, applies to both cases.

4.1 The general set-up and some general results

Let E be either $\mathbb{R}^k$ or $\mathbb{R}^k_+$ and let N be a finite set. Suppose that for any $y \in E$ there is a partial order $\succ_y$ defined on the set N. For every $i, j \in N$ denote by R_{ij} (resp. W_{ij}) the set of all $y \in E$ such that $i \succ_y j$ (resp. $i \succsim_y j$). Thus, $W_{ij} = E \backslash R_{ji}$. We assume that R_{ij} and W_{ij} are convex cones and that R_{ij} is open (in the relative topology in E). Thus, W_{ij} (which is the complement of an open set) is closed.

Lemma 2 *For every $i, j \in N$, if neither R_{ij} nor R_{ji} are empty, then $0 \notin R_{ij}$.*

Proof. If $0 \in R_{ij}$, then a neighborhood of 0 is in R_{ij} and must intersect R_{ji}, which is a non-empty cone. This is a contradiction. ∎

Define ℓ**NCC** similarly to how it was defined in the previous section. n**NCC** is also denoted as **NCC**.

In what follows we deal with points in subsets of $\mathbb{R}^{nk}$. Vectors in $\mathbb{R}^{nk}$ will be referred to as lists of n blocks, consisting of k coordinates each. For every $r \in \mathbb{R}^k$, i and j, denote by $\varphi(r, i, j)$ the point in $\mathbb{R}^{nk}$ whose i-th block is r, j-th block is $-r$ and all other blocks are zeros. Let $\mathcal{B} = \text{conv}\{\varphi(r, i, j);\ i, j \in N,\ r \in \mathbb{R}^k\}$. For any $\varepsilon > 0$, $\mathcal{U}_\varepsilon$ denotes the set $\text{conv}\{\varphi(r, i, j);\ i, j \in N,\ \|r\| < \varepsilon\}$. This is an open neighborhood of zero in the relative topology of $\mathcal{B}$.

For every pair $i, j \in N$ let $\mathcal{W}_{ij}$ be the subset of $\mathcal{B}$ defined as follows: $\mathcal{W}_{ij} = \text{conv}\{\varphi(r_{ij}, i, j);\ r_{ij} \in W_{ij}\}$. Set $\mathcal{W} = \text{conv} \cup \{\mathcal{W}_{ij};\ i, j \in N\}$. Note that $\mathcal{W}_{ij}$ is a closed convex cone for every i and j. Moreover, $\mathcal{W}$ is a closed and convex cone as a convex hull of closed convex cones.

Lemma 3 *If for every i and j, $W_{ij} + R_{ij} \subseteq R_{ij}$ and if **NCC** is satisfied, then for every $i, j \in N$ and $r_{ij} \in R_{ij}$, $\varphi(-r_{ij}, i, j) \notin \mathcal{W}$.*

Proof. Suppose to the contrary that for some $i, j \in N$ and $s_{ij} \in R_{ij}$, $\varphi(-s_{ij}, i, j) \in \mathcal{W}$. Since all the W_{ij} are convex cones, this means that there are $r_{ij} \in W_{ij} \cup \{0\}$ such that for every $m \notin \{i, j\}$, $\sum_\ell (r_{m\ell} - r_{\ell m}) = 0$. Moreover, $\sum_\ell (r_{i\ell} - r_{\ell i}) = -s_{ij}$ and $\sum_\ell (r_{j\ell} - r_{\ell j}) = s_{ij}$. However, if in the last two equations, s_{ij} and $-s_{ij}$ are transferred to the left side, we obtain (since by assumption, $r_{ij} + s_{ij}$ is in R_{ij}) a contradiction to **NCC**. ■

Theorem 4 *There are $\gamma_1, ..., \gamma_n$ such that $R_{ij} = \{r;\ \langle r, \gamma_i \rangle > \langle r, \gamma_j \rangle\}$ for every i and j if and only if for every i and j, $W_{ij} + R_{ij} \subseteq R_{ij}$ and **NCC** is satisfied.*

Remark 2 Note that if $\gamma_1, ..., \gamma_n$ satisfy the condition of Theorem 4, then for every $e \in \mathbb{R}^k$ and a positive constant c, $c\gamma_1 + e, ..., c\gamma_n + e$ also satisfy it. This is so because $\{r;\ \langle r, \gamma_i \rangle > \langle r, \gamma_j \rangle\} = \{r;\ \langle r, c\gamma_i + e \rangle > \langle r, c\gamma_j + e \rangle\}$.

Proof. Suppose that **NCC** is satisfied. We claim first that if $R_{\ell m} \cup R_{m\ell} \neq \emptyset$, then for every i, either $R_{i\ell} \cup R_{\ell i} \neq \emptyset$ or $R_{im} \cup R_{mi} \neq \emptyset$. Otherwise, we can assume that there is $r \in R_{\ell m} \cap W_{\ell i} \cap W_{i\ell} \cap W_{mi} \cap W_{im}$ for some i. We therefore obtain a cycle in $\succsim_r$, which contradicts (as in Lemma 1) **NCC**.

Fix now ℓ and m in N such that $R_{\ell m} \cup R_{m\ell} \neq \emptyset$. Assume that $r \in R_{\ell m}$ and consider the set $\mathcal{Y} = \text{conv}(\varphi(-r, \ell, m) + \mathcal{U}_\varepsilon) \cup \{0\}) \backslash \{0\}$. $\mathcal{Y}$ is the open cone generated by $\varphi(-r, \ell, m) + \mathcal{U}_\varepsilon$. By Lemma 3, $\mathcal{W} \cap (\varphi(-r, \ell, m) + \mathcal{U}_\varepsilon) = \emptyset$. As $\mathcal{W}$ is a cone, $\mathcal{W} \cap \mathcal{Y} = \emptyset$. Since, both $\mathcal{Y}$ and $\mathcal{W}$ are convex and since the relative interior of $\mathcal{Y}$ is not empty, there is a separating vector $\gamma^{(\ell m)}$ in R^{nk} that has the following separating property: $\inf_{b \in \mathcal{Y}} \langle b, \gamma^{(\ell m)} \rangle < 0 \leq \sup_{a \in \mathcal{W}} \langle a, \gamma^{(\ell m)} \rangle$ (see Rockafellar, 1970). Denote by $\gamma_i^{(\ell m)}$ the i-th block of $\gamma^{(\ell m)}$. For any $r_{ij} \in R_{ij}$, we obtain $\langle \varphi(r_{ij}, i, j), \gamma^{(\ell m)} \rangle \geq 0$. Thus, $\langle r_{ij}, \gamma_i^{(\ell m)} \rangle \geq \langle r_{ij}, \gamma_j^{(\ell m)} \rangle$ for every i and j. We may assume (due to Remark 2, by subtracting, if necessary, $\gamma_\ell^{(\ell m)}$ from all the blocks) that $\gamma_\ell^{(\ell m)} = 0$.

We show now that $\gamma_m^{(\ell m)} - \gamma_\ell^{(\ell m)} = \gamma_m^{(\ell m)} \neq 0$. Since, $\inf_{b \in \mathcal{Y}} \langle b, \gamma^{(\ell m)} \rangle < 0$, there is an i such that $\gamma_i^{(\ell m)} \neq 0$. By the previous observation, since $R_{\ell m} \neq \emptyset$, either $R_{\ell i} \cup R_{i\ell} \neq \emptyset$ or $R_{mi} \cup R_{im} \neq \emptyset$. We assume that $R_{\ell i} \cup R_{i\ell} \neq \emptyset$. (The case where $R_{mi} \cup R_{im} \neq \emptyset$ is treated similarly.)

Let $s \in R_{\ell i} \cup R_{i\ell}$ be such that $||s|| < \varepsilon$ (there exists such s since $R_{\ell i} \cup R_{i\ell}$ is a cone). By the definition of $\mathcal{Y}$ both $\varphi(-r, \ell, m) + \varphi(s, i, \ell)$ and $\varphi(-r, \ell, m) + \varphi(-s, i, \ell)$ are in $\mathcal{Y}$. Thus, $\langle \varphi(-r, \ell, m) + \varphi(s, i, \ell), \gamma^{(\ell m)} \rangle \leq 0$ and $\langle \varphi(-r, \ell, m) + \varphi(-s, i, \ell), \gamma^{(\ell m)} \rangle \leq 0$. Hence, if we show that $\langle \varphi(s, i, \ell), \gamma^{(\ell m)} \rangle \neq 0$, we show that $\langle \varphi(-r, \ell, m), \gamma^{(\ell m)} \rangle \neq 0$, which proves the desired inequality, $\gamma_m^{(\ell m)} \neq 0$.

Recall that $s \in R_{\ell i} \cup R_{i\ell}$ and that $\gamma_i^{(\ell m)} \neq 0$. Suppose, by negation, that $\langle \varphi(s, i, \ell), \gamma^{(\ell m)} \rangle$, which is equal to $\langle s, \gamma_i^{(\ell m)} \rangle - \langle s, \gamma_\ell^{(\ell m)} \rangle = \langle s, \gamma_i^{(\ell m)} \rangle$, is equal to 0. Since $R_{\ell i} \cup R_{i\ell}$ is open, there is an open ball around s which is in $R_{\ell i} \cup R_{i\ell}$. Thus, for all e in an open ball around 0, $\langle s + e, \gamma_i^{(\ell m)} \rangle$ have the same sign (due to the separation property). Thus, all have to be zero, meaning that $\gamma_i^{(\ell m)} = 0$. This contradicts the choice of i (i.e., $\gamma_i^{(\ell m)} \neq 0$). We conclude that $\langle \varphi(r, i, \ell), \gamma^{(\ell m)} \rangle \neq 0$ and thus $\gamma_m^{(\ell m)} \neq 0$, as desired.

As in the previous argument, if there is $r \in R_{\ell m} \cup R_{m\ell}$ such that $\langle r, \gamma_m^{(\ell m)} \rangle = 0$, then $\gamma_m^{(\ell m)} = 0$. Therefore, for any $r \in R_{\ell m}$, $\langle r, \gamma_m^{(\ell m)} \rangle < 0$ and for any $r \in R_{m\ell}$, $\langle r, \gamma_m^{(\ell m)} \rangle > 0$.

This conclusion is true only for the pair ℓ and m. However, having the same construction for any such pair and taking the summation $(\gamma_1, ..., \gamma_n) = \sum_{\ell m} \gamma^{(\ell m)}$ will give us the required separation property of the theorem. This concludes the proof of the "if" direction.

As for the inverse direction of the theorem, suppose now that there are $\gamma_1, ..., \gamma_n$ that satisfy $\langle r, \gamma_i \rangle > \langle r, \gamma_j \rangle$ if and only if $r \in R_{ij}$. In order to prove **NCC**, assume that there are $r_{ij} \in W_{ij} \cup \{0\}$ such that for any j, $\sum_i r_{ij} = \sum_i r_{ji}$. Thus, $\sum_i \langle r_{ij}, \gamma_i \rangle - \sum_i \langle r_{ji}, \gamma_i \rangle = 0$. Summing all these equations over all j provides the following: $\sum_{ij} r_{ij}(\gamma_i - \gamma_j) = 0$. Thus, no r_{ij}

is in R_{ij}, which proves **NCC**. The fact that for every i and j, $W_{ij}+R_{ij} \subseteq R_{ij}$, is clearly implied. ■

The following corollary is a particular implication of Theorem 4 to the case of three elements in N (3**NCC**).

Corollary 1 *For every distinct $i, j, m \in N$ there are $\gamma_i, \gamma_j, \gamma_m$ such that for every $a, b \in \{i, j, m\}$, $R_{ab} = \{r;\ \langle r, \gamma_a\rangle > \langle r, \gamma_b\rangle\}$ if and only if*
(a) *$r_{ij} - r_{ji} = r_{mi} - r_{im} = r_{jm} - r_{mj}$, where $r_{ab} \in W_{ab} \cup \{0\}$ for every $a, b \in \{i, j, m\}$, implies that none of r_{ab} is in R_{ab}; and* **(b)** *for every $a, b \in \{i, j, m\}$, $W_{ab} + R_{ab} \subseteq R_{ab}$.*

Proof. For every distinct $i, j, m \in N$, **3NCC** is precisely **(a)**. ■

4.2 The domain is the entire Euclidean space

Suppose that $E = \mathbb{R}^k$.

The following proposition states that when the domain is the entire Euclidean space, then **ORDER** implies **3NCC**.

Proposition 1 *If for every $y \in E$, $\succ_y$ is an order, then for every $i, j, m \in N$, $r_{ij} - r_{ji} = r_{mi} - r_{im} = r_{jm} - r_{mj}$, where $r_{ab} \in W_{ab} \cup \{0\}$ for every $a, b \in \{i, j, m\}$, implies that none of r_{ab} is in the respective R_{ab}.*

Proof. Assume to the contrary that there are $i, j, m \in N$ and $r_{ab} \in W_{ab} \cup \{0\}$, some in the respective R_{ab}, that satisfy $r_{ij} - r_{ji} = r_{mi} - r_{im} = r_{jm} - r_{mj}$. Without loss of generality, $r_{ij} \in R_{ij}$. Let $z = r_{ij} - r_{ji}$. The order $\succsim_z$ is defined.

The point z is not equal to zero because if $z = 0$, then either $r_{ij} = r_{ji} \neq 0$, which is impossible since $R_{ij} \cap W_{ji} = \emptyset$, or $r_{ij} = r_{ji} = 0$, which contradicts Lemma 2.

We claim that $i \succ_z j$. Since, $r_{ji} \in W_{ji} \cup \{0\}$, $j \succsim_z i$ would imply, by the convexity of W_{ji}, that $j \succsim_{z+r_{ji}} i$. However, $z + r_{ji} = r_{ij}$. Therefore,

$i \succ_z j$. By similar arguments, using that $z \neq 0$, $j \succsim_z m$ and $m \succsim_z i$. This contradicts the fact that $\succ_z$ is acyclic. ■

As in the previous chapter, we need here a condition similar to **No TRIUMVIRATE** (and we call it here by the same name). We say that x' is *insignificant* for a set of $L \subseteq N$, if for every x, the partial orders $\succ_{x+x'}$ and $\succ_x$ coincide over L. **No TRIUMVIRATE** is satisfied if whenever x' is insignificant for L, it is also insignificant for N.

Remark 3 Suppose that for every i and j, $W_{ij} + R_{ij} \subseteq R_{ij}$. Furthermore, assume that $R_{ij} \neq \emptyset$ implies $R_{ji} \neq \emptyset$. Then there is only one (up to multiplication with a positive constant) vector that separates R_{ij} and R_{ji} whenever these are not empty. We denote this vector by β_{ij}.

Example 1 In this example we show that when there is a triumvirate, namely, all β_{ij} are on the same plane, then there may be a linear representation of any triplet of N, but there is no linear representation for the entire set. Suppose that the dimension of the Euclidean space is 2. For any pair $i \in N$ attach a different vector γ_i in $\mathbb{R}^2$. These vectors induce an order for any $x \in \mathbb{R}^2$, that is, these vectors induce the separations β_{ij} between R_{ij} and R_{ji}. Slightly perturbing each of β_{ij} will retain the order, which means retaining **3NCC**, while spoiling the linear representation.

Lemma 4 *Suppose that there is a unique β_{ij} that separates R_{ij} and R_{ji}. Furthermore, suppose that for any triplet i, j, m, β_{ij} is linearly dependent on β_{im} and β_{jm}. Then,* ***No TRIUMVIRATE*** *implies that if there are at least four elements in N, then for every $p \in N$ there are three elements in N, called i, j, m, such that $\beta_{pi}, \beta_{pj}, \beta_{pm}$ are linearly independent.*

Proof. **No TRIUMVIRATE** implies that the linear span of all β_{ij}'s is of a dimension greater than 2. Since β_{ij} is linearly dependent on β_{im} and β_{jm} for every triplet i, j, m, for every m, the vectors β_{mj}, $j \in N$ span the same subspace: the one generated by all β_{ij}'s. In particular, for every $p \in N$

there are i, j and m such that β_{pi}, β_{pj} and β_{pm} are linearly independent, as desired. ■

Let $T = \{i, j, m\} \subseteq N$ be a triplet of agents. We say that T generates $\gamma_i^T, \gamma_j^T, \gamma_m^T$ such that for every $a, b \in T$, $R_{ab} = \{r;\ \langle r, \gamma_a^T \rangle > \langle r, \gamma_b^T \rangle\}$. Note that if T generates $\gamma_i^T, \gamma_j^T, \gamma_m^T$ it also generates $C\gamma_i^T + v$, $C\gamma_j^T + v$, and $C\gamma_m^T + v$ for every non-negative C and vector v. Let $T = \{i, j, m\}, S = \{i, j, t\}$ be two triplets of agents; T and S generate $\gamma_i, \gamma_j, \gamma_m, \gamma_t$ if T generates $\gamma_i, \gamma_j, \gamma_m$ and S generates $\gamma_i, \gamma_j, \gamma_t$.

Lemma 5 *Suppose that $T = \{i, j, m\}$ and $S = \{i, j, t\}$ generate $\gamma_i, \gamma_j, \gamma_m, \gamma_t$. Furthermore, suppose that $T' = \{j, t, m\}$ and $S' = \{i, t, m\}$ generate $\gamma'_i, \gamma'_j, \gamma'_m, \gamma'_t$. Then, either $\beta_{im}, \beta_{jm}, \beta_{t,m}$ are linearly dependent or there exist $v \in \mathbb{R}^k$ and $C \geq 0$ such that $\gamma_i = C\gamma'_i + v, \gamma_j = C\gamma'_j + v, \gamma_m = C\gamma'_m + v, \gamma_t = C\gamma'_t + v$, meaning that T' and S' also generate $\gamma_i, \gamma_j, \gamma_m, \gamma_t$.*

Proof. Due to the uniqueness of the $\beta's$, $\gamma_i - \gamma_m = C_1(\gamma'_i - \gamma'_m)$, $\gamma_m - \gamma_j = C_2(\gamma'_m - \gamma'_j)$, $\gamma_j - \gamma_t = C_3(\gamma'_j - \gamma'_t)$, $\gamma_t - \gamma_i = C_4(\gamma'_t - \gamma'_i)$, where $C_1, C_2, C_3, C_4 > 0$. Summing up the the four equations gives $0 = (C_1 - C_4)(\gamma'_i - \gamma'_m) + (C_3 - C_2)(\gamma'_j - \gamma'_m) + (C_4 - C_3)(\gamma'_t - \gamma'_m)$. Thus, either $\beta_{im}, \beta_{jm}, \beta_{tm}$ are linearly dependent or $C_1 = C_2 = C_3 = C_4 = C \neq 0$.

If the latter is true we obtain, $\gamma_i - \gamma_m = C(\gamma'_i - \gamma'_m)$, $\gamma_m - \gamma_j = C(\gamma'_m - \gamma'_j))$, $\gamma_j - \gamma_t = C(\gamma'_j - \gamma'_t))$ and $\gamma_t - \gamma_i = C(\gamma'_t - \gamma'_i)$. Thus, $C\gamma'_i - \gamma_i = C\gamma'_m - \gamma_m = C\gamma'_j - \gamma_j = C\gamma'_t - \gamma_t$. This proves the desired result. ■

Theorem 5 *There are $\gamma_1, ..., \gamma_n$ such that for every $i, j \in N$, $R_{ij} = \{r;\ \langle r, \gamma_i \rangle > \langle r, \gamma_j \rangle\}$ if and only if **(a)** for every $y \in E$, $\succ_y$ is an order; **(b)** for every i and j, $W_{ij} + R_{ij} \subseteq R_{ij}$; **(c)** $R_{ij} \neq \emptyset$ implies $R_{ji} \neq \emptyset$; and **(d) No TRIUMVIRATE**.*

Proof. The "only if" direction is simple. As for the "if" direction, by **(a)** and **(c)** we may assume, without loss of generality, that $R_{ij} \neq \emptyset$ for every i and j. This is so because if $R_{ij} = \emptyset$, then, by **(c)**, $W_{ij} \cap W_{ji} = E$ and therefore, due to **(a)**, for any m, $R_{im} = R_{jm}$ and $R_{mi} = R_{mj}$, and we can deal with $N \backslash \{j\}$ rather than with N.

In order to use Corollary 1, we need to verify that **(a)** and **(b)** of this corollary hold. By Proposition 1, assumption **(a)** implies **(a)** of Corollary 1; **(b)** of Corollary 1 is the same as assumption **(b)**.

Now, by Corollary 1, for every subset $S = \{i, j, m\}$ of N there are $\gamma_i^S, \gamma_j^S, \gamma_m^S$ such that for every $a, b \in S$, $R_{ab} = \{r;\ \langle r, \gamma_a^S \rangle > \langle r, \gamma_b^S \rangle\}$. In particular, $\gamma_i^S - \gamma_j^S$ separates R_{ij} and R_{ji}. By Remark 3, assumptions **(b)** and **(c)** imply that there is only one (up to multiplication with a constant) vector, denoted β_{ij}, that separates R_{ij} and R_{ji}. This means that if S' and S'' contain three elements each and $i, j \in S \cap S'$, then $\gamma_i^S - \gamma_j^S = C(\gamma_i^{S'} - \gamma_j^{S'})$ for some positive number C. Furthermore, for every triplet i, j, m, β_{ij} is linearly dependent on β_{im} and β_{jm}, as in the hypothesis of Lemma 4.

Using **(d)**, Lemma 4 ensures that there are four elements in N, say, $1, 2, 3, 4$, such that $\beta_{14}, \beta_{24}, \beta_{34}$ are linearly independent. Note that this independence implies other types of independence: that of $(\beta_{ij})_{j;\ j \neq i}$ for every i and of $(\beta_{ij})_{i;\ i \neq j}$ for every j.

We will show how this enables us to produce first γ_4, using $\{1, 2, 3, 4\}$, and then γ_m, using $\{1, 2, 3, 4, m\}$. Finally, we will show that there is no inconsistency, meaning that for every m and n in N, $\gamma_n - \gamma_m = C\beta_{mn}$, $C \geq 0$.

This will be done in three steps.

Step 1: creating γ_4

Set $T = \{1, 2, 4\}$ and $S = \{1, 2, 3\}$. S and T together generate $\gamma_1, \gamma_2, \gamma_3, \gamma_4$. Similarly, $T' = \{2, 3, 4\}$ and $S' = \{1, 3, 4\}$ generate $\gamma_1', \gamma_2', \gamma_3', \gamma_4'$. From Lemma 5, since $\beta_{14}, \beta_{24}, \beta_{34}$ are linearly independent, there exist $v \in \mathbb{R}^k$ and $C \in \mathbb{R}$ such that $\gamma_1 = C\gamma_1' + v, \gamma_2 = C\gamma_2' + v, \gamma_3 = C\gamma_3' + v, \gamma_4 = C\gamma_4' + v$, meaning that T' and S' also generate $\gamma_1, \gamma_2, \gamma_3, \gamma_4$. In particular we obtain that for every distinct i and j, $\gamma_i - \gamma_j = C\beta_{ij}$ for some positive C.

Step 2: creating γ_m using $1, 2, 3, 4, m$ for every $m > 4$.

Suppose that $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are obtained from Step 1 and that m is greater than 4.

In order to construct γ_m, we first find a permutation of $(1,2,3,4)$, (i,j,k,t), such that both $\{\beta_{ij},\beta_{ik},\beta_{im}\}$ and $\{\beta_{jm},\beta_{km},\beta_{tm}\}$ are linearly independent. We divide the proof that such a permutation exists into four cases.

Case 1: β_{1m} depends on some β_{1i}, $i\in\{2,3,4\}$. In this case $\{\beta_{ij},\beta_{ik},\beta_{i1}\}$ is linearly independent (because, $\{\beta_{12},\beta_{13},\beta_{14}\}$ is independent). Furthermore, $\{\beta_{1j},\beta_{1k},\beta_{1m}\}$ is linearly independent and so is $\{\beta_{jm},\beta_{km},\beta_{1m}\}$. Therefore, $(i,j,k,1)$ is the desired permutation.

Case 2: β_{1m} linearly depends on $\{\beta_{1i},\beta_{1j}\}$ for some $i,j\in\{2,3,4\}$, but does not depend on any β_{1j}, $j\in\{2,3,4\}$. Denote by k the member of $\{1,2,3,4\}\setminus\{1,i,j\}$. Since $\{\beta_{12},\beta_{13},\beta_{14}\}$ is independent, $\{\beta_{1k},\beta_{1j},\beta_{1m}\}$ is independent. Thus, $\{\beta_{1m},\beta_{km},\beta_{jm}\}$ is also independent. Furthermore, $\{\beta_{1i},\beta_{1k},\beta_{1m}\}$ is independent (otherwise, β_{1m} depends on β_{1i}, in contradiction to the assumption). Therefore, $\{\beta_{i1},\beta_{ik},\beta_{im}\}$ is independent and $(i,1,k,j)$ is the desired permutation.

Case 3: β_{1m} is a linear combination of all $\{\beta_{12},\beta_{13},\beta_{14}\}$ and not of any strict subset of it. In this case, both $\{\beta_{12},\beta_{13},\beta_{1m}\}$ and $\{\beta_{13},\beta_{14},\beta_{1m}\}$ are linearly independent. The first independence implies that $\{\beta_{21},\beta_{23},\beta_{2m}\}$ is linearly independent, while the second independence implies that $\{\beta_{1m},\beta_{3m},\beta_{4m}\}$ is linearly independent. Thus, the permutation $(2,1,3,4)$ ensures the desired claim.

Case 4: $\{\beta_{12},\beta_{13},\beta_{14},\beta_{1m}\}$ are linearly independent. Therefore, $\{\beta_{2m},\beta_{3m},\beta_{4m}\}$ and $\{\beta_{12},\beta_{13},\beta_{1m}\}$ are independent. Thus, the permutation $(1,2,3,4)$ ensures the desired claim.

Second, we will use the independence of $\{\beta_{ij},\beta_{ik},\beta_{im}\}$ and $\{\beta_{jm},\beta_{km},\beta_{tm}\}$ to construct γ_m. Using the independence of $\{\beta_{ij},\beta_{ik},\beta_{im}\}$, we construct $\gamma_i',\gamma_j',\gamma_k',\gamma_m'$ as in step 1.

There exist a positive scalar C and a vector v such that $\gamma_i=C\gamma_i'+v$, $\gamma_j=C\gamma_j'+v$. Normalizing all $\gamma_i',\gamma_j',\gamma_k',\gamma_m'$ by multiplying by C and adding v, we obtain $\gamma_i,\gamma_j,\gamma_k'',\gamma_m$. Since β_{ij} and β_{ik} are independent, so are β_{jk} and β_{ik}. The fact that $\gamma_i'-\gamma_k'\in span\{\beta_{ik}\}$implies that $\gamma_i-\gamma_k''\in span\{\beta_{ik}\}$

(normalization does not change it). The same argument holds for j and k. Hence, $\gamma_j - \gamma_k'' \in span\{\beta_{jk}\}$. This implies that γ_k'' is equal to γ_k.

Note that γ_m was constructed without using t. It is still left to show that there is some positive scalar C that satisfies $\gamma_t - \gamma_m = C\beta_{tm}$. Since $\beta_{jm}, \beta_{tm}, \beta_{km}$ are linearly independent, they generate $\gamma_j^*, \gamma_k^*, \gamma_t^*, \gamma_m^*$ (as in step 1). We can find a positive scalar C and a vector v such that $\gamma_j = C\gamma_j^* + v$, and $\gamma_k = C\gamma_k^* + v$, in order to transform $\gamma_j^*, \gamma_k^*, \gamma_t^*, \gamma_m^*$ to $\gamma_j, \gamma_k, \gamma_t^{**}, \gamma_m^{**}$. Since $\{\beta_{jm}, \beta_{km}, \beta_{tm}\}$ are independent, so are β_{jt} and β_{kt}. This implies that $\gamma_t^{**} = \gamma_t$. From the independence of β_{jm}, β_{km}, it implies that $\gamma_m^{**} = \gamma_m$. As $\gamma_t^* - \gamma_m^* = C\beta_{tm}$ for some $C > 0$, the same is true for $\gamma_m - \gamma_t$.

Step 3: For every $m, n \in N, \gamma_m - \gamma_n = C\beta_{mn}$, for some $C \geq 0$.

γ_m and γ_n were created using $\{1, 2, 3, 4, m\}$ and $\{1, 2, 3, 4, n\}$, respectively. To see why $\gamma_m - \gamma_n = C\beta_{mn}$, for some $C \geq 0$, we need to find $i, j \in \{1, 2, 3, 4\}$ such that $\beta_{im}, \beta_{jm}, \beta_{nm}$ are linearly independent. When such i and j are found, $\beta_{im}, \beta_{jm}, \beta_{nm}$ can be used to produce $\gamma_i^\sim, \gamma_j^\sim, \gamma_m^\sim, \gamma_n^\sim$. Due to the independence of β_{im}, β_{jm}, and of β_{in}, β_{jn} we can normalize $\gamma_i^\sim, \gamma_j^\sim, \gamma_m^\sim, \gamma_n^\sim$ to $\gamma_i, \gamma_j, \gamma_m, \gamma_n$, thus proving that $\gamma_m - \gamma_n = C\beta_{mn}$, for some $C \geq 0$.

It remains to show that such i, j can be found. If such i, j cannot be found, then each of $\{\beta_{1m}, \beta_{2m}, \beta_{nm}\}$, $\{\beta_{1m}, \beta_{3m}, \beta_{nm}\}$ and $\{\beta_{1m}, \beta_{4m}, \beta_{nm}\}$ is a set of linearly dependent vectors. This means that both $\{\beta_{1m}, \beta_{2m}, \beta_{3m}\}$ and $\{\beta_{1m}, \beta_{2m}, \beta_{4m}\}$ are linearly dependent. The first dependence implies that $\{\beta_{1m}, \beta_{12}, \beta_{13}\}$ are dependent, and the second implies that $\{\beta_{1m}, \beta_{12}, \beta_{14}\}$ are linearly dependent.

This may happen without a contradiction to the independence of $\{\beta_{12}, \beta_{13}, \beta_{14}\}$ only if β_{1m} and β_{12} are linearly dependent. In this case we can employ the dependence of $\{\beta_{1m}, \beta_{3m}, \beta_{nm}\}$ and of $\{\beta_{1m}, \beta_{4m}, \beta_{nm}\}$ to show that $\{\beta_{1m}, \beta_{3m}, \beta_{4m}\}$ are linearly dependent. The latter contradicts the independence of $\{\beta_{12}, \beta_{13}, \beta_{14}\}$.

■

4.3 The domain is the positive orthant

Suppose that $E = \mathbb{R}^k_+$. In the previous case, where the domain was the entire $\mathbb{R}^k$, the fact that every $\succ_y$ was an order implied that there was no trivial way to obtain $r_{ij} - r_{ji} = r_{mi} - r_{im} = r_{jm} - r_{mj}$. However, this is not the case when the domain is restricted, as illustrated by Example 2 below. Here, we need additional conditions.

Definition 6 The set $\{\succ_y\}_{y\in E}$ of partial orders is *extendable* if there is a system of orders $\{\succ_y\}_{y\in\mathbb{R}^k}$ that agrees with the originals.

The following theorem is an immediate implication of Theorem 4.

Theorem 6 *Suppose that* ***No TRIUMVIRATE*** *holds. There are* $\gamma_1, ..., \gamma_n$ *such that for every* $i, j \in N$, $R_{ij} = \{y;\ \langle y, \gamma_i\rangle > \langle y, \gamma_j\rangle\}$ *if and only if* ***(a)*** *the system* $\{\succ_y\}_{y\in E}$ *of partial orders is extendable;* ***(b)*** *with respect to the extension, for every* $i, j \in N$, $W_{ij} + R_{ij} \subseteq R_{ij}$*; and* ***(c)*** *with respect to the extension,* $R_{ij} \neq \emptyset$ *implies* $R_{ji} \neq \emptyset$.

The subject of extendability is related to **NCC** and is discussed in Appendix A. It turns out that **NCC** implies that the set $\{\succ_y\}_{y\in E}$ of partial orders is extendable and that **(b)** and **(c)** of Theorem 6 are satisfied.

Theorem 7 *Suppose that* $R_{ij} \neq \emptyset$ *for every* i, j *and* ***No TRIUMVIRATE****. Then, there exist* $\gamma_1, ..., \gamma_n$ *such that for every* $i, j \in N$, $R_{ij} = \{y;\ \langle y, \gamma_i\rangle > \langle y, \gamma_j\rangle\}$ *if and only if* ***(a)*** *for every* $i, j, m \in N$, $r_{ij} - r_{ji} = r_{mi} - r_{im} = r_{jm} - r_{mj}$, *where* $r_{ab} \in W_{ab} \cup \{0\}$ *for every* $a, b \in \{i, j, m\}$, *implies that none of* r_{ab} *is in* R_{ab}*; and* ***(b)*** *for every* $i, j \in N$, $W_{ij} + R_{ij} \subseteq R_{ij}$.

Proof. Suppose that **(a)** and **(b)** hold. These imply **(a)** and **(b)** of Corollary 1 for any i, j and m. Thus, by Corollary 1, for every subset $S = \{i, j, m\}$ of N there are $\gamma_i^S, \gamma_j^S, \gamma_m^S$ such that for every $a, b \in S$ the fact that $r \in R_{ab}$ implies $\langle r, \gamma_a^S\rangle > \langle r, \gamma_b^S\rangle$. Therefore, $\gamma_i^S - \gamma_j^S$ separates R_{ij} and R_{ji}. Assumption **(b)** guarantees that there is only one (up to multiplication with a constant) vector that separates R_{ij} and R_{ji}. As in the proof of Theorem 5, it proves the desired assertion. ∎

The following example shows that the assumption that every $\succ_y$ is an order may not imply **(a)** of Theorem 7.

Example 2 Define $R_{12} = \{r \in \mathbb{R}^3_+;\ \langle r, (1,2,-2)\rangle > 0\}$, $R_{13} = \{r \in \mathbb{R}^3_+;\ \langle r, (2,1,-6)\rangle > 0\}$ and $R_{23} = \{r \in \mathbb{R}^3_+;\ \langle r, (1,2,-14)\rangle > 0\}$. R_{12}, R_{31} and R_{32} are defined similarly way with respective reversed strict inequalities. It is easy to see that $R_{21} \subsetneqq R_{31} \subsetneqq R_{32}$. Thus, every point in $\mathbb{R}^3$ induces an order. Let, $r_{12} = (0, 1.25, 1) \in R_{12}$ and $r_{21} = (1, 0.25, 1) \in R_{21}$. Thus, $r_{12} - r_{21} = (-1, 1, 0)$. Also, let $r_{31} = (0, 5.75, 1) \in R_{31}$ and $r_{13} = (1, 4.75, 1) \in R_{13}$. Hence, $r_{31} - r_{13} = (-1, 1, 0)$. Finally, let $r_{23} = (0, 7.25, 1) \in R_{23}$ and $r_{32} = (1, 6.25, 1) \in R_{32}$. Thus, $r_{23} - r_{32} = (-1, 1, 0)$. This contradicts **(a)** of Theorem 7. In particular this is a contradiction to **3NCC**, which is a necessary condition for linear representation. Therefore, this example does not have a linear representation.

In $\mathbb{R}^2_+$ such an example is impossible. That is, if any point in $\mathbb{R}^2_+$ induces an order, then **3NCC** (or **(a)** of Theorem 7) is satisfied.

Proposition 2 *If for every $y \in E$, $\succ_y$ is an order, and if for every $i, j, m \in N$ there is a point s in the interior of E such that $i \sim_s j \sim_s m$, then for every $i, j, m \in N$, $r_{ij} - r_{ji} = r_{mi} - r_{im} = r_{jm} - r_{mj}$, where $r_{ab} \in W_{ab} \cup \{0\}$ for every $a, b \in \{i, j, m\}$, implies that none of r_{ab} is in R_{ab}.*

Proof. Suppose to the contrary that there are $i, j, m \in N$ and $r_{ab} \in W_{ab} \cup \{0\}$, with at least one in the respective R_{ab} satisfying $r_{ij} - r_{ji} = r_{mi} - r_{im} = r_{jm} - r_{mj} = z$. Without loss of generality, $r_{ij} \in R_{ij}$. By assumption, there is a point s in the interior of E such that $i \sim_s j \sim_s m$. Consider the point $s + tz$, where t is a small number. If t is small enough, the point $s + tz$ is still in E. As in the proof of Proposition 2 the order $\succsim_{s+tz}$ is cyclic, that is, $i \succ_{s+tz} j \succsim_{s+tz} m \succsim_{s+tz} i$. ■

This proposition implies the following theorem.

Theorem 8 *Suppose that for every $i, j, m \in N$ there is a point s in the inte-*

rior of E such that $i \sim_s j \sim_s m$. Suppose in addition that ***No TRIUMVIRATE*** *holds. Then, there exist $\gamma_1, ..., \gamma_n$ such that for every $i, j \in N$, $R_{ij} = \{y;\ \langle y, \gamma_i\rangle > \langle y, \gamma_j\rangle\}$ if* ***(a)*** *for every $y \in E$, $\succ_y$ is an order; and* ***(b)*** *for every $i, j \in N$, $W_{ij} + R_{ij} \subseteq R_{ij}$.*

Proof. Proposition 2 ensures, due to **(a)**, that for every subset $S = \{i, j, m\}$ of N there are $\gamma_i^S, \gamma_j^S, \gamma_m^S$ such that for every $a, b \in S$, $R_{ab} = \{r;\ \langle r, \gamma_a^S\rangle > \langle r, \gamma_b^S\rangle\}$. Therefore, $\gamma_i^S - \gamma_j^S$ separates R_{ij} and R_{ji}, whenever at least one of them is not empty. The uniqueness of the separation is needed in order to finish the proof.

The fact that for every $i, j, m \in N$ there is a point s in the interior of E such that $i \sim_s j \sim_s m$, guarantees that for every i and j, if $R_{ij} \neq \emptyset$, then $R_{ji} \neq \emptyset$. As in the proof of Theorem 5, we can assume without loss of generality that $R_{ij} \neq \emptyset$ for every i and j. The uniqueness of the separation, again as in the proof of Theorem 5, is now guaranteed by **(b)**. Due to **No TRIUMVIRATE** all the triplets produce $\gamma_1, ..., \gamma_n$ that satisfy the desired property. ■

The **No TRIUMVIRATE** is needed to extend the linear representation of any case of three elements to a linear representation of the partial orders over all N. In the case where **No TRIUMVIRATE** does not hold, there are three elements in N that determine the orders of all other elements in N. In this case, in order to extend the linear representation of any three elements in N to all N, we need the **NCC** condition.

4.4 The connection to Gilboa and Schmeidler (2001)

Gilboa and Schmeidler (2001) resort to a similar duality structure in a different context. In both discussions, vectors in a subset of the Euclidean space define partial orders or orders over a finite set of elements. There, the vectors, which are in the unit simplex of $\mathbb{R}_+^k$ represent empirical distributions, while here the vectors contain data about agents, such as needs, endowments etc. Moreover, while the finite set there is the set of alternatives, here it is the set of agents.

Gilboa and Schmeidler (2001) always assume that each vector induces a complete order on the finite set. They prove that if R_{ij} are open and convex cones, then an additional condition, the diversity axiom, ensures a linear representation.

The diversity axiom requires that for every list a, b, c, d of distinct elements (here, of agents and there, of alternatives) there exists a vector, say, x, such that $a \succ_x b \succ_x c \succ_x d$. This result is an implication of Theorem 8, since the diversity axiom implies **No TRIUMVIRATE** and the fact that for every triplet i, j and m there exists a point s in the interior of E that satisfies $i \sim_s j \sim_s m$, as assumed by Theorem 8. The reason why the diversity axiom implies **No TRIUMVIRATE** is that when the separating vectors are all in the same two-dimensional plane, there may be at most 12 orders of any four elements, while the diversity requires 4! orders.

No TRIUMVIRATE actually requires that there are four elements in N whose respective separating vectors are not on the same plane. Thus, a diversity axiom applied to only one set of four elements would be sufficient (but not necessary). In fact, requiring 13 orders of the four instead of the entire 24 would be sufficient to ensure **No TRIUMVIRATE**. Hence, the axiom **DIFF** (which is anyway a necessary condition) and the existence of 13 orders of one set of four elements in N imply the existence of linear representation.

Diversity of every three elements in N is sufficient to imply the assumption of Theorem 8 (for every triplet i, j and m there exists a point s in the interior of E that satisfies $i \sim_s j \sim_s m$). Hence, the diversity of every three elements, the existence of 13 orders of one set of four elements in N and **ORDER** imply the existence of linear representation.

In the case where all the separating vectors are in the same plane, we could not find a more intuitive condition than **NCC** to ensure linear representation.

5 The proofs of Theorems 1, 2 and 3

Proof of Theorem 1. This is an immediate implication of Theorem 4. (Recall that in the previous chapter we assumed that R_{ij} are all open and convex cones, which are equivalent to **OPEN** and **CONV**.)
Proof of Theorem 2. This is an immediate implication of Theorem 5. Note that **(a)** of Theorem 5 is equivalent to **ORDER**. Moreover, **(b)** of Theorem 5 is equivalent to **RECIPROCITY**. Finally, **(c)** of Theorem 5 and the assumption regarding the fact that R_{ij} are all open and convex cones are equivalent to **CONV**.
Proof of Theorem 3. This is an immediate consequence of Theorem 7, since **DIFF** implies **(a)** of the theorem.

6 Final comments

6.1 Refining the set of axioms

It appear that the **RECIPROCITY** axiom can be dispensed with, but we were unable to show it. The idea of finding the desired separation vectors should take the following direction. For every triplet, there is a triplet of separating vectors, as ensured by Theorem 4 (the **NCC** is satisfied by three elements). Thus the set of k separating vectors that separate three is non-empty and, moreover, convex. The objective then is to show that these convex sets (one for each triplet) have a non-empty intersection.

6.2 The NCC and extending the domain

Ashkenazi and Lehrer (2001) show that if for every vector in a subset of an abstract vector space there exists a partial order that satisfies the **NCC**, then an order can be defined on every vector in the vector space such that the orders extend beyond the existing partial orders and the **NCC** is satisfied.

It means, in particular, that the **NCC** enables one to extend partial orders to complete orders without losing the no cycles condition.

7 References

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