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On a representation of a relation by a measure

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Abstract: We give a necessary and sufficient condition for the existence of a representation for a relation by a positive measure for the general case in which the relation is defined on any set of subsets of Ω.

1. Introduction

The problem of describing the relations that are defined on an algebra of events, and can be represented by a probability measure, has been given a lot of attention: Kraft, Pratt and Seidenberg (1959), and Scott (1964), gave a characterization for the finite case. Savage (1954) gave a sufficient condition for a relation to be representable by a non-atomic measure. A full characterization of the representable relations defined on a Boolean algebra has been given by Chateauneuf (1985). Such relations are representable if and only if they are well-bounded (WB), weakly Archimedian (WA), and perfectly separable (S). The proof is not constructive and is based on Fan's theorem [Fan (1956)].

In some contexts we find relations defined on a certain collection of sets, not necessarily an algebra. Yaari (1987) in his essay on the dual utility theory discussed a relation defined on a set of random variables that attain values in the unit interval. These random variables are interpreted as lotteries which a decision maker might consider holding. Three of Yaari's axioms are phrased in terms of accumulative distribution functions which correspond to the former random variables. An alternative set-up is, therefore, a relation defined on the set of the non-increasing right continuous functions, G, from the unit interval to itself, that satisfy $G(1)=0$. However, instead of considering the functions G, we can consider the set of points enclosed between the axes and the graph of G: $a(G)=\{(x,t)\,|\,0\leqq t\leqq G(x)\}$. Define a relation on these

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sets by saying that $a(G) \succ a(G')$ if and only if $G \succ G'$. Notice that the collection of all $a(G)$ do not form an algebra. It is only natural to ask: Under what conditions this preference can be represented by a probability measure?

A decision maker might also face a situation where he should decide between leasing the portion A of a lot from time 0 to time t, and leasing the portion B from time 0 to time s. In this framework of time preferences the decision maker focuses only on alternatives of a certain sort: $A \times [0, t]$. It is clear that these sets do not form an algebra, and it is still interesting to ask when these preferences are representable.

As a consequence of a bounded rationality it might happen that a decision maker has preferences defined on some collection of events, not necessarily an algebra. Some events are not relevant, some demand time to compute and to consider. It seems natural and more realistic to get rid of the restricting assumption that a preference order is defined on an entire algebra.

We give here two characterizations of all the representable relations defined on any set of events (not necessarily an algebra). The proofs are based on a separating theorem as well as on the fact that any continuous functional on $L_\infty(\Omega)$ (the set of all bounded functions of Ω endowed with the maximum norm) can be represented by a finitely additive measure. In the first characterization we do not use any separability assumption of the kind used by Debreu (1964). In the second one we do use it, and we utilize the dense denumerable set to phrase our condition. The first characterization along with the main arguments of the paper are given in section 2. Section 3 is devoted to the second characterization. In this section we connect between Chateauneuf's result and ours. We show that if an order satisfies (WB), (WA) and (S), then our conditions are also satisfied. Thereby, we provide another proof to Chateauneuf's theorem.

In section 4 we give two examples. Both examples present preferences on the collection of leasing alternatives: to lease the fraction A for t periods of time starting now (i.e., in the interval $[0, t]$). In the first example, we give a non-representable preference. The argument is simple and it is presented only to illustrate the role of characteristic functions and their norm. In the second example, we show how to employ our technique in order to obtain an extension of a given partial order to a complete order.

Section 5 mentions an application of the main theorems to game theory. A coalition A is greater than coalition B if its value is greater than the value of B. This relation is representable by a probability measure μ if and only if the characteristic function of the game v is given by a monotonic function f composite with the measure μ, i.e., $v = f \circ \mu$. The result can be applied to games defined on an algebra of coalitions as well as to games defined on a general collection of coalitions. The latter is more natural in some political games. Due to ideological considerations it is reasonable to exclude some coalitions. For instance, the extremist parties in a parliament: the one from the right and the one from the left will not form a coalition, but it may happen that each one will create a coalition with the center party. Therefore, in some cases it is more realistic to deal with a characteristic function defined on the plausible coalitions which do not necessarily form an algebra.

In the last section we provide an explicit formula for the representing measure, in the non-atomic case, using the terms of the relation, $\succsim$.

2. The first characterization

Let Ω be a set, and $\mathscr{B}$ a field of subsets of Ω, and let $\succsim$ be a reflexive relation defined on a set $\mathscr{A} \subseteq \mathscr{B}$. We will assume that $\Omega \succsim \emptyset$. A strong relation $\succ$ can be derived from $\succsim$ as follows. $A \succ B$ ($A, B \in \mathscr{A}$) if $A \succsim B$ and not $B \succsim A$. Define $\mathscr{F} = \{(\chi_A, \chi_B) | A, B \in \mathscr{A}, A \succ B\}$ and $\bar{\mathscr{F}} = \{(\chi_A, \chi_B) | A, B \in \mathscr{A}, A \succsim B\}$, where χ_C is the characteristic function of the set C. For any finite string G of elements of $\chi(\mathscr{A})^2$, the set of pairs of characteristic functions of subsets in $\mathscr{A}$, $G = ((\chi_{A_1}, \chi_{B_1}), (\chi_{A_2}, \chi_{B_2}), \ldots, (\chi_{A_m}, \chi_{B_m}))$ define the number $n(G)$ to be

$$n(G) = \max_{\omega} \left|(1/m) \sum_{i=1}^{m} (\chi_{A_i}(\omega) - \chi_{B_i}(\omega))\right|.$$

In words, the function $\sum_{i=1}^{m} (\chi_{A_i}(\omega) - \chi_{B_i}(\omega))$ is the number of times ω is included in A_i's minus the number of times ω is included in B_i's. $n(G)$ is the maximal number that the absolute value of this function attains divided by the number of elements in G.

Let $\mathscr{F}' \subseteq \chi(\mathscr{A})^2$. Define

$$n(\mathscr{F}') = \inf \{n(G) | G \text{ is a finite string of elements of } \mathscr{F}'\}.$$

Notation 1. Let $\lambda \in \mathbb{R}$. Define $\bar{\mathscr{F}}_\lambda$ to be the set $\{(\chi_A + \lambda\chi_\Omega, \chi_B) | (\chi_A, \chi_B) \in \bar{\mathscr{F}}\} = (\lambda\chi_\Omega, 0) + \bar{\mathscr{F}}$.

Definition 1. A positive finitely additive measure (PFAM) μ defined on Ω is a $\mathscr{F}'$-representation of $\succsim$ if

(i) $A \succsim B \Leftrightarrow \mu(A) \geqq \mu(B)$,
(ii) $(\chi_A, \chi_B) \in \mathscr{F}' \Rightarrow \mu(A) > \mu(B)$.

Theorem 1. Let $\mathscr{F}' \subseteq \mathscr{F}$; and let $\lambda = n(\mathscr{F}')$.
If $n(\mathscr{F}' \cup \bar{\mathscr{F}}_\lambda) = \lambda > 0$, then there is an $\mathscr{F}'$-representation of $\succsim$ by a PFAM.

Proof. Define $C = \operatorname{conv}\{\chi_A - \chi_B | (\chi_A, \chi_B) \in \mathscr{F}' \cup \bar{\mathscr{F}}_\lambda\}$. We claim that $\lambda = n(\mathscr{F}' \cup \bar{\mathscr{F}}_\lambda\} = \inf_{g \in C} \|g\|_\infty$, where $\|\bullet\|_\infty$ is the maximum norm. It is clear that $\inf_{g \in C} \|g\|_\infty \leqq \lambda$. However, for any convex combination $\sum \alpha_i(\chi_{A_i} - \chi_{B_i})$ in C and for any $\varepsilon > 0$, we can find an ε-close rational convex combination $\sum(\beta_i/\ell)(\chi_{A_i} - \chi_{B_i})$, where β_i are integers that sum up to the integer ℓ. Notice that $n(G)$ of the finite string G consisting of β_i times the pair (χ_{A_i}, χ_{B_i}) for any i is ε-close to $\|\sum \alpha_i(\chi_{A_i} - \chi_{B_i})\|_\infty$. Since $\varepsilon > 0$ and the convex combination from C are both arbitrary, we get $\inf_G n(G) \leqq \inf_{g \in C} \|g\|_\infty$. The infimum at the left side is taken over all the finite strings of elements from $\mathscr{F}' \cup \tilde{\tilde{\mathscr{F}}}_\lambda$ which is exactly λ. Thus, $\lambda = \inf_{g \in C} \|g\|_\infty$, and our claim is established.

Let $D = \{f \in L_\infty(\Omega) \mid \|f\|_\infty < \lambda\}$, the λ-ball around the origin. D is open. Moreover, C and D are disjoint convex sets. Thus by the separating theorem [see Dunford and Schwartz (1958)], there is a continuous linear functional of $L_\infty(\Omega)$, x^*, and a number q such that $x^*(c) \geqq q$ for any $c \in C$ and $x^*(d) < q$ for any $d \in D$. Since any continuous linear functional of $L_\infty(\Omega)$ is defined by a PFAM, we conclude that there is a finitely additive measure, μ_0, such that $\int_\Omega c \, \mathrm{d}\mu_0 \geqq q > \int_\Omega d \, \mathrm{d}\mu_0$ for any $c \in C$ and $d \in D$. The function 0 is included in D. Thus, $0 < q$. It remains to show that μ_0 is positive. Otherwise there is a set $B \in \mathscr{B}$ such that $\mu_0(B) < 0$. For every $\lambda' < \lambda$ the function $\lambda' \chi_{\Omega - B}$ is included in D. Hence

$$\mu_0(\lambda' \chi_{\Omega - B}) = \lambda'(\mu_0(\Omega) - \mu_0(B)) < q.$$

Thus,

$$\mu_0(\Omega) \leqq q/\lambda + \mu_0(B). \tag{1}$$

As $\gtrsim$ is reflexive, $(\lambda \chi_\Omega, \chi_\varnothing) \in \tilde{\tilde{\mathscr{F}}}_\lambda$ and $\lambda \chi_\Omega \in C$. Therefore, $\lambda \mu_0(\Omega) \geqq q$. Since $\mu_0(B) < 0$, this contradicts (1).

We will prove now that μ_0 is a $\mathscr{F}'$-representation of $\gtrsim$. Let $A \gtrsim B$. Since $(\chi_A + \lambda \chi_\Omega, \chi_B) \in \tilde{\tilde{\mathscr{F}}}_\lambda$, $\chi_A + \lambda \chi_\Omega - \chi_B \in C$. Therefore, $\int \chi_A + \lambda \chi_\Omega - \chi_B \, \mathrm{d}\mu_0 \geqq q$. Hence, $\mu_0(A) - \mu_0(B) \geqq q - \lambda \mu_0(\Omega)$. In order to complete the proof that μ_0 satisfies (i) of Definition 1 it remains to show that $q - \lambda \mu_0(\Omega) \geqq 0$. But for every $\lambda' < \lambda$, $\lambda' \chi_\Omega \in D$. Thus $\lambda' \mu_0(\Omega) < q$, and therefore $\lambda \mu_0(\Omega) \leqq q$.

We will finish the proof by proving that for any $(\chi_A, \chi_B) \in \mathscr{F}'$, $\mu_0(A) - \mu_0(B) > 0$. However, if $(\chi_A, \chi_B) \in \mathscr{F}'$, then $\chi_A - \chi_B \in C$. Therefore, $\mu_0(A) - \mu_0(B) = \int \chi_A - \chi_B \, \mathrm{d}\mu_0 \geqq q > 0$. Q.E.D.

Lemma 1. If $\mathscr{F}$ is a union of subsets $\mathscr{F}^i$, $i \in \mathbb{N}$, the set of integers, such that for every i there exists an $\mathscr{F}^i$-representation of $\gtrsim$, then there is an $\mathscr{F}$-representation of $\gtrsim$.

Proof. Let μ_i be the $\mathscr{F}^i$-representation of $\gtrsim$. Set $\mu = \sum_{i=1}^\infty 2^{-i} \mu_i$. μ is a $\mathscr{F}$-representation of $\gtrsim$, because for any $(\chi_A, \chi_B) \in \mathscr{F}$ there is an i such that $(\chi_A, \chi_B) \in \mathscr{F}^i$. Thus, $\mu_i(B) - \mu_i(B)$ is positive. Since μ_j is an $\mathscr{F}^j$-representation, $\mu_j(A) - \mu_j(B)$ is non-negative for all $j \neq i$. Hence, $\mu(A) - \mu(B)$ is positive as a sum of non-negative summands and at least a positive one.

Theorem 2. Assume that $\mathscr{F} \neq \emptyset$. There is an $\mathscr{F}$-representation of $\succsim$ by a PFAM *iff there is a sequence of subsets $\{\mathscr{F}^i\}_{i\in\mathbb{N}}$ such that:*

(i) $\mathscr{F}$ is the union of $\{\mathscr{F}^i\}$;
(ii) $n(\mathscr{F}^i) = \lambda_i > 0$ for all $i \in \mathbb{N}$;
(iii) $n(\mathscr{F}^i \cup \tilde{\mathscr{F}}_{\lambda_i}) = \lambda_i$.

Proof. Assume first that there is an $\mathscr{F}$-representation of $\succsim$, and that it is given by the PFAM μ. Since $\mathscr{F} \neq \emptyset$, $\mu(\Omega) > 0$. Thus, without loss of generality, we can assume that $\mu(\Omega) = 1$. Define $\mathscr{F}^i = \{(\chi_A, \chi_B) \in \mathscr{F} \mid \mu(A) - \mu(B) \geqq 1/i\}$. We will prove that $n(\mathscr{F}^i) = 1/i$. Assume to the contrary that there is a finite string of elements from $\mathscr{F}^i$, $G = ((\chi_{A_1}, \chi_{B_1}), \ldots, (\chi_{A_n}, \chi_{B_n}))$ such that $|(1/n)\sum_{j=1}^{n} \chi_{A_j}(\omega) - \chi_{B_j}(\omega)| < 1/i$ for all $\omega \in \Omega$.

Hence, $(1/i)\chi_\Omega(\omega) > |(1/n)\sum \chi_{A_j}(\omega) - (1/n)\sum \chi_{B_j}(\omega)| \geqq |(1/n)\sum \chi_{A_j}(\omega)| - |(1/n)\sum \chi_{B_j}(\omega)| = (1/n)\sum \chi_{A_j}(\omega) - (1/n)\sum \chi_{B_j}(\omega)$. Thus, $(1/n)(\sum \chi_{A_j}(\omega) - \chi_{B_j}(\omega)) < 1/i\chi_\Omega(\omega)$. By integrating both sides with respect to μ we get $1/i < 1/i\mu(\Omega)$, a contradiction. The same technique is applied to show that $n(\mathscr{F}^i \cup \tilde{\mathscr{F}}_{1/i}) = 1/i$. The proof of the other direction is given by Theorem 1 and Lemma 1. Q.E.D.

Remark 1. By the proof of Theorem 2 it is clear that for the sequence $\{\mathscr{F}^i\}$ it can be required that it is an increasing sequence, i.e., $\mathscr{F}^i \subseteq \mathscr{F}^{i+1}$, $i = 1, 2, \ldots$.

Corollary 1. If Ω is finite then there is an $\mathscr{F}$-representation of $\succsim$ by a PFAM *iff*

$$n(\mathscr{F} \cup \tilde{\mathscr{F}}_{n(\mathscr{F})}) = n(\mathscr{F}) > 0.$$

Proof. Since Ω is finite, $\mathscr{F}$ is also finite. By Remark 1, one of the $\mathscr{F}_i$ should be $\mathscr{F}$ and the corollary follows.

3. Alternative characterization

3.1. Alternative characterization of preferences

We will present here an alternative way to characterize the preferences that are presentable by PFAM. We will use here a separability axiom, used first by Debreu (1964) to characterize the preferences that are presentable by a utility measure.

Definition 2. An order, $\succsim$, defined on $\mathscr{A}$ is *separable* (S) *by a sequence* $\{A_i\} \subseteq \mathscr{A}$ if for any B, $C \in \mathscr{A}$ s.t. $B \succ C$ there is an A_i which satisfies $B \succsim A_i \succsim C$. To such a sequence we will call a *dense sequence*.

Let $\{A_i\}$ be a dense sequence of the separable order $\succsim$. We can enlarge the sequence by adding the maximal element in $\mathscr{A}$ which is smaller than A_i if there exists such, and the minimal element in $\mathscr{A}$ which is greater than A_i if there is such. By doing so for any i we can get a *maximal dense sequence* $\{A_i'\}$ which satisfies: if $B \succ C$ (both are in $\mathscr{A}$), then there are A_i and A_j s.t. $B \succsim A_i \succ A_j \succsim C$.

Theorem 3. Let $\succsim$ be an order defined on $\mathscr{A}$. $\succsim$ is presentable by a PFAM *iff $\succsim$ is separable by a maximal dense sequence $\{A_i\}$ with the following property: for any pair (A_i, A_j) s.t. $A_i \succ A_j$ there exists a positive number α_{ij} which satisfies*[1] $n((\chi_{A_i}, \chi_{A_j}) \cup \mathscr{F}_{\alpha_{ij}}) = \alpha_{ij}$.

Proof. Necessity. If $\succsim$ is representable by μ then there is a maximal dense sequence $\{A_i\}$. Let $\alpha_{ij} = \mu(A_i) - \mu(A_j)$ for any $A_i \succ A_j$. By the proof of Theorem 2 we know that $n(\{(\chi_A, \chi_B) | \mu(A) - \mu(B) \geqq \alpha_{ij}\} \cup \tilde{\mathscr{F}}_{\alpha_{ij}}) = \alpha_{ij}$. However, the left side is smaller than or equal to $n((\chi_{A_i}, \chi_{A_j}) \cup \tilde{\mathscr{F}}_{\alpha_{ij}})$, which is obviously smaller than or equal to α_{ij}. Therefore,

$$n((\chi_{A_i}, \chi_{A_j}) \cup \tilde{\mathscr{F}}_{\alpha_{ij}}) = \alpha_{ij} > 0.$$

Sufficiency. For a fixed i and j, let $C = \operatorname{conv}(\{\chi_{A_i} - \chi_{A_j}\} \cup \tilde{\mathscr{F}}_{\alpha_{ij}})$. By the assumption $0 < \alpha_{ij} = \inf_{g \in C} \|g\|$. Define D as the open ball with radius α_{ij} around the origin. We can separate by a PFAM, say μ_{ij}, between C and D. By a similar argument that in the proof of Theorem 1, we infer that μ_{ij} is non-negative. Moreover, μ_{ij} is a (χ_{A_i}, χ_{A_j})-representation. Since $\{A_j\}$ is a maximal dense sequence, any positive combination of all μ_{ij} is a $\mathscr{F}$-representation.

Remark 2. The condition in the previous theorem can be written down without utilizing the maximal dense sequence. We can say that $\succsim$ is representable by a PFAM iff $\succsim$ is separable and if for any $A \succ B$ (in $\mathscr{A}$) there exists a positive number $\eta(A, B)$ which satisfies: $n((\chi_A, \chi_B) \cup \tilde{\mathscr{F}}_{\eta(A,B)}) = \eta(A, B)$. This formulation is similar to that presented in Chateauneuf (1985).

3.2. Another proof of Chateauneuf's theorem

We will use the notation and terminology used by Chateauneuf (1985). We will assume that $\succsim$ is defined on all the algebra $\mathscr{B}$.

Definition 3. We say that an order $\succsim$ is *weakly Archimedian* (WA) if for any $A \succ B$, $A, B \in \mathscr{B}$, there exists a positive integer $n(A, B)$ such that

[1] We identify a singleton with its single element.

$$k\chi_{\Omega} - n(\chi_A - \chi_B) = \sum_{i \in I} (\chi_{C_i} - \chi_{D_i}), \tag{2}$$

where k, n, $|I|$ are integers, and $C_i \succsim D_i$ for all $i \in I$ implies

$$k/n > 1/n(A, B). \tag{3}$$

In words, in any rational convex combination of functions $\chi_{C_i} - \chi_{D_i}$ and $\chi_A - \chi_B$ which is equal to χ_{Ω} the coefficient of $\chi_A - \chi_B$ is bounded by $n(A, B)$.

Definition 4. The order $\succsim$ is well bounded (WB) if any $C \in \mathscr{B}$ satisfies $\Omega \succsim C \succsim \varnothing$ and $\Omega \succ \varnothing$.

Proposition 1. Let $\succsim$ be a (WB) *order defined on $\mathscr{B}$ and let $A \succ B$. If $\succsim$ is* (WA), *then $n((\chi_A, \chi_B) \cup \bar{\bar{\mathscr{F}}}_{\eta}) = \eta$, where $\eta = 1/n(A, B)$.*

Proof. See appendix.

Now we are ready to prove Chateauneuf's theorem:

Theorem 4 [see Chateauneuf (1985)]. *An order $\succsim$ is representable by* PFAM *iff it is* (WB), (WA), *and* (S).

Proof. We will prove sufficiency. Assume that $\succsim$ satisfies all the conditions. Take a maximal dense sequence $\{A_j\}$. By Proposition 1 $n((\chi_{A_i}, \chi_{A_j}) \cup \bar{\bar{\mathscr{F}}}_{1/n(A_j, A_j)}) = 1/n(A_i, A_j)$ for every $A_i \succ A_j$. Therefore, Theorem 3's conditions are satisfied. Thus, $\succsim$ is representable by a PFAM.

4. Two examples

In the following examples we have a situation described as follows. A lot owned by a lessor is offered to a lessee for holding t periods of time. Not only the entire lot I is offered (say, the unit interval), but also its parts. The potential lessee should decide on the combination (A, t), i.e., the portion A of the lot to be held on lease from time 0 to time t, that he wants to hold.

Formally, we have a preference relation defined on the collection $\mathscr{A} = \{(A, t) = A \times [0, t]\}$ and not on an entire algebra. For the sake of simplicity we will say that $t \in [0, 1]$. The first example demonstrates the role of the characteristic function and not the power of the theorems. The second example exhibits how to extend the domain of a partial order, and it utilizes the technique formerly presented.

Example 1. Consider a case where only four offers are available: $(I,1)$, (A,t), (A,s), and (B,t), where $A \subsetneqq B$ and $t<s$. The preferences of the lessee are given by: $(I,1)$ is strictly preferred over any other offer, $(B,t)\succ(A,t)$ and all other pairs of offers are equivalent. Is this order, $\succ$, representable? The answer is, No. A simple calculation shows that $n(\mathscr{F})=1/2$. We will prove that $n(\mathscr{F}\cup\tilde{\mathscr{F}}_{1/2})<1/2$. By Corollary 1 it implies that the order, $\succ$, is not representable.

$(\chi_{(B,t)},\chi_{(A,t)})\in\mathscr{F}$ and both $(0.5\chi_{(I,1)}+\chi_{(A,s)},\chi_{(B,t)})$ and $(0.5\chi_{(I,1)}+\chi_{(A,t)},\chi_{(A,s)})$ are in $\tilde{\mathscr{F}}_{1/2}$. Thus, the norm of the function $f=(1-\alpha-\beta)(\chi_{(B,t)}-\chi_{(A,t)})+\alpha(0.5\chi_{(I,1)}+\chi_{(A,s)}-\chi_{(B,t)})+\beta(0.5\chi_{(I,1)}+\chi_{(A,t)}-\chi_{(A,s)})$ is at least $n(\mathscr{F}\cup\tilde{\mathscr{F}}_{1/2})$ for all non-negative α and β satisfying $\alpha+\beta\leqq 1$. However, if, for instance, $\alpha=\beta=1/3$, then f is equal to $1/3$. Thus, $1/3=\|f\|\geqq n(\mathscr{F}\cup\tilde{\mathscr{F}}_{1/2})$, and the order $\succ$ is not representable.

Example 2. We will define a partial order on the collection $\{(A,t)\}$ and we will see that by using our technique we can extend it to be a complete order. Let $\{\mathscr{P}_i\}$ be an increasing ($\mathscr{P}_{i+1}$ refines $\mathscr{P}_i$) sequence of finite partitions of I^2. Divide any $\mathscr{P}_i$ into two sets of atoms: $\mathscr{A}_i$ and $\mathscr{B}_i$ in such a way that any atom C of $\mathscr{P}_{i-1}$ intersects $\bigcup\mathscr{A}_i$. Precisely, $\mathscr{P}_i=\mathscr{A}_i\cup\mathscr{B}_i$, $\mathscr{A}_i\cap\mathscr{B}_i=\emptyset$, and for any $C\in\mathscr{P}_{i-1}$ there is a $\emptyset\neq C'\in\mathscr{A}_i$ s.t. $C'\subseteq C$. In words, a certain part of C is missing in $\mathscr{B}_i$. Define now a partial order. Say that $(B,s)\succ(A,t)$ if (1) $\chi_{(B,s)}$ is $\mathscr{P}_{i-1}$ measurable and (2) (A,t) is a union of atoms from $\mathscr{B}_i$. In other words, (B,s) is a union of atoms from $\mathscr{P}_{i-1}$, and (A,t) is a union of atoms from $\mathscr{B}_i$, and thus there exists a non-void set C, such that $C\subseteq(B,s)\setminus(A,t)$. The question is whether this partial order can be extended to all the measurable sets of I^2. The answer is, Yes.

Define $\mathscr{F}^i=\{((B,s),(A,t))\,|\,\chi_{(B,s)}$ is $\mathscr{P}_i$ measurable and $(B,s)\succ(A,t)\}$. Notice that in the definition of the partial order, $\succ$, we made sure (by missing a part of the $\mathscr{P}_i$-atoms in $\mathscr{B}_{i+1}$) that if (B,s) is a union of $\mathscr{P}_i$-atoms it cannot be covered by the union of all those sets (A,t) that are less preferred than it. For any $(\chi_{(B,s)},\chi_{(A,t)})\in\mathscr{F}^i$ there exists an atom $C\in\mathscr{P}^{i+1}$ s.t. $C\subseteq(B,s)\setminus(A,t)$. Thus, for any finite string of length ℓ, of elements from $\mathscr{F}^i$, $G=((\chi_{(B_j,s_j)},\chi_{(A_j,t_j)}))_{j=1}^{\ell}$, there exists an atom $C\in\mathscr{P}_{i+1}$ s.t. $|\{j\,|\,C\subseteq(B_j,s_j)\setminus(A_j,t_j)\}|\geqq\ell/|\mathscr{P}_i|$ ($=\ell$ divided by the number of atoms in $\mathscr{P}_i$). Therefore, $\lambda=n(\mathscr{F}^i)\geqq 1/|\mathscr{P}_i|>0$. Define $\tilde{\mathscr{F}}=\bigcup\mathscr{F}^i$. In the same fashion it can be confirmed that $\lambda=n(\mathscr{F}^i\cup\tilde{\mathscr{F}}_\lambda)$.

By applying our technique one can get a PFAM μ_i s.t. for any j and $(\chi_{(B,s)},\chi_{(A,t)})$ in $\mathscr{F}^j$, $\mu_i(B,s)\geqq\mu_i(A,t)$ with strict inequality whenever $j=i$. Any positive convex combination of μ_i, say μ, is an $\tilde{\mathscr{F}}$-presentation. The last step is to extend the partial order $\succ$ by using the measure μ, and saying that $D \succ D'$ (both are measurable sets in I^2) if $\mu(D) > \mu(D')$.

5. Application to game theory

A game in coalitional form is a triple $(\Omega, \mathscr{B}, v)$, where Ω is the set of players, $\mathscr{B}$ is a set of subsets of Ω (a set in $\mathscr{B}$ is called coalition), and v is a function $v: \mathscr{B} \to \mathbb{R}$, where $v(\emptyset) = 0$. Theorems 2 and 3 give two characterizations of those games which can be written as $v = f \circ \mu$, where f is a monotonic function. We can define an order $\succsim$ on the set $\mathscr{B}$ as follows: $S \succsim T$ if $v(S) \geqq v(T)$. If μ represents this order, then the function $\mu(S) \to v(S)$ is monotonic on the range of μ.

An interesting case is the case where $|\Omega| < \infty$, and $\mathscr{B} = 2^{\Omega}$. Here we can apply Corollary 1.

6. An explicit formula for the measure

In this part we also use the maximum norm of functions that are averages of characteristic functions. This number was used first by Kelly (1959) who characterized those algebras $\mathscr{A}$ such that there exists a measure μ defined on $\mathscr{A}$ that satisfies $\mu(A) > 0$ for every $A \in \mathscr{A}$. A variation of the same number was used also by Einy and Lehrer (1989) who applied it to cooperative game theory.

Savage's Theorem [Savage (1954)] characterizes those relations on an algebra of subsets which can be represented by a non-atomic probability measure. We provide here an explicit formula for this measure given the weak relation, $\succsim$, defined on an algebra $\mathscr{A}$ of subsets of Ω.

Notation. Let $A \in \mathscr{A}$. Denote

$$\sigma(A) = \inf_{\mathscr{E}} 1/|\mathscr{E}| \, \|\textstyle\sum_{A' \in \mathscr{E}} \chi_{A'}\|_{\infty},$$

where the infimum is taken over all $\mathscr{E} = \{A_1, \ldots, A_n | A_i \succsim A$ for all $i\}$. In words, for any $\mathscr{E}$ as above, $\|\sum_{A \in \mathscr{E}} \chi_A\|_{\infty}$ is the maximal number of sets from $\mathscr{E}$ that have a non-void intersection. The following theorem states that if $\mu(A)$ is a non-atomic probability measure that represents $\succsim$, then $\mu(A)$ is equal to $\sigma(A)$, which is the infimum over all the $\mathscr{E}$'s of the relative maximal number of sets from $\mathscr{E}$ which have non-void intersection.

Theorem 5. Let $\succsim$ be a complete relation defined on the algebra $\mathscr{A}$ of *subsets of Ω. which has a representation by the non-atomic probability measure μ. Then, $\mu(A)=\sigma(A)$.*

A simple and known corollary is the following:

Corollary 2 [Savage (1954)]. *If $\gtrsim$ has a representation by a non-atomic probability measure, then this measure is unique.*

Proof of the Theorem 5. Let μ be a non-atomic measure that represents $\gtrsim$, and let $A\in\mathscr{A}$. If $\mu(A)$ is a rational number, say $\mu(A)=p/q\leqq 1$, then divide Ω into q pairwise disjoint subsets $B_1,\dots,B_q$, each of which with probability $1/q$. It can be done because μ is non-atomic. Define $A_i=B_i\cup\cdots\cup B_{i+p-1}$, where if $i+p>q$ then $i+p$ represents $i+p \pmod q$, and let

$$\mathscr{E}=\{A_1,\dots,A_q\}.$$

By the definition, B_i is contained only in $A_{i-p+1},A_{i-p+2},\dots,A_i$ for every $1\leqq i\leqq q$. Thus,

$$1/|\mathscr{E}|\,\|\textstyle\sum_{i=1}^{q}\chi_{A_i}\|_\infty=p/q.$$

Therefore, $\sigma(A)\leqq\mu(A)$ for all A whose measure $\mu(A)$ is rational. Since μ is a probability measure (in particular a positive measure), if $C\gtrsim B\supseteq A$ $(A,B,C\in\mathscr{A})$ then $C\gtrsim A$. Therefore, the infimum is taken in the definition of $\sigma(A)$ over a broader collection than in the definition of $\sigma(B)$. Hence, $\sigma(A)\leqq\sigma(B)$. Thus, for all $A\in\mathscr{A}$ if $\mu(A)$ is irrational $\sigma(A)$ is less than any rational number which is greater than $\mu(A)$. Therefore $\sigma(A)\leqq\mu(A)$ for all $A\in\mathscr{A}$.

In order to show that $\mu(A)\leqq\sigma(A)$ for all $A\in\mathscr{A}$, assume, to the contrary, that there is $\mathscr{E}=\{A_1,\dots,A_n|A_i\gtrsim A \text{ for all } i\}$ so that

$$1/|\mathscr{E}|\left\|\sum_{i=1}^{n}\chi_{A_i}\right\|_\infty<\mu(A). \tag{4}$$

Define the following two functions:

$$f(\omega)=1/|\mathscr{E}|\sum_{i=1}^{n}\chi_{A_i}(\omega)\quad\text{and}\quad g(\omega)=\mu(A)\chi_\Omega(\omega).$$

By (4) we have

$$f(\omega)<g(\omega)\quad\text{for}\quad\text{all } \omega\in\Omega. \tag{5}$$

Integrate both sides of (5) with respect to μ and get (since μ represents $\gtrsim$):

$$\mu(A) \leqq 1/|\mathscr{E}| \sum_{i=1}^{n} \mu(A_i) < \mu(A).$$

By this contradiction, we get $\sigma(A) = \mu(A)$ as desired. Q.E.D.

Appendix

Proof of Proposition 1. Clearly, $n((\chi_A, \chi_B) \cup \tilde{\mathscr{F}}_\eta) \leqq \eta$. It remains to prove the inverse inequality. Take an arbitrary rational convex combination

$$f = \sum_{i=1}^{m} \alpha_i(\eta\chi_\Omega + \chi_{C_i} - \chi_{D_i}) + \beta(\chi_A - \chi_B),$$

where $\alpha_1, \ldots, \alpha_m$, β are positive rational numbers that sum up to 1. We have to show that $\|f\| \geqq \eta$.

$f \leqq 1$ therefore $g = f + (1 - \|f\|)\chi_\Omega$ also satisfies $g \leqq 1$. Since g is a step function, $h = \chi_\Omega - g$ is also a step function. Thus, it can be written as a positive combination (not necessarily convex) of characteristic functions: $h = \sum_{j=1}^{k} \gamma_j \chi_{E_j}$, where γ_j are positive rational numbers. We conclude that $\chi_\Omega = f + (1 - \|f\|)\chi_\Omega + h$. Hence, $\|f\|\chi_\Omega = f + h = \sum \alpha_i(\eta\chi_\Omega + \chi_{C_i} - \chi_{D_i}) + \sum \gamma_j \chi_{E_j} + \beta(\chi_A - \chi_B)$. Thus.

$$(\|f\| - \eta \textstyle\sum \alpha_i)\chi_\Omega = \sum \alpha_i(\chi_{C_i} - \chi_{D_i}) + \sum \gamma_j \chi_{E_j} + \beta(\chi_A - \chi_B). \tag{A.1}$$

The latter is a combination allowed in (2), because $C_i \gtrsim D_i$ and $E_j \gtrsim \varnothing$ [recall that $\gtrsim$ is defined on $\mathscr{B}$ and it is (WB)].

Claim. $\|f\| - \eta \sum \alpha_i > 0$.

Proof of the Claim. Otherwise, we can subtract $(\|f\| - \eta \sum \alpha_i)\chi_\Omega$ from both sides and get, by renaming the sets and the coefficients, $0 = \sum \delta_\ell(\chi_{F_\ell} - \chi_{H_\ell}) + \beta(\chi_A - \chi_B)$, where δ_ℓ are rational numbers and $F_\ell \gtrsim H_\ell$. We can multiply all the coefficients by any positive number c. We can also add χ_Ω to both sides to obtain:

$$\chi_\Omega = \sum c\delta_\ell(\chi_{F_\ell} - \chi_{H_\ell}) + c\beta(\chi_A - \chi_B) + \chi_\Omega.$$

Since $\Omega \gtrsim \varnothing$ we can use (3) and deduce that $c\beta < n(A, B)$. As $\beta > 0$ and c is an arbitrary positive number, we get a contradiction. This concludes the proof of the claim.

We return to (A.1) and use again (3) in order to infer that the coefficient of $(\chi_A - \chi_B)$ which is $\beta/(\|f\| - \eta \sum \alpha_i)$ is at most $n(A, B) = 1/\eta$. Since $\beta = 1 - \sum \alpha_i$, we get $\|f\| \geqq \eta$, as desired. Q.E.D.

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