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Signaling and mediation in games with common interests

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ABSTRACT

Players who have a common interest are engaged in a game with incomplete information. Before playing they get differential stochastic signals that depend on the actual state of nature. These signals provide the players with partial information about the state of nature and may also serve as a means of correlation.
Different information structures induce different outcomes. An information structure is *better than* another, with respect to a certain solution concept, if the highest solution payoff it induces is at least that induced by the other structure. This paper characterizes the situation where one information structure is better than another with respect to various solution concepts: Nash equilibrium, strategic-normal-form correlated equilibrium, agent-normal-form correlated equilibrium and belief-invariant Bayesian solution. These solution concepts differ from one another in the scope of communication allowed between the players. The characterizations use maps that stochastically translate signals of one structure to signals of another.

1. Introduction

In Bayesian games the amount of information players obtain about the actual payoffs crucially affects the outcome. This paper investigates how changes in information may influence the outcome of an interaction.

It has been noticed that one or more players obtaining better information about the state of nature (i.e., the actual game) does not necessarily mean that their payoffs are improved. Hirshleifer (1971) showed that in some circumstances better public information may reduce economic welfare. In the lemon market (see, Akerlof, 1970), providing the seller with private information might render any trade impossible, and thereby reduce social welfare. Neyman (1991) argued that the source of this phenomenon is that any improvement in the information that some players obtain generates a change (and not necessarily an improvement) in all the knowledge of all the others: the latter know that the former know more. Bassan et al. (2003) characterized the games in which getting more information about the state of the world (i.e., about the entire hierarchy of beliefs) always improves all players' payoffs. The class of games they identified contains the games with common interests.

The issue of whether or not information has a positive value is related to a broader subject: comparison of information structures based on the outcomes they induce. An information structure specifies the distribution over signals that each player receives conditional on any particular state of nature.

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Blackwell (1953) was the first to compare information structures in one-player games, that is, in one-person decision problems. Blackwell defined a partial order over information structures as follows. An information structure is said to be *better than* another, if in *any* decision problem the optimal value associated with the one structure is higher than the optimal value associated with the other. An information structure is said to be *more informative* than another if the signals provided by the latter can be reproduced (by using a map, called *garbling*) from the signals provided by the former. Blackwell proved that an information structure is better than another if and only if it is more informative than the other. This result demonstrates that in one-player decision problems, information has a positive value: a single agent would always prefer more information to less.

Matters are more intricate in multi-player interactions:

– First, the signal each player obtains contains more than just information about the state of nature. These signals may be correlated across players, and may partially convey what other players know about the game being played, and what they know about what others know about the actual game, etc. A particular information structure affects the outcome of the game by the direct information it provides about the state of nature, by information it provides about other players' information, and by correlations that might exist between the players' signals.
– Second, the outcome of a game also depends on the type of mediation the players may resort to. Nash equilibrium, for instance, needs no correlation device. On the other hand, all sorts of extensions of correlated equilibrium (Aumann, 1987; Forges, 1993) to Bayesian games do require some kind of mediation. Each solution concept induces a different set of outcomes, and in turn, results in a different criterion to compare information structures.
– Third, for any solution concept there is typically multiple outcome, and comparing sets of outcomes can be done in more than one reasonable way. For any solution concept one could define the notion of being a better outcome by solely referring to the payoff of one player, or to that of a subset of players, or simultaneously to the payoffs of all players, as we do here.
– Finally, there is more than one way to extend the definition of "being more informative than". Each way depends on the extent to which the particular garbling used changes one player's information about that of the others.

In this paper we extend Blackwell's comparison to games with common interests. There are several reasons for dealing separately with games with common interests. First, these games are closest in nature to one-player decision problems, in the sense that players would like to behave as one player if they could share their information perfectly and coordinate their actions. Second, the connection between existing non-cooperative solution concepts of Bayesian games and information structures is clearest in such games. Third, the feasible set of a general Bayesian game is determined by the games generated by identifying linear combinations of the players' payoffs with the payoffs of a common interest game. Finally, unlike the general case, in games with common interests there is *one* outcome, which is mostly desired by all players and can be regarded as the 'value' of the game.[1]

We say that one information structure is *better than* another with respect to (w.r.t.) Nash equilibrium if, in any game with identical payoffs, it ensures a Nash equilibrium payoff which dominates any Nash equilibrium payoff of the other. It turns out that one structure is better than another, if it is derived (a) by adding a public randomizing device; and (b) based on public signal, the players individually garble the information of the other. This procedure produces a garbling, called *coordinated*. This result implies, in particular, that under the more informative structure each player knows no more than under the less informative structure about what the other player knows. However, the converse is not true.

We extend this result to solution concepts different from Nash equilibrium. We refer to solution concepts that require some sort of communication between the players, such as correlated equilibrium, agent-normal-form correlated equilibrium, Bayesian solution and communication equilibrium (see Forges, 1993). These equilibrium concepts differ from one another in the amount of players' private information that the communication device is allowed to use.[2] For each of these concepts we provide a complete characterization of the information structures that are better than a given one. All the characterizations are stated in terms of maps, called garblings, that enable the players to reconstruct the signals of the inferior information structure from the signals of the superior one. Each solution concept allows a certain type of garblings. The specific type of garblings allowed by a certain solution concept reflect the amounts of correlation and communication between the players that this concept allows for.

It turns out that one information structure is better than another w.r.t. correlated and agent-normal-form correlated equilibria precisely when this is true w.r.t. Nash equilibrium. The question of when one information structure is better than another w.r.t. to the belief-invariant Bayesian solution is answered by means of a special kind of garbling, called

[1] Zero-sum games own a similar feature: there is a number, the value, that represents the outcome of the game.

[2] In a *correlated equilibrium* a mediator chooses a pair of strategies (of the Bayesian game), one for each player, and tells each player his selected strategy. In an *agent-normal-form correlated equilibrium* any player has many agents, one for each of his signals. A mediator chooses randomly an action for each player's agent and tells a player only the action selected for the specific signal he received. A *Bayesian solution* is described in terms of a larger state space than that containing the states of nature. In any state of the world each player knows a set of states that contains the actual one. This knowledge represented by a partition of the state space (in the spirit of Aumann, 1987), captures all he knows about the world, including his own private signal and action. Subject to this knowledge, each player plays his best response. In *communication equilibrium* players report to the mediator about their signals and then the mediator chooses randomly a recommended action for each player.

non-communicating, that translates one information structure into another without providing the players more information about each other's information or about the state of nature. We prove that a structure is better than another w.r.t. the belief-invariant Bayesian solution if the first is a garbled version of the other using non-communicating garbling. Finally, a structure in better than another w.r.t. to communication equilibrium if the one is a garbled version (with no restriction) of the other.

These results provide a complete picture of the relations between the amount of communication involved in each solution concept and the type of garbling used when comparing information structures according to the outcomes induced by this concept. For the sake of simplicity we state the model and the results in the case of two players. All the results readily extend to more-than-two-player games with common interests.

Gossner (2000) defined another partial order over information structure in n-person games. Despite the differences between the approaches, we use Gossner's result in the companion paper, Lehrer et al. (2007).

The paper is organized as follows. The next section defines the model of Bayesian games and information structures. Comparison of information structure w.r.t. Nash equilibrium is provided in Section 3, where we introduce the notion of 'being better than', and characterize the situation when an information structure is better than another w.r.t. Nash equilibrium. Section 4 is devoted to the extensions of this result to other solution concepts. We review the various concepts and we characterize when an information structure is better than another w.r.t. each one. The proofs are given in Section 5. Section 6 contains a few final comments.

2. Information structures and games

Two players participate in a finite Bayesian game. A *state of nature* k is randomly drawn from a set K according to a known distribution p. The players are not directly informed of the realized state. Rather, each player receives a stochastic signal that depends on k. The signals that the players receive are typically correlated.

Formally, an *information structure* consists of two finite sets of signals, S, T, and a function[3] $\sigma : K \to \Delta(S \times T)$ that assigns to every state of nature a joint distribution over signals. When the realized state is k, player 1 obtains the signal s and player 2 obtains the signal t with probability $\sigma(k)[s,t]$, which we usually denote $\sigma(s,t|k)$. Information structures will be referred to as triples of the kind (S,T,σ) and will be denoted by $\mathcal{I}$.

At the beginning of the game, a state $k \in K$ is chosen according to p and then the players receive signals according to σ. Upon receiving a signal, each player takes an action and receives a payoff that depends on both players' actions and on the state of nature. Formally, let A be player 1's finite set of actions and B be that of player 2. Note that these sets are common to all states. If the state is k, player 1 plays a and player 2 plays b, then the payoff player i receives is $r_k^i(a,b)$. A *strategy* x of player 1 assigns a mixed action to every signal in S. When player 1 plays according to strategy x, the action $a \in A$ is played with probability $x(a|s)$ if he observes the signal s. A strategy y of player 2 is defined in a similar manner.

It is worth noting that the players hold a common prior p over K before receiving any signal. This common prior is not a part of the information structure.

The expected payoff of player i when the strategy profile (x,y) is played is therefore,

$$r^i(x,y) = \sum_{k\in K} p(k) \sum_{(s,t)\in S\times T} \sigma(s,t|k) \sum_{(a,b)\in A\times B} x(a|s)y(b|t)r_k^i(a,b).$$

In this paper we focus on the class of games with common interests.

Definition 2.1. The game is with *common interests* if for any $k \in K$, $r_k^1 = r_k^2$.

Common interest games are called *team games* in Marschak and Radner (1972).

3. Comparison of information structures and equilibria

3.1. Better information structures w.r.t. Nash equilibrium

This section is devoted to the comparison of information structures as far as *Nash equilibrium* is concerned. The strategy profile (x,y) is a Nash equilibrium if no player has an incentive to deviate. That is, $r(x,y) \geqslant r(x',y)$ and $r(x,y) \geqslant r(x,y')$ for all strategies x' and y' (recall that r is the payoff to each player).

In two player zero-sum games, there is a unique Nash equilibrium payoff, the value. From the maximizer's point of view, an information structure can be said to be *better than* another in the class of zero-sum games if it induces a higher value in any game in this class. However simple to describe, the question of characterizing the information structures that are better than a given one in the class of zero-sum games is still open.

It is obvious that, in complete information games with common interests, the pair of actions that gives the maximal payoff to the player is a Nash equilibrium. A Bayesian game can also be formulated as a complete information normal-form

[3] $\Delta(D)$ denotes the set of distributions over a set D.

game. This means that in a Bayesian game with common interests there is a Nash equilibrium expected payoff that is maximal among all feasible expected payoffs. Therefore, in games with common interests, although there is typically more than one Nash equilibrium payoff, one is clearly more appealing than the others.

Since the set of Nash equilibria and the set of Nash equilibrium payoffs are typically not singletons, there are *a priori* various ways to extend the notion of 'being better than'. The existence of a Pareto dominating equilibrium enables us to choose the following definition from among these possible notions. Since we later discuss solution concepts other than Nash equilibrium it is written in general terms.

Definition 3.1. Fix an equilibrium concept and let $\mathcal{I}, \mathcal{I}'$ be two information structures. $\mathcal{I}$ is *better than* $\mathcal{I}'$ w.r.t. this equilibrium concept in common interest games if, for any common interest game, every equilibrium payoff under $\mathcal{I}'$ is weakly Pareto dominated by an equilibrium payoff under $\mathcal{I}$.

An example of a structure that is better than another w.r.t. Nash equilibrium in common interest games is provided in Example 3.7.

Our first result characterizes when one structure is better than another w.r.t. Nash equilibrium in common-interest games. The characterization hinges upon the notion of garbling, used first by Blackwell (1953).

3.2. Garbling of information

Let $\mathcal{I} = (S, T, \sigma)$ be an information structure. Suppose that a joint signal (s, t) in $S \times T$ is produced (i.e., is randomly selected according to σ). However, instead of sending signals to the players, a pair of new signals, say (s', t'), is randomly selected from new sets of signals, say S' and T', according to a distribution $q(s, t)$. Players 1 and 2 are then informed of s' and t' respectively. This procedure generates a new information structure, $\mathcal{I}' = (S', T', \sigma')$, which is said to be a garbled version of $\mathcal{I}$. Formally,

Definition 3.2. Let $\mathcal{I} = (S, T, \sigma)$ and $\mathcal{I}' = (S', T', \sigma')$ be two information structures. $\mathcal{I}'$ is a *garbled version* of $\mathcal{I}$ if there is a map q from[4] $S \times T$ to $\Delta(S' \times T')$ such that the distribution induced by the composition $q \circ \sigma$ coincides with[5] σ'.

The map q is called a *garbling* that transforms $\mathcal{I}$ to $\mathcal{I}'$.

In the one-player case, Blackwell (1953) defined $\mathcal{I}$ as being *more informative* than $\mathcal{I}'$ if there exists a garbling that transforms $\mathcal{I}$ into $\mathcal{I}'$. He then proved that $\mathcal{I}$ is better than $\mathcal{I}'$ if $\mathcal{I}$ is more informative than $\mathcal{I}'$.

Let us now move to two-player games. Imagine a fictitious agent who knows the signals received by both players. Note that if $\mathcal{I}'$ is a garbled version of $\mathcal{I}$, then this agent would be better informed, in the sense of Blackwell, getting the signal through $\mathcal{I}$, than through $\mathcal{I}'$.

While in a one-player decision problem the signal may only convey some information about the actual state, in games things are much more involved. A signal contains not only information about k, but also about the other player's information about k, and about the other's information about his own information about k and so forth. Moreover, the signals the players receive may be correlated, which may, in turn, enrich the set of possible outcomes. This explains why only specific garblings will be used to characterize the information structures that are better than a given one.

Definition 3.3. (i) A garbling $q : S \times T \to \Delta(S' \times T')$ is said to be *independent* if there are maps $q_1 : S \to \Delta(S')$ and $q_2 : T \to \Delta(T')$ such that for every s, t, s', t',

$$q(s', t'|s, t) = q_1(s'|s) \cdot q_2(t'|t).$$

(ii) A garbling q is *coordinated* if it is in the convex hull of independent garblings, that is,

$$q = \sum_i \lambda(i) q^i,$$

with $q^i : S \times T \to \Delta(S' \times T')$ being an independent garbling and $\lambda(i) \geqslant 0$ for every i, and $\sum_i \lambda(i) = 1$.

Note that independent garbling can be implemented without any mediation or communication between the players (each player manipulates his signal independently of the other) and a coordinated garbling can be implemented by a public signaling which is independent of the players' signals. We are not aware of a simple method that checks whether an information structure can be obtained by a coordinated garbling of another.

[4] We still denote by q the linear extension of q to a function from $\Delta(S \times T)$ to $\Delta(S' \times T')$.

[5] That is, for every k and every $(s', t') \in S' \times T'$, $\sigma'(s', t'|k) = \sum_{(s,t) \in S \times T} \sigma(s, t|k) q(s', t'|s, t)$, where $q(s', t'|s, t)$ is the probability that the output signals will be s', t' given the input signals s, t.

Example 3.4. Let $S = T = S' = T' = \{0, 1\}$. A garbling will be denoted as a $(S \times T) \times (S' \times T')$ matrix. Let

$$q_1 = \begin{pmatrix} 4/9 & 2/9 & 2/9 & 1/9 \\ 2/9 & 4/9 & 1/9 & 2/9 \\ 2/9 & 1/9 & 4/9 & 2/9 \\ 1/9 & 2/9 & 2/9 & 4/9 \end{pmatrix}, \qquad q_2 = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix},$$

where the rows and the columns correspond to the possible pairs of signals ordered from top to bottom (or from left to right) as $(0, 0), (0, 1), (1, 0), (1, 1)$.

The garbling q_1 takes a joint signal in $S \times T$ and garbles it to a joint signal in $S' \times T'$. For instance, the signal $(0, 1)$ is garbled by q_1 to $(0, 0)$ with probability 2/9, to $(0, 1)$ with probability 4/9, to $(1, 0)$ with probability 1/9, and to $(1, 1)$ with probability 2/9. q_1 is the product of two garblings of player 1 and player 2, both are given by the matrix $\left(\begin{smallmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{smallmatrix}\right)$. The garbling q_1 is, therefore, independent. On the other hand, q_2 is coordinated but not independent. It can be written as $q_2 = \frac{1}{2}q + \frac{1}{2}q'$ where q and q' are the independent garblings given by

$$q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad q' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

3.3. A first characterization

Our first result characterizes the situations where one information structure is better than another w.r.t. Nash equilibrium in the class of games with common interests.

Theorem 3.5. *Let $\mathcal{I}$ and $\mathcal{I}'$ be two information structures. $\mathcal{I}$ is Nash-better than $\mathcal{I}'$ in the class of games with common interests if and only if there exists a coordinated garbling that transforms $\mathcal{I}$ to $\mathcal{I}'$.*

The proof of this theorem is postponed to Section 5.

The intuition of this result is as follows. If each player can (independently) mimic $\mathcal{I}'$ by using the signal he got from $\mathcal{I}$, then in any game with common interests the players can ensure in $\mathcal{I}$ whatever they can in $\mathcal{I}'$. Furthermore, in games with common interests, any correlation that does not depend on payoff-relevant information is worthless in the sense that it may not increase the best Nash equilibrium payoff. Thus, even if the players use a public coordination device (which is independent of their information) in order to choose the particular way in which they independently garble their signals, the highest equilibrium payoff does not increase. Therefore, if $\mathcal{I}'$ is a coordinated garbling of $\mathcal{I}$, then $\mathcal{I}$ is better than $\mathcal{I}'$.

Example 3.6. Here we provide two information structures $\mathcal{I}$ and $\mathcal{I}'$ such that $\mathcal{I}'$ is a garbled version of $\mathcal{I}$ but with a garbling that is non-coordinated. Furthermore, we exhibit a game in which there is a Nash equilibrium payoff with $\mathcal{I}'$ that is strictly higher than any Nash equilibrium payoff with $\mathcal{I}$.

Let $K = \{0, 1\}^2$, S and T be equal to $\{0, 1\}$ and let $\mathcal{I}$ be the information structure in which player 1 knows the first coordinate of k and player 2 knows the second coordinate. Formally, for each $k = (s, t) \in \{0, 1\}^2$, $\sigma(s, t|k) = 1$.

Consider the game with common interests in which the prior distribution over K is uniform, the action sets are $A = B = \{0, 1\}$ and payoff functions are given by[6] $r_k(a, b) = (-1)^{a+b+st}$ for every $k = (s, t) \in K$. By examining all pairs of pure strategies in this game, one can verify that the maximal payoff achievable under $\mathcal{I}$ is $\frac{1}{2}$, obtained when each player, regardless of the signal, always plays action 1. Now consider the garbling,

$$q = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

and denote by $\mathcal{I}'$ the garbled version of $\mathcal{I}$ with this garbling.

If the players are informed through $\mathcal{I}'$, and each player plays his signal, then the payoff is $(1, 1)$ (which is the maximal feasible payoff, and therefore an equilibrium). Theorem 3.5 implies that the garbling q is not coordinated. This means, in particular, that although $\mathcal{I}'$ is a garbled version of (S, T, σ), (S, T, σ) is not better than $\mathcal{I}'$ w.r.t. Nash equilibrium.

[6] This game, as well as the related concept of non-communicating garblings, are well known in the quantum physics literature (see, for example Cleve et al., 2004 and Barret et al., 2005).

Example 3.7. In this example we show a game with common interests and two information structures $\mathcal{I}$ and $\mathcal{I}'$ such that $\mathcal{I}$ is better than $\mathcal{I}'$. Consider $K = S = T = S' = T' = \{0, 1\}$ with probability $(1/2, 1/2)$ over K and the following common interest game,

$$\underset{k=0}{\begin{pmatrix} 1,1 & 0,0 \\ 0,0 & 1,1 \end{pmatrix}}, \qquad \underset{k=1}{\begin{pmatrix} -1,-1 & 0,0 \\ 0,0 & -1,-1 \end{pmatrix}}.$$

Let $\mathcal{I}$ be the information structure described by the following signaling matrices

$$\underset{k=0}{\begin{pmatrix} 1/2 & 1/2 \\ 0 & 0 \end{pmatrix}}, \qquad \underset{k=1}{\begin{pmatrix} 0 & 0 \\ 1/2 & 1/2 \end{pmatrix}}.$$

For instance, if $k = 0$, then with probability 1/2 players 1 and 2 receive the signal 0 and with probability 1/2 they receive, respectively, the signals 0 and 1; and if $k = 1$, with probability 1/2 players 1 and 2 receive, respectively, the signals 1 and 0, and with probability 1/2 they receive the signal 1. Thus, in $\mathcal{I}$ player 1 knows k and player 2 knows nothing. With this information structure, if the states are equally probable, then the best equilibrium payoff is 1/2 which can be achieved with player 2 always playing left and player 1 playing top if $k = 0$ and bottom if $k = 1$.

Denote by $\mathcal{I}'$ the garbled version of $\mathcal{I}$ with the garbling q of the previous example. The information structure $\mathcal{I}'$ is

$$\underset{k=0}{\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}}, \qquad \underset{k=1}{\begin{pmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}}.$$

Although the garbling q is not coordinated (see Example 3.6), $\mathcal{I}'$ is a garbled version of $\mathcal{I}$ with a coordinated garbling. Indeed, denote by $\delta_{s_0,s_1;t}$ the information structure under which, with probability 1, player 1 gets the signal s_0 if $k = 0$, the signal s_1 if $k = 1$, and player 2 gets the signal t (independently of k). Obviously, $\delta_{s_0,s_1;t}$ is a garbled version of $\mathcal{I}$ with an independent garbling. Furthermore, $\mathcal{I}' = \frac{1}{4}\delta_{0,0;0} + \frac{1}{4}\delta_{0,1;0} + \frac{1}{4}\delta_{1,1;1} + \frac{1}{4}\delta_{1,0;1}$. Theorem 3.5 implies that $\mathcal{I}$ is better than $\mathcal{I}'$ w.r.t. Nash equilibrium and therefore, the maximal Nash equilibrium payoff under $\mathcal{I}'$ is not greater than 1/2.

This example demonstrates that $\mathcal{I}'$ may be a garbled version of $\mathcal{I}$ with a coordinated garbling and at the same time a garbled version of $\mathcal{I}$ with a non-coordinated garbling.

4. Comparison of information structures with respect to other solution concepts

In this section we discuss various solution concepts that allow for a certain amount of communication between the players before or after obtaining signals. All extend the notion of correlated equilibrium and are equivalent to it when the game is with complete information. They differ in the extend to which correlation and/or communication depends on the signals. The three first notions are defined by means of a mediator, while the fourth is based on an epistemic model. The equivalence between all approaches is proved for games with complete information in Aumann (1987).

In the description of the solution concepts we follow Forges (1993). The reader is referred to her paper for formal definitions and the relation between the various concepts. For each of these concepts we prove an analogous result to Theorem 3.5.

4.1. Solution concepts that require a mediator

In a *normal-form correlated* equilibrium the mediator randomly selects a profile of strategies, (x, y) and then, he tells x to player 1 and y to player 2. The strategies x and y are interpreted as recommendations made by the mediator to the players. Each player then chooses an action as a function of his information and the recommendation of the mediator. The incentive compatible conditions are that no player has an incentive to deviate from the recommended strategy. For instance, in equilibrium, when player 1 has been told x and his signal is s, he plays the action a with probability $x(a|s)$.

Note that here the mediator does not know the signals the players received. In other words, the mediator provides the players with a correlation that is independent of their signals.

In an *agent-normal-form* correlated equilibrium (introduced by Samuelson and Zhang, 1989), the mediator is assumed to *know* the signals received by the players. He then sends recommendations, one to each pair (player i, signal s_i). The recommendation to the pair (player i, signal s_i) is an action recommended to player i when receiving the signal s_i. Note that after receiving the signal s_i player i is not aware of the recommendation he would have received upon getting signal s_i'. One possible interpretation of this mediation scheme is that each player is replaced by a group of agents, one agent for each possible signal. An agent-normal-form correlated equilibrium is a correlated equilibrium of the non-Bayesian game played by the agents. The payoff of a player in the original game is the payoff of the realized agent (see Myerson, 1991 and Forges, 1993 for further comments).

More precisely, the mediator chooses its recommendation in two steps: (i) before knowing the signals of the players it chooses a correlation device, i.e., with probability λ_i it chooses to use the mechanism ε_i in step two; (ii) then after being aware of the signals of the players, the mechanism ε_i chooses independently a recommendation for player 1 with signal s and a recommendation for player 2 with signal t. This solution concept can be interpreted as a normal-form correlated equilibrium in the game in which each pair (player i, signal s_i) is considered a separate player.

A *communication equilibrium* (introduced by Forges, 1986) is implemented by a mediator to which each player is asked to send his signal (but he can of course lie). Then, as a function of the reported signals the mediator sends to each player a recommended action. The equilibrium condition states that no player has an incentive to lie nor to deviate from the recommended action. Note that a communication equilibrium may involve a stronger dependence between the signals of the players and the recommended actions than in the previous equilibrium notions, because the mediator is allowed to use both signals in order to choose the recommendations he makes.

4.2. The epistemic approach

The previous equilibrium notions required the presence of a mediator. They differ one another in the information the mediator has about the private signals. In this section we adopt the epistemic approach introduced in Aumann (1987), and used by Forges (1993), and Bassan et al. (2003). This approach is based on a probability space $(\Omega, \mathbb{P})$, with Ω being rich enough to reflect the state of nature, the signals and the actions of players.

The epistemic model is described by a probability space, $(\Omega, \mathbb{P})$, two partitions $\mathfrak{A}_1, \mathfrak{A}_2$ of Ω and a few random variables over Ω: (i) κ takes values in K (i.e., κ is the state of nature), (ii) ς and τ take values in S and T, resp. (i.e., ς and τ are the signals); and (iii) α and β take values in A and B, resp. (i.e., α and β are the actions). The partitions $\mathfrak{A}_1, \mathfrak{A}_2$ represent the information available to player 1 and player 2, respectively.

A *belief-invariant Bayesian solution*[7] under the information structure $\mathcal{I} = (S, T, \sigma)$ is an epistemic model that satisfies the following conditions:

1. The distribution of κ over K is p and for every $k \in K$, the joint distribution of ς, τ given that $\kappa = k$ is $\sigma(k)$. I.e., the joint distribution induced by κ, ς and τ coincides with the distribution that p and σ induce on $K \times S \times T$.
2. ς, α (resp. τ, β) are $\mathfrak{A}_1$-measurable (resp. $\mathfrak{A}_2$-measurable). I.e., each player knows his signal and action.
3. For every k, a, the signal τ of player 2 completely summarizes his information on player 1's signal.
$$\mathbb{P}(\varsigma = s|\mathfrak{A}_2) = \mathbb{P}(\varsigma = s|\tau).$$
I.e., the information embedded in $\mathfrak{A}_2$ does not give player 2 more knowledge about s than his signal t.
A similar condition holds for player 1.
4. For every k, the joint signals of the players completely summarize their joint information on the state of the world:
$$\mathbb{P}(\kappa = k|\mathfrak{A}_1, \mathfrak{A}_2) = \mathbb{P}(\kappa = k|\varsigma, \tau).$$
5. Incentive compatibility conditions: any deviation of player 1 (resp. 2) from playing α (resp. β) is not profitable. (For a formal expression of this condition (5) the reader is referred to Forges, 1993.)

We say that a distribution π over $K \times A \times B$ can be achieved by a belief-invariant Bayesian solution if π is the joint distribution of κ, α, β in some belief-invariant Bayesian solution.

The following example illustrates the idea behind the epistemic approach and the belief-invariant Bayesian solution.

Example 4.1. Consider the game with common interests and the information structure $\mathcal{I}$ presented in Example 3.6. Recall that under $\mathcal{I}$, the maximal payoff achievable in any pair of pure strategies is $\frac{1}{2}$, which is therefore also the maximal agent-normal-form correlated equilibrium payoff.

Now consider the probability space $\Omega = S \times T \times S' \times T'$ with the following distribution: $\mathbb{P}(s, t, s', t') = \frac{1}{4} \cdot q(s', t'|s, t)$, where q is as in Example 3.7. The knowledge of players and the interrelation between the players' knowledge is captured by this space.

For $\omega = (s, t, s', t') \in \Omega$ let $\kappa(\omega) = (s, t), \sigma(\omega) = s, \tau(\omega) = t, \alpha(\omega) = s', \beta(\omega) = t'$. The variable κ represents the state. Finally, ς is equal to s and τ is equal to t. Suppose that player 1 knows s and s' (this defines $\mathfrak{A}_1$) and player 2 knows t and t' (this defines $\mathfrak{A}_2$). In other words, at the point $\omega = (s, t, s', t')$ the state is (s, t), player 1's signal is s and he plays s' and player 2's signal is t and he plays t'. In particular, player 1 knows one component, s, of the state and player 2 knows the other component, t. One can verify that all the conditions in the definition of a belief-invariant Bayesian solution are satisfied by these items.

Suppose, for instance, that the point realized is $\omega = (0, 1, 1, 1)$. Then, the state is $\kappa(\omega) = 01$; player 1 knows $s = 0$ and his action, $s' = 1$; and player 2 know $t = 1$ and his action, $t' = 1$. From player 1's point of view, given that $s = 0$ and $s' = 1$, the probability of $t' = 1$ is 1, and moreover, the states 00 and 01 are equally likely. Thus, the action 1 is player 1's best response. On the other hand, given player 2's information, with probability $\frac{1}{2}$ the state is 01 and player 1 plays 1 and with probability $\frac{1}{2}$ the state is 11 and player 1 plays 0. To this belief player 2's best response is indeed $t' = 1$. The payoff induced

[7] We use this terminology that was coined by Forges (2006).

by this belief-invariant Bayesian solution is $(1, 1)$. See Forges (2006) for further discussion of this example and extensions of correlated equilibrium.

4.3. Comparison of information structures

Theorem 3.5 extends to all solution concepts. Each solution concept involves a type of garbling that reflects the amount of correlation allowed in it.

Theorem 4.2. *The information structure $\mathcal{I}$ is better than $\mathcal{I}'$ w.r.t. correlated equilibrium, or w.r.t. agent-normal-form correlated equilibrium in games with common interests if and only if there exists a coordinated garbling that transforms $\mathcal{I}$ to $\mathcal{I}'$.*

The proof of this theorem relies on the Pareto dominant correlated equilibrium payoff (or agent-normal-form correlated equilibrium payoff) being a Nash equilibrium payoff. This is so because in a common interest game the players can directly coordinate on the best pure Nash equilibrium without resorting to any external correlation device: the best Nash equilibrium payoff is the best feasible payoff.

Before stating the results concerning the epistemic approach, we need the following definition. We say that a garbling is non-communicating if no information has passed between the players through the garbling. This means that the garbled signal, s' of player 1, does not give him more information about the original signal t of player 2 than he had knowing s.

Definition 4.3. A garbling $q : S \times T \to \Delta(S' \times T')$ is *non-communicating* if for every $s \in S$, $s' \in S'$ and $t \in T$,

(i) $\sum_{t'} q(s', t'|s, t)$ does not depend on t; and
(ii) for every $t \in T$, $t' \in T'$ and $s \in S$, $\sum_{s'} q(s', t'|s, t)$ does not depend on s.

Let (s, t) be a pair of random signals generated according to some distribution $\pi \in \Delta(S \times T)$ and let (s', t') be the random garbling according to q. If q is non-communicating, then the posterior distribution of t given s, s' equals the posterior distribution over t given s. Indeed,

$$\mathbb{P}(t|s, s') = \frac{\mathbb{P}(s, s', t)}{\mathbb{P}(s, s')} = \frac{\mathbb{P}(s'|s, t)\mathbb{P}(s, t)}{\mathbb{P}(s'|s)\mathbb{P}(s)} = \frac{\mathbb{P}(s'|s)\mathbb{P}(s, t)}{\mathbb{P}(s'|s)\mathbb{P}(s)} = \frac{\mathbb{P}(s, t)}{\mathbb{P}(s)} = \mathbb{P}(t|s).$$

In other words, if a non-communicating garbling is performed by a mediator, although this mediator is allowed to use the information of the players, he is not allowed to give a player more information than he had before about the signal of the other player.

Example 4.4. Let $S = T = S' = T' = \{0, 1\}$. Consider the garbling q of Example 3.6. This garbling is not a coordinated garbling but it is non-communicating.

Using game theoretical terminology, we can say that Examples 3.6 *and* 4.1 *show the possibility of improving the outcome of a game with common interests by coordination without communication* (*that is, without any information exchange between the players*).

The following theorems provide a complete characterization of the order 'being better' w.r.t. all remaining solution concepts in games with common interests. They establish a strong relation between the various types of garblings and the amount of communication allowed in each concept.

Theorem 4.5. *Let $\mathcal{I}$ and $\mathcal{I}'$ be two information structures. $\mathcal{I}$ is better than $\mathcal{I}'$ w.r.t. belief-invariant Bayesian solution in common interest games if and only if there exists a non-communicating garbling that transforms $\mathcal{I}$ to $\mathcal{I}'$.*

Theorem 4.6. *The information structure $\mathcal{I}$ is better than $\mathcal{I}'$ w.r.t. communication equilibrium in games with common interests if and only if there exists a garbling that transforms $\mathcal{I}$ to $\mathcal{I}'$.*

The intuition underlying Theorem 4.6 is that in games with common interests players have an incentive to share their information. Therefore, they have an incentive to correctly report their signals to the mediator, which reduces the situation to a single player decision problem.

5. Proofs

Before proving the aforementioned results we need some notations for strategies and payoffs.

5.1. Strategies

Let S and T be the signals sets of players 1 and 2, respectively. A *global strategy* is a function from $S \times T$ to $\Delta(A \times B)$. That is, a global strategy attaches a distribution over $A \times B$ to every pair of signals (s, t).

If x is a strategy of player 1 and y is a strategy of player 2, then $x \otimes y : S \times T \to \Delta(A \times B)$ denotes the global strategy played if the players play independently of each other. Formally, $x \otimes y(a, b|s, t) = x(a|s)y(b|t)$. Such a strategy is called *independent* global strategy.

Let ε be a global strategy of the form $\varepsilon = \sum_{i \in I} \lambda(i)\varepsilon_i$ with I being a finite set, $\lambda \in \Delta(I)$ and $\varepsilon_i = x_i \otimes y_i$ is an independent global strategy, for any $i \in I$. The global strategy ε is obtained by the players observing first a public signal i, which is randomly selected according to the probability distribution λ, and then independently playing x_i and y_i. Such a strategy is called *coordinated*. Note that the set of coordinated strategies is a convex set whose extreme points are the independent global strategies.

A global strategy ε such that $\sum_a \varepsilon(a, b|s, t)$ is independent of s (i.e., $\sum_a \varepsilon(a, b|s, t) = \mathbb{P}(b|t)$) and $\sum_b \varepsilon(a, b|s, t)$ is independent of t, it is called *non-communicating*.

5.2. Nash, correlated and agent-normal-form-correlated equilibria

Here we prove Theorems 3.5 and 4.2. We first need the following lemma.

Lemma 5.1. *In a game with common interests, the maximal possible payoff achievable with global coordinated strategies is a Nash equilibrium payoff. It is also a normal-form correlated equilibrium payoff and an agent-normal-form correlated equilibrium payoff.*

Proof. Since the set of global coordinated strategies is compact, the maximum payoff (which is a linear function) is achieved on this set. Since the set of coordinated strategies is convex, the maximum is achieved at an extreme point, which is an independent strategy, say $\varepsilon^* = x^* \otimes y^*$. Since (x^*, y^*) achieves the maximal payoff possible, no player has a profitable deviation, and is therefore a Nash equilibrium. Since any Nash equilibrium is in particular a normal-form correlated equilibrium and an agent-normal-form correlated equilibrium, the proof is complete. □

We now turn to prove the result.

Proof of Theorem 3.5. Assume that $\mathcal{I} = (S, T, \sigma)$ is better than $\mathcal{I}' = (S', T', \sigma')$. We prove that $\mathcal{I}'$ is a garbled version of $\mathcal{I}$ with a coordinated garbling. Let $\mathcal{G}$ be the set of maps σ'' from K to $\Delta(S' \times T')$ such that (S', T', σ'') is a garbled version of $\mathcal{I}$ with a coordinated garbling. The set $\mathcal{G}$ is closed and convex in $\mathbb{R}^{K \times S' \times T'}$.

Suppose that σ' does not belong to $\mathcal{G}$. By the separation theorem σ' can be separated from $\mathcal{G}$ by a hyperplane: $r \in \mathbb{R}^{K \times S' \times T'}$ (r can be thought of also as $|K|$ functions, $r_k : S' \times T' \to \mathbb{R}$, $k \in K$). The separation by r means that for any coordinated garbling q,

$$\sum_{k \in K} \sum_{\substack{s' \in S' \\ t' \in T'}} \sum_{\substack{s \in S \\ t \in T}} r_k(s', t')\sigma(s, t|k)p(k)q(s', t'|s, t) < \sum_{k \in K} \sum_{\substack{s' \in S' \\ t' \in T'}} r_k(s', t')\sigma'(s', t'|k)p(k). \tag{1}$$

Consider the game with the action sets $A = S'$ and $B = T'$ and the payoff function r. The left-hand side of (1) is the payoff associated with the coordinated strategy q when the information structure is $\mathcal{I}$. The right-hand side is the payoff associated with the independent strategy according to which every player plays his signal when the information structure is $\mathcal{I}'$.

Lemma 5.1 implies that $\mathcal{I}$ is not better than $\mathcal{I}'$ w.r.t. Nash, normal-form correlated and agent-normal-form equilibria.

As for the converse, assume that $\mathcal{I}'$ is a garbled version of $\mathcal{I}$ with a coordinated garbling q. We prove that $\mathcal{I}$ is better than $\mathcal{I}'$ w.r.t. all three solution concepts.

Let ε be a global strategy in some game with common interests with information structure $\mathcal{I}'$ and action sets A and B. Assume that the payoff function is $(r_k)_{k \in K}$, and that ε is a Nash equilibrium. As a Nash equilibrium, ε is an independent strategy $x \otimes y$. The payoff associated with ε is

$$\sum_{k \in K} \sum_{\substack{s' \in S' \\ t' \in T'}} \sum_{\substack{a \in A \\ b \in B}} p(k)\sigma'(s', t'|k)x(a|s')y(b|t')r_k(a, b). \tag{2}$$

The garbling q is coordinated and therefore in the convex hull of the independent garblings. Thus, there is a finite set J and a probability μ on J, such that $q(s', t'|s, t) = \sum_{j \in J} \mu_j q_j^1(s'|s) q_j^2(t'|t)$ for every $(s, t) \in S \times T$ and $(s', t') \in S' \times T'$. Using this garbling the payoff in (2) can be rewritten as

$$\sum_{k \in K} \sum_{\substack{s \in S \\ t \in T}} \sum_{\substack{a \in A \\ b \in B}} \sum_{j \in J} p(k)\sigma(s, t|k)\mu_j \left(\sum_{s' \in S'} q_j^1(s'|s)x(a|s') \right) \left(\sum_{t' \in T'} q_j^2(t'|t)y(b|t') \right) r_k(a, b). \tag{3}$$

This is the payoff associated with a certain coordinated strategy in the game with information structure $\mathcal{I}$. By Lemma 5.1, the maximal equilibrium payoff in the game with information structure $\mathcal{I}$ is greater than or equal to that in (3). Therefore, $\mathcal{I}$ is better than $\mathcal{I}'$ w.r.t. Nash equilibrium.

Lemma 5.1 ensures that $\mathcal{I}$ is also better than $\mathcal{I}'$ w.r.t. normal-form correlated and agent-normal-form equilibria. □

5.3. A belief-invariant Bayesian solution and communication equilibria

We first prove an analog of Lemma 5.1.

Lemma 5.2. *In a game with common interests,*

(i) *The maximal payoff achievable with a non-communicating strategy is a belief-invariant Bayesian solution payoff. Moreover, every belief-invariant Bayesian solution payoff is achievable by a non-communicating strategy.*
(ii) *The maximal payoff achievable with a global strategy is a communication equilibrium payoff.*

Proof. Let A and B be the action sets and $\mathcal{I}$ be an information structure. Fix a payoff function r_k and let ε be a non-communicating global strategy that achieves the maximal payoff. We prove that ε is a belief-invariant Bayesian solution.

Define, $\Omega = K \times S \times T \times A \times B$, and the probability $\mathbb{P}$ by $\mathbb{P}(k,s,t,a,b) = p(k)\sigma(s,t|k)\varepsilon(a,b|s,t)$. Let κ, ς, τ, α, β be the projections from Ω to K, S, T, A, B respectively. Then, the joint distribution of $(\kappa,\varsigma,\tau,\alpha,\beta)$ is $\mathbb{P}$. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be the partitions generated, respectively, by (ς,α) and (τ,β). Conditions 1, 2, 4 of the definition of belief invariant Bayesian solution in Section 4.2 are obviously satisfied. As for 3, we have

$$\mathbb{P}(\kappa = k, \varsigma = s|\mathfrak{A}_2) = \mathbb{P}(\kappa = k, \varsigma = s|\tau, \beta)$$

but we would like to have

$$\mathbb{P}(\kappa = k, \varsigma = s|\mathfrak{A}_2) = \mathbb{P}(\kappa = k, \varsigma = s|\tau).$$

However,

$$\begin{aligned}\mathbb{P}(\beta = b|\kappa = k, \varsigma = s, \tau = t) &= \sum_{a\in A}\mathbb{P}(\beta = b, \alpha = a|\kappa = k, \varsigma = s, \tau = t)\\ &= \sum_{a\in A}\mathbb{P}(\beta = b, \alpha = a|\varsigma = s, \tau = t) = \sum_{a\in A}\varepsilon(a,b|s,t).\end{aligned}$$

By the definition of a non-communicating strategy, $\sum_{a\in A}\varepsilon(a,b|s,t) = \mathbb{P}(\beta = b|\tau = t)$ and therefore,

$$\mathbb{P}(\beta = b|\kappa = k, \varsigma = s, \tau = t) = \mathbb{P}(\beta = b|\tau = t).$$

Hence,

$$\mathbb{P}(\kappa = k, \varsigma = s|\tau = t, \beta = b) = \frac{\mathbb{P}(\kappa = k, \varsigma = s, \tau = t)\mathbb{P}(\beta = b|\kappa = k, \varsigma = s, \tau = t)}{\mathbb{P}(\beta = b|\tau = t)\mathbb{P}(\tau = t)} = \mathbb{P}(\kappa = k, \varsigma = s|\tau = t),$$

which is condition 3. As in Lemma 5.1, the equilibrium condition follows from the fact that the payoff is maximum.

As for the converse, for a belief-invariant Bayesian solution defined over the space $(\Omega,\mathbb{P})$, consider the garbling q given by $q(a,b|s,t) = \mathbb{P}(\alpha = a, \beta = b|\varsigma = s, \tau = t)$. We claim first that q is non-communicating. Indeed, for every s,a,

$$\begin{aligned}\sum_b q(a,b|s,t) = \mathbb{P}(\alpha = a|\varsigma = s, \tau = t) = \frac{\mathbb{P}(\alpha = a, \varsigma = s, \tau = t)}{\mathbb{P}(\varsigma = s, \tau = t)} &= \frac{\mathbb{P}(\alpha = a, \varsigma = s)\cdot\mathbb{P}(\tau = t|\alpha = a, \varsigma = s)}{\mathbb{P}(\varsigma = s)\cdot\mathbb{P}(\tau = t|\varsigma = s)}\\ &= \frac{\mathbb{P}(\alpha = a, \varsigma = s)}{\mathbb{P}(\varsigma = s)},\end{aligned}$$

independently of t. For the last equality note that by condition 3 of belief-invariant Bayesian solution,

$$\mathbb{P}(\tau = t|\alpha = a, \varsigma = s) = \mathbb{P}(\tau = t|\varsigma = s).$$

Next, we show that the joint distribution of (κ,α,β) is indeed distribution that is induced by applying q over σ. For every k,a,b,

$$\begin{aligned}&\mathbb{P}(\kappa = k, \alpha = a, \beta = b) = \mathbb{P}(\kappa = k)\cdot\mathbb{P}(\alpha = a, \beta = b|\kappa = k)\\ &\quad = \mathbb{P}(\kappa = k)\sum_{s,t}\mathbb{P}(\varsigma = s, \tau = t|\kappa = k)\cdot\mathbb{P}(\alpha = a, \beta = b|\kappa = k, \varsigma = s, \tau = t)\end{aligned}$$

$$= \mathbb{P}(\kappa = k) \sum_{s,t} \mathbb{P}(\varsigma = s, \tau = t | \kappa = k) \frac{\mathbb{P}(\alpha = a, \beta = b | {}^{\varsigma = s}_{\tau = t}) \mathbb{P}(\kappa = k | {}^{\alpha = a}_{\beta = b}, {}^{\varsigma = s}_{\tau = t})}{\mathbb{P}(\kappa = k | \varsigma = s, \tau = t)}$$

$$= \mathbb{P}(\kappa = k) \sum_{s,t} \mathbb{P}(\varsigma = s, \tau = t | \kappa = k) \cdot \mathbb{P}(\alpha = a, \beta = b | \varsigma = s, \tau = t) = p(k) \sum_{s,t} \sigma(s, t | k) q(a, b | s, t).$$

Note that in the third equality we used Bayes' theorem and in the fourth we used the fact that, by condition 4 of belief invariant Bayesian solution,

$$\mathbb{P}(\kappa = k | \alpha = a, \beta = b, \varsigma = s, \tau = t) = \mathbb{P}(\kappa = k | \varsigma = s, \tau = t).$$

Now consider ε, a global strategy in a game with action sets A, B and information structure $\mathcal{I}$, that achieves the maximal payoff. We prove that ε is a communication equilibrium. This global strategy is clearly feasible in the game in which the players are asked to report their signals to a mediator. The mediator chooses a couple (a, b) with probability $\varepsilon(a, b|s, t)$ and recommends player 1 to play a and player 2 to play b. The fact that the payoff is maximal implies the equilibrium condition. Indeed, any wrong reports of the players (for instance player 1 reporting $\bar{s}$ when his true signal is s) also induce a payoff that is compatible with some global strategy and therefore smaller than the payoff associated to ε. □

Proof of Theorems 4.5 and 4.6. The proofs follow closely that in the previous section. We therefore provide only a sketch of them. The details are omitted.

First let us assume that $\mathcal{I}$ is better than $\mathcal{I}'$ for Bayesian (resp. communication) equilibrium. Let $\mathcal{G}'$ (resp. $\mathcal{G}''$) denote the set of functions σ'' from K to $\Delta(S' \times T')$ such that (S', T', σ'') is a garbled version of $\mathcal{I}$ with a non-communicating garbling (resp. a general garbling). Assume by contradiction that σ' is not in $\mathcal{G}'$ (resp. $\mathcal{G}''$), then there is a separating hyperplane, and as in the previous section this hyperplane defines a game. By Lemma 5.2, this game contradicts the fact that $\mathcal{I}$ is better than $\mathcal{I}'$.

Assume now that $\mathcal{I}'$ is a garbled version of $\mathcal{I}$ with a non-communicating garbling (resp. any garbling). As in the previous section, simple computation shows that any Bayesian equilibrium payoff (resp. communicating equilibrium payoff) in any game with information structure $\mathcal{I}'$ is a feasible payoff in the same game with information structure $\mathcal{I}$. This follows from Lemma 5.2 and the fact that composition of non-communicating garblings is a non-communicating garbling. □

6. Final comments

6.1. Information structures and the hierarchy of beliefs

When one information structure is better than another and vice versa, we say that they are *equivalent*. A natural question that arises is whether two equivalent structures w.r.t. Nash equilibrium in games with common interests induce the same hierarchy of beliefs? Ely and Peski (2006) and Dekel et al. (2007) give an example that answers this question in the negative. They showed two information structures that induce the same distribution over players' hierarchies of beliefs, and nevertheless, have different sets of Nash equilibria.

In the class of zero-sum games two equivalent structures induce the *same* value in any zero-sum game. Gossner and Mertens (2001) state that two information structures are equivalent in the class of zero-sum games if and only if they induce the same distribution over players' hierarchies of beliefs.

In a companion paper (Lehrer et al., 2007) we show that one direction of the question posed above is true. That is, if two information structures are equivalent in games with common interests, then they induce the same distribution over players' hierarchies of beliefs.

Ely and Peski (2006) introduced the notion of Δ-hierarchies, which are hierarchies of beliefs over conditional (interim) posteriors. They show that Δ-hierarchies are sufficient to predict rationalizable behavior in two-player games.

It is important to mention that Δ-hierarchies are not sufficient to predict the set of Nash equilibrium payoffs. Consider, for example the following variation of Example 3.6. Suppose that under $\mathcal{I}'$ the players are informed of the q-garbled signals and in addition remember their original signals. All signals in $\mathcal{I}$ and $\mathcal{I}'$ induce the same conditional belief (for every state of nature $k \in K$ the conditional belief will be δ_k with probability $\frac{1}{4}$). And yet, the two structures induce different Nash equilibrium payoffs since $\mathcal{I}$ cannot be transformed to $\mathcal{I}$ by coordinated-independent garbling. Note however, that $\mathcal{I}$ and $\mathcal{I}'$ induce the same belief-invariant Bayesian solution payoffs in games with common interest.

In Lehrer et al. (2007) we argue that if two information structures can be transformed to each other by non-communication garblings, then they induce the same distribution over Δ-hierarchies.

6.2. Positive value of information

Bassan et al. (2003) dealt with Bayesian games in a different setting than ours. While here players receive a partial information about the realized state of nature, in Bassan et al. (2003) the information of a player is represented by a σ-field of the set of states of the world. An information structure where player i knows the σ-field $\mathcal{S}_i$ is *more informative* than an information structure where player i knows $\mathcal{S}'_i$, if $\mathcal{S}_i$ refines $\mathcal{S}'_i$ for every i.

Bassan et al. (2003) characterized the games that have what they call the positive-value-of-information property. This means that more informative information structures entail higher Nash equilibrium payoffs for all players. The class of games they define contains games with common interests.

In Bassan et al. (2003) the notion of being more informative is set theoretical: $\mathcal{S}_i$ refines $\mathcal{S}'_i$ if $\mathcal{S}'_i$ is a subset of $\mathcal{S}_i$. In contrast, in our model, being more informative is probabilistic: it involves a stochastic map (garbling). It is not completely clear how the results in the two models relate.

6.3. The standard model of information

In this paper we followed Blackwell's approach of presenting information structures using stochastic signals. This allows for a neat separation between information structures, games and garblings. In this section we address the standard information model, and the extent to which our results apply to it. Recall that the set K of states of nature is fixed, and that a stochastic information structure is given by (S, T, σ), where S, T are finite sets of signals and $\sigma : K \to \Delta(S \times T)$.

The following model of information is frequently used in literature.

Definition 6.1. A *standard information model* is given by $(\Omega, \mathcal{A}, \mathbb{P}, \mathfrak{A}_1, \mathfrak{A}_2, \kappa)$, where

1. $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space;
2. $\mathfrak{A}_1, \mathfrak{A}_2$ are partitions representing the information sets of the players; and
3. $\kappa : \Omega \to K$ is a random variable over Ω with values in K.

The elements of Ω are called *states of the world*.

Every standard model can be represented as a stochastic model in a natural way. The sets S, T are the atoms of $\mathfrak{A}_1, \mathfrak{A}_2$, and for $k \in K, s \in S, t \in T$, the conditional probability $\sigma(s, t|k) = \mathbb{P}(s \cap t|\kappa = k)$ (i.e., is equal to the conditional distribution of $s \cap t$ given the state k). It means that the signal of each player is the atom of his information partition that contains the realized state.

Given actions sets A, B and a payoff function $r : A \times B \to \mathbf{R}$, one can define a pure strategy of player i as a $\mathfrak{A}_i$-measurable function $\alpha_i : \Omega \to A$ and the payoff associated with a pair of strategies is the expectation (w.r.t. $\mathbb{P}$) of $r(\alpha_1, \alpha_1)$. Thus, every standard model and every A, B, r induce a normal-form game.

As far as Nash equilibrium is concerned, the equivalence between the standard and the stochastic models is based on the following observation. Let $\sigma : K \to \Delta(S \times T)$ be a stochastic model induced by the standard model $(\Omega, \mathcal{A}, \mathbb{P}, \mathfrak{A}_1, \mathfrak{A}_2, \kappa)$. Then, for every action sets A, B and every payoff function $r : A \times B \to \mathbf{R}$, the set of strategies in the game defined by the information structures is the same as the set of strategies defined in the standard model, and every pair of strategies yields the same payoff in both models. In particular, the sets of Nash equilibrium payoffs coincide. Regarding correlated equilibrium, it can be defined in the standard model by introducing an exogenous randomization device.

While the similarity of the standard model and the belief-invariant solution is clear, the approaches are different. The standard model $(\Omega, \mathcal{A}, \mathbb{P}, \mathfrak{A}_1, \mathfrak{A}_2, \kappa)$ that appears in Section 4.2 is a part of the *solution* along with the variables that specify the actions, α and β. In particular, it might contain correlations between player's actions and the state of nature, beyond the correlation between the signals and the state of nature. On the other hand, the standard model in this section is part of the game's description. Under this formulation any correlation between actions and the state of nature can be obtained from the correlation between the signals and the state of nature.

It is not clear how to define belief-invariant Bayesian solution in the standard model. If one tries to apply the definition of Section 4.2 to the probability space of the standard model, one obtains in return Nash equilibrium.

As for transformations between information structures, one might think that the analog to garbling is that partitions of one standard model refine those of another. It turns out that an additional auxiliary randomization device is needed to appropriately define the transformations from one model to another. The following proposition introduces the notion equivalent to coordinated garbling when comparing two standard models. Similar results concerning independent garblings and non-communicating garblings are omitted.

Denote by $\mathcal{B}$ the Borel σ-algebra of $[0, 1]$ and by λ the Lebesgue measure.

Proposition 6.2. *Let $(\Omega, \mathcal{A}, \mathbb{P}, \mathfrak{A}_1, \mathfrak{A}_2, \kappa)$ and $(\Omega', \mathcal{A}', \mathbb{P}', \mathfrak{A}'_1, \mathfrak{A}'_2, \kappa')$ be two standard models and let σ, σ' be the induced stochastic models. Then the following conditions are equivalent*:

1. *There exists a coordinated garbling q such that $q \circ \sigma = \sigma'$.*
2. *There exists a measurable maps $\phi : \Omega \times [0, 1] \to \Omega'$ such that*:
   - *ϕ is measure-preserving, that is, $(\mathbb{P} \otimes \lambda)(\phi^{-1}(A)) = \mathbb{P}'(A)$ for every $A \in \mathcal{A}'$.*
   - *ϕ is $\mathfrak{A}_1 \otimes \mathcal{B} \to \mathfrak{A}'_1$ measurable, that is, $\phi^{-1}(A) \in \mathfrak{A}_1 \otimes \mathcal{B}$ for every $A \in \mathfrak{A}'_1$.*
   - *ϕ is $\mathfrak{A}_2 \otimes \mathcal{B} \to \mathfrak{A}'_2$ measurable.*

Remark 6.3. Let $\mathcal{I} = (\Omega, \mathcal{A}, \mathbb{P}, \mathfrak{A}_1, \mathfrak{A}_2, \kappa)$ and $\mathcal{I}' = (\Omega, \mathcal{A}, \mathbb{P}, \mathfrak{A}'_1, \mathfrak{A}'_2, \kappa')$ be two standard models with the same underlying probability space $(\Omega, \mathcal{A}, \mathbb{P})$. If $\mathfrak{A}_1$ is finer than $\mathfrak{A}'_1$ and $\mathfrak{A}_2$ is finer than $\mathfrak{A}'_2$, then the map $\pi : (\omega, x) \in \Omega \times [0,1] \mapsto \omega$ satisfies condition 2 of Proposition 6.2. However, it might be that condition 2 of Proposition 6.2 is satisfied although $\mathfrak{A}_i$ is not finer than $\mathfrak{A}'_i$. For example, let $\Omega = \{0,1\}^2$ with the uniform distribution, $\kappa(w_1, w_2) = \kappa'(w_1, w_2) = w_1$, in $\mathcal{I}$ the partitions are trivial and in $\mathcal{I}'$ both players know the second coordinate, w_2, of w. Note that in this case $\mathcal{I}$ is indeed (weakly) better than $\mathcal{I}'$ w.r.t. Nash/correlated equilibrium in games with common interests.

6.4. The results cannot be extended to general games — an example

The following example is adapted from Forges (1993).

Example 6.4. Let the state space consist of two states: $K = \{1, 2\}$. In the first structure, player 1 knows k and player 2 knows nothing. In the second structure, player 1 knows k and receives in addition the signal s and player 2 receives the signal t, both are independent of k. The signals s and t have the following joint distribution: If $k = 1$ then s, t are identical and randomly chosen from $1, 2, \ldots, n$. If $k = 2$ then s, t are independent and are chosen randomly according to the uniform distribution from $1, 2, \ldots, n$. The two structures are equivalent in the class of games with common interests. That is, they induce the same Pareto dominant equilibrium payoff. This means that each is a garbled version of the other with a coordinated garbling. We now provide an example of a game without common interests where these two structures do not induce the same Nash equilibrium payoffs.

Consider the Bayesian game in which $A = \{1, 2, \ldots, n\}$, $B = \{b_1, b_2\}$, and the two states are equally likely. Suppose that the payoffs are determined only by the state and player 2's action as follows:

$$\underset{k=1}{\begin{pmatrix} b_1 & b_2 \\ 1,2 & 0,0 \end{pmatrix}}, \qquad \underset{k=2}{\begin{pmatrix} b_1 & b_2 \\ 1,0 & 0,4 \end{pmatrix}}.$$

With the first structure the only Nash and strategic normal-form correlated equilibrium payoff is $(0, 2)$. However, in the second structure there is a Nash equilibrium payoff close to $(1/2, 3)$. Such a payoff is obtained by player 1 sending s to player 2, and player 2 playing b_1 if $s = t$ and b_2, otherwise. Thus, although each structure is a garbled version of the other with a coordinated garbling, they induce different Pareto efficient Nash equilibrium payoffs and strategic-normal-form correlated equilibrium payoffs.

Note, however, that the sets of agent-normal-form correlated equilibria under both structures coincide. In a companion paper (Lehrer et al., 2007) we show that if $\mathcal{I}$ and $\mathcal{I}'$ are two information structures such that each is a garbled version of the other with a coordinated garbling, then they induce the same set of agent-normal-form correlated equilibria.

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