Signaling and mediation in Bayesian games

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September 29, 2006

Abstract

Two players participate in a Bayesian game. Before they take any actions each receives a stochastic signal that depends on the actual state of nature. The signals the players receive are determined by the information structure, which in turn, determines the equilibria of the game.

Two information structures are *equivalent* with respect to a certain solution concept, if the equilibria they generate induce the same distributions over outcomes. We characterize when two information structures are equivalent with respect to three solution concepts: Nash equilibrium, agent-normal-form correlated equilibrium and the Bayesian solution.

Keywords: Bayesian game; Information structure; Nash equilibrium; agent-normal-form correlated equilibrium; Bayesian solution; independent garbling; coordinated garbling; non-communicating garbling

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This research was partially supported by the French-Israeli Grant for Mathematics, 2005-2006.

1 Introduction

Agents that interact in an incompletely known environment might be willing to pay for extra information. How much they would be willing to pay depends on the impact of extra information on the final outcome of the interaction. The *information structure* of the interaction determines the "amount" of information each agent gets on the actual parameters of the game, and the extend to which this information is correlated across agents.

Even from a single agent's point of view, it is hard to rank all information structures, for two reasons. First, what might be considered an outcome depends on the specific solution concept adopted; and second, each structure typically induces a multiple outcome. However, once a certain solution concept is adopted, it is obvious that agents will be indifferent between information structures that induce the same set of outcomes. The goal of this paper is to characterize when two information structures induce the same set of outcomes. We perform the analysis w.r.t. three plausible solution concepts.

The model we use to accomplish this goal is a Bayesian game with incomplete information. A state of nature is randomly selected according to a known distribution. The players are not fully informed of the selected state. Rather, each player obtains a signal, typically stochastically dependent of the realized state, that provides him with partial information. This signal not only contains partial information about the actual state, it may also be correlated with other players' signals and may also partially tell what others know about the actual state. The specific way the signals depend on the realized state of nature, and the extent of correlation embedded in them, is determined by the information structure of the game. The correlation embedded in the signals may sometimes serve as a means of coordination between the players.

An *outcome* in a Bayesian game is given by a state and a joint action. The information players obtain about the actual game has a crucial role in determining the outcome. Different information structures typically induce different sets of outcomes.

We refer to three solution concepts: Nash equilibrium, agent-normal-form correlated equilibrium (Samuelson and Zhang ,1989) and the Bayesian solution (Forges, 1993). We then define three equivalence relations between information structures, one for each solution concept. Two information structures are *equivalent* w.r.t. a given solution concept, if they induce the same set of distributions over outcomes, when this solution is invoked.

The question asked in this paper is when are two information structures equivalent w.r.t. a given solution concept? A more general question was answered by Blackwell (1953) in a one-player model. Blackwell (1953) defined two partial orders between information structures: one in terms of the game's outcomes and one in probabilistic terms.

One information structure is said to be *better than* another if it induces a higher optimal payoff in every one-player decision problem. And one information structure is *more informative than* another, if there exists a map that stochastically associates to each of its signals a new signal in the second structure, so that the distribution of the signals so obtained is exactly the distribution of signals in the second structure. Such a map is called *garbling* of information. Using garbling, the agent can simulate the information obtained through the second structure when receiving information through the original structure.

Blackwell (1953) showed that one structure is better than another if and only if it is more informative. It follows that two structures are *equivalent* in the sense that they induce the same optimal payoff in any one-player decision problem, if and only if, each can be transformed into the other by garbling of information.

While in the one-player model there is only one kind of natural garbling, in a multi-player model there are a few. These kinds differ from one another over how separate individuals' garblings might depend on each other. There are three natural kinds of garblings: independent, coordinated and non-communicating. Two information structures are informationally equivalent w.r.t. a certain kind of garbling if each can be transformed into the other by a garbling of this kind. As in Blackwell (1953), we are interested in the connection between the probabilistic terms of garbling and the game's terms.

The objective of this paper is to find the game-theoretic solution concept that corresponds to each kind of informational garbling. In other words, we seek to explore which solution concept renders two information structures equivalent in the game theoretical sense whenever they are probabilistically equivalent w.r.t. a certain kind of garbling.

Two information structures are garbled versions of each other with *independent* garblings if each player can translate the signals he obtains through one structure to signals he obtains through the other structure, independently of other players' signals. That is, the translation of signals (garbling) is done by individuals, separately and independently of each other. It turns out that two structures are Nash-equivalent (i.e., they induce the same sets of Nash equilibrium distributions), if and only if, they are garbled versions of each other with independent garblings.

The second result links coordinated garblings and agent-normal-form correlated equilibrium. Two information structures are garbled versions of each other with *coordinated* garblings if the players can translate the signals they obtain through one structure to signals they obtain through the other structure in two steps. At the first, a joint dictionary is chosen randomly. In the second step, as in the Nash-equivalence case, this dictionary is used to translate the signals. Each player does it separately and independently of other players. Thus, in a coordinated garbling the players first choose an independent garbling (possibly by using a public signal) and then each player translates his signal independently.

Like in correlated equilibrium (Aumann, 1974), the agent-normal-form correlated equilibrium involves a mediator. In correlated equilibrium the mediator gives each player a full menu of recommendations: one for each possible signal. This recommendation can be given to a player *ex ante,* before obtaining any signal. By contrast, in an agent-normal-form correlated equilibrium the mediator randomly selects a full menu, but the recommendation is made *interim.* It is required that in the interim phase the mediator gets to know the signals of the agents. Upon getting this information, she recommends each player only one action: the one relevant to the signal he actually obtained. This way, the mediator avoids providing the players with unnecessary information. Here, a player cannot not tell what the recommendation of the mediator would have been had he received a different signal. In other words, a player receives only the information relevant to the signal he actually received. Any non-relevant information that would potentially render the recommended action incentive incompatible is eliminated.

The recommendation of the mediator may contribute some additional coordination beyond that provided by the information structure. It is therefore natural to expect that an information structure would be equivalent (w.r.t. agent-normal-form correlated equilibrium) to more information structures than in the Nash case. The second result confirms this intuition. It states that two information structures are garbled versions of each other with coordinated garblings, if and only if, they induce the same sets of agent-normal form correlated equilibrium distributions.

The third result links non-communicating (Lehrer et al., 2006) garblings and the Bayesian solution. Two information structures are garbled versions of each other with *non-communicating* garblings if signals obtained through one structure can be garbled to signals of the other structure, without providing the players with extra information about the other players' original signals. That is, knowing the garbled signal on top of the original, does not change the belief of a player about others' signals. This kind of garbling has roots in quantum theory (see for instance, Barret et al. (2005)) and is related to the locality of the physical universe - it takes time for information to travel from one spot to another.

The Bayesian solution is an extension of the correlated equilibrium as described in Aumann (1987). The state of the world contains, among other things, the precise description of the game played, the signals of each player, his action and his general knowledge. The notion 'general knowledge' refers to what a player knows beyond what is relevant to the game under consideration. In particular, it provides no further information about other players' signals, beyond the information already embedded in the player's own signal. This knowledge may serve as a correlation device between the players. A Bayesian solution is a situation where the action of each player is his best response to his belief about the state of nature and other players' actions, given his knowledge.

This solution can be implemented by an omniscient mediator who knows the signals each player received. Based on this knowledge, she recommends, in the interim phase, each player an action. This action depends on both, the identity of the player and the signal he received. However, the mediator is restricted to make recommendations that give any player no further knowledge about other players' signal beyond what he knew before.

Two information structures are garbled versions of each other with non-communicating garblings if and only if, they are equivalent with respect to the Bayesian solution.

A characterization of equivalent information structures in the framework of zero-sum games, i.e. information structures that induce the same value of the game, has been provided by Gossner and Mertens (2001). This characterization is not stated in terms of garbling of information but rather involves the players' hierarchy of beliefs. In a companion paper (Lehrer et al., 2006) we restricted attention to games with identical payoffs and characterized when, w.r.t. each of the solution concepts treated here, one information structure induces a higher best payoff than another.

Gossner (2000) investigated general games and found when an information structure is richer than another in the sense that it induces a larger set of Nash equilibrium distributions over outcomes. We use this result in the proof of our first result.

The paper is organized as follows. In a first section we introduce the model of Bayesian games compounded with information structures. Section 3 introduces the definitions related to garblings of information structures. Section 4 states the main results which are later proved in Section 5. Section 6 links between information structures and the hierarchy of beliefs. We prove, in particular, that if two structures are equivalent with respect to Nash equilibrium or to agent-normal-form equilibrium, then they necessarily induce the same hierarchy of beliefs. By means of an example we show that the converse is false.

2 The model

We deal with Bayesian games in which the players get some information about the state of nature according to a signalling function. Such a model is defined by an information structure and a game structure.

Games: Throughout the paper we fix a finite set K whose elements are called *states of nature*. Two players are engaged in a Bayesian game, in which the payoffs depend on the state of nature and on the actions taken by the players. A *game* is given by a probability distribution p over K (the common prior), a finite set of actions for each player (A and B) and a payoff function , $r^i : K \times A \times B \to \mathbb{R}$, for each player i ($i = 1, 2$). We usually use the notation $r^i_k(a, b) := r^i(k, a, b)$.

Information structures: The players are not directly informed of the realized state. Rather, each player obtains a stochastic signal that depends on k. The signals that the players obtain are typically correlated.

Formally, an *information structure* consists of two finite sets of signals, S, T, and a function[1] $\sigma : K \to \Delta(S \times T)$ that assigns a joint distribution over signals to every state of nature. When the realized state is k, player 1 obtains the signal s and player 2 obtains the signal t with probability $\sigma(k)[s, t]$, which we usually denote as $\sigma(s, t|k)$. Information structures will be later referred to as triples of the kind (S, T, σ) and will be denoted as $\mathcal{I}$.

Mixed strategies and expected payoffs: Upon receiving a signal, a player takes an action and receives a payoff that depends on both players' actions and on the state of nature. Formally, when the state is k, player 1 plays a and player 2 plays b, the

[1] $\Delta(D)$ denotes the set of distributions over a set D.

payoff player i receives is $r_k^i(a,b)$. A *strategy* x of player 1 assigns to every signal in S a mixed action. When player 1 plays according to strategy x, and he observes the signal s, the action $a \in A$ is played with probability $x(a|s)$. A strategy y of player 2 is defined in a similar manner.

This definition of a mixed strategy assumes that each player, after receiving his signal, takes his action independently of the other player.

When the strategy profile (x,y) is played, the expected payoff of player i is,

$$r^i(x,y) = \sum_{k \in K} p(k) \sum_{\substack{(s,t)\in S\times T\\(a,b)\in A\times B}} \sigma(s,t|k)x(a|s)y(b|t)r_k^i(a,b).$$

A pair of mixed strategies x^*, y^* is a *Nash equilibrium* if for any strategy x of player 1 and any strategy y of player 2, $r^1(x,y^*) \leq r^1(x^*,y^*)$ and $r^2(x^*,y) \leq r^2(x^*,y^*)$.

One can define other solution concepts in which the players do not necessarily play independently of each other, as in Nash equilibrium, but rather play correlatively. Such concepts are extensions of the correlated equilibrium defined by Aumann (1974, 1987), and will be discussed in Section 4. In order to define such concepts one needs a more general notion of strategy: a global strategy. The definition makes use of stochastic maps.

Stochastic maps: The following notation will be used extensively throughout the paper. Let X, X' be two finite sets. A *stochastic map* from X to X' is a function from X to $\Delta(X')$. We denote by $\mathcal{S}(X,X')$ the set of such maps. Note that when S and A are, respectively, player 1's sets of signals and actions, $\mathcal{S}(S,A)$ is player 1's set of strategies. Similarly, when K is the set of states and S and T are, respectively, player 1's and player 2's sets of signals, then $\mathcal{S}(K, S\times T)$ is an information structure.

Any stochastic map from X to X' induces a natural linear function from $\Delta(X)$ to $\Delta(X')$: its linear extension. In the sequel we do not distinguish between a stochastic map from X to X' and its linear extension. The set $\mathcal{S}(X,X')$ is a polygon and its extreme points are the *pure maps*. Every pure map in $\mathcal{S}(X,X')$ can be described as a function $f: X \to X'$. Here too, we do not distinguish between f and its linear extension to the domain $\Delta(X)$.

Let X, X', Y, Y' be finite sets and let $\varphi \in \mathcal{S}(X,X')$ and $\psi \in \mathcal{S}(Y,Y')$. Their product, denoted $\varphi \otimes \psi$, is a stochastic map from $X \times Y$ to $X' \times Y'$, defined by

$$\varphi \otimes \psi : (x,y) \mapsto \varphi(x) \otimes \psi(y) \tag{1}$$

for every $x, y \in X \times Y$. Here, $\varphi(x) \otimes \psi(y)$ is the product distribution of $\varphi(x)$ and $\psi(y)$.

Let X, Y, Z be finite sets and let $\varphi \in \mathcal{S}(X, Y)$ and $\psi \in \mathcal{S}(Y, Z)$. The composition of ϕ and ψ induces a stochastic map in $\mathcal{S}(X, Z)$. Formally, for every $x \in X$ and for every $z \in Z$, the probability assigned to z by $\psi \circ \phi(x)$ is $\psi \circ \phi(z|x) = \sum_{y \in Y} \psi(z|y)\phi(y|x)$. The composition notation agrees with our convention not to distinguish between ψ and its linear extension from $\Delta(Y)$ to $\Delta(Z)$.

Global strategies: A *global strategy* attaches a distribution over $A \times B$ to every pair of signals (s, t) . Thus, a global strategy is a member of $\mathcal{S}(S \times T, A \times B)$. For every global strategy ε, if player 1 receives the signal s and player 2 the signal t, the probability over $A \times B$ is $\varepsilon(s, t)$ and $\varepsilon(a, b|s, t)$ is the probability that the pair (a, b) is played.

When the two players play independently the mixed strategies x and y, the global strategy induced is $x \otimes y$.

Let $\mathcal{I} = (S, T, \sigma)$ be an information structure and let ε be a global strategy. For every state of nature k, $\sigma(k)$ is a probability distribution over $S \times T$. When the global strategy ε and σ are composed together they induce the distribution $\varepsilon \circ \sigma(k)$ over $A \times B$. When the global strategy ε is played, the expected payoff of player i is:

$$R^i(\sigma, \varepsilon) = \sum_{k \in K} p(k) \sum_{\substack{(s,t) \in S \times T \\ (a,b) \in A \times B}} \sigma(s, t|k)\varepsilon(a, b|s, t) r^i_k(a, b). \tag{2}$$

Finally, every information structure $\sigma : K \to S \times T$ and a global strategy $\varepsilon \in \mathcal{S}(S \times T, A \times B)$ induce a probability distribution $D(\sigma, \varepsilon)$ over $K \times A \times B$. Elements of $K \times A \times B$ are called *plays.* $D(\sigma, \varepsilon)$ is the distribution over plays when the players play according to this global strategy. It is given by

$$D(\sigma, \varepsilon)(k, a, b) = p(k) \cdot \varepsilon \circ \sigma(a, b|k) = p(k) \sum_{s,t} \sigma(s, t|k)\varepsilon(a, b|s, t). \tag{3}$$

Note that $R^i(\sigma, \varepsilon)$ is the expectation of r^i w.r.t. $D(\sigma, \varepsilon)$.

Solution concepts: A global equilibrium[2] is a global strategy ε that has the feature that none of the players has an incentive to deviate, given his signal. This condition

[2]Forges (1993) calls this notion a 'universal Bayesian solution' and Forges (2006) names it a 'Bayesian solution'. What we earlier called a Bayesian solution Forges (2006) calls a *belief invariant Bayesian solution.* We prefer to stick with the name we adopted in our previous paper (Lehrer et al, 2006) that was borrowed (without the restriction 'partial') from Forges (1993).

is formally expressed as follows. A global strategy $\varepsilon \in \mathcal{S}(S \times T, A \times B)$ is a *global equilibrium* if, for every deviation function $v : S \times A \to \Delta(A)$ of player 1,

$$R^1(\mathcal{I}; \varepsilon'_v) \leq R^1(\mathcal{I}; \varepsilon), \tag{4}$$

where

$$\varepsilon'_v(a, b|s, t) = \sum_{a' \in A} \varepsilon(a', b|s, t) v(a|s, a').$$

And a similar condition holds for player 2.

A global equilibrium can be defined in terms of a mediator that may take into account both players' signals when choosing recommended actions. In particular, the mediator is allowed to transmit information between the players. She might even completely reveal to a player the other player's signal.[3]

In the sequel we will use various solution concepts that extend the Nash equilibrium concept. A solution concept (also called equilibrium concept) is a correspondence that associates to each game and information structure a set of global equilibria. Nash equilibrium, for instance, is a global equilibrium ε that can be written as $\varepsilon = \phi \otimes \psi$ with $\phi \in \mathcal{S}(S, A)$ and $\psi \in \mathcal{S}(T, B)$.

Equilibrium and Equivalence of information structures: We now fix a solution concept. Depending on the chosen concept we define when two information structures are equivalent i.e. when they lead to the same equilibrium distributions. Our goal is to give a characterization of equivalent information structures for various solution concepts.

Definition 2.1 *Let $\mathcal{I}$ and $\mathcal{I}'$ be two information structures. Fix an equilibrium concept.*

1. *$\mathcal{I}$ is* larger *than $\mathcal{I}'$ w.r.t. this equilibrium concept if the set of distributions over plays, induced by equilibria associated with this concept under $\mathcal{I}$, contains that induced by $\mathcal{I}'$.*

[3] Note that the notion of global equilibrium is different from the more natural notion of communication equilibrium, defined by Forges (1986). In a communication equilibrium the mediator is not omniscient. The players are asked to reveal their signals to the mediator. Based on these reports, the mediator makes her recommendation.
In a communication equilibrium players strategically choose the signals they decide to report on. They communicate information to the mediator and thereby to each other, only if it in their best interest to do so.

2. $\mathcal{I}$ is equivalent *to $\mathcal{I}'$ w.r.t. this equilibrium concept, if $\mathcal{I}$ is larger than $\mathcal{I}'$ and $\mathcal{I}'$ is larger than $\mathcal{I}$ w.r.t. this equilibrium concept.*

The results of this paper concern equivalent information structures. The first part of the Definition 2.1, which follows Gossner (2000), is used only in the proof of Theorem 4.1.

3 Garbling of information

Following Blackwell (1953), we characterize equivalent information structures for various solution concepts using several kinds of garblings of information for two player games.

3.1 Garbling of information in two player games

Let $\mathcal{I} = (S, T, \sigma)$ be an information structure. Suppose that a joint signal (s, t) in $S \times T$ is produced (i.e., is randomly selected according to σ). Instead of sending the signals to the players, a pair of new signals, say (s', t'), is randomly selected from new sets of signals, say S' and T'. This selection is done according to a distribution $q(s, t)$. Players 1 and 2 are then informed of s' and t', respectively. This procedure generates a new information structure, $\mathcal{I}' = (S', T', \sigma')$, which is said to be a *garbled version* of $\mathcal{I}$. Formally,

Definition 3.1 *Let $\mathcal{I} = (S, T, \sigma)$ and $\mathcal{I}' = (S', T', \sigma')$ be two information structures. $\mathcal{I}'$ is a* garbled version *of $\mathcal{I}$ if there is a map $q \in \mathcal{S}(S \times T, S' \times T')$ such that the distribution induced by the composition $q \circ \sigma$ coincides with σ'. This means that, for every $s' \in S', t' \in T'$,*

$$\sigma'(s', t'|k) = \sum_{s \in S, t \in T} \sigma(s, t|k) q(s', t'|s, t).$$

The map q is called a garbling *that transforms $\mathcal{I}$ to $\mathcal{I}'$.*

Using the language of stochastic maps, when S, S', T, T' are sets of signals, a garbling is an element of $\mathcal{S}(S \times T, S' \times T')$.

In the case of one-player decision problems, Blackwell (1953) defined a structure, $\mathcal{I}$, as *more informative* than another, $\mathcal{I}'$, if $\mathcal{I}'$ is a garbled version of $\mathcal{I}$. However, in general games, players act individually and they typically do not share their signals with others. In particular, they are unable to join forces in trying to get new signals from old. This means that players are not able to garble their information with a general garbling. Only restricted classes of garblings are available.

Definition 3.2 . *(i) A garbling q is said to be* independent *if there are maps $q_1 : S \to \Delta(S')$ and $q_2 : T \to \Delta(T')$ such that for every s, t, s', t',*

$$q(s', t'|s, t) = q_1(s'|s) \cdot q_2(t'|t). \tag{5}$$

(ii) A garbling q is coordinated *if it is a convex combination of independent garblings.*

Using the tensor product notation (1), Eq. (5) can be equivalently written as $q = q_1 \otimes q_2$. Moreover, q is a coordinated garbling if $q = \sum_{\ell=1}^{L} \alpha^\ell q_1^\ell \otimes q_2^\ell$, where $\alpha^\ell \geq 0$, $q_1^\ell \otimes q_2^\ell$ is an independent garbling, $\ell = 1, ..., L$, and $\sum_{\ell=1}^{L} \alpha^\ell = 1$,

Note that independent garbling can be implemented without communication (every player manipulates his signal independently of the other). A coordinated garbling, on the other hand, can be implemented by a public signaling which is independent of the players' signals. In particular, a coordinated garbling does not transmit is information from one player to the other.

Not every garbling that does transmit information from one player to the other is a coordinated garbling. A garbling is *non-communicating* if the garbled signal s' of player 1 does not give him more information about the original signal t of player 2. This means that no information has passed between the players through the garbling. Formally,

Definition 3.3 *A garbling q that transforms $\mathcal{I} = (S, T, \sigma)$ to $\mathcal{I}' = (S', T', \sigma')$ is* non-communicating, *if it preserves the information available to each player in the following sense.*

(i) For every $s \in S$, $s' \in S'$ and $t \in T$, $\sum_{t'} q(s', t'|s, t)$ does not depend on t; and

(ii) For every $t \in T$, $t' \in T'$ and $s \in S$, $\sum_{s'} q(s', t'|s, t)$ does not depend on s.

Let (s, t) be a pair of random signals generated according to some distribution $\pi \in \Delta(S \times T)$ and let (s', t') be their random garbling according to q. If q is non-communicating, then the posterior distribution over t, given s, s', coincides with the posterior distribution over t, given s alone. In other words, if the garbling is performed by a mediator, she is allowed to use the information of both players to perform it.

However, she is not allowed to give a player more information about the signal of his opponent than what he had before. It is important to note that any coordinated garbling is, in particular, non-communicating but the converse is wrong (see e.g., Lehrer et al. (2006)).

The leading question in this paper can be now restated as follows. For each of the three classes of garblings (independent, coordinated and non-communicating) we introduced assume that $\mathcal{I}$ and $\mathcal{I}'$ are garbled versions of each other with the corresponding garbling. What is then the relevant game theoretic solution concept w.r.t. which $\mathcal{I}$ and $\mathcal{I}'$ are equivalent?

3.2 Example

In the following example we introduce two information structures and show that they are equivalent in the sense of non-communicating garbling (meaning that there is a non-communicating garbling that translates one to the other), but they are not equivalence in the sense of coordinated garbling. In particular, the non-communicating garblings that transform one structure to the other are not coordinated.

Let $K = S = T = S' = T' = \{1, 2, 3, 4\}$. Suppose that σ is given by the following distributions.

	1	2	3	4
1	1/4	0	0	0
2	0	1/4	0	0
3	0	0	1/4	0
4	0	0	0	1/4

if $k = 1$

	1	2	3	4
1	0	1/4	0	0
2	0	0	1/4	0
3	0	0	0	1/4
4	1/4	0	0	0

if $k = 2$

	1	2	3	4
1	0	0	1/4	0
2	0	0	0	1/4
3	1/4	0	0	0
4	0	1/4	0	0

if $k = 3$

	1	2	3	4
1	0	0	0	1/4
2	1/4	0	0	0
3	0	1/4	0	0
4	0	0	1/4	0

if $k = 4$

For instance, when $k = 2$, player 1 obtains the signal 3 and player 2 obtains the signal 4 with probability 1/4. Note that when player 1 obtains a signal, the conditional distributions over K and over T are uniform. A similar situation holds for player 2. Denote, $\phi(s,t) = s+t-1 (\text{mod } 4)$, where 0 is identified with 4. The pair (s,t) uniquely identifies the state $k = \phi(s,t)$, with probability 1.

Now consider $S' = T' = \{1,2,3,4\}$. Let σ' be given by the following distributions.

	1	2	3	4
1	1/4	0	0	0
2	0	1/4	0	0
3	0	0	1/4	0
4	0	0	0	1/4

if $k=1$

	1	2	3	4
1	0	1/4	0	0
2	1/4	0	0	0
3	0	0	0	1/4
4	0	0	1/4	0

if $k=2$

	1	2	3	4
1	0	0	1/4	0
2	0	0	0	1/4
3	1/4	0	0	0
4	0	1/4	0	0

if $k=3$

	1	2	3	4
1	0	0	0	1/4
2	0	0	1/4	0
3	0	1/4	0	0
4	1/4	0	0	0

if $k=4$

Here too, the pair (s',t') uniquely identifies k, say $k = \psi(s',t')$. The second structure is a garbled version of the first: for every (s,t), the pair (s',t') is selected according to the bottom distribution that corresponds to $k = \phi(s,t)$. It is easy to check that this is a non-communicating garbling: the distribution over T given s, s' is also uniform. That is, knowing s alone, and knowing both s and s', induce the same distribution over player 2's signals. By a similar argument, σ is a garbled version of σ' via a non-communicating garbling. Thus, σ and σ' are equivalent in the sense that there is a non-communicating garbling that transforms σ to σ' and there is a non-communicating garbling that transforms σ' to σ.

To show that there is no coordinated garbling that translates σ' to σ, it is sufficient, by Theorem 4.5 in Lehrer et al (2006), to demonstrate a game with common interests in which σ induces a higher Nash-equilibrium payoff than σ'. Consider the game with common interests, where the action set of each player is $\{1,2,3,4\}$ and the payoffs are identical to the probabilities in the first diagram. For instance, when $k=2$, if player 1 plays 3 and player 2 plays 4, the payoff of both players is 1/4. With the information structure σ, the players can ensure the payoff 1/4, while with σ' they can ensure at most 3/16.

We further discuss this example in Subsection 6.2.

4 Equilibrium and equivalence of information structures

In this section we provide the main results of the paper. Each subsection is devoted to a specific solution concept. We define a concept and state the corresponding result that characterizes when two information structures are equivalent.

The solution concepts we analyze are extensions of the notion of correlated equilibrium in games with complete information (Aumann 1974, 1987). All notions are equivalent to correlated equilibrium when restricted to such games. They differ from one another in the way the correlation and/or the communication between the players is allowed to depend on players' private information. In the description of the various solution concepts we follow Forges (1993) and refer to her paper for formal definitions and the relation between them.

Each of our results is an extension of the result in games with identical payoffs that corresponds to the same solution concept and appears in Lehrer et al. (2006).

4.1 Nash Equilibrium

This section is devoted to the comparison of information structures as far as Nash equilibrium is concerned.

Theorem 4.1 *Let $\mathcal{I}$ and $\mathcal{I}'$ be two information structures. There exist independent garblings q, q' such that q transforms $\mathcal{I}$ to $\mathcal{I}'$ and q' transforms $\mathcal{I}'$ to $\mathcal{I}$, if and only if, $\mathcal{I}$ and $\mathcal{I}'$ are Nash-equivalent.*

The intuition of this result is the following. In Nash equilibrium the actions played by the players are independent of each other. Suppose that (x, y) is a Nash equilibrium with the signals of $\mathcal{I}$. Furthermore, suppose that $q' = q'_1 \otimes q'_2$ is an independent garblings that transforms $\mathcal{I}'$ to $\mathcal{I}$. Define (x', y') as $x' = x \circ q'_1$, $y' = y \circ q'_2$. The strategies x' and y' are independent of each other in the game with information $\mathcal{I}'$, they form a Nash equilibrium and induce the same payoff as (x, y) does.

The proof of this theorem makes use of the following notion, introduced by Gossner (2000).

Definition 4.2 *Let $\mathcal{I} = (S, T, \sigma)$ and $\mathcal{I}' = (S', T', \sigma')$ be two information structures and let q be a garbling that transforms $\mathcal{I}$ to $\mathcal{I}'$. We say that q acts* faithfully *if no player loses information about the state of nature and the other player's garbled signal if he forgets his original signal. This means that for every k, s', t', s, t, $P(k, t'|s') = P(k, t'|s, s')$, $P(k, s'|t') = P(k, s'|t, t')$. Here P is the distribution over $K \times S \times T \times S' \times T'$ given by $P(k, s, t, s', t') = p(k)\sigma(s, t|k)q(s', t'|s, t)$.*

Note that faithfulness is not an intrinsic property of the garbling, but a property of the action of a garbling on a specific information structure. Gossner (2000) defined the notion of faithfulness in the case of independent garbling. He proved that an information structure $\mathcal{I}$ is Nash-larger (recall Definition 2.1) than $\mathcal{I}'$ if and only if there exists an independent garbling q that faithfully transforms $\mathcal{I}$ to $\mathcal{I}'$. One direction of Theorem 4.1 follows directly from Gossner's theorem. Indeed, if $\mathcal{I}$ and $\mathcal{I}'$ are Nash-equivalent, then each is Nash-larger than the other and therefore, by Gossner's Theorem, can be faithfully transformed to the other by an independent garbling. For the other direction we will prove that if q and q' are independent garblings such that q transforms $\mathcal{I}$ to $\mathcal{I}'$ and q' transforms $\mathcal{I}'$ to $\mathcal{I}$, then both q and q' act faithfully.

4.2 Agent-normal-form correlated equilibrium

The following solution concept is due to Samuelson and Zhang (1989). It is implemented by a mediator.

In an *agent-normal-form* correlated equilibrium, the mediator is assumed to act in two steps. First, before knowing the signals of the players she chooses a recommendation vector, i.e., one action for each possible signal of each player. Then, the mediator learns the signals received by the players, and sends a recommendation: one action to every player. The action recommended to a player depends on his signal. In an agent-normal-form correlated equilibrium a player does not get a full menu of recommendations (one for each possible signal). A player is recommended only the action relevant to the signal he actually received.

More precisely, in step 1, the mediator chooses a pair of strategies (x, y) according to a probability λ (the pair (x, y) is selected with probability $\lambda(x, y)$). In step 2, after learning the signals of the players, the mediator recommends player 1 to play $x(s)$ and player 2 to play $y(t)$.

Note that an agent-normal-form correlated equilibrium can equivalently be defined as a global equilibrium $\varepsilon \in \mathcal{S}(S \times T, A \times B)$ which is a coordinated garbling of $S \times T$ to $A \times B$.

A pair (r^1, r^2) of payoffs is an *agent-normal-form correlated equilibrium payoff* in a game with an information structure $\mathcal{I}$ if there is an agent-normal-form correlated equilibrium ε such that $R^i(\mathcal{I}, \varepsilon) = r^i$ for $i = 1, 2$.

The concept of agent-normal-form correlated equilibrium differs from the concept of strategic normal form correlated equilibrium. As in the former, in correlated equilibrium the mediator chooses a pair (x, y) according to a distribution λ. However, unlike the former, in a correlated equilibrium the mediator sends x to player 1 and y to player 2. That is, the mediator sends to each player a recommendation which is a strategy: a *function* from his signals to actions.

Agent-normal-form correlated equilibrium can be interpreted as a normal form correlated equilibrium in the game where each pair (player i, signal s_i) is regarded as a separate player (see Forges (1993), for an elaboration on this point). This concept has been criticized (see in particular Myerson (1991)) on the grounds that the two assumptions on which it is built are unrealistic.

The first assumption is that the mediator may make a recommendation to a player that depends on this player's signal, but not on the other player's signal. The second assumption is that the mediator can send signal-dependent messages, while the agents cannot send any information to the mediator.

It seems to us that the next theorem may serve as a firm justification of the heavily criticized concept of agent-normal-form correlated equilibrium. Coordinated garblings sounds like a natural notion of informational garbling. It is the agent-normal-form correlated equilibrium that naturally emerges when trying to understand equivalence of information structure w.r.t. coordinated garbling. It is precisely this solution concept that renders two structures equivalent, if they are garbled versions of each other with coordinated garbling. The more natural concept of normal form correlated equilibrium does not come up in this context, nor in any natural garbling context that we are aware of.

Definition 4.3 *Two information structures $\mathcal{I}$ and $\mathcal{I}'$ are* payoff equivalent w.r.t. agent-normal-form correlated equilibrium *if the sets of agent-normal-form correlated equilibrium payoffs they induce coincide in any game.*

The first major contribution of this paper is a full characterization of when two structures are coordinated garbling versions of each other:

Theorem 4.4 *Let $\mathcal{I}$ and $\mathcal{I}'$ be two information structures. There exist two coordinated garblings q and q' such that q transforms $\mathcal{I}$ into $\mathcal{I}'$ and q' transforms $\mathcal{I}'$ into $\mathcal{I}$, if and only if, $\mathcal{I}$ and $\mathcal{I}'$ are equivalent (resp. payoff-equivalent) w.r.t. agent-normal-form correlated equilibrium.*

Note that this theorem characterizes not only equivalence of information structures but also payoff equivalence. The proof of this theorem differs from the proof of Theorem 4.1, because in the case of Theorem 4.4 the garblings q and q' need not act faithfully.

4.3 The Bayesian solution

The previous notion of equilibrium requires a mediator. In this section we adopt a different approach. This one was introduced in Aumann (1987), used by Forges (1993), and adopted by Bassan et al. (2003). It can be viewed as a general epistemic approach to express Bayesian rationality.

The epistemic model is described by a probability space, $(\Omega, \mathbb{P})$ (with Ω being rich enough to reflect the state of nature, the signals and the actions of players), two partitions $\mathfrak{A}_1, \mathfrak{A}_2$ of Ω and a few random variables over Ω: (i) κ takes values in K (i.e., κ is the state of nature), (ii) ς and τ take values in S and T, resp. (i.e., ς and τ are the signals); and (iii) α and β take values in A and B, resp. (i.e., α and β are the actions). The partitions $\mathfrak{A}_1, \mathfrak{A}_2$ represent the information available to player 1 and player 2, respectively.

The ς and τ are meant to represent any information that the players might have about the parameters of the game (i.e., the state and the other player's signal). The partitions $\mathfrak{A}_1, \mathfrak{A}_2$, on the other hand, represent the entire knowledge of the players. This include any game-relevant information and more. But any game-relevant information contained in $\mathfrak{A}_1, \mathfrak{A}_2$, is already embedded in ς and τ. Any extra information in $\mathfrak{A}_1, \mathfrak{A}_2$ might serve as a correlation device.

A *Bayesian solution*[4] under the information structure $\mathcal{I} = (S, T, \sigma)$ is an epistemic model that satisfies the following conditions:

[4]We use this terminology that has been coined by Forges (1993).

1. The distribution of κ over K is p and for every $k \in K$, the joint distribution of ς, τ given that $\kappa = k$ is $\sigma(k)$. I.e., the joint distribution of κ, ς and τ is the distribution that p and σ induce on $K \times S \times T$.

2. ς, α (resp. τ, β) are $\mathfrak{A}_1$-measurable (resp. $\mathfrak{A}_2$-measurable). I.e., each player knows his signal and action.

3. The signal τ of player 2 completely summarizes his information on the state of the world and player 1's signal.

$$\mathbb{P}(\kappa = k, \varsigma = s | \mathfrak{A}_2) = \mathbb{P}(\kappa = k, \varsigma = s | \tau) \text{ for every } k, s.$$

   I.e., the information embedded in $\mathfrak{A}_2$ does not give player 2 more knowledge about s and k than his own signal.

   A similar condition holds for player 1.

4. The joint signals of the players completely summarize their joint information on the state of the world:

$$\mathbb{P}(\kappa = k | \mathfrak{A}_1, \mathfrak{A}_2) = \mathbb{P}(\kappa = k | \varsigma, \tau) \text{ for every } k.$$

5. Incentive compatibility conditions: any deviation of player 1 (resp. 2) from playing α (resp. β) is not profitable. (For a formal expression of this condition (5) the reader is referred to Forges (1993).)

We say that a distribution π over outcomes (i.e., $\pi \in \Delta(K \times A \times B)$) can be achieved by a Bayesian solution if π is the joint distribution of κ, α, β in some Bayesian solution. We say that $(r^1, r^2) \in \mathbb{R}^2$ is a Bayesian solution payoff if (r^1, r^2) is the expected payoff under some Bayesian solution.

Theorem 4.5 *$\mathcal{I}$ and $\mathcal{I}'$ are equivalent w.r.t. Bayesian solution if and only if there exist non-communicating garblings q, q' such that q transforms $\mathcal{I}$ to $\mathcal{I}'$ and q' transforms $\mathcal{I}'$ to $\mathcal{I}$.*

5 Proofs of the theorems

5.1 Preliminaries: The geometric structure of garblings

We will use the following notation. The set of coordinated garblings is called the *small* polygon and is denoted by $\mathrm{SP}(S,T,S',T')$. That is,

$$\mathrm{SP}(S,T,S',T') = \mathrm{Conv}\{\varphi \otimes \psi;\ \varphi \in \mathcal{S}(S,S'), \psi \in \mathcal{S}(T,T')\}.$$

The fact that this is a polygon is a consequence of the following proposition that summarizes basic properties of operations over stochastic maps.

Proposition 5.1 *Let X, X', X'', Y, Y', Y'' be finite sets.*

1. *If $\varphi_i \in \mathcal{S}(X,X')$ and $\psi_i \in \mathcal{S}(Y,Y')$, then*

$$(\sum_i \lambda_i \varphi_i) \otimes (\sum_j \mu_j \psi_j) = \sum_{i,j} \lambda_i \mu_j \varphi_i \otimes \psi_j. \tag{6}$$

2. *If $\varphi \in \mathcal{S}(X,X'), \varphi' \in \mathcal{S}(X',X''), \psi \in \mathcal{S}(Y,Y'), \psi' \in \mathcal{S}(Y',Y'')$, then*

$$(\varphi' \otimes \psi') \circ (\varphi \otimes \psi) = (\varphi' \circ \varphi) \otimes (\psi' \circ \psi).$$

It follows from (6) that $\mathrm{SP}(S,T,S',T')$ is a polygon, the vertices of which are tensor products of deterministic maps. The set of non-communicating garblings is called the *middle* polygon and is denoted by $\mathrm{MP}(S,T,S',T')$. The fact that this is a polygon follows from Definition 3.3, in which the set of non-communicating garblings is defined as the feasible set of a linear programming problem. The set of all garblings from $S \times T$ to $S' \times T'$ is called the *large* polygon and is denoted by $\mathrm{LP}(S,T,S',T')$. That is,

$$\mathrm{LP}(S,T,S',T') = \mathcal{S}(S \times T, S' \times T').$$

The proofs of all theorems rely on the fact that all the polygons are closed under map-compositions. This result is summarized in the following claims. The first follows from Proposition 5.1.

Claim 5.2 *Let X, X', X'', Y, Y', Y'' be finite sets and let $\eta \in SP(X,Y,X',Y')$ and $\eta' \in SP(X',Y',X'',Y'')$. Then, $\eta' \circ \eta \in SP(X,Y,X'',Y'')$.*

The following claim follows from the definition of the middle polygon.

Claim 5.3 *Let X, X', X'', Y, Y', Y'' be finite sets and let $\eta \in MP(X, Y, X', Y')$ and $\eta' \in MP(X', Y', X'', Y'')$. Then, $\eta' \circ \eta \in MP(X, Y, X'', Y'')$.*

In what follows we use the terms SP, MP and LP garbling and refer to elements of the small, middle and large polygons, respectively.

Definition 5.4 *Let $\sigma : K \to \Delta(S \times T)$ and $\sigma' : K \to \Delta(S' \times T')$ be two information structures.*
*(i) σ and σ' are SP-*equivalent *if there exist garblings $q \in SP(S, T, S', T')$ and $q' \in SP(S', T', S, T)$ be such that q transforms σ to σ' and q' transforms σ' to σ.*
*(ii) σ and σ' are MP-*equivalent *if there exist garblings $q \in MP(S, T, S', T')$ and $q' \in MP(S', T', S, T)$ such that q transforms σ to σ' and q' transforms σ' to σ.*

Claim 5.5 *Let $\mathcal{I}'$ be a garbled version of $\mathcal{I}$ with the garbling q. Fix the sets of actions A, B. Then, for every global strategy ε in the $\mathcal{I}'$-game,*

$$D(\mathcal{I}', \varepsilon) = D(\mathcal{I}, \varepsilon \circ q).$$

Here $\varepsilon \circ q$ is a global strategy in the $\mathcal{I}$-game.

Similarly, the following claim follows from (2).

Claim 5.6 *Let $\mathcal{I}'$ be a garbled version of $\mathcal{I}$ with the garbling q. Fix the action sets A, B and the payoff functions $r_k : A \times B \to \mathbb{R}$. Then, for every global strategy ε in the $\mathcal{I}'$-game,*

$$R(\mathcal{I}', \varepsilon) = R(\mathcal{I}, \varepsilon \circ q).$$

5.2 Proof of Theorem 4.1

We saw (just after Definition 4.2) that in order to prove Theorem 4.1, it is sufficient to check the following.

Proposition 5.7 *Let $\sigma : K \to \Delta(S \times T)$ and $\sigma' : K \to \Delta(S' \times T')$ be two information structures and let $\varphi \in \mathcal{S}(S, S')$, $\varphi' \in \mathcal{S}(S', S)$, $\psi \in \mathcal{S}(T, T')$, $\psi' \in \mathcal{S}(T, T')$ such that $\varphi \otimes \psi$ transforms σ to σ' and $\varphi' \otimes \psi'$ transforms σ' to σ. Then, $\varphi \otimes \psi$ (resp. $\varphi' \otimes \psi'$) acts faithfully on σ (resp. σ').*

Proof. Consider the probability distribution P over $K \times S \times T \times S' \times T'$, given by $P(k,s,t,s',t') = p(k)\sigma(s,t|k)\phi(s'|s)\psi(t'|t)$. We have to prove that $P(k,t'|s,s') = P(k,t'|s')$. For this purpose it suffices to prove that $I_P(k,t'|s,s') = I_P(k,t'|s')$, where I_P is the mutual information[5] calculated w.r.t. the distribution P.

We claim that

$$I_P(k,t'|s') \leq I_P(k,t'|s,s') = I_P(k,t'|s) \leq I_P(k,t,t'|s) = I_P(k,t|s). \tag{7}$$

The first equality follows from the fact that the conditional distribution (and therefore the conditional entropy) of k, t' given s, s' depends only on s. Indeed,

$$P(k,t'|s,s') = \frac{\sum_{t\in T} p(k)\sigma(s,t|k)\phi(s'|s)\psi(t'|t)}{\sum_{k\in K}\sigma(s|k)\phi(s'|s)} = \frac{\sum_{t\in T} p(k)\sigma(s,t|k)\psi(t'|t)}{\sum_{k\in K}\sigma(s|k)} = P(k,t'|s).$$

The second equality follows from the fact that the conditional distribution (and therefore the conditional entropy) of s given k,t,t' depends only on k,t. Indeed,

$$P(s|k,t,t') = \frac{p(k)\sigma(s,t|k)\psi(t'|t)}{p(k)\sigma(t|k)\psi(t'|t)} = \frac{p(k)\sigma(s,t|k)}{p(k)\sigma(t|k)} = P(s|k,t).$$

We therefore get, $I_P(k,t,t'|s) = I_P(s|k,t,t') = I_P(s|k,t) = I_P(k,t|s)$. This implies, $I_P(k,t'|s') \leq I_P(k,t|s)$.

Similarly, we define a probability P' over $K \times S \times T \times S' \times T'$ by $P'(k,s,t,s',t') = p(k)\sigma'(s',t'|k)\phi'(s|s')\psi'(t|t')$. Note that we do not know whether $P = P'$. Nevertheless, with the probability P', the same argument as above yields $I_{P'}(k,t|s) \leq I_{P'}(k,t'|s')$.

Since the conditional of P on k,t,s coincides with the conditional of P' on k,t,s, we obtain $I_P(k,t|s) = I_{P'}(k,t|s)$. For a similar reason, $I_P(k,t'|s') = I_{P'}(k,t'|s')$. Therefore, $I_P(k,t'|s') = I_P(k,t|s)$ and all the inequalities in (7) are actually equalities. ■

5.3 The proof of Theorem 4.5

The main tool of the proof is the following lemma, proved in Lehrer et al. (2006). It relates between non-communicating stochastic maps and the Bayesian solution.

[5]The *mutual information* of two random variables X, Y is given by $I(X|Y) = H(X) - H(X|Y) = H(Y) - H(Y|X)$, where H is Shannon's entropy. We use two facts: (i) $I(X|Y) \leq I(X|Y,Y')$ with equality iff the conditional distribution of X given Y equals the conditional distribution of X given Y, Y'; and (ii) $I(X|Y) \leq I(X,Y'|Y)$. See, for example, Cover and Thomas (1991).

Lemma 5.8 *A distribution π_0 over $K \times A \times B$ is a Bayesian solution iff there exists a global equilibrium ε and a non-communicating $\sigma \in \mathcal{S}(S \times T, A \times B)$ such that π_0 coincides with the distribution $D(\sigma, \varepsilon)$ (recall Eq. (3)).*

Assume first that σ and σ' are equivalent w.r.t. the Bayesian solution. Then, they induce the same Bayesian solution payoffs in any game with common interests (i.e., games where $r^1 = r^2$). In particular, they induce the same maximal expected payoff of a Bayesian solution in such games. Theorem 4.6 in Lehrer et al. (2006) implies that they are MP-equivalent.

We now assume that $\sigma : K \to \Delta(S \times T)$ and $\sigma' : K \to \Delta(S' \times T')$ are two MP-equivalent and prove that they are equivalent with respect to the Bayesian solution.

There exist non-communicating garblings $q : S \times T \to \Delta(S' \times T')$ and $q' : S' \times T' \to \Delta(S \times T)$ such that for every $k \in K$, $q(\sigma(k)) = \sigma'(k)$ and $q'(\sigma'(k)) = \sigma(k)$. Let $\xi = q' \circ q$. Then, ξ is a non-communicating garbling (by Claim 5.3). We view ξ as a Markov transition matrix over $S \times T$. Let d be its period. By the theory of Markov chains (see e.g., Feller (1968)), the limit $\xi^\infty = \lim_{n\to\infty} \xi^{nd}$ exists. By replacing q' with $\xi^\infty \circ q'$ we can assume w.l.o.g. that $\xi = \xi^\infty$; i.e., that ξ is idempotent. We are thus in a position to apply Lemma 8.1 (see Appendix) with $Z = S \times T$ for every $k \in K$.

Let $\pi \in \Delta(K \times A \times B)$ be a Bayesian solution in the σ'-game. By Lemma 5.8 there exists a non-communicating map $\varepsilon' : S' \times T' \to A \times B$ such that ε' is a global equilibrium in the σ'-game. Let $\varepsilon : S \times T \to A \times B$ be given by

$$\varepsilon = \varepsilon' \circ q \circ \xi. \tag{8}$$

Then ε is a non-communicating garbling and, for every $k \in K$,

$$\varepsilon(\sigma(k)) = \varepsilon'(q(\xi(\sigma(k)))) = \varepsilon'(q(\sigma(k))) = \varepsilon'(\sigma'(k)).$$

Thus $D(\sigma, \varepsilon) = D(\sigma', \varepsilon')$ (i.e., the distributions over plays induced by ε in the σ-game equals the distribution over plays induced by ε' in the σ'-game). In particular $R^1(\sigma, \varepsilon) = R^1(\sigma', \varepsilon')$. By Lemma 5.8, it remains to show that ε is a global-equilibrium in the σ-game. Indeed, let $v : S \times A \to A$ be a deviation of player 1.

$$\begin{aligned}
&\sum_k p(k) \sum_{s,t,a,b} \sigma(s,t|k)\varepsilon(a,b|s,t) r^1(k, v(s,a), b) = \\
&\sum_k p(k) \sum_{s,t,s',t',a,b} \sigma(s,t|k)(q \circ \xi)(s',t'|s,t)\varepsilon'(a,b|s',t') r^1(k, v(s,a), b) = \\
&\sum_k p(k) \sum_{s,t,s',t',a,b} \sigma'(s',t'|k)(\xi \circ q')(s,t|s',t')\varepsilon'(a,b|s',t') r^1(k, v(s,a), b) = \\
&\sum_k p(k) \sum_{s,s',t',a,b} \sigma'(s',t'|k)\varepsilon'(a,b|s',t')(\xi \circ q')(s|s') r^1(k, v(s,a), b) \le \\
&R^1(\varepsilon', \sigma') = R^1(\varepsilon, \sigma).
\end{aligned} \tag{9}$$

The first equality follows from (8). The second follows from Lemma 8.1 with $z = (s,t), z' = (s',t'), p = \sigma(k), p' = \sigma'(k)$. In the third equation we use the fact that $\xi \circ q'$ is non-communicating, which implies that $\sum_t \xi \circ q'(s,t|s',t')$ is independent of t'. We denote this sum by $(\xi \circ q')(s|s')$. The inequality follows from the fact that ε is a global equilibrium in the game with the structure σ'. That is, player 1 has no incentive to randomize s according to $\xi \circ q'(s')$, and then deviate to $v(s,a)$.

5.4 The proof of Theorem 4.4

The proof of Theorem 4.5 applies, *mutatis mutandis*, to Theorem 4.4. For the 'if' part we use Theorem 4.5 in Lehrer et al. (2006) instead of Theorem 4.6 in that paper. For the 'only if' part, we carry the same construction. This time, all the garblings are in the small polygon.

Instead of Lemma 5.8 we use the definition of agent-normal-form correlated equilibrium to determine the following. A distribution π_0 over $K \times A \times B$ is induced by an agent-normal-form correlated equilibrium iff there exists a global equilibrium ε and a coordinated $\sigma \in \mathcal{S}(S \times T, A \times B)$ such that π_0 coincides with the distribution $D(\sigma, \varepsilon)$.

Instead of Claim 5.3 we use Claim 5.2. Note that in the last equality of Equation (9) we used the fact that the garbling $\xi \circ q'$ is non-communicating. This remains valid also here, since every coordinated garbling is in particular non-communicating.

6 The hierarchy of beliefs and garblings

In this section we elaborate on the comparison of the result of Gossner and Mertens (2001) on zero-sum games. They prove that two structures are equivalent for zero-sum games if they induce the same hierarchy of beliefs. Here we investigate the relation between this property and equivalence with respect to some garblings.

6.1 Hierarchy of beliefs and coordinated garblings

Let (S, T, σ) be an information structure. Consider the probability space $\Omega = K \times S \times T$ endowed with the probability distribution $I\!P$, where $I\!P(k, s, t) = p(k)\sigma(s, t|k)$ and p is the prior distribution over states. For every $s \in S$ denote by $\rho_1(s)$ the posterior distribution over $K \times T$ given s. Similarly, for every $t \in T$ $\rho_2(t) \in \Delta(K \times S)$ is the posterior distribution over $K \times S$ given t. $\rho_1(k, t|s)$ is the probability the the state is k and the signal of player 2 is t given that the signal of player 1 is s.

Let $B_1 = \Delta(K)$, and let $f_1 : S \to \Delta(K)$ (resp. $g_1 : T \to B_1$) be the map that assigns to every signal s (resp. t) player 1's (resp. player 2's) posterior belief over K given the signal s (resp. t). B_1 is the set of first order beliefs. Denote[6] $B_2 = \Delta(K \times B_1)$. This is the space of second order beliefs (that is, beliefs over the state of nature and the other player's first order belief).

Before we proceed we need the following notation. If $\phi : X \to Y$ and $q : Z \to \Delta(X)$, then $\phi \circ q$ is a function from Z to $\Delta(Y)$ defined as follows. For every $z \in Z$,

$$\phi \circ q(y|z) = \sum_{x, \phi(x)=y} q(x|z).$$

This agrees with our previous convention of denoting by ϕ the linear extension of ϕ from $\Delta(X)$ to $\Delta(Y)$.

¿From player 1's point of view, when he receives the signal s, he assigns the probability $\rho_1(k, t|s)$ to the event that the state is k and that player 2 believes that the distribution over states is $g_1(t)$. For every s, $f_2(s)$ is the second order belief given s: it is the belief of player 1 over the first order beliefs of player 2.

More formally, recall that $\rho_1(s)$ is a distribution in $\Delta(K \times T)$ and that $g_1(t)$ is in B_1. Let id denote the identity map from K to K. Thus, the function f_2, defined as $(\mathrm{id} \otimes g_1) \circ \rho_1$, assigns to every s a point in $\Delta(K \times B_1)$ (i.e., $f_2 : S \to B_2$).

[6]For every space C, $\Delta(C)$ denotes the space of all probability measures over C that have a final support.

We define inductively,

$$B_{n+1} = \Delta(K \times B_n)$$
$$f_{n+1}(s) = (\mathrm{id} \otimes g_n) \circ \rho_1(s)$$
$$g_{n+1}(t) = (\mathrm{id} \otimes f_n) \circ \rho_2(t).$$

For every $s \in S$ and $t \in T$, $(f_n(s), g_n(t))$ is a point in $B_n \times B_n$. It is the pair of the n-th order beliefs of players 1 and 2, when they receive the signals s and t, respectively. Let $\mu_n = \mu_n(\sigma)$ be the distribution over $B_n \times B_n$ induced by σ, f_n and g_n. That is, the probability of the point $(f_n(s), g_n(t))$ is

$$\sum_{\substack{(s',t')\in S\times T \text{ s.t.} \\ (f_n(s'),g_n(t'))=(f_n(s),g_n(t))}} \sum_k p(k)\sigma(s',t'|k).$$

We call μ_n the distribution of n-th order beliefs induced by σ.

The following theorem establishes the linkage between garbling and the hierarchy of beliefs. It states that two information structures that are MP-equivalent induce the same hierarchy of beliefs.

Theorem 6.1 *If two information structures $\mathcal{I} = (S, T, \sigma)$ and $\mathcal{I}_* = (S_*, T_*, \sigma_*)$ are MP-equivalent, then $\mu_n(\sigma) = \mu_n(\sigma_*)$ for every n.*

The proof is based on the following claim:

Claim 6.2 *Let $\sigma : K \to \Delta(S \times T)$ be an information structure and $q : S \times T \to \Delta(S' \times T')$ a non-communicating garbling. If $\sigma_q : K \to \Delta\big((S \times S') \times (T \times T')\big)$ is the information structure given by*

$$\sigma_q(s, s', t, t'|k) = \sigma(s, t|k)q(s', t'|s, t),$$

then σ_q induces the same hierarchy of beliefs as σ.

Note that according to σ_q the players receive the signals s, t and, in addition, the garbled signals s', t'. Since q is non-communicating the garbled signal doesn't change the player's posterior belief about the state of nature and also the player's posterior belief about the other player posterior belief, etc.

Returning to the proof of the theorem, let q, q' be non-communicating garblings such that $q \circ \sigma = \sigma'$ and $q' \circ \sigma = \sigma$. Let $\xi = q' \circ q$. Then Lemma 8.1 states that $\sigma_{q\circ\xi} = \sigma'_{\xi\circ q'}$. From the previous claim it follows that σ and $\sigma_{q\circ\xi}$ induce the same hierarchy and that σ' and $\sigma'_{\xi\circ q'}$ induce the same hierarchy. It follows that σ and σ' induce the same hierarchy.

6.2 Hierarchy of beliefs and Nash equivalence

Ely and Peski (2004) construct a "larger" hierarchy of beliefs based on conditional beliefs: the Δ-hierarchy. As opposed to the regular hierarchy of beliefs which is based on individual signals, the Δ-hierarchy is based on the beliefs conditional on the *joint* signal.

In order to illustrate the main idea (without elaborating on the formal definition) recall Example 3.2. Imagine that the σ-signal of player 1 is, say 3. He then believes that the distribution over T is uniform. Thus, the joint signal is $(3, 1)$, $(3, 2)$, $(3, 3)$ or $(3, 4)$ with probability 1/4 each. Since the joint signal uniquely identifies the state, in each case the probability over K is a mass point. Moreover, this is common knowledge. This is true for both σ and σ', and thus, they induce the same Δ-hierarchy.

Example 3.2 demonstrates a game in which σ induces a Nash-payoff which is higher than any Nash-payoff induced by σ'. In this game the two information structures induce the same hierarchy of beliefs. This shows, in particular, that two information structures can induce the same hierarchy of beliefs and the same Δ-hierarchy without being equivalent with respect to Nash or to agent-normal-form-correlated equilibrium.

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8 Appendix: some relevant properties of Markov Chains

Let Z be a finite set. A stochastic maps $\xi : Z \to \Delta(Z)$ can be viewed as a Markov transition matrix over Z. A probability measure $p \in \Delta(Z)$ is called ξ-invariant if

$$p(z) = \sum_{z' \in Z} p(z')\xi(z|z')$$

for every $z \in Z$. Recall that ξ is called *idempotent* if $\xi \circ \xi = \xi$. We will use some classical propositions from the theory of Markov chains that can be found in Feller (1968).

For a stochastic map $q : Z \to \Delta(Z')$, we denote by $q(z'|z)$ the probability of z' according to the distribution $q(z)$.

Lemma 8.1 *Let Z and Z' be two finite sets. And let $q : Z \to \Delta(Z')$, $q' : Z' \to \Delta(Z)$ be a pair of garblings, $p \in \Delta(Z)$ and $p' \in \Delta(Z')$ such that $q(p) = p'$ and $q'(p') = p$ (recall that we identify q and q' with their linear extensions). Assume that $\xi = q' \circ q$ is idempotent. Then, for every $z \in Z$ and $z' \in Z'$,*

$$p(z) \cdot \sum_{w \in Z} \xi(w|z) q(z'|w) = p'(z') \cdot \sum_{w \in Z} q'(w|z') \xi(z|w).$$

Equivalently, this equation can be written as

$$p(z) \cdot q \circ \xi(z'|z) = p'(z') \cdot \xi \circ q'(z|z'). \tag{10}$$

Proof.

1. Let $Z_1, \dots, Z_k$ be the ergodic sets of ξ, p_i their corresponding invariant distributions (such that the support of p_i is Z_i) and $p'_i = q(p_i)$.

   Since $\xi(p) = q'(q(p)) = q'(p') = p$ it follows that p is ξ-invariant and therefore,

   $$p = \sum_i \lambda_i p_i, \tag{11}$$

   for some $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$. From the linearity of the garbling it follows that,

   $$p' = \sum_i \lambda_i p'_i. \tag{12}$$

2. Since ξ is idempotent, it is reduced to its invariant distribution on each ergodic class:

   $$\text{if } z \in Z \text{ then } \xi(z) = p_i, \tag{13}$$

   and therefore $q \circ \xi(z) = p'_i$. Using (11) it follows that the left-hand side of (10) equals $\sum_i \lambda_i p_i(z) \cdot p'_i(z')$.

3. Since $\xi = q' \circ q$ is idempotent, it follows that if $z \in Z_i$ and $z' \in Z'$ such that $q(z'|z) > 0$, then the support of $q'(z')$ is contained in Z_i. Indeed, if $q'(z''|z') > 0$ for some $z'' \in Z$, then $\xi(z''|z) \geq q(z'|z) \cdot q'(z''|z') > 0$, and since Z_i is ergodic, it follows that $z'' \in Z_i$. In particular, it follows from (13) that if $q(z'|z) > 0$ then $\xi \circ q'(z') = p_i$. Using (12), the right-hand side of (10) equals $\sum_i \lambda_i p_i(z) \cdot p'_i(z')$.

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