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# Notes, Comments, and Letters to the Editor

## Some General Results on the Metric Rationalization for Social Decision Rules*

EHUD LERER

*Department of Mathematics, Hebrew University of Jerusalem, 91905 Jerusalem, Israel*

AND

SHMUEL NITZAN

*Department of Economics, Bar-Ilan University, 52100 Ramat Gan, Israel*

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The purpose of this paper is twofold. First, to provide a transparent characterization of the family of metrizable social decision rules. Second, to provide the necessary and sufficient conditions for a reasonable metric rationalization. Theorem 1 establishes that the class of metrizable social decision rules is uniquely characterized by a variant of the well-known Pareto condition. Theorem 2 establishes that positional rules can be characterized in terms of a special class of additively decomposable quasi-metric rationalizations. Theorem 3 characterizes strong positional rules in terms of reasonable metric rationalizations. *Journal of Economic Literature*. Classification Number: 025. 

### I. INTRODUCTION

Standard voting procedures can be represented by means of an additively decomposable metric which is defined on preference profiles whereby the socially selected outcomes are precisely those alternatives which are the closest to being unanimously most preferred. In general, the purpose of a metric is to define distance and the metric generated by a social decision rule defines the notion of an alternative being close to

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unanimously preferred. Additive decomposability requires that the distance between any two preference profiles be the sum of the distances between the respective individual preference relations comprising the two profiles. (Nitzan [5], Farkas and Nitzan [2].)

If some alternative is most preferred by all individuals then surely it should be declared the consensus "social choice." This is the so-called unanimity principle which is naturally very appealing and has been accepted as axiomatic since Arrow's work [1]. Of course, a unanimously preferred alternative generally does not exist. A significant problem therefore is to find out in what precise sense different social decision rules attempt (if at all) to approximate or respect the political ideal of using the unanimity rule (see Wicksell [6]). Put differently, the problem is that of clarifying the meaning of the consensus obtained by various decision rules.

A reasonable additively decomposable metric rationalization according to the unanimity criterion certainly suggests a natural interpretation for the social compromise attained by certain social decision rules. Furthermore, it is desirable for a social decision rule to be metrizable in the above sense because otherwise it cannot satisfy a weak version of the Pareto criterion. Specifically, our first theorem establishes that a social decision rule is metrizable according to unanimity if and only if it is Paretian. Such a rule, however, need not have an additively decomposable metric rationalization. Our second theorem establishes that any positional rule (sometimes referred to as weighted summation rules or point voting schemes) is uniquely characterized by a reasonable symmetric additively decomposable quasi-metric rationalization. The rationalizations of the common plurality and Borda rules reported by Farkas and Nitzan [2] and Nitzan [5] can be derived as special cases of this general result. Theorem 3 characterizes strong positional rules in terms of reasonable symmetric additively decomposable metric rationalizations. We conclude by demonstrating that a rule which has a symmetric additively decomposable metric rationalization is not necessarily a positional rule.

## II. Metrizable Social Decision Rules

Let $X$ be a nonempty finite set of alternatives. $N = \{1, \ldots, n\}$ is a finite set of individuals (decision makers). The set of linear orders (complete, transitive, and asymmetric relations) on $X$ is denoted by $P$. The set of all preference profiles is denoted $\Omega = P^N$. For each profile $P = (P_1, \ldots, P_n) \in \Omega$, $P_i \in P (i \in N)$ is individual $i$'s preference relation on $X$. A social decision rule is a function $h: \Omega \to 2^X - \{\varnothing\}$, where $2^X - \{\varnothing\}$ is the set of all nonempty subsets of $X$.

Consider the set of profiles $U(x)$ in $\Omega$ having alternative $x$ as a unanimous best outcome, $U(x) = V(x) \times V(x) \times \cdots \times V(x)$, where $V(x) = \{P_i \in P \mid xP_i y \ \forall y \in X, y \neq x\}$. Note that $x \neq y \Rightarrow U(x) \cap U(y) = \varnothing$.

DEFINITION 1. A social decision rule $h$ is *Paretian* if $\forall x \in X$, $P \in U(x) \Rightarrow h(P) = \{x\}$.

Let $\delta$ be a metric on $\Omega$. This metric defines a distance function $d$ between a profile $P$ and a non-empty set of profiles $Y \subset \Omega$, $d(P, Y) = \mathrm{Min}_{Q \in Y} \delta(P, Q)$.

DEFINITION 2. The metric $\delta$ on $\Omega$ is a *rationalization* according to unanimity (henceforth a *rationalization*) for the social decision rule $h$, if $\forall P \in P$, $h(P) = \{x \in X \mid d(P, U(x)) \leqslant d(P, U(y)) \ \forall y \in X\}$.

That is, the metric $\delta$ rationalizes $h$ according to the unanimity criterion whenever for any given profile $P$ the social outcome is an alternative which is unanimously preferred according to the profile "nearest" to the profile $P$ and for which ananimity exists. The characterization of the family of social decision rules having such a metric rationalization is provided by the following:

THEOREM 1. *A social decision rule h has a metric rationalization if and only if it is Paretian.*

*Proof.* (i) The reader can easily verify that if $\delta$ rationalizes $h$, then $h$ is Paretian.

(ii) Suppose that $h$ is Paretian. Define $\delta$ as follows: $\forall P, Q \in \Omega$,

$$\delta(P, Q) = \begin{cases} 0 & P = Q \\ 1 & P \neq Q, h(P) \cap h(Q) \neq \varnothing \\ 2 & P \neq Q, h(P) \cap h(Q) = \varnothing. \end{cases}$$

$\delta$ is obviously symmetric, non-negative, and $\delta(P, Q) = 0 \Leftrightarrow P = Q$. It also satisfies the triangle inequality and so $\delta$ is a metric. Let us conclude the proof by showing that the metric $\delta$ is a rationalization for $h$. Let $\bar{d}(P, Y) = \mathrm{Min}_{Q \in Y} \delta(P, Q)$, where $P \in \Omega$, $Y \subset \Omega$. Define $\bar{h}\colon \Omega \to 2^X - \{\varnothing\}$ as follows: $\forall P \in \Omega$, $\bar{h}(P) = \{x \in X \mid \bar{d}(P, U(x)) \leqslant \bar{d}(P, U(y)) \ \forall y \in X$. We now show that $h = \bar{h}$. If $P \in U(x)$ for some $x \in X$, then $\bar{d}(P, U(x)) = \delta(P, P) = 0$. Hence, $z \in \bar{h}(P) \Rightarrow \bar{d}(P, U(z)) = 0$ and so, $P \in U(z) \Rightarrow x = z$. Therefore, $\bar{h}(P) = \{x\} = h(P)$. Suppose $P \notin U(x), \forall x \in X$. Then $\forall y \in X, \bar{d}(P, U(y)) \neq 0$. If $x \in h(P)$, then $\delta(P, Q) = 1$ for any $Q \in U(x)$. Therefore, $\bar{d}(P, U(x)) = 1$ and $x \in \bar{h}(P)$ from the definition of $\bar{h}$. Therefore, $h(P) \subseteq \bar{h}(P)$. Suppose $x \in \bar{h}(P)$. If

$x \notin h(P)$, then $\exists y \in h(P)$ and $\bar{d}(P, U(y)) = 1$. Therefore, $\bar{d}(P, U(x)) = 2$ and, by the definition of $\bar{h}$, $x \notin \bar{h}(P)$, a contradiction. Therefore, $x \in h(P)$ and $\bar{h}(P) \subseteq h(P)$. Therefore, $\bar{h}(P) = h(P)$. Q.E.D.

## III. Reasonable Metric Rationalizations

The metric used in the proof of Theorem 1, $\bar{\delta}(P, Q)$, is induced by the choices determined according to the social decision function. That is, loosely speaking, the metric is based on what the profile does and not on what the profile is. Also, although the triangle inequality is satisfied by $\bar{\delta}(P, Q)$ it does not play any role in the proof and therefore the theorem could be stated as follows: a social decision rule $h$ has a quasi-metric rationalization if and only if it is Paretian ($\delta$ is a quasi-metric if it is symmetric, non-negative, and $\delta(P, Q) = 0$ if and only if $P = Q$). The lack of dependence on the internal structure of the profiles and the inessentiality of the triangle unequality suggest that some unreasonable social decision rules may be rationalized by unreasonable quasi-metrics.

In particular, the social decision rule need not be monotonic, anonymous or neutral[1] and the distance function between preference profiles need not be reasonably related to the individual distances between the respective preference relations comprising the profiles. The normative appeal of metrizable rules as well as their interpretation are considerably strengthened once the distance function between profiles is "appropriately related" to the structure of the profiles. In this section, we suggest four particular restrictions on quasi-metrics. A quasi-metric (metric) satisfying these requirements is considered as reasonable. We then demonstrate that only (strong) positional rules do have reasonable quasi-metric (metric) rationalizations.

Definition 3. A social decision rule $h$ has an *additively decomposable quasi-metric rationalization* $\delta$ if:

(i) $\delta$ rationalizes $h$;

(ii) there exist $n$ metrics on $P$, $\delta_i$, $i = 1, \ldots, n$, such that for any two profiles $P = (P_1, \ldots, P_n)$, $Q = (Q_1, \ldots, Q_n)$, $P, Q \in \Omega$, $\delta(P, Q) = \sum_{i=1}^{n} \delta_i(P_i, Q_i)$.

[1] Anonymity ensures that the social outcome is independent of the way in which individuals are labelled. Neutrality requires independence of the labelling of the alternatives. Monotonicity guarantees that as some alternative becomes more widely favored by individuals, it does not become more difficult to sustain as a social outcome. For more formal definitions of these standard conditions which appear to be fundamental to democratic decision-making, see Fishburn [3, pp. 100–101]. Example 1, below, confirms that a metrizable rule need not be neutral.

DEFINITION 4. A social decision rule $h$ has a *symmetric additively decomposable quasi-metric rationalization* $\delta$ if (i) and (ii) of Definition 3 are satisfied and $\delta_i = \delta_j = \delta$, $\forall i, j \in N$.

A metrizable rule need not however have an additively decomposable metric rationalization. A particular example is presented below.

EXAMPLE 1. Let $X = \{x, y\}$ and $N = \{1, 2\}$. A profile can be represented by a pair, such as $(x, y)$ implying $xP_1 y$, $yP_2 x$. Define, $h(y, y) = y$ and $h(\cdot) = x \, \forall P \neq (y, y)$; $h$ is Paretian and therefore it has some metric rationalization $\delta$. Suppose that $\delta(P, Q) = \delta_1(P_1, Q_1) + \delta_2(P_2, Q_2) \, \forall P, Q$. Since $\delta((x, y), (x, x)) < \delta((x, y), (y, y))$, we have $\delta_2(y, x) = \delta_2(x, y) < \delta_1(x, y)$. Since $\delta((y, x), (x, x)) < \delta((y, x), (y, y))$, we have $\delta_1(y, x) = \delta_1(x, y) < \delta_2(x, y)$, a contradiction. Therefore, $\delta$ is not additively decomposable.

The well-known Borda method and the common plurality rule do have additively decomposable metric rationalizations which are symmetric (cf. Farkas and Nitzan [2] and Nitzan [5]). These two social decision rules are typical positional rules (which are occasionally referred to as weighted summation schemes, scoring methods, or point voting methods). We demonstrate below that positional rules are uniquely charactrized by a special class of symmetric additively decomposable metric rationalizations.

Let $|X| = s$ and denote by $t(x, P_i)$ the position of alternative $x$ according to individual $i$'s linear order $P_i$. That is, $\forall P_i \in P$, $x \in X$, $t(x, P_i) = |\{y \in X: yP_i x\}| + 1$. For $1 \leqslant j \leqslant s$ and $P_i \in P$ define the function $r$, $r(j, P_i) = x \Leftrightarrow t(x, P_i) = j$.

*Remark* 1. By definition of $r$ and $t$:

(i) $t(r(j, P_i), P_i) = j$;

(ii) $r(t(x, P_i), P_i) = x$;

(iii) for any $P_i \in P$, $P_i \in V(r(1, P_i))$.

Let us denote by $\sigma$ a permutation of $X$. For a linear order $P_i \in P$ and a permutation $\sigma$ denote by $P_i^\sigma$ the permuted linear order. That is, $\forall x, y \in X$, $xP_i y \Leftrightarrow \sigma(x) \, P_i^\sigma \, \sigma(y)$.

DEFINITION 5. A quasi-metric (metric) $\delta$ on $P$ is *neutral* if ($\forall P_i, Q_i \in P$ and any permutation $\sigma$ of $X$) $\delta(P_i, Q_i) = \delta(P_i^\sigma, Q_i^\sigma)$.[2]

[2] This axiom is similar to Axiom 2 in Kemeny and Snell [4, Chap. II] who first introduced metrics into social choice theory.

DEFINITION 6. A quasi-metric (metric) $\delta$ on $P$ is *monotonic* if $(\forall x, y \in X, \forall P_i \in P)\, xP_i y \Rightarrow \delta(P_i, V(x)) \leqslant \delta(P_i, V(y))$.

DEFINITION 7. A quasi-metric (metric) $\delta$ on $P$ is *strongly monotonic* if $(\forall x, y \in X, \forall P_i \in P)\, xP_i y \Rightarrow \delta(P_i, V(x)) < \delta(P_i, V(y))$.

DEFINITION 8. A social decision rule $h$ has a *reasonable* quasi-metric rationalization if it has a symmetric additively decomposable quasi-metric rationalization $\bar{\delta}$ ($\bar{\delta} = \Sigma\delta$) and $\delta$ is neutral and monotonic.

DEFINITION 9. A social decision rule $h$ has a *strongly reasonable* metric rationalization $\bar{\delta}$ ($\bar{\delta} = \Sigma\delta$) and $\delta$ is neutral and strongly monotonic.

DEFINITION 10. A social decision rule $h$ is a (*strong*) *positional rule* if $\forall P \in \Omega, h(P) = \{x \in X: T(x, P) \geqslant T(y, P)\, \forall y \in X\}$, where $T(x, P) = \sum_{i=1}^{n} \alpha(t(x, P_i))$ and $\alpha$ is a real function defined on the positive integers that satisfies

$$\alpha(1) > \alpha(2) \geqslant \alpha(3) \geqslant \cdots \geqslant \alpha(s) \qquad \text{and for i} > s, \alpha(i) = 0$$

$$(\alpha(1) > \alpha(2) > \alpha(3) > \cdots > \alpha(s) \qquad \text{and for } i > s, \alpha(i) = 0).$$

THEOREM 2. *A social decision rule h has a reasonable quasi-metric rationalization if and only if it is a positional rule.*

*Proof.* (i) Suppose that the social decision rule $h$ has a reasonable quasi-metric rationalization. For $P_i \in P$ define

$$\varepsilon(j) = \delta(P_i, V(r(j, P_i))) \qquad (j = 1, ..., s).$$

LEMMA 1. *The definition of $\varepsilon(j)$ is independent of the choice of $P_i$.*

*Proof.* Let $P_i, Q_i \in P$. Consider the permutation $\sigma$ on $X$ such that $P_i^\sigma = Q_i$. Since, by assumption, $\delta$ is neutral, $\forall x \in X$ and $\forall \sigma$ on $X$, $\delta(P_i, V(x)) = \delta(P_i^\sigma, V(\sigma(x)))$. By the definition of $r, r(j, P_i) = x \Leftrightarrow t(x, P_i) = j$. But, $t(x, P_i) = j$ and $t(\sigma(x), P_i^\sigma) = j$. Hence, $r(j, P_i^\sigma) = \sigma(x)$ or $\sigma(r(j, P_i)) = r(j, P_i^\sigma)$. Therefore, $\delta(P_i, V(r(j, P_i))) = \delta(P_i^\sigma, V(\sigma(r(j, P_i)))) = \delta(Q_i, V(r(j, P_i^\sigma))) = \delta(Q_i, V(r(j, Q_i)))$. Define

$$\alpha(j) = \varepsilon(s) - \varepsilon(j) \qquad (j = 1, ..., s).$$

LEMMA 2. $\alpha(1) > \alpha(2) \geqslant \cdots \geqslant \alpha(s)$.

*Proof.* (i) We need to prove that $\varepsilon(s) - \varepsilon(1) > \varepsilon(s) - \varepsilon(2) \geqslant \cdots \geqslant$

$\varepsilon(s)-\varepsilon(s)$ or $\varepsilon(1)<\varepsilon(2)\leqslant \cdots \leqslant\varepsilon(s)$. By the monotonicity of $\delta$, if $xP_i y$, then $\delta(P_i, V(x))\leqslant\delta(P_i, V(y))$. Therefore, for $1\leqslant j\leqslant s$,

$$\varepsilon(j-1)=\delta(P_i, V(r(j-1, P_i)))\leqslant\delta(P_i, V(r(j, P_i)))=\varepsilon(j).$$

Now suppose that $\varepsilon(1)=\varepsilon(2)$. That is,

$$\delta(P_i, V(r(1, P_i)))=\delta(P_i, V(r(2, P_i)))=0.$$

Hence, there exist $Q_i\in V(r(2, P_i))$ such that $\delta(P_i, Q_i)=0$, but, by Remark 1(iii), $P_i\notin V(r(2, P_i))$, and so $\delta$ is not a quasi-metric, a contradiction. We thus obtain that $\varepsilon(1)<\varepsilon(2)\leqslant \cdots \leqslant\varepsilon(s)$. For $P\in\Omega$ let

$$\bar{h}(P)=\left\{x\in X\colon \sum_{i=1}^{n}\alpha(t(x, P_i))\geqslant\sum_{i=1}^{n}\alpha(t(y, P_i))\ \forall y\in X\right\}.$$

We now show that $\bar{h}(P)=h(P)$. By definition,

$$\sum_{i=1}^{n}\alpha(t(x, P_i))=\sum_{i=1}^{n}\varepsilon(s)-\varepsilon(t(x, P_i))=n\varepsilon(s)-\sum_{i=1}^{n}\varepsilon(t(x, P_i))$$

and so,

$$\sum_{i=1}^{n}\alpha(t(x, P_i))\geqslant\sum_{i=1}^{n}\alpha(t(y, P_i))\Leftrightarrow\sum_{i=1}^{n}\varepsilon(t(x, P_i))\leqslant\sum_{i=1}^{n}\varepsilon(t(y, P_i))$$

and therefore $\bar{h}$ can be defined as

$$\bar{h}(P)=\left\{x\in X\colon \sum_{i=1}^{n}\varepsilon(t(x, P_i))\leqslant\sum_{i=1}^{n}\varepsilon(t(y, P_i))\ \forall y\in X\right\}.$$

By Remark 1(ii) and Lemma 1,

$$\varepsilon(t(x, P_i))=\delta(P_i, V(r(t(x, P_i), P_i)))=\delta(P_i, V(x)).$$

Hence,

$$\bar{h}(P)=\left\{x\in X\colon \sum_{i=1}^{n}\delta(P_i, V(x))\leqslant\sum_{i=1}^{n}\delta(P_i, V(y))\ \forall y\in X\right\}=h(P).$$

(ii) Suppose that $h$ is a positional rule. We have to show that $h$ has a reasonable quasi-metric rationalization.

For $P_i, Q_i\in P$, let us denote by $\gamma(P_i, Q_i)$ the inversion metric—the minimum number of steps needed to transform $P_i$ into $Q_i$, where a single step consists of interchanging the relative ranking of two neighboring alternatives.

Employing the real-valued function associated with the positional rule $h$ (see Definition 9), let us define the following quasi-metric on $P$. For $P_i, Q_i \in P$,

$$\delta(P_i, Q_i) = \alpha(1) - \alpha(1 + \gamma(P_i, Q_i)). \tag{1}$$

Note that $\delta$ is neutral and monotonic. Also note that

$$\min\{\gamma(P_i, Q_i): Q_i \in V(x)\} = t(x, P_i) - 1. \tag{2}$$

(1), (2), and the monotonicity of $\alpha$ imply that

$$\min\{\delta(P_i, Q_i): Q_i \in V(x)\} = \alpha(1) - \alpha(t(x, P_i)). \tag{3}$$

We now define the quasi-metric $\bar{\delta}$, on $\Omega$. For any two profiles $P, Q$ in $\Omega$ $\bar{\delta}(P, Q) = \sum_{i=1}^{n} \delta(P_i, Q_i)$. We prove below that the quasi-metric $\bar{\delta}$ rationalizes the positional rule $h$. For $P \in \Omega$. $\phi \neq Y \subset \Omega$ let $\bar{d}(P, Y) = \min[\bar{\delta}(P, Q): Q \in Y]$. For $x \in X$ and $P \in \Omega$ we therefore obtain

$$\begin{aligned} \bar{d}(P, U(x)) &= \min\{\bar{\delta}(P, Q): Q \in U(x)\} \\ &= \min\left\{\sum_{i=1}^{n} \delta(P_i, Q_i): Q_i \in V(x), i = 1,..., n\right\} \\ &= \sum_{i=1}^{n} \min\{\delta(P_i, Q_i): Q_i \in V(x)\} \end{aligned}$$

and so, by (3)

$$\bar{d}(P, U(x)) = \sum_{i=1}^{n} (\alpha(1) - \alpha(t(x, P_i))). \tag{4}$$

By assumption, for $P \in \Omega$,

$$\begin{aligned} h(P) &= \left\{x \in X: (\forall y \in X)\left[\sum_{i=1}^{n} \alpha(t(x, P_i)) \geqslant \sum_{i=1}^{n} \alpha(t(y, P_i))\right]\right\} \\ &= \left\{x \in X: (\forall y \in X)\left[\sum_{i=1}^{n} (\alpha(1) - \alpha(t(x, P_i)) \right.\right. \\ &\left.\left. \leqslant \sum_{i=1}^{n} [\alpha(1) - \alpha(t(y, P_i))\right]\right\}. \end{aligned}$$

By (6),

$$h(P) = \{x \in X: (\forall y \in X)[\bar{d}(P, U(x)) \leqslant \bar{d}(P, U(y))]\}.$$

Finally, note that $h(P)$ is Paretian, that is, for $P \in U(x)$, $h(P) = \{x\}$, since $\alpha(1) > \alpha(2)$. So $\bar{\delta}$ is indeed a reasonable rationalization for $h$.

*Remark* 2. If $\forall i, j, \alpha_1 + \alpha_{i+j-1} \geqslant \alpha_i + \alpha_j$, then $\bar{\delta}$ satisfies the triangle inequality. In other words, the above inequality defines a subclass of positional rules that have a symmetric, additively decomposable neutral and monotonic *metric* rationalization.

THEOREM 3. *A social decision rule h has a strongly reasonable metric rationalization if and only if it is a strong positional rule.*

*Proof.* (i) Suppose that the social decision rule $h$ has a strongly reasonable metric rationalization. Define $\alpha(j)$ $(j = 1,..., s)$ as in the proof of Theorem 2.

LEMMA 3. $\alpha(1) > \alpha(2) > \cdots > \alpha(s)$.

*Proof.* (i) Use the definition of $\alpha(j)$, $\varepsilon(j)$, the strong monotonicity of $\delta$, and a similar proof to that of Lemma 2. Using Lemma 3 and the proof of part (i) of Theorem 2, one directly obtains that $h$ is a strong positional rule.

(ii) Suppose that the social decision rule $h$ is a strong positional rule. Employing the function $\alpha$ associated with the positional rule $h$, let $\phi(j) = \alpha(j-1) - \alpha(j)$ $(j = 1,..., s)$. Since $h$ is a strong positional rule $\phi(j) > 0$ $(j = 2,..., s)$.

Define $\Pi_j$ to be the transposition that interchanges the relative ranking of the neighboring alternatives, $x$, $y$, such that $t(r(j, P_i), \Pi_j(P_i)) = j-1$ and $t(r(j-1, P_i), \Pi_j(P_i)) = j$. Let us denote by $\Pi_j(P_i)$ the permuted preference relation of individual $i$. Note that $\Pi_j(\Pi_j(P_i)) = P_i$ for any $P_i \in P$ and $2 \leqslant j \leqslant s$. Consider the following metric $\delta$ on $P$: For $P_i, Q_i \in P$,

$$\delta(P_i, Q_i) = \left\{ \text{Min} \sum_{j=1}^{l} \phi(\alpha_j) \colon \Pi_{\alpha_l}(\Pi_{\alpha_{l-1}}(\cdots(\Pi_{\alpha_1}(P_i))\cdots) = Q_i \right\}.$$

We verify below that $\delta$ satisfies the triangle inequality. The other requirements (see (1), (2), (3) in part (ii) of Theorem 1) can be readily verified. Let $P_i, Q_i, R_i \in P$,

$$\delta(P_i, Q_i) = \sum_{j=1}^{l} \phi(\alpha_j) \qquad \text{where} \quad \Pi_{\alpha_l}(\cdots(\Pi_{\alpha_1(P_i)}))\cdots) = Q_i$$

$$\delta(Q_i, R_i) = \sum_{j=1}^{k} \phi(\beta_j) \qquad \text{where} \quad \Pi_{\beta_k}(\cdots(\Pi_{\beta_1}(Q_i))\cdots) = R_i$$

so

$$\Pi_{\beta_k}(\cdots(\Pi_{\beta_1}(\Pi_{\alpha_l}(\cdots(\Pi_{\alpha_1}(P_i))\cdots)=R_i$$

and therefore, by the definition of $\delta$,

$$\delta(P_i, R_i) \leqslant \sum_{j=1}^{l} \phi(\alpha_j) + \sum_{j=1}^{k} \phi(\beta_j) = \delta(P_i, Q_i) + \delta(Q_i, R_i).$$

Now for any $P, R \in \Omega$ let $\bar{\delta}(P, Q) = \sum_{i=1}^{n} \delta(P_i, R_i)$. The proof that $\bar{\delta}$ rationalizes $h$ is similar to the proof of Theorem 2(ii).

*Remark* 3. In Theorem 2 the quasi-metric indicating the distance between $P_i$ and $Q_i$ is a function of the minimal number of steps needed to transform $P_i$ into $Q_i$. In Theorem 3 the metric measuring the minimal "cost" of moving from $P_i$ to $Q_i$ is a function of the particular steps made and not merely their number. In general, the cost of transposition $\Pi_j$ differs from that of $\Pi_k$.

We conclude by showing that a Paretian social decision rule $h$ need not be a positional rule even if it has a symmetric additively decomposable metric rationalization. Consider the following:

EXAMPLE 3[3]. Let $X = \{x, y, z\}$, $N = \{1, 2\}$, and $P = \{a, b, c, d, e, f\}$, where

$$a\colon xyz, \quad b\colon xzy, \quad c\colon yxz$$
$$d\colon yzx, \quad e\colon zxy, \quad f\colon zyx.$$

For each individual $i \in N$, consider the metric $\hat{\delta}_i$ on $P$ which is defined in the following symmetric matrix (rows and columns refer to elements in $P$ and the entries indicate the corresponding distance between the linear orders).

| | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|
| $a$ | 0 | 2 | 2 | 2 | 1 | 2 |
| $b$ | 2 | 0 | 2 | 2 | 2 | 2 |
| $c$ | 2 | 2 | 0 | 2 | 2 | 2 |
| $d$ | 2 | 2 | 2 | 0 | 1 | 2 |
| $e$ | 1 | 2 | 2 | 1 | 0 | 2 |
| $f$ | 2 | 2 | 2 | 2 | 2 | 0 |

[3] Note that constructing a simpler example with $|X| = 2$ and $n = 2$, as in Example 1, is impossible here. If $|X| = 2$ and $h$ has an additively decomposable metric rationalization, then $h$ must be a positional rule. This is a direct corollary of Theorem 3 as in such a case the metric is neutral and strongly monotonic.

Suppose that $\forall P \in \Omega$,

$$\hat{h}(P) = \{v \in X \mid \hat{d}(P, U(v)) \leqslant \hat{d}(P, U(z))\ \forall z \in X\}$$

where

$$\hat{d}(P, Y) = \underset{Q \in Y}{\text{Min}}\ \hat{\delta}(P, Q) \qquad \text{and} \qquad \hat{\delta}(P, Q) = \sum_{i=1}^{2} \hat{\delta}_i(P_i, Q_i);$$

$$\hat{d}((a, d), U(z)) = \hat{\delta}((a, d), (e, e)) = \hat{\delta}_1(a, e) + \hat{\delta}_2(d, e) = 1 + 1 = 2$$

$$\hat{d}((a, d), U(x)) = \hat{\delta}((a, d), (a, a)) = \hat{\delta}_2(d, a) = 2$$

$$\hat{d}((a, d), U(y)) = \hat{\delta}((a, d), (c, d)) = \hat{\delta}_1(a, c) = 2.$$

Therefore, $\hat{h}((a, d)) = \{x, y, z\}$.

We now show that $\hat{h}$ cannot be a positional rule. Suppose to the contrary that there exist $\alpha(1), \alpha(2), \alpha(3), \alpha(1) > \alpha(2) \geqslant \alpha(3)$, such that

$$\forall P \in \Omega,\ \hat{h}(P) = \left\{ v \in X \mid \sum_{i=1}^{2} \alpha(t(v, P_i)) \geqslant \sum_{i=1}^{2} \alpha(t(z, P_i))\ \forall z \in X \right\}$$

$$T(x, (a, d)) = \alpha(t(x, a)) + \alpha(t(x, d)) = \alpha(1) + \alpha(3)$$

$$T(z, (a, d)) = \alpha(t(z, a)) + \alpha(t(z, d)) = \alpha(3) + \alpha(2).$$

Since

$$\hat{h}((a, d)) = \{x, y, z\},\ T(x, (a, d)) = T(z, (a, d))$$

or

$$\alpha(1) + \alpha(3) = \alpha(1) + \alpha(2) \Rightarrow \alpha(3) = \alpha(2)$$

$$\alpha(1) + \alpha(2) = \alpha(2) + \alpha(3) \Rightarrow \alpha(1) = \alpha(3).$$

Hence, $\alpha(1) = \alpha(2) = \alpha(3)$, a contradiction.

## References


1. K. J. Arrow, "Social Choice and Individual Values," 2nd ed., Wiley, New York, 1963.
2. D. Farkas and S. Nitzan, The Borda rule and pareto stability: A comment, *Econometrica* **47** (1979), 1305–1306.
3. P. Fishburn, "The Theory of Social Choice," Princeton Univ. Press, Princeton, N.J., 1972.
4. J. G. Kemeny and J. L. Snell, "Mathematical Models in the Social Sciences," MIT Press, Cambridge, Mass., 1962.
5. S. Nitzan, Some measures of closeness to unanimity and their implications, *Theory and Decision* **13** (1981), 129–138.
6. K. Wicksell, "Finanztheoretische Untersuchungen und das Steuerwesen Schweden's," Jena, Germany, 1896; English transl. R. A. Musgrave and A. T. Peacock (Eds.), "Classics in the Theory of Pubic Finance," Int. Econ. Assoc., Macmillan & Co., London, 1958.