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ON STRICTLY ERGODIC MODELS WHICH ARE NOT ALMOST TOPOLOGICALLY CONJUGATE

BY

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ABSTRACT

Answering a question raised by Glasner and Rudolph (1984) we construct uncountably many strictly ergodic topological systems which are metrically isomorphic to a given ergodic system $(X, \mathcal{B}, \mu, T)$ but not almost topologically conjugate to it.

1. Introduction

In [2], M. Denker and M. Keane showed that, given an ergodic almost topological dynamical system $(X, \mathcal{B}, \mu, T)$ which is not strictly ergodic, there exists an almost topological dynamical system $(X', \mathcal{B}', \mu', T')$ such that T and T' are metrically isomorphic but not finitarily isomorphic. This proof relies on realizing (i) the invariance of strict ergodicity under finitary isomorphism and (ii) every ergodic topological process is metrically isomorphic to a uniquely ergodic topological process [1], [5], [6], and [7]. They end their proof by asking the question: Is the finitary isomorphism class for an ergodic topological process strictly smaller than the metrical isomorphism class?

In what follows, a *topological process* would be a system $(X, \mathcal{B}, \mu, T)$ where T is a homeomorphism of the compact metric space X which preserves the probability measure μ defined on the Borel σ-algebra $\mathcal{B}$. We say that two such topo-

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logical processes $(X,\mathcal{B},\mu,T)$ and (Y,C,ν,S) are *almost topologically conjugate* if there exist residual invariant Borel sets X_0 and Y_0 where $X \supset X_0$ and $Y \supset Y_0$ with $\mu(X_0) = \nu(Y_0) = 1$ and a bimeasurable, bicontinuous isomorphism, $\phi : (X_0,\mu,T) \to (Y_0,\nu,S)$. So almost topological conjugacy is the same as finitary isomorphism.

In response to the question asked in [2], S. Glasner and D. Rudolph showed in [4] that there always exist uncountably many topological processes, all metrically isomorphic such that any two are not almost topologically conjugate. They then pose the question whether these processes can be made strictly ergodic, that is: Do there always exist uncountably many non-almost topologically conjugate, strictly ergodic, topological processes which are metrically isomorphic to a given ergodic topological process?

We answer the above question in the affirmative.

2. Sketch of the proof

Our proof is based on techniques different from those in [4] and uses some methods developed in [8]. The proof is essentially in two steps. In Theorem 1 we have a technique by which a given partition of the space X can be perturbed, by a small amount, such that the intersection of appropriate iterates of the union, of some fixed atoms of the new partition, can be made arbitrarily small. That is: given a set D, $\mu(D) > 0$, $P = \{P_1,\dots,P_{r-1}\}$, a partition of space X and a fixed set $\{n_1,n_2,\dots,n_t\}$ of integers where $\{n_1,n_2,\dots,n_t\} \subsetneqq \{0,1,2,\dots,r-1\}$, then there exist integers n and k and a new partition $P' = \{P'_0,P'_1,\dots,P'_{r-1}\}$, which is a small perturbation of P, such that

$$\mu(R \cap T^nR \cap \cdots \cap T^{nk}R) < \mu(D \cap T^nD \cap \cdots \cap T^{nk}D)$$

where $R = \bigcup_{i=1}^t P'_{n_i}$.

In the next step, we combine the technique of Theorem 1 with the technique of [8].

Let $\{D_j\}$ be a base for the topology of X. We build a sequence of partitions $\{Q_j\}$ and a set A which, for every j, is the union of atoms of Q_j. Furthermore,

(1) $Q_i \subseteq Q_{i+1}$ for all $i \geq 1$,

(2) $\bigvee_{i=1}^{\infty} Q_i = \mathcal{B}$,

(3) for every $i \geq 1$, Q_i is a uniform partition (see [5]),

(4) there exist two sequences of positive integers $\{n_j\}$ and $\{k_j\}$,

satisfying

$$\mu(A \cap T^{n_j}A \cap \cdots \cap T^{n_jk_j}A) < \mu(D_j \cap T^{n_j}D_j \cap \cdots \cap T^{n_jk_j}D_j) \qquad \forall j.$$

If (Y_i, U_i) is the symbolic system corresponding to (X, Q_i, T) then (Y_i, U_i) is uniquely ergodic and (Y, ν, U) is defined as the inverse limit of $\{(Y_i, U_i)\}$. Then (Y, ν, U) is uniquely ergodic, metrically isomorphic to (X, μ, T) but not almost topologically conjugate.

The reason why (Y, ν, U) is not almost topologically conjugate to (X, μ, T) is provided by property (4). The set A corresponds to a certain open set, say, $\hat{A}$, in Y. If, to the contrary, (Y, ν, U) and (X, μ, T) are almost topologically conjugate, then there is a bimeasurable, bicontinuous isomorphism $\phi : X_0 \to Y_0$, where X_0 and Y_0 are full measure subsets of X and Y, respectively. Thus, A' contains, up to measure zero, an image of at least one D_j. This is inconsistent with (4).

For strict ergodicity it suffices to assume that the measure ν has full support. The case of uncountably many models is an extension of these techniques.

3. Definitions

Let $P = \{P_0, \ldots, P_{r-1}\}$ be a partition of X and $q \in \mathbf{N}$. We denote by $F_q(P)$ the set of all P-q-names, i.e.,

$$F_q(P) = \left\{(a_1, \ldots, a_q) \in \{0, \ldots, r-1\}^q \,\middle|\, \mu(P_{a_1} \cap T^{-1}P_{a_2} \cap \cdots \cap T^{-q+1}P_{a_q}) > 0\right\}.$$

Definition 1. Let P be a partition. A word $z = (z_1, z_2, \ldots, z_l)$ has *good P-(ϵ, q, n) statistics* if for every $(x_0, \ldots, x_{q-1}) = x \in F_q(P)$ and $1 \le s \le l - n$ the following holds:

$$\left| 1/(n-q)\#\{i/(z_i, \ldots, z_{i+q}) = x,\ s \le i \le s+n-q\} - \mu\left(\bigcap_{i=0}^{q-1} T^{-i}P_{x_i}\right) \right| < \epsilon.$$

That is, for any word $x \in F_q(P)$, the distribution of x in any n-subword of z is close to its measure in the symbolic system produced by (X, T, P). A set of words has good P-(ϵ, q, n) statistics if every word in the set has good P-(ϵ, q, n) statistics.

Definition 2. Given two partitions, P and Q, $|P| = |Q| = n$, then $d(P, Q) = \sum_{i=1}^{n} \mu(P_i \Delta Q_i)$ is a measure of the distance between the partitions.

Definition 3. Let Q be a partition. Q is said to be a *uniform partition* if, for every $\epsilon > 0$ and integer k, there is an integer n such that for each $x \in X$ and $A \in \bigvee_{i=0}^{k-1} T^{-i}Q$

$$|(1/n)\#\{t/T^t x \in A;\ 0 \le t \le n-1\} - \mu(A)| < \epsilon.$$

Definition 4. Given a partition P and an integer m, P is *m-universal* if there is a set S in $\bigvee_{i=0}^{m-1} T^{-i}P$ and two relatively prime integers t and q satisfying

$$\mu(T^t S \cap S) > 0 \quad \text{and} \quad \mu(T^q S \cap S) > 0.$$

4. Previously known results

Denote by $F_m(P)$ the set of all the P m-names; the term *word* is used to denote a finite string of symbols. If w is a word, then $|w|$ denotes its length.

PROPOSITION 1 (Furstenberg's Multiple Recurrence Theorem [3]). *If $T_1, T_2, \ldots, T_l$ are commuting measure preserving transformations of a measure space $(X, \mathcal{B}, \mu)$ and $A \in \mathcal{B}$ with $\mu(A) > 0$, then*

$$\liminf_{N \to \infty} (1/N) \sum_{n=1}^{N} \mu(T_1^{-n} A \cap T_2^{-n} A \cap \cdots \cap T_l^{-n} A) > 0.$$

PROPOSITION 2 (see [8]). *For every pair of integers, $m, k \in \mathbf{N}$, m-universal partition P and $\epsilon > 0$, there is a set of words $E = E(P, \epsilon, k, m)$ and two integers $l = l(P, \epsilon, k, m)$ and $n = n(P, \epsilon, k, m)$ such that:*

(i) *For every $x, y \in F_m(P)$ and $l' \geq l$ there is a $z \in E$ and a word w such that $|w| = l'$ and $z = xwy$.*

(ii) *For each $z \in E$ every m-subword of z is in $F_m(P)$.*

(iii) *Each $z \in E$ has good P-(ϵ, k, n) statistics.*

PROPOSITION 3 (see [8]). *Given two integers m and k, two reals $\delta > 0$, $\epsilon > 0$ and an m-universal partition P, there exist an integer $\bar{n} = \bar{n}(P, \epsilon, \delta, m)$ and a partition P' such that*

(i) $F_m(P') \subseteq F_m(P)$.

(ii) *Each word in $F_{\bar{n}}(P')$ has good P'-$(\epsilon, k, \bar{n})$ statistics.*

(iii) *The distance, in the sense of d, between P and P' is less than δ.*

(iv) *P' is an $\bar{n}$-universal partition.*

The last statement we quote from [8] is:

PROPOSITION 4. *If P is an m-universal partition, Q is any partition and $\epsilon > 0$, then there is a partition Q^* such that:*

(i) $d(Q, Q^*) < \epsilon$,

(ii) *$P \vee Q^*$ is an m-universal partition.*

5. The perturbation theorem

In this section we show how to perturb a given partition $P = \{P_0, P_1, \ldots, P_{r-1}\}$ by a small amount to obtain a new partition, P'. In doing so we make sure that no new names (of a fixed length) are created and the intersection of appropriate iterates of the union of some fixed atoms of P' is arbitrarily small.

THEOREM 1. *Suppose that $P = \{P_0, \ldots, P_{r-1}\}$ is a partition, D is a measurable set, β and δ are positive numbers, and, finally, $n_1, \ldots, n_t, m, q, k, n \in \mathbf{N}$ satisfy the following*:

(1) *P is m-universal and each word in $F_m(P)$ has good P-(β, q, m) statistics. Let $l = l(P, \beta, q, m)$ as in Proposition 2.*

(2) (i) $\mu(D \cap T^n D \cap T^{2n} D \cap \cdots \cap T^{kn} D) = \alpha > 0$.

(ii) *There is a prime number $p < k$ satisfying $p/2 > 2(m + l)/\delta + 1$.*

(3) $\{n_1, \ldots, n_t\} \subsetneqq \{0, \ldots, r-1\}$.

(4) $\beta/q > \delta$.

Then, there exists a new partition $P' = \{P'_0, P'_1, \ldots, P'_{r-1}\}$ satisfying the following:

(a) $F_m(P') \subset F_m(P)$;

(b) $d(P, P') < \delta$;

(c) *$F_m(P')$ has good P'-$(2\beta, q, m)$ statistics; and*

(d) $\mu(R \cap T^n R \cap \cdots \cap T^{kn} R) < \mu(D \cap T^n D \cap \cdots \cap T^{kn} D)$, *where*

$$R = \bigcup_{i=1}^{t} P'_{n_i}.$$

Since the proof of Theorem 1 is the key idea of this paper, let us review it. We first take a Rohlin tower which covers at least $1 - \alpha/2$ of the space. Moreover, the height of the Rohlin tower, denoted by v, is bigger than $2(n+1)k/\alpha\delta$. Denote the base of the tower by B.

We define the new partition P' by changing the P-names of points in the Rohlin tower. For instance, if we take level s of the tower and say that all the points in the $T^{s-1}B$ having the name a (namely, included in the atom P_a of P) will own a new name, b, we mean that $T^{s-1}B \cap P_a$ will be included in the atom P'_b of P'.

Let p be a prime number satisfying $p < k$ and $p/2 > 2(m + l)/\delta + 1$. Suppose first that n is not a multiple of p. The main goal of the construction is to rename all the levels, $p - 1$, $2p - 1$, $3p - 1$, and so on, so that $T^{ip-1}B$ will not have a name in $\{n_1, \ldots, n_t\}$. By doing so we ensure that a measurable set in the main body of the tower (i.e., not in the $(n+1)k$ lowest levels), say, E, has the following property. For all $x \in E$ at least one of the points, $T^{-n}x, \ldots, T^{-nk}x$, has a name outside of $\{n_1, \ldots, n_t\}$. In particular, if we denote $E = R \cap \bigcup_{i=(n+1)k}^{v-1} T^i B$ (recall, v is the height of the tower), then $(T^n R \cap \cdots \cap T^{nk} R) \cap E = \varnothing$. However, $\bigcup_{i=(n+1)k}^{v-1} T^i B$ covers at least $1 - (\alpha/2 + (n+1)k\mu(B))$ of the space. Since $\mu(B) \le 1/v < \alpha/2(n+1)k$, $\bigcup_{i=(n+1)k}^{v-1} T^i B$ covers at least $1 - \alpha$ of the space.

In view of the above,

$$R \cap T^n R \cap \cdots \cap T^{nk} R$$
$$= (E \cap T^n R \cap \cdots \cap T^{nk} R) \cup ((R \setminus E) \cap T^n R \cap \cdots \cap T^{nk} R) \subseteq R \setminus E.$$

Since this set is included in $X \setminus \bigcup_{i=(n+1)k}^{v-1} T^i B$ we conclude that it has a measure of at most α, and (d) is satisfied.

So far we have not taken care of (a)–(c). Instead of just renaming the levels $ip-1$ we will rename a few levels above and below each one of these levels so that the P'-m-names will be included in $F_m(P)$, i.e., (a) is satisfied. Since p is much larger than l and m and since P' differs from P only on the Rohlin tower, (b) will be satisfied. It follows from (4) that (a) and (b) imply (c).

In a case when n is a multiple of p, we still want to replace names of levels in the tower so that for every $x \in E$ at least one of the points $T^{-n}x, \ldots, T^{-nk}x$ has a name outside of $\{n_1, \ldots, n_t\}$. We say that the level $T^i B$ of the Rohlin tower has an index i, $0 \le i \le v-1$.

Define the set of indices $J = \{p, 2p, \ldots, gp\}$, where $n = gp$. We replace the name of all the levels with an index in J, namely, the levels $T^j B$, $j \in J$. Then we replace the levels with indices in $n + J + 1$, i.e., in the set $\{n+p+1, n+2p+1, \ldots, 2n+1\}$; then in the sets $2n + J + 2$, $3n + J + 3$, and so on, until the set $en + J + e$, where $e = [p/2]$. In other words, we always shift the previous set of indices by $n+1$. We do so $e-1$ times. From that point on we go downward from $p-1$ to $e+1$, i.e., we replace the names of the levels with indices in $(e+1)n + J + (p-1), (e+2)n + J + (p-2), \ldots, 2en + J + (e+1)$. Then we start again with a block of g levels whose indices are $0 (\mathrm{mod}\, p), 1 (\mathrm{mod}\, p), \ldots, e (\mathrm{mod}\, p)$, $(p-1)(\mathrm{mod}\, p), (p-2)(\mathrm{mod}\, p), \ldots, (e+1)(\mathrm{mod}\, p)$. And we continue with this procedure until the tower ends.

In this procedure of replacing names we start with $0(\mathrm{mod}\, p)$ and go up to $e(\mathrm{mod}\, p)$ and then proceed with $(p-1)(\mathrm{mod}\, p)$, and go down to $(e+1)(\mathrm{mod}\, p)$ and continue again with $0(\mathrm{mod}\, p)$, $1(\mathrm{mod}\, p)$, and so on. In so doing we make sure that the distance (in the tower) between two levels whose names are replaced is at least e, which is, by 2(ii), greater than $2(m+l)/\delta$. Therefore, we can use the method of the first case to replace the names of these levels without enlarging the set of m-names.

In the following proof we denote by $N^s(x)$ the word in $F_s(P)$ that corresponds to $(x, Tx, \ldots, T^{s-1}x)$, where s is an integer.

Proof of Theorem 1: Let B be a base of the Rohlin tower of height $v = [2(n+1)k/\delta\alpha] + 1$ satisfying $\mu(\bigcup_{i=0}^{v-1} T^i B) > 1 - \alpha/2$.

We change names of length $m + 2l$ on a Kakutani–Rohlin tower built on a base

B. Next we define the new partition P' according to the new names. During our construction we take care to change the names in a way to ensure that certain levels in the new tower (the one with changed names) would not have any element from R. This enables us to make the intersection of the appropriate iterates of R arbitrarily small. Let p satisfy 2(ii). We divide the analysis into two cases. Since the proof of the second case (n is a multiple of p) is similar to that of the first one (n is not a multiple of p), we provide a full proof of the first case and only a detailed description of the construction in the second.

Case I: n is not a multiple of p. For each x in the base B (which is level 0) and for any level $ip-1$ satisfying $ip < v-(m+l)$: choose $y \in F_m(P)$ such that its first symbol is not in the set $\{n_1, n_2, \ldots, n_t\}$. Such a choice is possible because we have made sure in our hypothesis that $\{n_1, n_2, \ldots, n_t\} \subsetneqq \{0,1,2,\ldots,r-1\}$.

We want to replace the name $N^{m-1}(T^{ip-1}x)$ by y. Using Proposition 2 twice it follows that there exist words $\underline{w}$ and w such that $|\underline{w}| = |w| = l$ and every m-subword of the concatenation

$$z = \underline{u}\underline{w}ywu \qquad \text{is in } F_m(P),$$

where $\underline{u} = N^{m-1}(T^{-l-m+ip-1}x)$ and $u = N^{m-1}(T^{l+m-1+ip-1}x)$. Replace $N^{3m+2l-1}(T^{-l-m+ip-1}x)$ by z. Since the head $\underline{u}$ and the tail u of z are not changed (they are also names of the former partition P), we actually replace words of length $2l+m$.

The property that every m-subword of z is in $F_m(P)$ ensures that we do not enlarge the vocabulary of the m-names. Thus conclusion (a) is satisfied.

It remains to show (b)–(d).

CLAIM 1. $d(P,P') < \delta$.

PROOF. Observe that we change names of l levels below and $l+m$ levels above each of the levels ip, $1 \le i < (v-m-l)/p$. Thus we have changed names of at most $2(l+m)(v-m-l)/p$. The measure of each level is $\mu(B) \le 1/v$. Thus, we have changed names of points in a set of at most

$$[2(l+m)(v-m-l)/p]\,(1/v) < 2(l+m)/p < \delta.$$

The last inequality follows from hypothesis 2(ii). ∎

Hypothesis (4) and the previous claim imply that conclusion (c) holds. Recall $R = \bigcup_{j=1}^{t} P'_{n_j}$. The following claim will take care of (d).

CLAIM 2. $\mu(R \cap T^nR \cap \cdots \cap T^{nk}R) < \mu(D \cap T^nD \cap \cdots \cap T^{nk}D)$.

PROOF. Denote $E = R \cap (\bigcup_{i=(n+1)k}^{v-1} T^i B)$. In words, E is the part of R contained in the tower but not in one of the lowest $(n+1)k$ levels. Since n is not a multiple of p, every point $x \in E$ satisfies the following: at least one of the points $T^{-n}x, T^{-2n}x, \ldots, T^{-np}x$ lies in a level of the type $T^{ip-1}B$. However, all these levels are, by the construction, outside of R. Therefore (recall that $p < k$), $E \cap T^n R \cap \cdots \cap T^{nk} R = \emptyset$. Thus,

$$R \cap T^n R \cap \cdots \cap T^{nk} R \subseteq (R \backslash E) \cap T^n R \cap \cdots \cap T^{nk} R \subseteq R \backslash E.$$

However, the measure of $R \backslash E$ is less than the measure of the set not covered by the tower plus the measure of the $(n+1)k$ lowest levels.

Precisely,

$$\mu(R \backslash E) < \mu\left(X \Big\backslash \bigcup_{i=0}^{v-1} T^i B\right) + \mu(B)(n+1)k < \alpha/2 + (n+1)k/v \leq \alpha,$$

and the claim follows. This completes the proof of (d) and thereby the proof of Theorem 1, if Case I holds.

Case II: n is a multiple of p. Let $n/p = g$ and $e = [p/2]$. The idea is again to replace names of a few levels in the Rohlin tower so that for every $x \in E$ at least one of the points, $T^{-n}x, \ldots, T^{-nk}x$ has a name outside $\{n_1, \ldots, n_t\}$. Recall that J was defined as $\{p, \ldots, gp\}$.

Instead of replacing the names of the level of the type $ip - 1$, as was done in the previous case, we replace here the names of the following levels:

i. the first g (except for the base) levels whose indices are $0 (\operatorname{mod} p)$ (i.e., the names of the levels $T^p B, T^{2p} B, \ldots, T^n B$);

ii. the next g levels whose indices are $1 (\operatorname{mod} p)$ (i.e., the next g level with indices in the set $\{n + p + 1, n + 2p + 1, \ldots, 2n + 1\} = n + J + 1$);

iii. the next g levels whose indices are $2 (\operatorname{mod} p)$ (i.e., the indices in $2n + J + 2$).

Continue this way to levels whose indices are $3 (\operatorname{mod} p)$, $4 (\operatorname{mod} p)$ and so on until indices that are $e (\operatorname{mod} p)$. From that point on we replace levels with indices that are $(p-1)(\operatorname{mod} p)$ and go backward to $(p-2)(\operatorname{mod} p)$ until $(e+1)(\operatorname{mod} p)$.

In other words, we continue replacing those levels whose indices are in $(e+1)n + J + (p-1), (e+2)n + J + (p-2), (e+3)n + J + (p-3), \ldots, 2en + J + (e+1)$.

To sum up, for every $0 \leq i \leq p - 1$, we replaced a block of g consecutive levels whose indices are $i (\operatorname{mod} p)$. Moreover, the distance between two changed levels is at least $p - 1$. We went up from 0 to e and down from $p - 1$ to $e + 1$ because it enables us to continue with the following g levels with indices that are $0 (\operatorname{mod} p)$

while taking care of (a) of the theorem. The reason is that, after replacing the names of those levels with indices in $2en + J + (e + 1)$, we continue replacing names of the next g levels whose indices are $0(\bmod p)$, i.e., indices in $n^2 + J$. Thus, the distance between the last level in the former (i.e., the level with the index $2en + gp + e + 1$) and the first level in the latter (i.e., with index $n^2 + p$) is $e > p/2 - 1$. When the distance between any two levels whose names are changed is at least $p/2 - 1$, we may employ the method used in the previous case (because, by 2(ii), $p/2 > 2(2m + l)/\delta + 1$) to replace names in the level described above without enlarging the set of the m-names.

We proceed then by replacing, inductively, names of g consecutive levels whose indices are $1(\bmod p), 2(\bmod p), \ldots, e(\bmod p)$, $(p - 1)(\bmod p), \ldots,$ $(p - 2)(\bmod p)(e + 1)(\bmod p)$, and so forth until the highest level of the Rohlin tower.

The main feature of this construction is that, for every $0 \le i \le p - 1$ and for any p consecutive blocks (of g changed levels), there is a block of changed levels whose indices are $i(\bmod p)$. Thus, when one starts with $x \in E$ and goes with it k steps downstairs the tower in paces of length n, one of the steps should fall (since $p < k$) into one of the changed levels, i.e., one of $T^{-n}x, \ldots, T^{-nk}x$ is contained in one of the changed levels, and therefore outside R.

The proofs of Claims 1 and 2 in Case II are similar to their proofs in Case I, and are therefore omitted. ∎

The following lemma ensures that by perturbing P by a small δ we do not lose the m-universality property.

LEMMA 1. *For any m-universal partition* $P = \{P_0, P_1, \ldots, P_{r-1}\}$ *there exists a* $\delta > 0$ *such that* $d(P, P') < \delta$ *implies that* P' *is also m-universal.*

PROOF. Since P is m-universal there exists $S \in V_{i=0}^{m-1} T^{-i}P$ and two relatively prime integers, t and r, for which $\mu(T^tS \cap S)\mu(T^rS \cap S) > 0$. Choose δ satisfying $0 < 2m\delta < (1/2)\min\{\mu(T^tS \cap S), \mu(T^rS \cap S)\}$ to ensure the lemma.

6. Construction of a uniquely ergodic model

In the main result of this chapter we construct, using Theorem 1 and the methods developed in [8], a uniquely ergodic, topological model which is metrically isomorphic but not almost topologically conjugate to a given ergodic model.

THEOREM 2. *Given an ergodic topological process* $(X, \mathcal{B}, \mu, T)$ *and a sequence* $\{D_j\}$ *of nontrivial sets in* X, *we can construct:*

(i) *a uniquely ergodic topological system* (Y, C, ν, U) *which is metrically isomorphic to* $(X, \mathcal{B}, \mu, T)$,

(ii) *a clopen set* $\hat{A}$ *in* Y, *and*

(iii) *increasing sequences of positive integers* $\{n_j\}$ *and* $\{k_j\}$ *such that the following inequalities,*

$$\nu(\hat{A} \cap U^{n_j}\hat{A} \cap \cdots \cap U^{n_jk_j}\hat{A})$$
$$< \mu(D_j \cap T^{n_j}D_j \cap \cdots \cap T^{n_jk_j}D_j) \quad \textit{for all } j \textit{ in } \mathbf{N},$$

are true.

Outline of the Proof. We will construct rows of partitions $\{Q_i^j\}$, $1 \le i \le j$ and $j \in \mathbf{N}$:

$$\begin{array}{ccccc} Q_1^1 & & & & \\ Q_1^2 & Q_2^2 & & & \\ Q_1^3 & Q_2^3 & Q_3^3 & & \\ & \vdots & & & \\ Q_1^k & Q_2^k & Q_3^k & \cdots & Q_k^k. \end{array}$$

Further, we will show that $\forall i$, the sequence Q_i^k converges to a partition Q_i. The sequence $\{Q_i\}$ of partitions will have the following properties:

(P1) $Q_i \subseteq Q_{i+1}$ for all $i \ge 1$.

(P2) $\bigvee_{i=1}^{\infty} Q_i = \mathcal{B}$.

(P3) For each $i \ge 1$, Q_i is a uniform partition.

We will be interested in a set A, which, for every i, will be the union of certain atoms of Q_i and will satisfy the inequality:

(P4) $\mu(A \cap T^{n_j}A \cap \cdots \cap T^{n_jk_j}A) < \mu(D_j \cap T^{n_j}D_j \cap \cdots \cap T^{n_jk_j}D_j)$ for all j in $\mathbf{N}$.

For every (X, Q_i, T) we consider the corresponding symbolic system (Y_i, U_i). The space (Y, U) will be defined as the inverse limit of the sequence $\{(Y_i, U_i)\}$.

Definition 5. Let $\{D_j\}$ be a sequence of positive measure sets in $\mathcal{B}$, and $P = \{P_0, \ldots, P_{r-1}\}$ be a partition of X, A be a union of some atoms of P and, finally, $\{k_j\}_{j=1}^t$ and $\{n_j\}_{j=1}^t$ be two sequences of integers such that

$$\mu(A \cap T^{n_j}A \cap \cdots \cap T^{n_jk_j}A) < \mu(D_j \cap T^{n_j}D_j \cap \cdots \cap T^{n_jk_j}D_j),$$
$$\text{for } 1 \le j \le t.$$

We say that partition Q is an improvement of P *in the sense of the intersection property* if there exists a set B which is a union of certain atoms of Q and two integers n_{t+1}, k_{t+1} such that

$$\mu(B \cap T^{n_j}B \cap \cdots \cap T^{n_j k_j}B) < \mu(D_j \cap T^{n_j}D_j \cap \cdots \cap T^{n_j k_j}D_j),$$
$$\text{for } 1 \le j \le t+1.$$

Proof of Theorem 2: We will use the following terminology:

(i) Let R be a perturbation of the partition Q and A be a union of certain atoms of Q. We will say that A', which is a union of R-atoms, *corresponds to* A if it is a union over the same indices as A.

(ii) Let V, U', U be partitions of X, where $V \subseteq U$ and U' is a perturbation of U (in particular $|U'| = |U|$), then U' induces a natural perturbation V' of V (for details see [8]).

Remark 1. If $U \supseteq V'$, U' a perturbation of U and V' is the natural perturbation of V then $d(V, V') \le d(U, U')$.

Construction. Start with a partition $R_1^1 = \{K_0, \ldots, K_{q-1}\}$ which is 1-universal, and let $\{\underline{Q}_i\}$ be a sequence of partitions of X such that $V\underline{Q}_i = \mathcal{B}$.

Stage 1: Given

1. R_1^1, which is 1-universal and $F_1(R_1^1)$ has a good R_1^1-(1,1,1) statistics, and
2. the set D_1 with $\mu(D_1) > 0$.

Let $\delta > 0$ be the one satisfying Lemma 1 with $P = R_1^1$ and $m = 1$. By using Theorem 1 (getting n_1 and k_1 from Proposition 1 so that hypothesis (2) is satisfied for $D = D_1$, $m_1 = l_1 = 1$, and the set $\{0\}$ for hypothesis (3)), we obtain a new partition $Q_1^1 = \{K_0', \ldots, K_{q-1}'\}$ satisfying

(a) $F_1(Q_1^1) \subseteq F_1(R_1^1)$.
(b) $d(Q_1^1, R_1^1) < \delta$.
(c) Q_1^1 is 1-universal and $F_1(Q_1^1)$ has good Q_1^1-(2,1,1) statistics.
(d) $\mu(K_0' \cap T^{n_1}K_0' \cap \cdots \cap T^{n_1k_1}K_0') < \mu(D_1 \cap T^{n_1}D_1 \cap \cdots \cap T^{n_1k_1}D_1)$.

Label this K_0' as A_1 and let

$$\mu(D_1 \cap T^{n_1}D_1 \cap \cdots \cap T^{n_1k_1}D_1) - \mu(A_1 \cap T^{n_1}A_1 \cap \cdots \cap T^{n_1k_1}A_1) = \eta_1.$$

Choose a sequence $\{\epsilon_j\}$, of real numbers, satisfying $\Sigma_{j>i}\epsilon_j < \epsilon_i$ and $\forall i$, $i\epsilon_i \to 0$. It is to be noted that the only use of hypothesis (3) of Theorem 1 is to identify the set $\bigcup_{i=1}^{t} P_{n_i}$, and hence the set R which corresponds to it after perturbation.

Let $\delta_1 = 1$.

Stage 2: We refine and perturb Q_1^1 in two steps to obtain Q_2^2 which is an improvement both in the sense of uniformity as well as in the sense of the intersection property.

Part 1: In Part 1 we refine the partition Q_1^1 and then perturb it slightly to obtain partition R_2^2 which is an imporvement of Q_1^1 in the sense of uniformity. Following Proposition 4 we can refine Q_1^1 to obtain $P_2 = Q_1^1 \vee Q_2^*$, where

(i) $d(Q_2^*, \underline{Q}_2) < \epsilon_2$, and

(ii) P_2 is 1-universal.

Define $\delta_2 = \min\{\epsilon_2, \delta_1, \eta_1/k_1\}/((2)2^3)$.

We are defining δ_2 in this way to make sure that perturbation by δ_2 will not destroy the intersection property of set A_1, obtained in stage 1.

For partition Q_1^1, $m_1 = 1$, $k = 2$, ϵ_2, and δ_2, we apply Proposition 3 to obtain a new partition R_2^2 and an integer m_2 such that the following hold:

(c1) $F_{m_1}(R_2^2) \subseteq F_{m_1}(P_2)$,

(c2) each word in $F_{m_2}(R_2^2)$ has good R_2^2-$(\epsilon_2, 2, m_2)$ statistics,

(c3) $d(R_2^2, P_2) < (1/2)\delta_2$,

(c4) R_2^2 is an m_2-universal partition.

Because of (c3) and the choice of δ_2 we have:

(c5) $\mu(A_1' \cap T^{n_1}A_1' \cap \cdots \cap T^{n_1k_1}A_1') < \mu(D_1 \cap T^{n_1}D_1 \cap \cdots \cap T^{n_1k_1}D_1)$, where A_1' is the set in R_2^2 which corresponds to A_1 in Q_1^1. (See the note on terminology before construction.)

Part 2: We want to use Theorem 1 to perturb the partition R_2^2 to obtain another partition, Q_2^2, which is an improvement of R_2^2 in the sense of the intersection property. We have

1. An m_2-universal partition R_2^2 where $F_{m_2}(R_2^2)$ has good R_2^2-$(\epsilon_2, 2, m_2)$ statistics.
2. The set D_2, $\mu(D_2) > 0$.

Let δ satisfy Lemma 1 with $m = m_2$ and $P = R_2^2$. Moreover, $\delta \leq \text{Min}\{\delta_2, \epsilon_2\}/2$.

By Proposition 1 we can choose two large integers k_2 and n_2 such that:

(i) $\mu(D_2 \cap T^{n_2}D_2 \cap \cdots \cap T^{n_2k_2}D_2) = \alpha_2 > 0$;

(ii) there exists a prime number $p_2 < k_2$ such that $p_2/2 > 2(l_2 + m_2)/\delta + 1$, where $l_2 = l_2(R_2^2, \epsilon_2, 2, m_2)$ as in Proposition 2.

By Theorem 1 and Lemma 1 there exists a new partition Q_2^2 such that

(a) $F_{m_2}(Q_2^2) \subseteq F_{m_2}(R_2^2)$,

(b) $d(R_2^2, Q_2^2) < \delta$,

(c) $F_{m_2}(Q_2^2)$ has Q_2^2-$(2\epsilon_2,2,m_2)$ good statistics,

(d) if A_1'' is the set in Q_2^2 which corresponds to A_1' then

$$\mu(A_1'' \cap T^{n_2}A_1'' \cap \cdots \cap T^{n_2k_2}A_1'') < \mu(D_2 \cap T^{n_2}D_2 \cap \cdots \cap T^{n_2k_2}D_2),$$

(e) Q_2^2 is m_2-universal.

Rename A_1'' as A_2.

The set of integers, needed for hypothesis (3) of Theorem 1, is the set of indices which the atoms of A_1' have in R_2^2. The role of set R of conclusion (d) in Theorem 1 is played by A_1''.

These properties, (a)–(d), imply that properties, similar to (c1)–(c5) listed before in Part 1, hold for Q_2^2 and A_2 also. That is, since $m_1 \leq m_2$, by (c1) of Part 1 and (a) we have:

(c1′) $F_{m_1}(P_2) \supset F_{m_1}(R_2^2) \supset F_{m_1}(Q_2^2)$,

(c2′) $F_{m_2}(Q_2^2)$ has good Q_2^2-$(2\epsilon_2,2,m_2)$ statistics.

Combining (b) and (c3) we get (recall $\delta \leq \delta_2/2$):

(c3′) $d(Q_2^2,P_2) < \delta_2$,

(c4′) Q_2^2 is an m_2-universal partition.

Because of (c5) of Part 1, (d), (c3′) and because of the appropriate choice of δ_2 we have that A_2 retains the intersection property corresponding to n_1 and k_1, and therefore:

(c5′) For $j \in \{1,2\}$

$$\mu(A_2 \cap T^{n_j}A_2 \cap \cdots \cap T^{n_jk_j}A_2) < \mu(D_j \cap T^{n_j}D_j \cap \cdots \cap T^{n_jk_j}D_j).$$

Let $\eta_2 = \mu(D_2 \cap T^{n_2}D_2 \cap \cdots \cap T^{n_2k_2}D_2) - \mu(A_2 \cap T^{n_2}A_2 \cap \cdots \cap T^{n_2k_2}A_2)$. Observe that Q_2^2 is a perturbation of P_2 which refines Q_1^1. So denote by Q_1^2 the natural perturbation of Q_1^1 that Q_2^2 induces. By Remark 1, $d(Q_1^1,Q_1^2) < \delta_2$. We now have the second row of partitions with the following desirable properties:

(d1) property (c1′) implies $F_{m_1}(Q_1^2) \subseteq F_{m_1}(Q_1^1)$,

(d2) from property (c2) it follows that each word in $F_{m_2}(Q_i^2)$ has good Q_i^2-$(2\epsilon_2,2,m_2)$ statistics, $i = 1,2$,

(d3) $d(Q_1^2,Q_1^1) < \delta_2$,

(d4) Q_2^2 is m_2-universal,

(d5) for $j \in \{1,2\}$,

$$\mu(A_2 \cap T^{n_j}A_2 \cap \cdots \cap T^{n_jk_j}A_2) < \mu(D_2 \cap T^{n_j}D_2 \cap \cdots \cap T^{n_jk_j}D_2),$$

(d6) by construction, $Q_1^2 \subseteq Q_2^2$.

Notice that, for every i, A_2 is a union of certain atoms of Q_i^2, $1 \leq i \leq 2$.

Continuing inductively in the same way we obtain partitions $\{Q_i^s\}$, $s \in \mathbf{N}$, $1 \le i \le s$, sets A_s which are Q_j^s-measurable for all $1 \le j \le s$ and integers m_s, k_s, n_s such that (we do not mention the analogy of (d4) because it had importance only during the construction):

(d1′) $F_{m_s}(Q_i^{s+1}) \subset F_{m_s}(Q_i^s)$, where $i \le s$, $s \in \mathbf{N}$,

(d2′) $F_{m_s}(Q_i^s)$ has good $Q_i^s - (2\epsilon_s, s, m_s)$ statistics, $1 \le i \le s$, i and s in $\mathbf{N}$,

(d3′) $d(Q_i^{s+1}, Q_i^s) < \delta_{s+1}$ for $i \le s$,

(d5′) for $1 \le j \le s$, $s \in \mathbf{N}$,

$$\mu(A_s \cap T^{n_j} A_s \cap \cdots \cap T^{n_j k_j} A_s) < \mu(D_s \cap T^{n_j} D_s \cap \cdots \cap T^{n_j k_j} D_s),$$

(d6′) $Q_i^s \subset Q_{i+1}^s$, $i < s$,

(d7′) A_{s+1} corresponds to A_s; thus $\mu(A_s \Delta A_{s+1}) < \delta_{s+1}$.

By (d3′), $\{Q_i^s\}_s$ is a Cauchy sequence. Thus, it converges to a partition denoted by Q_i. Furthermore, by (d7′), the sequence $\{A_i\}$ converges to a set A which is Q_i-measurable for every i.

Note that at this point of (P1)–(P4), mentioned in the outline, only (P4) needs a proof. Fix j. By construction, for every $p > j+1$, $\mu(A_j \Delta A_p) < (1/2)\delta_{j+1}$. Choose a large p such that $\mu(A \Delta A_p) < (1/2)\delta_{j+1}$. Thus, $\mu(A \Delta A_j) < \delta_{j+1}$. Because of (d5′) (for $j = s$) and since A is a perturbation of A_j by less than δ_{j+1}, which is smaller than

$$(\mu(D_j \cap T^{n_j} D_j \cap \cdots \cap T^{n_j k_j} D_j) - \mu(A_j \cap T^{n_j} A_j \cap \cdots \cap T^{n_j k_j} A_j))/k_j,$$

we get (P4).

Now we are ready to conclude:

Claim 5. There exists a uniquely ergodic system (Y, C, ν, U) which is metrically isomorphic to $(X, \mathcal{B}, \mu, T)$ and a clopen set $\hat{A}$ in Y which satisfies:

$$\nu(\hat{A} \cap U^{n_j} \hat{A} \cap \cdots \cap U^{n_j k_j} \hat{A}) < \mu(D_j \cap T^{n_j} D_j \cap \cdots \cap T^{n_j k_j} D_j)$$

for every $j \in \mathbf{N}$.

Proof of the Claim. Define (Y_i, ν_i, U_i) to be the symbolic system induced by (X, Q_i, T). Since Q_i is uniform, (Y_i, ν_i, U_i) is uniquely ergodic. Let (Y, C, ν, U) be the inverse limit of

$$\cdots \to (Y_2, U_2) \to (Y_1, U_1).$$

By setting $\{D_j\}$ as a base for the topology of X and using the method of [4], we use Theorem 2 to obtain a uniquely ergodic topological system (Y, C, ν, U) which is metrically isomorphic but not almost topologically conjugate to $(X, \mathcal{B}, \mu, T)$. Moreover, by assuming that measure ν has full support we get (Y, C, ν, U) to be strictly ergodic.

This concludes the proof of Theorem 2.

Starting with countably many models (Y_i, C_i, U_i), we can apply the same technique to the countable collection of the topological bases of these models to construct one more strictly ergodic model which is isomorphic but not almost topologically conjugate to any Y_i.

Hence, we state:

THEOREM 3. *There exist uncountably many strictly ergodic topological models which are metrically isomorphic to a given ergodic model $(X, \mathcal{B}, \mu, T)$ but not almost topologically conjugate to $(X, \mathcal{B}, \mu, T)$. Furthermore, no two are almost topologically conjugate.*

We conclude by noting that our technique fails to apply to Z^2 action. B. Weiss notified us that a combination of the idea presented here and the one given by Rosenthal [9] may work. After checking it, we think that an extension of our result to Z^2 action requires just Rosenthal's technique and ours and no new one. Therefore, we decided not to elaborate on this direction.

REFERENCES

1. A. Bellow and H. Furstenberg, *An application of number theory to ergodic theory and the construction of uniquely ergodic models*, Isr. J. Math. **33** (1979), 231–240.

2. M. Denker and M. Keane, *Almost topological dynamical systems*, Isr. J. Math. **34** (1979), 139–160.

3. H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press, Princeton, 1981.

4. S. Glasner and D. Rudolph, *Uncountably many topological models for ergodic transformations*, Ergodic Theory and Dynamical Systems **4** (1984), 233–236.

5. G. Hansel and J. P. Raoult, *Ergodicity, uniformity and unique ergodicity*, Indiana Univ. Math. J. **23** (1974), 221–237.

6. R. I. Jewett, *The prevalence of uniquely ergodic systems*, J. Math. Mech. **19** (1970), 717–729.

7. W. Krieger, *On unique ergodicity*, in *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability*, University of California Press, Berkeley, 1972, pp. 327–346.

8. E. Lehrer, *Topological mixing and uniquely ergodic systems*, Isr. J. Math. **57** (1987), 239–255.

9. A. Rosenthal, *Strictly ergodic models for commuting ergodic transformations*, mimeo, 1985.