Subjective independence and concave expected utility ☆

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Abstract

When a potential hedge between alternatives does not reduce the exposure to uncertainty, we say that the decision maker considers these alternatives *structurally similar*. We offer a novel approach and suggest that *structural similarity is subjective* and should be different across decision makers. Structural similarity can be recovered through a property of the individual's preferences referred to as *subjective codecomposable independence*. This property characterizes a class of event-separable models and allows us to differentiate between perception of uncertainty and attitude towards it. In addition, our approach provides a behavioral foundation to *Concave Expected Utility* preferences.
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1. Introduction

In the heart of decision theory, and economics in general, lies the Expected Utility model. It is well-known, however, that this model does not capture observed behavior in environments of subjective uncertainty. In such environments perception of uncertainty changes when an opportunity of hedging arises; Agents often exhibit strict preference for hedging since that reduces their exposure to uncertainty. This well studied phenomenon is referred to as *uncertainty aversion*. Less attention, however, was given to understanding when a potential hedge does not change the agent's perception of her exposure to uncertainty, and more importantly, how this varies across agents.

We offer a novel approach to valuation of hedging and suggest that the value of a hedge depends on a notion of *structural similarity*. From a decision maker's perspective, hedging between structurally similar alternatives does not reduce the amount of uncertainty she is exposed to. Hence, she reveals no preference for hedging between such alternatives. In contrast with existing literature, we suggest that *structural similarity* is *subjective* and is in the 'eyes of the beholder'; What one decision maker deems structurally similar, might not seem structurally similar to another. Our approach gives rise to a class of preferences represented by a subjective non-additive probability (*capacity*) capturing the decision maker's belief, and a general integration scheme according to which expected utility is calculated. This class of preferences includes expected utility (Savage, 1954 and Anscombe and Aumann, 1963) and Choquet expected utility (Schmeidler, 1989) as special cases, both of which are based on objective structural similarity. We exemplify how subjective structural similarity leads to different predictions from existing models. Combining our approach with uncertainty aversion provides a behavioral foundation to a class of preferences termed *Concave Expected Utility* (*CavEU*) – a capacity-based model of uncertainty aversion that employs the concave integral (Lehrer, 2009) for evaluating alternatives. This model can accommodate recent Ellsberg-like paradoxes emerging from Choquet preferences (Machina, 2009). Lastly, our approach provides sufficient conditions for subjective and Choquet expected utility models; These conditions are weaker than the previous standard formulations.

1.1. Overview

Ellsberg's urn example (Ellsberg, 1961) raised conceptual issues with Savage's (1954) and Anscombe and Aumann's (1963) Subjective Expected Utility (*SEU*) theory. The example shows that the model cannot capture preferences for hedging exhibited in environments of subjective uncertainty. These difficulty stems from the sure-thing principle (in Savage) or the independence axiom (in Anscombe–Aumann). A solution was offered by Schmeidler (1989) who presented *Choquet Expected Utility* (*CEU*) theory, in which beliefs are represented by non-additive probabilities and expected utilities are calculated according to the Choquet integral (Choquet, 1955).

Schmeidler assumed that independence is maintained between alternatives that are comonotonic. Two functions over a state space are comonotonic if both induce the same ordering when the states are ordered according to their associated outcomes. Mixing between comonotonic alternatives yields an alternative in the same comonotonicity class. Thus, it is implicit in Schmeidler's approach that hedging between comonotonic acts should not reduce the amount of uncertainty the decision maker is exposed to, and no decision maker should have a strict preference for hedging between such alternatives.

We propose that valuation of hedging against uncertainty depends on a notion of structural similarity. Alternatives are *structurally similar* if hedging (between them) does not alter the perception of exposure to uncertainty. Thus, *structural similarity should be considered subjective*, while objective considerations need not dictate attitudes toward uncertainty. For instance, according to our approach the decision maker is let the liberty to have strict preferences for hedging between comonotonic acts.[2] We offer an example for a subjective-independence axiom reflecting our approach.

Any act f can be decomposed as a mixture (or lottery), $f = \alpha b_E + (1-\alpha)g$, between some bet b on an event E and a 'complementary' act g. Typically, there are different possibilities to represent an act this way. The independence axiom we postulate requires that any act can be decomposed in at least one way for which the decision maker exhibits no strict preference for hedging.[3] Note that according to this axiom, independence is required to hold for some bet-act decomposition, not for all of them (as suggested by the classic independence axiom), and not for bets and acts are comonotonic (as suggested by comonotonic independence). This particular decomposition depends on the decision maker and typically differs from one decision maker to another. We therefore refer to this axiom as *subjective codecomposable independence*.

Our approach gives rise to a class of *event-separable* preferences including, among other, *SEU* and *CEU*. These preferences all admit a utility form expressed by a subjective non-additive probability (*capacity*), capturing the decision maker's belief, and a general integration scheme according to which expected utility is calculated. More formally, a bet-decomposition of an alternative f is a collection of bets $\{b_E\}_E$ such that $\sum b_E = f$. The value of such a bet-decomposition with respect to a belief v is $\sum b_E v(E)$. Our theory suggests that the utility of an alternative is the value (with respect to non-additive belief) of *some* decomposition of the alternative into bets. It could be interpreted that the decision maker wishes to simplify the task of evaluating an act, doing so by decomposing it to elementary alternatives, that is bets. This decomposition into basic ingredients alleviates the evaluation of and the comparison between alternatives. Unlike *CEU*, our model allows decompositions that do not have to follow a particular structure.

The subjective independence axiom also implies that one can differentiate between the decision maker's perception of uncertainty and her attitude towards it. The perception of uncertainty can be described by the subjective belief, while the attitude towards it depends on the decompositions according to which the decision maker is evaluating the different alternatives. It is possible that two decision makers would perceive uncertainty in the same way (that is, entertain the same belief) but decompose acts differently, exhibit different attitudes towards uncertainty and thereby different preferences.

In order to obtain more structure and some understanding as to how alternatives are being evaluated in the presence of a particular attitude towards uncertainty, we impose (in addition to subjective independence) the classical uncertainty aversion axiom and refer to such preferences as *Concave Expected Utility* (*CavEU*). Such preferences admit a representation that employs the

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[2] There are other weakening of independence in the literature. The issue of similarity as perceived by a decision maker and that, as such, should be subjective has not been discussed. For example, constant independence introduced by Gilboa and Schmeidler (1989) postulates that the constants are similar to every act. This is an objective notion of similarity. Siniscalchi (2009) introduced an independence axiom across acts that are 'complementary' in some sense, but given the underlying utils, this notion of similarity as perception of uncertainty is also objective. Catagnoli and Maccheroni (2000) study independence within an abstract collection of cones of alternatives (see Section 6 for more details).

[3] More formally, any act can be decomposed in such a way that the decision maker exhibits uncertainty neutrality among acts represented by the same bet and complementary act. For brevity, we use the former wording throughout the Introduction.

concave integral (Lehrer, 2009). Thus, subjective independence allows one to provide a behavioral foundation for this class of preference relations.

Among the class of uncertainty averse preferences, *CavEU* are more flexible than *CEU* preferences in the sense that, the acts among which uncertainty neutrality applies are subjectively determined and are not dictated by pre-specified structure. Thus, *CavEU* preferences are less vulnerable to 'paradoxes' such as those introduced by Machina (2009). This is interesting because, while there are indeed some non-separable models that accommodate the Machina examples,[4] *CavEU* is an event-separable model just like *CEU* preferences and can still accommodate Machina's examples.

1.2. Organization

The rest of the paper is organized as follows. The formal framework of choice under uncertainty is presented in Section 2.1. Subjective codecomposability and the emergence of a capacity are presented in Sections 2.2 and 2.3. Section 3 provides a discussion of *Choquet Expected Utility*, *Concave Expected Utility* and the essential differences between the two approaches. Uncertainty aversion and the characterizations of *CavEU* preferences are formally discussed in Section 4. The relation of codecomposability to *SEU* and *CEU* is presented in Section 5. Section 6 concludes with a discussion of the related literature and additional aspects of the model we present. All the proofs are in Appendix A.

2. Subjective decomposability

2.1. Environment

Consider a decision making framework in which an object of choice is an act from the state space to utility outcomes. More formally, let S be a finite non-empty set of *states of nature*.[5] An act is a function from S to $\mathbb{R}_+$. The collection of acts is denoted by $\mathcal{F}$ with typical elements being f, g, h. We interpret $f(s)$ as the payoff induced by act $f \in \mathcal{F}$ in state $s \in S$ and assume it is the utility exerted by the decision maker if f is chosen and s is the realized state. Note that we assume the decision maker's vNM utility has already been identified.

Remark 1. 1. One can also consider the restatement by Fishburn (1970) of the classical Anscombe–Aumann (Anscombe and Aumann, 1963) set-up. In that case, standard axioms imply that the vNM utility index can be identified and that the formulation of alternatives as utility acts, as we resort to here, is well defined. Such results have been established repeatedly and we rely on these results. Note, however, that the axioms presented throughout do not rely on the assumption that the utility has been identified, and are formulated in a manner so they could be applied directly to Anscombe–Aumann acts as well.

2. The assumption that utils are non-negative an unbounded can be omitted and similar results as presented in the following sections would be obtained. Section 6.1 discusses briefly the implications of weakening the structural assumptions on utils.
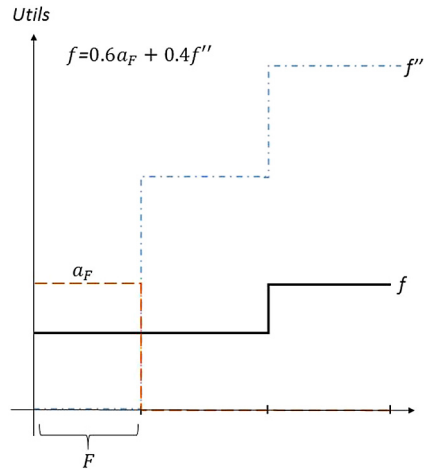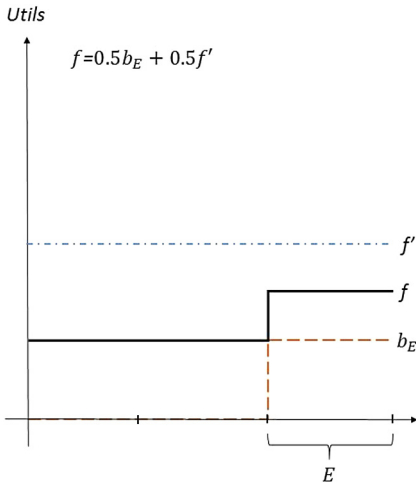

Fig. 1. Decomposition of f into f' and a bet on event E.

Fig. 2. Decomposition of f into f'' and a bet on event F.

Abusing notation, for an act $f \in \mathcal{F}$ and a state $s \in S$, we denote by $f(s)$ the constant act that assigns the utility $f(s)$ to every state of nature. Utils (and constant acts) will be typically denoted by a, b, c. Mixtures (convex combinations) of acts are performed pointwise. That is, if $f, g \in \mathcal{F}$ and $\delta \in [0, 1]$, then $\delta f + (1 - \delta)g$ is the act in $\mathcal{F}$ that yields $\delta f(s) + (1 - \delta)g(s)$ utility for every $s \in S$. Mixture coefficients will be denoted by δ, α, etc.

In our framework, a decision maker is associated with a binary relation $\succeq$ over $\mathcal{F}$ representing his ranking. $\succ$ is the asymmetric part of the relation. That is $f \succ g$ if $f \succeq g$ but it is not true that $g \succeq f$. $\sim$ is the symmetric part, that is $f \sim g$ if $f \succeq g$ and $g \succeq f$. A binary relation $\succeq$ is *complete* if for every $f, g \in \mathcal{F}$, either $f \succeq g$ or $g \succeq f$. It is *transitive* if for $f, g, h \in \mathcal{F}$, $f \succeq g$ and $g \succeq h$ imply $f \succeq h$. As discussed in the Introduction, an important property of preferences is *independence*. It will be useful for later discussion to present the formal definition at this point. We say that a binary relation $\succeq$ satisfies the independence axiom over a (convex) collection of acts $\mathcal{F}' \subseteq \mathcal{F}$ if for every $f, g, h \in \mathcal{F}'$ and every $\delta \in (0, 1)$, $f \succeq g$ if and only if $\delta f + (1 - \delta)h \succeq \delta g + (1 - \delta)h$.

2.2. Codecomposable independence

A *bet* is an act that yields some positive utility $b > 0$ over a non-empty event $E \subseteq S$ and the utility 0 over the complement event. Such a bet will be denoted by b_E. An act which is not a bet can always be represented as a convex combination, or a decomposition, of some bet and another act. That is, for $f \in \mathcal{F}$ we can find a bet b_E, an act f', and $\delta \in [0, 1]$ such that $f = \delta b_E + (1 - \delta) f'$. This is exemplified in Fig. 1. As can be seen from Fig. 2, there are many decompositions of this sort for an act.

Pick one decomposition of f, say, to the bet b_E and the complementary act f' (as appears in Fig. 1). In particular, $f \in [b_E, f'] = \{\alpha b_E + (1 - \alpha) f' : \alpha \in [0, 1]\}$. Now, as can be seen in Figs. 1 and 3, every act $g \in [b_E, f']$ can be decomposed, similarly to f, to the bet b_E and the act f'. A decision maker might (but not necessarily) perceive the acts f and g, and any other act in $[b_E, f']$, to have a "similar" structure. If so, it is reasonable to assume, as in Schmeidler
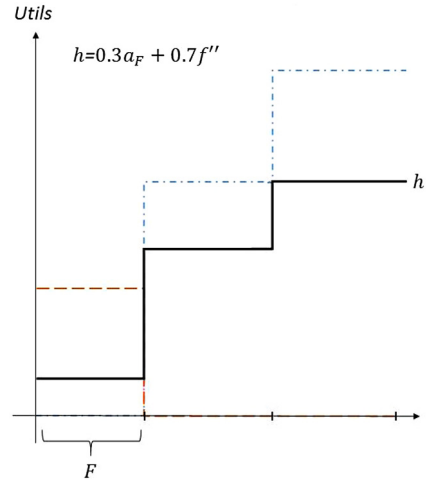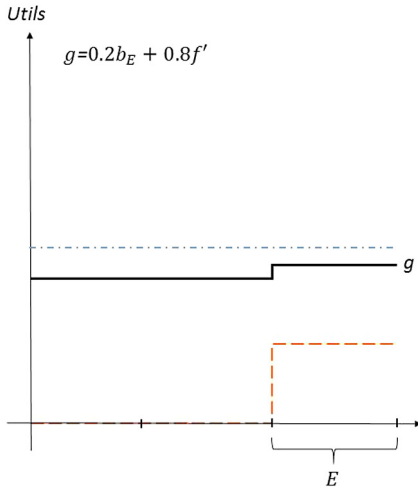
Fig. 3. Decomposition of g into f' and a bet on event F.   Fig. 4. Decomposition of h into f'' and a bet on event F.

(1989), that the decision maker will not have a strict preference for hedging (in other words, the decision maker will exhibit uncertainty neutrality when mixing) between such acts.

Note, however, that a different decision maker may not find g structurally similar to f. It is plausible that she entertains a different notion of structural similarity, and finds $h \notin [b_E, f']$ and f similar since they share a(nother) decomposition structure (see, for example, Figs. 2 and 4). While one decision maker exhibits uncertainty neutrality when mixing between f and g due to his own subjective notion of structural similarity, another decision maker finds f and h structurally similar, exhibiting uncertainty neutrality when mixing between f and h while exhibiting other uncertainty attitudes for mixtures between f and g. Also note that the bet a_F and the act f'' (in Figs. 2 and 4) are not comonotonic, hence mixing the two yields acts that are not necessarily comonotonic with f.

Now, structural similarity and the particular decomposition of an act (whether it is as in Fig. 1, Fig. 2, or rather a third alternative) is a subjective matter. Our main axiom, *Subjective Codecomposable Independence*, postulates that a similarity structure exists but is subjective: every act can be decomposed to a bet on some event and a complementary act such that, the decision maker exhibits uncertainty neutrality across all acts that can be decomposed to a bet on the same event and the complementary act.

Formally, for a bet b_E and an act f', let

$$\text{cone}(b_E, f') = \{\alpha b_E + \beta f'; \alpha, \beta \geq 0\}$$

be the cone generated by b_E and f'.

Subjective Codecomposable Independence. For every non-bet act f, there exist a bet b_E and f' such that $f \in [b_E, f')$ and $\succeq$ satisfies independence over $\text{cone}(b_E, f')$.

The axiom states that the preferences satisfy independence in particular domains of acts. For every act one can find (at least) one event and a complementary act such that independence holds between all acts in the cone generated by the complementary act and a bet on that event. There are finitely many events, so it is possible to verify what event is the one according to which we decompose the act. However, the complementary act could be one of infinitely many acts (all

in the same interval). Therefore, it is possible to find a complementary act, if such exists, with finitely many tests. Nevertheless, the axiom is not falsifiable.

The next axiom can be considered as complementary to *subjective codecomposable independence*, which does not have any bite when the act under consideration is a bet.

Worst-Outcome Bet Independence. For every two bets b_E and d_G, $b_E \succeq d_G$ if and only if $\alpha b_E \succeq \alpha d_G$ for every $\alpha \in (0, 1)$.

Worst-outcome bet independence allows us to directly compare different bets on different events, and states that whenever a bet b_E is preferred to a bet d_F, then mixing both with the worst bet 0_S does not reverse the preference. A stronger version which compares any two acts and their mixtures with the worst outcome, referred to as worst-outcome independence, can be found in Chateauneuf and Faro (2009). Thus, our axiom is also implied by *c*-independence (see Gilboa and Schmeidler, 1989) and comonotonic independence (Schmeidler, 1989).

2.3. A capacity emerges

To explore the implications of *subjective codecomposable independence* we need to present some notations and definitions. A *capacity* v over S is a function $v : 2^S \to [0, 1]$ satisfying: (i) $v(\phi) = 0$ and $v(S) = 1$; and (ii) $K \subseteq T \subseteq S$ implies $v(K) \leq v(T)$.

Fix and act f. A collection $\{(a_E, E) : E \subseteq S, a_E \geq 0\}$ is a *decomposition* of f if $\sum a_E \mathbb{1}_E = f$.[6] Clearly, every act has many decompositions. Consider for example the act g that induces a utility on event A and b utility on the complement event, A^c, where $b > a$. Two of many decompositions of g are $a\mathbb{1}_S + (b - a)\mathbb{1}_{A^c}$ and $a\mathbb{1}_A + b\mathbb{1}_{A^c}$.

Similarly to the classical expected utility theory, the *value* of the decomposition $\{(a_E \mathbb{1}_E)\}_E$, with respect to a capacity v, is simply $\sum a_E v(E)$. When a capacity is a probability distribution over the states S, then the values of all decompositions of an alternative coincide and equal to its expected utility with respect to that probability. However, when the capacity is not a probability distribution (that is, not additive) then this fact is no longer true, and different decompositions obtain different values.

Definition 1. We say that a binary relation $\succeq$ over all acts $\mathcal{F}$ admits an *event-separable representation* if there exist a homogeneous functional $V : \mathcal{F} \to \mathbb{R}$ and a capacity $v : 2^S \to [0, 1]$ such that:

1. V represents $\succeq$, that is $V(f) \geq V(g) \Leftrightarrow f \succeq g$ for every $f, g \in \mathcal{F}$;
2. $V(b_E) = b \cdot v(E)$ for every bet b_E; and
3. for every act $f \in \mathcal{F}$,

$$V(f) = \sum a_E v(E) \text{ for some } \sum a_E \mathbb{1}_E = f, \tag{1}$$

where $a_E \geq 0$.

Thus, a binary relation admits an event-separable representation if an act is ranked according to the value, with respect to the capacity, of one of its decompositions into bets. Choquet Expected Utility (Schmeidler, 1989) is one example, out of many, for such preferences. We will

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[6] $\mathbb{1}_E$ is the characteristic function of the event E.

discuss this theory, and the alternative presented in this paper, in more detail in the following sections.

In addition to the axioms above, which rise from the new approach we present, we will postulate throughout the following standard assumptions:

Weak Order. $\succeq$ is complete and transitive.

Monotonicity. For every $f, g \in \mathcal{F}$, $f(s) > g(s)$ for all $s \in S$ implies $f \succ g$.

Continuity. For every $f \in \mathcal{F}$ the sets $\{g \in \mathcal{F} : g \succeq f\}$ and $\{g \in \mathcal{F} : g \preceq f\}$ are closed.

Theorem 1. *Let $\succeq$ be a binary relation over $\mathcal{F}$ satisfying* weak order, monotonicity, continuity, worst-outcome bet independence, *and* subjective codecomposable independence. *Then, $\succeq$ admits an event-separable representation. Moreover, if both (V, v) and (V', v') represent $\succeq$, then $V' = V$ and $v' = v$.*

Theorem 1 states that given standard assumptions and *subjective codecomposable independence*, a binary relation admits an event-separable representation. The axioms are sufficient to (a) identify a unique non-additive belief representing how the decision maker perceives uncertainty; and (b) state that alternatives are ranked according to the value of one of their decompositions, which describes the decision maker's attitude towards uncertainty. Note that the belief can be identical across different decision makers while their attitudes towards uncertainty, and therefore their decisions, could be different.

Remark 2. The theorem can be rephrased as an 'if and only if' statement. Let $\mathcal{C}$ be a collection of bet-act cones of the form $\text{cone}(b_E, g)$. We say that $\mathcal{C}$ is a *bet-act cone cover* (of $\mathcal{F}$) if every non-bet act f is a member of the relative interior of at least one cone in $\mathcal{C}$. The axioms in Theorem 1 are necessary and sufficient for the existence of a bet-act cone cover and a continuous and monotonic representing functional V, where V is linear within each cone in $\mathcal{C}$. Theorem 1 implies that linearity over some bet-act cone cover is a consequence of the axioms. Furthermore, every representation of this kind is event separable.

It is important to note that it is not true that every preferences admitting an event-separable representation as in Eq. (1) satisfies our axioms. Such preferences need not imply that the representation is linear within each cone in some bet-act cone cover (see Example 3 in Appendix A). This is why Theorem 1 cannot be readily phrased as a logical equivalence statement.

The next example shows a difference between MMEU preferences (Gilboa and Schmeidler, 1989) and event-separable preferences. It also shows that preferences could be represented by a linear V over cones and yet, not be event-separable.

Example 1. Consider preferences over alternatives over three states that can be represented by the following value function.[7] For every $f = (f_1, f_2, f_3)$, $V(f) = \frac{f_1 + f_2}{2}$ if $\frac{f_1 + f_2}{2} \le f_3$, and $V(f) = f_3$ if $\frac{f_1 + f_2}{2} \ge f_3$. Note that every act is a member of one of the cones $\{f : \frac{f_1 + f_2}{2} \le f_3\}$ and $\{f : \frac{f_1 + f_2}{2} \ge f_3\}$. Moreover, the preferences satisfy independence within each of these cones.

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[7] These are Gilboa–Schmeidler (Gilboa and Schmeidler, 1989) preferences where the priors are taken to be (the convex hull of) $(0.5, 0.5, 0)$ and $(0, 0, 1)$.

Do these preferences satisfy our axioms and admit an event-separable representation? The answer in no.

To show why, consider the alternative $g = (2, 0, 1)$ and let us assume that the preferences are indeed event-separable as in Theorem 1. That means there is a decomposition $g = \sum_E a_E \mathbb{1}_E$, where $V(g) = \sum_E a_E v(E)$. Note however that the only event with *positive* value is the grand event itself. This event cannot be involved in any decomposition of g, because the second coordinate of g is zero. This means that $\sum_E a_E v(E) = 0$, which contradicts the fact that $V(g) > 0$. Thus, these preferences do not admit an event-separable representation and do not satisfy subjective codecomposable independence.

Note that *subjective codecomposable independence* is a weak assumption; it is not possible to determine exactly what is the decomposition according to which an alternative is ranked. A question is whether making stronger behavioral assumptions can help identify the integration mechanisms, and whether such integration mechanisms are natural and interesting while being different than Choquet? We investigate this direction in Section 4 below. We impose the classical uncertainty aversion axiom and show that the only decision model that is based on a capacity, as in Theorem 1, is the *Concave Expected Utility* model discussed in Section 3 in the context of the Machina examples.

2.4. Proof sketch: Theorem 1

The detailed proof of Theorem 1 appears in Appendix A. We here present an intuitive sketch. *Subjective codecomposeable independence* guarantees that every act f is contained in a cone generated by some bet and another act, where an affine function over this cone represents the preferences. Due to *worst-outcome bet independence* every bet can be directly compared to the constants, which guarantees that all these affine functions (one for every cone) can be calibrated with the one defined on the constant acts and thereby can be merged together in a consistent way. This yields a functional V over all acts that represents the preferences and is affine over the subjective cones. In particular, part 1 of Definition 1 is satisfied.

Now, due to *worst-outcome bet independence*, V is homogeneous over bets. That is, $V(\alpha b_E) = \alpha V(b_E)$. In particular $V(0) = 0$. Thus, normalizing $V(\mathbb{1}_S) = 1$ and defining $v(E) = V(\mathbb{1}_E)$, we obtain that v is a capacity due to *monotonicity*. This implies that part 2 of Definition 1 is satisfied.

In case f is a non-bet act the main idea is the following. *Subjective codecomposeable independence* assures us that f is a mixture (convex combination) of some bet b_{E_f} and a 'residual' act g such that V is affine over cone(b_{E_f}, g). In particular, $V(f) = V(\alpha b_{E_f} + (1 - \alpha)g) = \alpha V(b_{E_f}) + (1 - \alpha)V(g)$, and since V is homogeneous over bets, we get that $V(f) = \alpha b V(\mathbb{1}_{E_f}) + (1 - \alpha)V(g)$. By repeating this argument for g and substitute $V(g)$ into $V(f)$, one obtains a summation of bets and a smaller residual, the one obtained from decomposing g. This procedure is reiterated over and over again. However, and this is the delicate part of the proof, the sequence of residuals obtained from this process might not vanish. That is, it might converge to an act h which is not a bet and satisfies $V(h) > 0$.

When decomposing f such that $V(f) = \alpha b V(\mathbb{1}_{E_f}) + (1-\alpha)V(g)$, we show that it is possible to choose the decomposition in a way that $a_f = \alpha b$ is maximal. Then, when decomposing the residual, say g, it is impossible that g is decomposed in an affine fashion to a bet on E_f. This will contradict the maximality of a_f. Since the state space is finite, there are finite number of

events. This means that there must be a finite step in this process of decomposing the residuals. From part 2 of Definition 1, which was established above, the proof is complete.

3. Non-additive beliefs and expected utility

This section provides an informal discussion and (partial) comparison between the classic Choquet Expected Utility model and an alternative theory considered in this paper. The aim of this section is to present and motivate some of the key concepts relevant to the point the current paper is trying to make. The formal decision theoretic foundation for the alternative theory appears in the following section.

Following Ellsberg's urn example (Ellsberg, 1961), Schmeidler (1989) was the first to present an alternative to the classical subjective expected utility theory by incorporating non-additive beliefs. Schmeidler weakened the independence axiom and introduced comonotonic independence which serves as a behavioral foundation for *Choquet Expected Utility (CEU)*. The Choquet expected utility (see, Choquet, 1955) of g takes the form

$$V_{CEU}(g) = \sum_i b_i \cdot v(E_i), \tag{2}$$

where $\sum_i b_i \cdot \mathbb{1}_{E_i}$ is the unique decomposition of g such that $\{E_i\}_i$ form a chain (that is, $E_{i+1} \subsetneq E_i$ for every i). We refer to such a decomposition as the *Choquet decomposition*. For example, if g induces utility a on event A and utility b, where $b > a$ on the complement event A^c, then the Choquet decomposition of g is $(b - a)\mathbb{1}_{A^c} + a\mathbb{1}_S$ and the Choquet expected utility of g with respect to a capacity v is $(b - a)v(A^c) + av(S)$. The intuition as to why comonotonic independence (along with the standard axioms) implies that preferences are represented by *CEU* is discussed in Section 5. In fact, we will show that a version of the axiom in the spirit of code-composability is sufficient for this task.

As mentioned in the previous section, every alternative has more than one decomposition. Evaluating acts by using the Choquet decomposition is only one of many ways to do so. Consider, for instance, an alternative theory in which an act is evaluated according to the maximum value over *all* of its decompositions. The decomposition in which the maximum is obtained is referred to as the *optimal decomposition*. Such valuations are denoted by V_{CAV} and preferences that admit such a utility form are referred to as *Concave Expected Utility (CavEU)*. The term hints that such preferences always exhibit (weak) affinity for hedging.

To illustrate how *CavEU* may be different than *CEU*, consider the following example. Let the state space be $S = \{s_1, \ldots, s_4\}$ and define a capacity v over the state space as follows: $v(s) = \frac{1}{12}$ for every state s, $v(\{s_1, s_2\}) = v(\{s_1, s_3\}) = v(\{s_2, s_3\}) = v(\{s_1, s_4\}) = \frac{1}{6}$, $v(\{s_2, s_4\}) = v(\{s_3, s_4\}) = \frac{3}{12}$, $v(\{s_1, s_2, s_3\}) = v(\{s_1, s_3, s_4\}) = v(\{s_2, s_3, s_4\}) = \frac{1}{3}$, $v(\{s_1, s_2, s_4\}) = \frac{5}{6}$ and $v(S) = 1$. Note that the contribution of the state s_2 to any event that contains neither s_1 nor s_2 is greater than the contribution of s_1. Formally, for any event E that does not contain the states s_1, s_2, $v(E \cup \{s_1\}) \leq v(E \cup \{s_2\})$. Moreover, the inequality is strict when $E = \{s_4\}$. In this sense, if we interpret v as how the decision maker perceives uncertainty, then s_2 is more likely than s_1.

Now, consider the acts $f = (0, 1, 2, 3)$ and $g = (1, 0, 2, 3)$. Note that f and g differ only in states s_1 and s_2. The act f returns the lower outcome in the less likely state and the higher outcome in the more likely one. It is the opposite case for g; it returns the higher outcome in the less likely state. It is plausible then, that a decision maker perceiving uncertainty through v would rank f over g. Nevertheless, expected utility according to *CEU* does not support that;

V_{CEU} of both f and g is $\frac{8}{12}$: $V_{CEU}(f) = v(\{s_2, s_3, s_4\}) + v(\{s_3, s_4\}) + v(\{s_4\}) = v(\{s_1, s_3, s_4\}) + v(\{s_3, s_4\}) + v(\{s_4\}) = V_{CEU}(g)$. That is, the Choquet expected utility, holding the belief v, is the same for both f and g and the decision maker is indifferent between the two. However, *CavEU* ranks f strictly higher than g: $V_{CAV}(f) = v(\{s_2, s_4\}) + 2v(\{s_3, s_4\}) = \frac{9}{12} > \frac{8}{12} = v(\{s_1, s_3, s_4\}) + v(\{s_3, s_4\}) + v(\{s_4\}) = V_{CAV}(g)$.

Through the example above we can also demonstrate the idea of subjective structural similarity. We have seen that the optimal decomposition of f with respect to v is $\mathbb{1}_{\{s_2,s_4\}} + 2 \cdot \mathbb{1}_{\{s_3,s_4\}}$, implying that a decision maker following such preferences finds $\mathbb{1}_{\{s_2,s_4\}}$ and $\mathbb{1}_{\{s_3,s_4\}}$ structurally similar in the sense that mixing between them does not expose him less to uncertainty. Notice also that $\mathbb{1}_{\{s_2,s_4\}}$ and $\mathbb{1}_{\{s_3,s_4\}}$ are not comonotonic. Thus, the similarity structure associated with such preferences are different than that associated with *CEU* preferences. On the other hand, consider any convex capacity v'.[8] *CavEU* preferences with respect to v' coincide with *CEU* with respect to v' (Lehrer, 2009). In that case, bets on comonotonic events are structurally similar. Thus, the capacity discussed above and v' provide an example of different *CavEU* preferences associated with different similarity structures. This is unlike *CEU* preferences for which the similarity structure is determined by comonotonicity regardless of the beliefs.

4. Uncertainty aversion

Since Schmeidler (1989) and Gilboa and Schmeidler (1989) uncertainty aversion has been one of the most studied phenomenon in the theory of decision making. Unlike Schmeidler (1989) who focused on comonotonic-independence, following our discussion regarding subjective structural similarity, we here wish to impose the weaker *subjective codecomposable independence*, and add structure by postulating uncertainty aversion.

Uncertainty Aversion. For every $f, g \in \mathcal{F}$, if $f \sim g$ then $\delta f + (1 - \delta)g \succeq g$ for every $\delta \in [0, 1]$.

Before we state the result, we make a formal definition of the *concave integral* (Lehrer, 2009) with respect to capacities. The concave integral of an act $f : S \to \mathbb{R}_+$ with respect to a capacity $v : 2^S \to [0, 1]$ is defined by

$$\overset{Cav}{\int} f\,dv = \max \left\{ \sum a_E v(E) : \sum a_E \mathbb{1}_E = f, a_E > 0 \right\}. \tag{3}$$

The integral considers all possible decompositions of an act and evaluates it according to the decomposition with the maximal value (with respect to the capacity). It is immediate that the concave integral is indeed a concave functional. Indeed, fix a capacity v. While $\sum a_E \mathbb{1}_E$ and $\sum b_F \mathbb{1}_F$ may be optimal decompositions of f and g respectively, there might be a decomposition of $\alpha f + (1 - \alpha)g$ with a value (with respect to the capacity v) greater than that of $\sum \alpha a_E \mathbb{1}_E + \sum (1 - \alpha)b_F \mathbb{1}_F$.

We refer to preferences $\succeq$ over all acts $\mathcal{F}$ as *CavEU* if there exist a capacity v, such that for all $f, g \in \mathcal{F}$

$$f \succeq g \iff \overset{Cav}{\int} f\,dv \geq \overset{Cav}{\int} g\,dv. \tag{4}$$

---

[8] The notion of a convex capacity is due to Shapley (1971).

We know from Theorem 1 that along with the standard axioms, *subjective codecomposable independence* implies that preferences admit an event-separable representation. It turns out that from this family of preferences, there is only one class adhering to uncertainty aversion. This is the class of *CavEU* preferences.

Theorem 2. *Let $\succeq$ be a binary relation over $\mathcal{F}$. Then the following are equivalent:*

1. $\succeq$ *satisfies* weak order, monotonicity, continuity, worst-outcome bet independence, subjective codecomposable independence *and* uncertainty aversion*; and*
2. $\succeq$ *is CavEU.*

We know, due to Theorem 1, that the belief (or, capacity) representing the preferences is unique. However, the Theorem allows for very general preferences and there is not much more that can be said about such beliefs. Since now we have restricted attention to preferences that adhere to *uncertainty aversion*, it is possible to identify a particular structure for the decision maker's beliefs.

Proposition 1. *v can represent a* CavEU *preference relation (in the sense of Definition 1) if and only if v can be written as a minimum of finitely many measures over S (that is, $v = \min_i \mu_i$).*

The proposition states that *CavEU* preferences can be represented by a belief that is a lower envelope of finitely many measures. That is, if preferences are event-separable and uncertainty aversion is satisfied, then the capacity, representing the DM's perception of uncertainty, can be modeled as the minimum of a (finite) collection of measures over the state space.

Note that the concave integral is well defined for every capacity v, even when it does not satisfy this property. However, the propositions states that in this case Definition 1, and in particular point 2 of the definition, does not hold. Indeed, let v be a capacity and an event E such that $\int^{Cav} \mathbb{1}_E dv > v(E)$. Define the capacity $\hat{v}$ by $\hat{v}(F) = \int^{Cav} \mathbb{1}_F dv$ for every $F \subseteq S$. Of course, $\hat{v} \geq v$ where $\hat{v}(E) > v(E)$. Nevertheless, by Lehrer and Teper (2008) we have that the integral with respect to both v and $\hat{v}$ represent the same preferences. In particular, the content of Proposition 1 is that $v = \hat{v}$ if and only if v is the minimum of finitely many additive measures. This result is due to Kalai and Zemel (1982).

5. Codecomposable independence and expected utility models

It is interesting to see the links between the codecomposable independence approach to existing models. Clearly, both *SEU* and *CEU* are particular classes of preferences admitting an event-separable representation. It turns out that providing stronger versions of our independence axiom yields exactly *SEU* and *CEU*. Note that both versions resort to objective structural considerations.

For an act f and a utility level $a \in \mathbb{R}_+$, let $E_a^f = \{s \in S : f(s) \geq a\}$ be the event in which f performs better that a. A stronger codecomposable independence axiom can be formulated taking into account decompositions of acts to comonotonic bets.

Cumulative Codecomposable Independence. For every act f, and every comonotonic b_{Ef} and f' such that $f \in [b_{Ef}, f')$, $\succeq$ satisfies independence over cone(b_{Ef}, f')

The axiom postulates that if $f, g, h \in \mathcal{F}$ can all be expressed as a linear and positive combination of the comonotonic bet b_{Ef} and act f', then independence involving f, g, h holds. Like

Schmeidler's approach, this again is an objective structural similarity assumption. Note that, in this case, f, g and h are comonotonic, and hence this axiom is weaker than Schmeidler's comonotonic-independence. Resulting from such a strengthening of *subjective codecomposable independence* is the following proposition. Note that *worst-outcome bet independence* is implied by *cumulative codecomposable independence*.

Proposition 2. *The following two statements are equivalent:*

1. $\succeq$ *satisfies* weak order, continuity, monotonicity, *and* cumulative codecomposable independence*;*
2. $\succeq$ *admits a* CEU *representation.*

The following example provides intuition as to why *cumulative codecomposable independence* is enough to obtain that alternatives are evaluated by their Choquet decomposition. The detailed proof appears in Appendix A.

Example 2. Assume V is a homogenous functional representing preferences. Define the capacity v as follows: $v(E) = V(\mathbb{1}_E)$ for every event E. Let g be the act that induces utility a on event A and utility b on the complement event A^c, where $b > a$. The act g can be rewritten as $g = a\mathbb{1}_S + (b - a)\mathbb{1}_{A^c} = \frac{a}{b}(b\mathbb{1}_S) + \frac{b-a}{b}(b\mathbb{1}_{A^c})$. Thus, g is a convex combination of two comonotonic bets, b_S and b_{A^c}. From *cumulative codecomposable independence* we infer that $V(g) = \frac{a}{b}V(b\mathbb{1}_S) + \frac{b-a}{b}V(b\mathbb{1}_{A^c})$, and since V is homogenous, that $V(g) = \frac{a}{b}bV(\mathbb{1}_S) + \frac{b-a}{b}bV(\mathbb{1}_{A^c}) = aV(\mathbb{1}_S) + (b - a)V(\mathbb{1}_{A^c}) = av(S) + (b - a)v(A^c)$. We thus obtain that $V(g)$ is precisely the Choquet integral of g with respect to v.

Lastly, we explore a further strengthening of our approach by postulating that for every decomposition of f to a bet b_E and a complementary act f', the preference relation satisfies independence over $[b_E, f']$.

Codecomposable Independence. For every bet b_E and act f', $\succeq$ satisfies independence over $\text{cone}(b_E, f')$.

Assuming *codecomposable independence* along with the axioms specified above allows us to formulate the following result.

Proposition 3. *The following two statements are equivalent:*

1. $\succeq$ *satisfies* weak order, continuity, monotonicity, worst-outcome bet independence, *and* codecomposable independence*;*
2. $\succeq$ *admits an* SEU *representation.*

Proposition 3 states that given the standard axioms, *codecomposable independence* allows us to identify a subjective probability with respect to which the decision maker calculates the expected utility of the different alternatives and ranks them accordingly. Note that *worst-outcome bet independence*, is, again, not needed as it is implied by *codecomposable independence*.

It should be noted that recently Borah and Kops (2014) suggest that the independence axiom can be substantially weakened while still maintain a subjective expected utility representation. Roughly speaking, they show that it is enough to require independence across acts that differ only in one state.

6. Additional comments

6.1. On the assumptions on utils

We assume that the vN–M utility is non-negative. It is possible to assume the utility is negative while strengthening the axiomatic structure by postulating *c*-independence (instead of the weaker *worst-outcome bet independence*). This will imply that the functional form is translation covariant and any non-negative utility act can then be analyzed by translating it to a non-negative one.

We have also assumed throughout that the utility is unbounded. *Subjective codecomposable independence* is stronger when the utility is bounded. It might be impossible to obtain some decompositions for an act since the complementary act to a particular bet may require levels of utility that are not specified (or identified) by the decision maker's preferences. That implies more structure on our representation. Consider the least monotonic capacity v over the state space $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ that satisfies $v(s_1, s_4, s_5) = v(s_2, s_5, s_6) = v(s_3, s_4, s_6) = \frac{2}{3}$ and $v(S) = 1$. Note that $v(E) \leq \frac{2}{3}$ whenever $E \neq S$. It is easy to verify that v is totally balanced. Consider the act $f = (0.5, 0.5, 0.5, 1, 1, 1)$. It can be decomposed as $f = \frac{1}{2}\mathbb{1}_{\{s_1, s_4, s_5\}} + \frac{1}{2}\mathbb{1}_{\{s_2, s_5, s_6\}} + \frac{1}{2}\mathbb{1}_{\{s_3, s_4, s_6\}}$, which implies that $\int^{Cav} f dv \geq 3 \cdot \frac{1}{2} \cdot \frac{2}{3} = 1$. Let the utility be bounded in $[0, 1]$ and assume, adhering to *subjective codecomposeable independence*, that $\int^{Cav} f dv = \sum a_E v(E)$ for some decomposition $f = \sum a_E \mathbb{1}_E$, where $\sum a_E = 1$. Thus, $\sum a_E v(E) = \sum_{E \neq S} a_E v(E) + a_S v(S) \leq \frac{2}{3} \sum_{E \neq S} a_E + a_S = \frac{2}{3}(1 - a_S) + a_S \leq \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{2} = \frac{5}{6} < \int^{Cav} f dv$ (the second inequality is due to that a_S cannot exceed $\frac{1}{2}$ because $f = \sum a_E \mathbb{1}_E$). This is a contradiction. Hence, it cannot be that the capacity v represents preferences that adhere to *subjective codecomposeable independence* and *uncertainty aversion* when the utility is bounded. This illustrates how *subjective codecomposeable independence* entails more structure on preferences when utility is bounded relative to unbounded utility. In a separate note (see Lehrer and Teper, forthcoming) we show that a property termed the *Sandwich property* is necessary and sufficient for the representation of *CavEU* preferences when utility is bounded.

6.2. On examples by Machina

In a recent paper, Machina (2009) introduced two "paradoxes" for the Choquet Expected Utility model. He "exploits" comonotonic independence exhibited by such preferences and constructs several examples in which such preferences cannot accommodate choices that may be considered natural. For *CavEU* preferences, however, he optimal decomposition depends on the decision maker's subjective similarity structure and is not pre-specified structurally as in Eq. (2). Hence, *CavEU* preferences are less vulnerable to such "paradoxes".

The Reflection Example. An urn contains one hundred balls of four different colors a, b, c, and d. All you know is that there are fifty balls that either a or b, and fifty balls that are either c or d. A decision maker chooses an act, then a ball is randomly drawn and a reward in utils[9] is given to the decision maker according to the color of the ball and the chosen act. Table 1 summarizes the rewards related to four acts.

9　Even though the analysis would go through if entries are monetary, we consider utils for brevity and simplicity.

Table 1
The Reflection Example.

Bet	a	b	c	d
f_1	400	800	400	0
f_2	400	400	800	0
f_3	0	800	400	400
f_4	0	400	800	400

Table 2
The 50–51 Example.

Bet	a	b	c	d
f_5	800	800	400	400
f_6	800	400	800	400
f_7	1200	800	400	0
f_8	1200	400	800	0

One can see that f_3 and f_4 are mirror images of f_2 and f_1 respectively. It is then plausible that, if a decision maker strictly prefers f_1 to f_2 (i.e. $f_1 \succ f_2$) we would expect that she should strictly prefer f_4 to f_3 (i.e. $f_4 \succ f_3$). This is inconsistent with *CEU* preferences, as explained in Machina (2009). Indeed, the Choquet decompositions of $f_1, .., f_4$ are such that $V_{CEU}(f_1) > V_{CEU}(f_2)$ if and only if $V_{CEU}(f_3) > V_{CEU}(f_4)$.

We show now that the reversal as discussed in Machina is possible under *CavEU* preferences. Consider the least monotonic capacity v such that $v(a) = v(b) = v(c) = v(d) = 0$, $v(bc) = \frac{1}{100}$ and $v(ab) = v(cd) = \frac{1}{2}$. The optimal decomposition of f_1 is $400 \cdot \mathbb{1}_{ab} + 400 \cdot \mathbb{1}_{bc}$. However the optimal decomposition of f_2 is $400 \cdot \mathbb{1}_{ab}$. Thus, according to *CavEU* with respect to the described capacity, $V_{CAV}(f_1) > V_{CAV}(f_2)$. The reason is that $400 \cdot v(ab) + 400 \cdot v(bc) > 400 \cdot v(ab)$. For a similar calculation $V_{CAV}(f_4) > V_{CAV}(f_3)$. Indeed, an optimal decomposition for f_3 is $400 \cdot \mathbb{1}_{cd}$, while an optimal decomposition for f_4 is $400 \cdot \mathbb{1}_{cd} + 400 \cdot \mathbb{1}_{bc}$. Hence, *CavEU* is consistent with $f_1 \succ f_2$ and $f_4 \succ f_3$.

The 50–51 Example. The second example refers to a similar urn as above, only now there are 101 balls, out of which 51 are c or d. Table 2 summarizes the rewards related to four acts.

Note that f_7 and f_8 are obtained from f_5 and f_6 by increasing the rewards related to color a (which are the highest possible) by 400 and by reducing the rewards related to color d (which are the lowest possible) by 400. Machina argues that it is reasonable for uncertainty averse decision makers to prefer f_5 to f_6 (denoted $f_5 \succ f_6$) and at the same time to prefer f_8 to f_7 ($f_8 \succ f_7$). However, the Choquet decompositions of $f_5, \ldots, f_8$ are such that $V_{CEU}(f_5) > V_{CEU}(f_6)$ if and only if $V_{CEU}(f_7) > V_{CEU}(f_8)$.

Consider now the least monotonic capacity that takes the following values: $v(ab) = \frac{50}{101}$, $v(ac) = .5$, $v(cd) = \frac{51}{101}$, $v(abc) = .75$ and $v(abd) = .76$. It is possible to show that given this capacity, $V_{CAV}(f_5) > V_{CAV}(f_6)$ and that $V_{CAV}(f_8) > V_{CAV}(f_7)$.

Remark 3. Note that in both examples the capacities are not symmetric with respect to the exogenous "information" provided regarding the underlying uncertainty. It is impossible to construct

an informationally-symmetric capacity for which *CavEU* can explain the behavior pointed out by Machina.

6.3. How does it fit in the lit?

Catagnoli and Maccheroni (2000) notice that many of the weakening of independence (e.g., Schmeidler, 1989 and Gilboa and Schmeidler, 1989) take a form of independence within a collection of cones with a particular structure. They study the implication of independence within an abstract class of cones and show that preferences can be represented with an abstract function which is linear within every cone.

In terms of models of choice under uncertainty, the most related ones are *confidence preferences* presented by Chateauneuf and Faro (2009), maxmin expected utility (*MEU*) that were introduced by Gärdenfors and Sahlin (1982) and axiomatized by Gilboa and Schmeidler (1989) and, of course, *CEU* preferences.

CavEU is clearly a particular case of confidence preferences, but requires more structure since not every confidence preferences satisfy the decomposability property.[10] To see that, consider *MEU* preferences, which are a particular case of confidence preferences. Not every *MEU* preference relation can be represented as a (concave) integral (as seen in Example 1); *MEU* satisfies the c-independence axiom, while it is clear from *subjective codecomposable independence* that it does not have to be satisfied by *CavEU*. Since these two axioms are not nested, the two models are not nested. The subclass of *CavEU* preferences that do admit an *MEU* representation (or equivalently, satisfy *c*-independence) are those that can be represented with a capacity having a *large core* (see, Lehrer, 2009).[11] In other words, a representation in the style of Eq. (4) of preferences over acts ranged to the entire real line (i.e., utils could be negative and positive) can be carried out by a capacity having a large core. As for the axiomatization of such preferences, *worst-outcome bet independence* appearing in Theorem 2 would have to be strengthen to *c*-independence.

This brings us to *CEU* preferences. Lovasz (1983) (pp. 246–249) and Schmeidler (1989) show that the Choquet integral is a concave functional if and only if the capacity is *convex*. Hence we have that when the capacity is not convex *CavEU* and *CEU* differ. In addition, due to Lehrer (2009) and Lehrer and Teper (2008), *CEU* and *CavEU* preferences coincide if and only if the capacity representing the preferences is convex. In this case it is also *MEU*. The latter point emphasizes that within the class of uncertainty averse preferences, the class of *CavEU* preferences is more general than that of *CEU*.

Appendix A

Proof of Theorem 1. For an act $f \in \mathcal{F}$, let $c(f)_S \sim f$ be a constant act that is indifferent to f. Due to *monotonicity*, *continuity* and *weak order*, $c(f)$ exists and is unique. Define the real valued function $V : \mathcal{F} \to \mathbb{R}$ by $V(f) = c(f)$. Clearly, V represents the preferences. That is, for every $f, g \in \mathcal{F}$, $f \succeq g$ if and only if $V(f) \geq V(g)$. Note that $V(\alpha \mathbb{1}_S) = \alpha$ for every $\alpha \geq 0$. Now,

$$V(a_E) = V(ac(\mathbb{1}_E)_S) = ac(\mathbb{1}_E) = aV(\mathbb{1}_E) \tag{A.1}$$

---

[10] In particular, *CavEU* are those confidence preferences where the support of the confidence function has finitely many extreme points that constitute the set of measures dominating the capacity.

[11] The definition of large core is due to Sharkey (1982).

for every $a \geq 0$ and $E \subseteq S$, where the left equality is due to *worst-outcome bet independence* (applied to the bets $c(\mathbb{1}_E)_S$ and $\mathbb{1}_E$). This homogeneity property implies that $V(0) = 0$. Define $v : 2^S \rightarrow [0, 1]$ by $v(E) = V(\mathbb{1}_E)$. v is a capacity. Indeed, $v(\emptyset) = V(0) = 0$, $v(S) = V(\mathbb{1}_S) = 1$ and v is monotonic since V is monotonic.[12] Now, if $f = a_E$ is a bet, then the claim in the theorem is immediate due to Eq. (A.1).

Pick a non-bet act $f \in \mathcal{F}$. The main idea of the proof lies behind the following 4 claims.

Claim 1. *There is a bet b_E and an act f' such that $f \in \text{cone}(b_E, f')$ and V is affine over $\text{cone}(b_E, f')$. In particular, V is homogeneous, that is, $V(\alpha f) = \alpha V(f)$ for every $\alpha \geq 0$.*

Proof. Indeed, pick a non-bet act $f \in \mathcal{F}$. *Subjective codecomposable independence* implies that there exist an event $E_f \subseteq S$ and an act f' such that $f \in [b_{E_f}, f']$ (i.e., $f = (1 - \delta)b_{E_f} + \delta f'$ for some $\delta \in (0, 1)$) and $\succeq$ respects independence over $\text{cone}(b_{E_f}, f')$. Note that since f is not a bet, f' and b_{E_f} are algebraically independent. Since $\succeq$ respects independence over $\text{cone}(b_{E_f}, f')$ it can be represented (over $\text{cone}(b_{E_f}, f')$) by a unique affine function that agrees with V on $0, b_{E_f}$ and f'. In other words, for every $g = a_1 b_{E_f} + a_2 f' \in \text{cone}(b_{E_f}, f')$, $V(g) = a_1 V(b_{E_f}) + a_2 V(f')$ represents $\succeq$. In particular, $V(\alpha f) = \alpha V(f)$ for every $\alpha \geq 0$. $\quad\square$

The following claim is an immediate corollary of Claim 1.

Claim 2. *There exists an event E_f, an act $f' \in \mathcal{F}$, $a > 0$ and $\delta \in [0, 1]$ such that $f = a_{E_f} + \delta f'$ and $V(f) = V(a_{E_f}) + \delta V(f')$.*

Proof. There exist an event $E_f \subseteq S$ and an act f' such that $f \in [b_{E_f}, f']$ (i.e., $f = (1 - \delta)b_{E_f} + \delta f'$ for some $\delta \in (0, 1)$) and $V(f) = (1 - \delta)V(b_{E_f}) + \delta V(f')$. Now, letting $a = (1 - \delta)b$ we have that $(1 - \delta)V(b_{E_f}) = aV(\mathbb{1}_{E_f}) = V(a_{E_f})$. $\quad\square$

Claim 3. *There is a maximal $a \geq 0$ such that $f = \delta g + a_{E_f} + \sum_{E \neq E_f} b_E$ and $V(f) = \delta V(g) + V(a_{E_f}) + \sum_{E \neq E_f} V(b_E)$.*

Proof. Enumerate all events but E_f and consider now

$$
\mathcal{E}_f = \left\{ (b_{E_f}, b_{E_1}, \ldots, b_{E_{2|S|-1}}, g, \delta) : g \in \mathcal{F}, \delta \in [0, 1], f = \delta g + \sum_E b_E, \right.
$$
$$
\left. V(f) = V(g) + \sum_E V(b_E) \right\}.
$$

This set is not empty due to Claim 2, closed[13] since V is continuous, and bounded since f is. Hence $\mathcal{E}_f$ is a non-empty compact set. Let

$$
a^{*f} = \text{argmax}\left\{ a : (a_{E_f}, b_{E_1}, \ldots, b_{E_{2|S|-1}}, g, \delta) \in \mathcal{E}_f \right\}.
$$

Since $\mathcal{E}_f$ is compact, a^{*f} is well defined. $\quad\square$

---

[12] Note that due to *monotonicity* and *continuity*, $f(s) \geq g(s)$ for every s implies $f \succeq g$.

[13] In the product topology over $\mathbb{R}^{2|S|-1} \times \mathcal{F} \times [0, 1]$.

From Claim 3 there is a residual act h and a collection of bets $\{b_E\}_{E \neq E_f}$ such that $f = \delta h + a_{E_f}^{*f} + \sum_{E \neq E_f} b_E$ and

$$V(f) = \delta V(h) + V(a_{E_f}^{*f}) + \sum_{E \neq E_f} V(b_E). \tag{A.2}$$

Now, if the residual act h is a bet, then from Eq. (A.1) the theorem is proved. Assume h is not a bet. The following claim is a twist on Claim 2.

Claim 4. *There exist an event $E_h \neq E_f$, an act $f' \in \mathcal{F}$, $a > 0$ and $\delta \in [0, 1]$ such that $h = a_{E_h} + \delta f'$ and $V(h) = V(a_{E_h}) + \delta V(f')$.*

Proof. The proof is similar to that of Claim 2, but we need to show that $E_h \neq E_f$. Assuming $E_h = E_f$ contradicts the maximality of a^{*f}. Indeed, if $E_h = E_f$ then $f = \delta f' + a_{E_f} + a_{E_f}^{*f} + \sum_{E \neq E_f} b_E$ and $V(f) = \delta V(f') + V(a_{E_f}) + V(a_{E_f}^{*f}) + \sum_{E \neq E_f} V(b_E)$, and $a + a^{*f} > a^{*f}$. □

Claim 4 suggests that it is not possible to apply *subjective codecomposable independence* to decompose h while resorting to E_f. The same arguments apply when we decompose h according to Claim 3; if E_f appears in such a decomposition of h, and we substitute the decomposition of $V(h)$ with that of $V(f)$ appearing in Eq. (A.2), we again obtain a contradiction to the maximality of a^{*f}.

Following this argument, decompose h as $V(h) = \gamma V(k) + V(a_{E_h}^{*h}) + \sum_{E \neq E_h, E_f} V(d_E)$, and substitute its decomposition with that of f appearing in Eq. (A.2) to obtain

$$V(f) = \delta \gamma V(k) + V(\delta a_{E_h}^{*h}) + \sum_{E \neq E_h, E_f} V(\delta d_E) + V(a_{E_f}^{*f}) + \sum_{E \neq E_f} V(b_E). \tag{A.3}$$

Again, if k is a bet then the theorem is proved. Otherwise, repeat the procedure we did for h. We decompose k as in Claim 2, where following similar arguments as before, $E_h \neq E_h, E_f$, and so does every event in the decomposition of k as in Claim 3, which we substitute with $V(k)$ in the decomposition of f. We repeat this procedure for residual acts of each step. If at any point the residual act is a bet, then the theorem is proved. Other wise, it must be that this procedure ends after at most $2^{|S|-1}$ steps. Indeed, since at each step the associated event (as in Claim 2 for f) according to which we decompose the residual act from the prior step, must be different from all events resorted to in previous steps, at the $2^{|S|-1}$th step the residual act must be a bet. Otherwise we can decompose it using *subjective codecomposable independence*, which contradicts the fact that we already exhausted all possible event.

By Claim 1 V is homogeneous. To prove uniqueness, suppose that there are two homogeneous $V, V' : \mathcal{F} \to \mathbb{R}$ that represent $\succeq$ and satisfy $V(\mathbb{1}_S) = V'(\mathbb{1}_S) = 1$. Fix $f \in \mathcal{F}$. Without loss of generality $V(f) \leq V'(f)$. Let $b \in \mathbb{R}_+$ such that $V(f) \leq V(b_S) = b = V'(b_S) \leq V'(f)$. Since both represent $\succeq$, $f \preceq b_S \preceq f$. Thus, $V(f) = b = V'(f)$. Since this is true for every f, $V = V'$ and by definition $v = v'$. □

Proof of Theorem 2. The concave integral satisfies *subjective codecomposable independence* due to Proposition 5 in Even and Lehrer (2014) (and it is immediate that the rest of the axioms are implied by integral).[14]

Following Theorem 1, we have that $\succeq$ is represented by a homogeneous and continuous V such that $V(\mathbb{1}_E) \geq v(E)$. *Ambiguity aversion* implies that V is a concave functional. By Lemma 1 in Lehrer (2009) we have that $V(\cdot) \geq \int^{Cav}(\cdot)dv$. However, for every $f \in \mathcal{F}$ concavity of V implies that $V(f) \leq \sum \alpha_E V(\mathbb{1}_E)$ for all decompositions of f, implying that $V(f) \leq \int^{Cav} f \, dv$. Therefore $V(\cdot) = \int^{Cav}(\cdot)dv$. $\square$

Proof of Proposition 1. Due to Lemma 1 in Lehrer and Teper (2008), without loss of generality we can assume that v is totally balanced. Thus, the proposition is proved due to Theorem 1 in Kalai and Zemel (1982). $\square$

Proof of Proposition 2. It is clear that the axioms are satisfied by the *CEU* preferences. As for the inverse direction, we show first that *cumulative codecomposable independence* implies *worst-outcome bet independence*. Assume *cumulative codecomposable independence* and consider the two following bets: b_E and d_G. Without loss of generality $b_E \succeq d_G$. Due to *continuity* there is $c \geq 0$ such that $b_E \sim c_{E\cup G}$. In particular, $b_E \succeq c_{E\cup G} \succeq d_G$. We show that for every $\alpha \geq 0$, $\alpha b_E \succeq \alpha c_{E\cup G} \succeq \alpha d_G$, implying $\alpha b_E \succeq \alpha d_G$ and *worst-outcome bet independence*.

Consider the act $f = \mathbb{1}_E + \mathbb{1}_{E\cup G}$. By *cumulative codecomposable independence*, $f \in [1_{E\cup G}, 1_E]$, and $\succeq$ satisfies independence over cone$(1_{E\cup G}, 1_E)$. In particular, $b_E \succeq c_{E\cup G}$ implies that for every $\alpha \geq 0$, $\alpha b_E \succeq \alpha c_{E\cup G}$. We can now apply the same reasoning to $f = \mathbb{1}_G + \mathbb{1}_{E\cup G}$ and conclude that since $c_{E\cup G} \succeq d_G$, then $\alpha c_{E\cup G} \succeq \alpha d_G$ for every $\alpha \geq 0$. Thus, *worst-outcome bet independence* is satisfied.

It remains to show that given *cumulative codecomposable independence* (which obviously implies *subjective codecomposable independence*), the decomposition of any act obtained in the proof of Theorem 1 is the Choquet one. To see that, pick an act $f \in \mathcal{F}$ and, let $a_1 = \max\{f(s) : s \in S\}$ and $E_1 = \{s \in S : f(s) = a_1\}$. Also denote $a_2 = \max\{f(s) : s \in E_1^c\}$. Let f' be the act defined by $f'(s) = f(s)$ whenever $s \in E_1^c$ and a_2 otherwise (that is, f' coincides with f over the complement of E_1, and over E_1 it is defined as the second highest value f attains). Now, $f = f' + (a_1 - a_2)\mathbb{1}_{E_1} = \frac{a_2}{a_1}(\frac{a_1}{a_2}f') + \frac{a_1-a_2}{a_1}(a_1_{E_1})$. Note that E_1 and f' are comonotonic, hence by *cumulative codecomposable independence* we have that $V(f) = \frac{a_2}{a_1}V\left(\frac{a_1}{a_2}f'\right) + \frac{a_1-a_2}{a_1}V\left(a_1_{E_1}\right) = V(f') + (a_1 - a_2)V(\mathbb{1}_{E_1}) = V(f') + (a_1 - a_2)v(E_1)$. Repeating the same procedure to f' we obtain the desired result. $\square$

Proof of Proposition 3. It is clear that the axioms are satisfied by the *EU* preferences. As for the other implication, *codecomposable independence* implies *cumulative codecomposable independence*, which by the previous proof, implies *worst-outcome bet independence*. Thus, all that is needed to show is that given *codecomposable independence* the capacity obtained in the proof of Theorem 1 is additive, hence a probability.

Pick any event $E \subset S$ and state $s \in S \setminus E$ and consider an act of the form $f = 2\mathbb{1}_{\{s\}} + \mathbb{1}_E$. On one hand, from the proof of Proposition 2 we know that $V(f) = v(E \cup \{s\}) + v(\{s\})$. On the other

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[14] Note that the additivity property presented in Proposition 5 in Even and Lehrer (2014) is not satisfied by every functional form as in Definition 1 and is not a characteristic property of such preferences. It is strictly stronger than *subjective codecomposable independence*.

hand, we can write $f = \frac{1}{2}(4\mathbb{1}_{\{s\}}) + \frac{1}{2}(2\mathbb{1}_E)$ and due to *codecomposable independence* we have that $V(f) = \frac{1}{2}(4\mathbb{1}_{\{s\}}) + \frac{1}{2}(2\mathbb{1}_E) = 2v(\{s\}) + v(E)$. Thus, $v(\{s\} + v(E \cup \{s\}) = 2v(\{s\}) + v(E)$, implying that $v(\{s\}) + v(E) = v(E \cup \{s\})$. Since E is an arbitrary event, we get that $v(F) = \sum_{s \in F} v(s)$ for any event $F \subset S$, implying that v is a probability over S. $\quad\square$

Example 3. In this example we specify preferences admitting an event-separable representation that do not satisfy our main axiom. Let $S = \{1, 2\}$. Define the capacity v as follows. $v(S) = 2$ and $v(i) = 0$, $i = 1, 2$. We first define V over acts in the simplex (i.e., acts whose coordinates sum up to 1). Let $f = (x, 1 - x)$ be an act in the simplex. Define

$$V(f) = \begin{cases} 0 & \text{if } x < \frac{1}{4}; \\ 0 & \text{if } \frac{3}{4} < x; \\ 1 - 4|\frac{1}{2} - x| & \text{if } \frac{1}{4} \le x \le \frac{3}{4} \end{cases}$$

For instance, $V(1, 0) = 0$, $V(\frac{1}{2}, \frac{1}{2}) = 1$, $V(\frac{1}{4}, \frac{3}{4}) = 0$ and $V(\frac{1}{3}, \frac{2}{3}) = \frac{1}{3}$. Now extend V to the entire set of acts in a homogeneous way. For instance, $V(3, 0) = 0$, $V(1, 1) = 2$, and $V(1, 2) = 1$. Finally, define $\succeq$ accordingly.

V is clearly continuous. We verify first that V is event separable with respect to v. Any $(x, 1 - x)$ in the simplex such that either $x < \frac{1}{4}$ or $\frac{3}{4} < x$, satisfies $(x, 1 - x) = x(1, 0) + (1-x)(0, 1)$ and $V(x, 1 - x) = 0 = xV(1, 0) + (1 - x)V(0, 1)$. This means that V is event separable in this case. Now consider $\frac{1}{4} \le x \le \frac{1}{2}$. Here, $(x, 1 - x) = (2 - 4x)(\frac{1}{4}, \frac{3}{4}) + (4x - 1)(\frac{1}{2}, \frac{1}{2}) = \frac{2-4x}{4}(1, 0) + \frac{3(2-4x)}{4}(0, 1) + \frac{4x-1}{2}(1, 1)$ and $V(x, 1 - x) = 1 - 4|\frac{1}{2} - x|$ which indeed equals $4x - 1 = \frac{2-4x}{4}V(1, 0) + \frac{3(2-4x)}{4}V(0, 1) + \frac{4x-1}{2}V(1, 1)$. Thus, V is event separable also when $\frac{1}{4} \le x \le \frac{1}{2}$. The case where $\frac{1}{2} \le x \le \frac{3}{4}$ is similar and therefore omitted. We conclude that V is event separable on the simplex and since V is homogeneous, it is an event-separable representation of $\succeq$. However, $\succeq$ violates Subjective Codecomposable Independence, because when decomposing $(\frac{1}{4}, \frac{3}{4})$, for instance, one obtains the bets $(1, 0)$ and $(0, 1)$, but V is not linear over the cone generated by them.

References

Anscombe, F.J., Aumann, R.J., 1963. A definition of subjective probability. Ann. Math. Stat. 34, 199–295.

Borah, A., Kops, C., 2014. The Anscombe–Aumann representation and event-separability of preferences: a reconsideration. University of Mainz. Mimeo.

Catagnoli, E., Maccheroni, F., 2000. Restricting independence to convex cones. J. Math. Econ. 34, 215–223.

Chateauneuf, A., Faro, J.H., 2009. Ambiguity through confidence functions. J. Math. Econ. 45, 535–558.

Choquet, G., 1955. Theory of capacities. Ann. Inst. Fourier 5, 131–295.

Dillenberger, D., Segal, U., 2015. Recursive ambiguity and Machina's examples. Int. Econ. Rev. 56, 55–61.

Ellsberg, D., 1961. Risk, ambiguity, and the savage axioms. Quart. J. Econ. 75, 643–669.

Even, Y., Lehrer, E., 2014. Decomposition-integral: unifying Choquet and the concave integrals. Econ. Theory 56, 33–58.

Fishburn, C.P., 1970. Utility Theory for Decision Making. Wiley, New York.

Gärdenfors, P., Sahlin, N.E., 1982. Unreliable probabilities, risk taking and decision making. Synthese 53, 361–386.

Gilboa, I., Schmeidler, D., 1989. Maxmin expected utility with a non-unique prior. J. Math. Econ. 18, 141–153.

Kalai, E., Zemel, E., 1982. Totally balanced games and games of flow. Math. Oper. Res. 7, 476–478.

Lehrer, E., 2009. A new integral for capacities. Econ. Theory 39, 157–176.

Lehrer, E., Teper, R., 2008. The concave integral over large spaces. Fuzzy Sets Syst. 16, 2130–2144.

Lehrer, E., Teper, R., forthcoming. Sandwich games. Math. Program. http://dx.doi.org/10.1007/s10107-014-0796-7.

Lovasz, L., 1983. Submodular functions and convexity. In: Bachem, A., et al. (Eds.), Mathematical Programming: The State of the Art. Springer-Verlag, New York, pp. 235–257.

Machina, M.J., 2009. Risk, ambiguity and the rank-dependent axiom. Amer. Econ. Rev. 99, 385–392.

Savage, L.J., 1954. The Foundations of Statistics. Wiley, New York.

Schmeidler, D., 1989. Subjective probabilities and expected utility without additivity. Econometrica 57, 571–587.

Shapley, L.S., 1971. Cores of convex games. Int. J. Game Theory 1, 11–26.

Sharkey, W.W., 1982. Cooperative games with large cores. Int. J. Game Theory 11, 175–182.

Siniscalchi, M., 2009. Vector expected utility and attitudes toward variation. Econometrica 77, 801–855.