ISRAEL JOURNAL OF MATHEMATICS, Vol. 57, No. 2, 1987

TOPOLOGICAL MIXING AND UNIQUELY ERGODIC SYSTEMS

BY

EHUD LEHRER[†]

Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem, Israel

ABSTRACT

Every ergodic transformation $(X, T, \mathscr{B}, \mu)$ has an isomorphic system $(Y, U, \mathscr{C}, \nu)$ which is uniquely ergodic and topologically mixing.

1. Introduction

Following the works of Jewett [3] and Krieger [4] about uniquely ergodic systems, the question arises as to whether one can impose additional topological properties on models of ergodic systems. Our aim is to establish the following theorem:

THEOREM. *Let there be given an invertible ergodic measure preserving transformation of a Lebesgue space $(X, T, \mathscr{B}, \mu)$, then there is a dynamical system $(Y, U, \mathscr{C}, \nu)$, where Y is a compact space and U is a homeomorphism of Y to itself, which is uniquely ergodic, topologically mixing, and isomorphic to $(X, T, \mathscr{B}, \mu)$.*

(Y, U) is *topologically mixing* if for any two open sets R, S in Y, there is $l \in \mathbf{N}$ so that $U^{l'}S \cap R \neq \emptyset$ for $l' \geqq l$. The dynamical system (Y, U) will be constructed as an inverse limit of systems $\{(Y_i, U_i)\}_{i=1}^{\infty}$. (Y_i, U_i) will be defined as the symbolic system produced by (X, T, Q_i), $i = 1, 2, \ldots$, where Q_i is a partition of X.

2. Sketch of the Proof

The model $(Y, U, \mathscr{C}, \nu)$ will be constructed by using a sequence of partitions $\{Q_i\}_{i=1}^{\infty}$. In order to describe the relevant properties of this sequence we need the following Notation and Definition.

[†] This work is a part of an M.Sc. thesis written at The Hebrew University of Jerusalem under the supervision of Professor B. Weiss to whom the author is greatly indebted.

Received June 27, 1985 and in revised form June 1, 1986

NOTATION 2.1. Let Q and P be two partitions; then $Q \vee P$ will denote their common refinement.

DEFINITION 2.1. Let Q be a partition. Q is said to be a *uniform* partition if, for every $\varepsilon > 0$ and integer k, there is an integer n such that for each $x \in X$ and $A \in \bigvee_{i=0}^{k-1} T^{-i}Q$

$$\left| \frac{1}{n} \#\{l \mid T^l x \in A; 0 \leqq l \leqq n-1\} - \mu(A) \right| < \varepsilon.$$

The sequence $\{Q_i\}_{i=1}^{\infty}$ has the following four properties:

(p1) $Q_i \subset Q_{i+1}$ for all $i \geqq 1$, which means that Q_{i+1} refines Q_i. This ensures that the inverse limit can be defined.

(p2) $\bigvee_{i=1}^{\infty} Q_i = \mathcal{B}$, which means that the inverse limit is isomorphic to $(X, T, \mathcal{B})$.

(p3) For each $i \geqq 1$, Q_i is a uniform partition.

Whenever Q is a uniform partition the symbolic system produced by (X, T, Q) is uniquely ergodic [2]. In Proposition 3.5, it is proved that the inverse limit of uniquely ergodic systems is also uniquely ergodic. This ensures that (Y, U) is uniquely ergodic.

(p4) For each $i \leqq 1$, $k \in \mathbf{N}$, and $R, S \in \bigvee_{j=0}^{k-1} T^{-i}Q_i$, there is an integer l such that for all $l' \geqq l$

$$\mu(T^{l'}R \cap S) > 0.$$

This means that the symbolic system produced by (X, T, Q_i) is topologically mixing. In Proposition 3.6 it is proved that the inverse limit of topologically mixing systems is also topologically mixing. This ensures that (Y, U) is topologically mixing too.

Each partition of the sequence $\{Q_i\}_{i=1}^{\infty}$ is defined as a limit of a sequence $\{Q_i^j\}_{j \geqq i}$. This sequence will be built by repeated use of Proposition 3.3. In this proposition there is a construction of a partition P which is an "improvement" of a given partition $\bar{P}$, in the sense of uniformity and topological mixing. For Proposition 3.3 the m-universal property is required.

DEFINITION 2.2. Let m be an integer and $\bar{P}$ a partition. $\bar{P}$ is *m-universal* if there is a set S in $\bigvee_{i=1}^{m-1} T^{-i}\bar{P}$ and two relatively prime integers p, q, so that $\mu(T^p S \cap S) > 0$ and $\mu(T^q S \cap S) > 0$.

When $\bar{P}$ has the m-universal property, one can improve $\bar{P}$ by changing it into P without enlarging the set of the m-names. The fact that the $P-m$-names are included in the set of $\bar{P}-m$-names ensures that the former are as "good" as the latter, while there is a certain integer which is greater than m, for which the P-names are "better" than the $\bar{P}$-names.

The improvement of $\bar{P}$ in the proof of Proposition 3.3 is achieved in three steps. In the first step we improve $\bar{P}$ (in the sense of uniformity) and get the partition Q which is improved (in the sense of topological mixing) by R without spoiling its uniformity. Finally, in the third step R is modified by a small amount in order to get the partition P which has the good properties of R, but also an additional property, namely the $\bar{n}$-universal property of some integer $\bar{n}$. It is this additional property that makes possible the repeated use of Proposition 3.3.

Each one of the improvements is achieved by replacing "good" words for "bad" words. Proposition 3.1 ensures the existence of "good" words with sufficient length. Its proof is mainly based on the m-universal property and on the Ergodic Theorem.

Proposition 3.2 is needed for the second step of Proposition 3.3. This proposition ensures that R can improve Q in the sense of topological mixing without enlarging the set of m-names.

3. Notations and Proofs of Propositions

In all the following notations P and Q are partitions of X, w, x, y are words, $m, n \in \mathbf{N}$ and $\varepsilon > 0$.

NOTATIONS

3.1. $F_m(P)$ is the set of all the $P-m$-names, and $P^{(m)}$ is the partition $\bigvee_{j=0}^{m-1} T^{-j}P$.

3.2. $E(P) = \bigcup_{i=1}^{\infty} F_i(P)$.

3.3. $|w|$ is the length of the word w.

3.4. If $|w| = m$, w is an m-word.

3.5. xwy is the joining together of the words x, w and y to form one word.

3.6. w is a subword of z whenever there are two words x, y such that either $xwy = z$, $xw = z$, or $wy = z$.

3.7. w is an m-subword of x, if w is a subword of x and w is an m-word.

3.8. $P \subseteq Q$ means that Q refines P.

3.9. If $|P| = |Q| = n$, then $d(P, Q) = \sum_{i=1}^{n} \mu(P_i \triangle Q_i)$.
This is the distance between P and Q.

3.10. If P and Q are finite then $P \subseteq_\varepsilon Q$ means that there is a partition $Q' \subseteq Q$ such that $d(P, Q') \leqq \varepsilon$.

3.11. If P is finite, then $P \subseteq_\varepsilon Q$ means that for every finite partition $R \subseteq Q$ that satisfies $|R| \leqq |P|$ there is a partition $R' \subseteq_\varepsilon P$ such that $R' \subseteq R$.

In Proposition 3.1 it will be shown that when P is an m-universal partition one can find for any $x, y \in F_m(P)$ words of any sufficiently large length, with head x and tail y, say xwy, such that any m-subword of xwy is in $F_m(P)$. Furthermore, xwy can be chosen so as to ensure that its statistics will be "good".

Fix a partition P.

Definition 3.1. A word $z = (z_1, \dots, z_l)$ has *good* $P-(\varepsilon, k, n)$ *statistics* if for every $(x_0, \dots, x_{k-1}) = x \in F_k(P)$ and $0 \leqq s \leqq l - n + 2 - k$ the following holds:

$$\left| \frac{1}{n-k} \#\{i \mid (z_i, \dots, z_{i+k-1}) = x, s \leqq i \leqq s + n - k\} - \mu\left(\bigcap_{i=0}^{k-1} T^{-i} P_{x_i} \right) \right| < \varepsilon.$$

In other words, for any word $x \in F_k(P)$ the distribution of x in any n-subword of z is close to its measure in the symbolic system produced by (X, T, P).

Remark 3.1. Q is a uniform partition of $(X, T, \mathcal{B})$ if for any $\varepsilon > 0$ and $k \in \mathbf{N}$, there is an integer n such that all the words in $F_n(Q)$ have good $Q-(\varepsilon, k, n)$ statistics.

Proposition 3.1. *For every, $m, k \in \mathbf{N}$, m-universal partition P, and $\varepsilon > 0$, there is a set of words $E = E(P, \varepsilon, k, m)$, and two integers $l = l(P, \varepsilon, k, m)$ and $n = n(P, \varepsilon, k, m)$ such that*

(i) *For every $x, y \in F_m(P)$ and $l' \geqq l$ there is $z \in E$ and w, such that $|w| = l'$ and $z = xwy$.*

(ii) *For each $z \in E$ every m-subword of z is in $F_m(P)$.*

(iii) *Each $z \in E$ has good $P-(\varepsilon, k, n)$ statistics.*

Proof. P is an m-universal partition, so that there are two relatively prime numbers p, q and three words: $x \in F_m(P)$, z (where $|z| = p - m$), and w (where $|w| = q - m$), such that $xzx \in F_{p+m}(P)$ and $xwx \in F_{q+m}(P)$.

Because p and q are relatively prime, there is an integer i, so that for any $s \geqq i$ one can get an s-word v_s by concatenating xz and xw together. v_s will have the property that all its m-subwords are in $F_m(P)$. The ergodicity of T and the Ergodic Theorem ensure the existence of a word y, so that the word $xyx \in F_{r+m}$, for some r, and the word xy has good $P-(\varepsilon/4, k, r)$ statistics. Construct now the words

$$v_i xy,\ v_{i+1}xy,\ v_{i+2}xy, \ldots,\ v_{i+r-1}xy,$$

$$v_i xyxy,\ v_{i+1}xyxy, \ldots,\ v_{i+r-1}xyxy,$$

$$v_i xyxyxy, \ldots,\ v_{i+r-1}xyxyxy,$$

and so on. We can find a number t such that each of these listed words with length greater than t has good $P-(\varepsilon/2, k, t)$ statistics. This will occur when the number of times that xy appears is much larger than $\mathrm{Max}_{i\leqq l\leqq i+r-1} |v_l|$. Call these words $u_j, u_{j+1}, \ldots$ $(j \geqq t)$, where $|u_j| = j$.

Since T is ergodic, it follows that

(a) For every $u \in F_m(P)$ there is a word u', so that $uu'x \in E(P)$ (recall Notation 3.2).

(b) For every $w \in F_m(P)$ there is a word w', so that $xw'w \in E(P)$.

Fix two words u and w in $F_m(P)$. For any $s \geqq t$, u_s has good $P-(\varepsilon/2, k, t)$ statistics. Therefore, there are two integers $h(u, w)$ and $t(u, w)$ s.t. if $h \geqq h(u, w)$ then the word $uu'u_hxw'w$ has good $P-(\varepsilon/4, k, t(u, w))$ statistics. Choose an n larger than $\mathrm{Max}_{u,w\in F_m(P)} t(u, w)$ and big enough to ensure that for all $u, w \in F_m(P)$ the words $uu'u_hxw'w$ have good $P-(\varepsilon, k, n)$ statistics whenever $h > \mathrm{Max}_{u,w\in F_m(P)} h(u, w) = \bar{h}$. Let l' be the maximum between n and $\bar{h}$. Finally define

$$E = \{uu'u_hxw'w \mid u, w \in F_m(P), h \geqq l'\}$$

and

$$l = \mathrm{Max}\{|u'| + l' + m + |w'| \mid u, w \in F_m(P)\}.$$

The set E and the integer l satisfy the requirement of the Proposition. Q.E.D.

REMARK 3.2. The m-universal property of P is required to get (i) of Proposition 3.1. This means that for *all* $l' \geqq l$ there is $z \in E$, etc. If P lacks this property one can employ the same technique in order to get a weaker version of the proposition. Conditions (ii) and (iii) will remain valid without any change and (i) will take the following form: there are *infinitely many* $l' \geqq l$ for which there is $z \in E$, etc. This weaker form is an equivalent to the key lemma of Jewett [3], and of Bellow–Furstenberg [1].

Proposition 3.2 is a technical one. Its conclusion will be needed in the proof of Part II of Proposition 3.3.

PROPOSITION 3.2. *For every integer* n, *and* $\varepsilon > 0$, *there are sets* $A_n, A_{n+1}, A_{n+2}, \ldots$ *such that*

(i) $\mu(A_s) > 0$ $(s \geqq n)$,

(ii) $\mu(\bigcup_{m=n}^{\infty} \bigcup_{i=0}^{m-1} T^i A_m) < \varepsilon$,

(iii) *the sets* $T^j A_m$ $(n \leqq m < \infty,\ 0 \leqq j \leqq m-1)$ *are pairwise disjoint.*

PROOF. The sets A_m $(n \leqq m)$ will be defined as a limit of a sequences of sets $\{A^i_m\}_{i > m-n}$. The construction will be made inductively. At the i-th stage we will build the sets $\{A^i_m\}_{m=n}^{n+i-1}$.

At the first stage let A^1_n be the base of a Rohlin tower of height $[2n/\varepsilon]+1$. We have that the sets $A^1_n, TA^1_n, \ldots, T^{n-1}A^1_n$ are pairwise disjoint and that $\mu(\bigcup_{i=0}^{n-1} T^i A^1_m) < \varepsilon/2$. Assume that we have already built for each $j \leqq i$ the sets $A^j_n, A^j_{n+1}, \ldots, A^j_{n+j-1}$, which satisfy the following.

(i) If $0 \leqq k \leqq j-1$ then

$$\mu\left(\bigcup_{l=0}^{n+k-1} T^l A^j_{n+k}\right) < \varepsilon/2^{k+1}.$$

(ii) The sets $T^l A^j_{n+k}$, $j \leqq i$, $0 \leqq k \leqq j-1$, $0 \leqq l \leqq n+k-1$ are pairwise disjoint.

(iii) If $0 \leqq k \leqq j-2$, then $A^j_{n+k} \subseteq A^{j-1}_{n+k}$ and $\mu(A^j_{n+k}) > (1-2^{j+1})\mu(A^{j-1}_{n+k})$.

Define

$$G = \bigcup_{m=n}^{n+i-1} \bigcup_{j=0}^{m-1} T^j A^i_m \quad \text{and} \quad \delta = \mathrm{Min}\{\mu(A^i_n), \mu(A^i_{n+1}), \ldots, \mu(A^i_{n+i-1})\}.$$

Take a positive-measure set $L \subseteq X \setminus G$, with a measure less than $\delta \cdot \mathrm{Min}\{1, \varepsilon\}/((n+i+1)\cdot 2^{i+2})$.

Thus,

$$\mu\left(\bigcup_{j=-n-i}^{n+i} T^j L\right) < \delta \cdot \mathrm{Min}\{\varepsilon, 1\}/2^{i+1}.$$

For each $0 \leqq s \leqq i-1$, define A^{i+1}_{n+s} as follows:

$$A^{i+1}_{n+s} = A^i_{n+s} \setminus \bigcup_{j=-n-i}^{n+i} T^j L$$

and define

$$A^{i+1}_{n+i} = L \setminus \bigcup_{j=1}^{n+i} T^{-j} L.$$

From this we get: (1) that the sets $T^l A^{i+1}_{n+k}$, $0 \leqq k \leqq i$, $0 \leqq l \leqq n+k-1$ are pairwise disjoint; (2) that $0 < \mu(A^{i+1}_{n+i}) < \varepsilon/2^{i+1}(n+1)$; and finally (3) that for every $0 \leqq s \leqq i-1$

$$\mu(A^{i+1}_{n+s}) > \mu(A^{i}_{n+s}) - \frac{\delta}{2^{i+1}} > \left(1 - \frac{1}{2^{i+1}}\right)\mu(A^{i}_{n+s}).$$

Hence,

$$\mu\left(\bigcup_{m=n}^{n+i}\bigcup_{j=0}^{m-1} T^j A^{i+1}_m\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \cdots + \frac{\varepsilon}{2^{i+1}}.$$

At the end of the inductive process we shall have the sets A^{i}_{n+s} ($s \geqq 0$, $i = s+1, s+2, \ldots$), so that whenever $i > s \geqq 0$ then

$$A^{i+1}_{n+s} \subseteq A^{i}_{n+s}$$

and

$$\mu(A^{i+1}_{n+s}) > \left(1 - \frac{1}{2^{i+1}}\right)\mu(A^{i}_{n+s}).$$

Furthermore, the sets $T^l A^{i}_{n+k}$, $0 \leqq k \leqq i$, $0 \leqq l \leqq n+k-1$ are pairwise disjoint. Define

$$A_{n+s} = \bigcap_{i=s+1}^{\infty} A^{i}_{n+s}.$$

It can be immediately shown that these sets satisfy the proposition. Q.E.D.

The next proposition deals with partition improvement in the sense of topological mixing and uniformity, without spoiling the universality which is helpful in Proposition 3.1.

PROPOSITION 3.3. *Given two integers m, k, two reals $\delta > 0$, $\varepsilon > 0$ and an m-universal partition $\bar{P}$, there are an integer $\bar{n} = \bar{n}(P, \varepsilon, \delta, k, m)$ and a partition P such that*:

(i) $F_m(P) \subseteq F_m(\bar{P})$.

(ii) *Each word in $F_{\bar{n}}(P)$ has a good $P-(\varepsilon, k, \bar{n})$ statistics.*

(iii) *The distance between $\bar{P}$ and P is less than δ.*

(iv) *P is an $\bar{n}$-universal partition.*

(v) *If $l' > l = l(P, \varepsilon, k, m) + 2m$, then for every $S, R \in P^{(m)}$*

$$\mu(S \cap T^{l'} R) > 0,$$

where $l(P, \varepsilon, k, m)$ is that of Proposition 3.1.

PROOF. Let $\bar{\delta}$ be the minimum between δ and $\varepsilon/8k$.

Part I. In this part we will modify $\bar{P}$ by a small amount in order to obtain a partition Q and some fixed integer n, so that all the words of $F_n(Q)$ have good $Q-(\varepsilon, k, n)$ statistics. Let, for every $s \in \mathbf{N}$,

$$A_s = \{x \in X; | T_r \chi_{W_i}(x) - \mu(W_i)| < \varepsilon/8,\ 1 \leqq i \leqq |P^{(k)}| \text{ and } r \geqq s\},$$

where

$$T_r f = \frac{f + Tf + \cdots + T^{r-1}f}{r}$$

for a function f, and

$$P^{(k)} = \bigvee_{i=0}^{k-1} T^{-i}P = \{W_1, \ldots, W_{|P^{(k)}|}\}.$$

According to the pointwise Ergodic Theorem and to the fact that $|P^{(k)}| < \infty$,

$$\mu(A_s) \to 1 \qquad \text{as } s \to \infty.$$

Let f be an integer such that $\mu(A_t) > 1 - \bar{\delta}/100$ and $t > l(\bar{P}, \varepsilon/8, m, k) + 2m$ (of Proposition 3.1). Build now a Rohlin tower $\{T^i H\}_{i=0}^{t-1}$, namely, sets which are pairwise disjoint, with total measure greater than $1 - \bar{\delta}/8t$, and with a base H that is independent of the partition $\{X \setminus A_t, A_t\}$.

The base H is divided into $H_t, H_{t+1}, \ldots$ in such a way that for $i = 0, 1, \ldots$ the following holds: $H_{t+i}, TH_{t+i}, \ldots, T^{t+i-1}H_{t+i}$ are pairwise disjoint, $T^{t+i}H_{t+i} \subseteq H$, and $T^j H_{t+i} \cap H = \varnothing$ for every $0 \leqq j < t + i$ (see Fig. 1). Define

$$H^* = \bigcup \left\{ T^{s't} H_u \,;\ \text{all } u \geqq t \text{ and } 0 \leqq s' \leqq \left[\frac{u}{t}\right] - 1 \right\}$$

(see Fig. 2). The Katutani tower over H^* does not exceed the height of $2t-1$, and it induces a partition of H^* into $H_t^*, H_{t+1}^*, \ldots, H_{2t-1}^*$ (in a similar way to H). Notice that

$$\mu(H^* \cap (X \setminus A_t)) \leqq \mu(H \cap (X \setminus A_t)) + \mu\left(X \setminus \bigcup_{i=0}^{t-1} T^i H\right)$$

$$\leqq \mu(H) \cdot \mu(X \setminus A_t) + \bar{\delta}/8t \leqq \frac{1}{t} \cdot \frac{\bar{\delta}}{100} + \bar{\delta}/8t < \bar{\delta}/6t.$$

Now define Q as follows:

(1) On $T^j(H_{t+i} \cap A_t)$ when $0 \leqq i \leqq t-1$ and $0 \leqq j \leqq t + i - 1$, Q coincides with $\bar{P}$.

For example: if the $\bar{P} - (t+i)$-name of a point in H_{t+i}^* is w, then so is the $Q-(t+i)$-name of that point.

(2) The names of the points in $H^* \setminus A_t$ will be changed. That means that we will change the 1-names of points in a set of at most the measure of $(2t-1) \cdot \bar{\delta}/6t$,

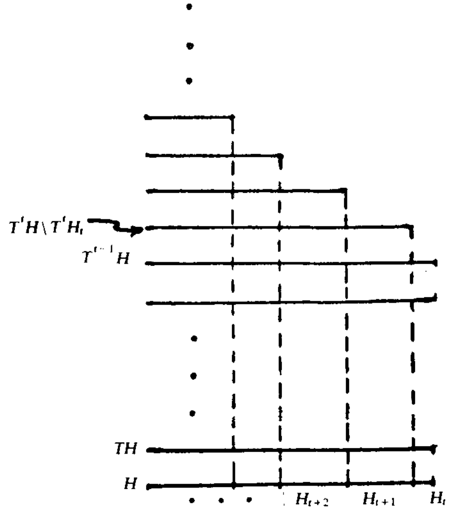

Fig. 1.

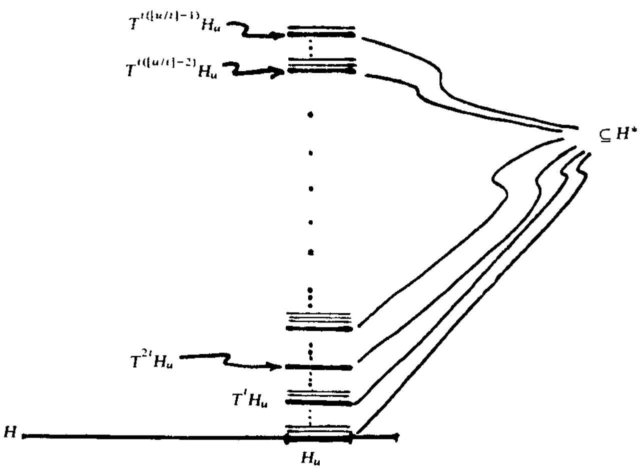

Fig. 2.

which is less than $\bar{\delta}/3$. Let $h \in H_{t+i} \setminus A_t$ for some $0 \leqq i \leqq t-1$, x be the $\bar{P}-m$-name of h, and y be the $\bar{P}-m$-name of $T^{t+i-m}h$. By Proposition 3.1, there is a word w that satisfies:

(a) $|w| = t+i-2m$ (because $t > l(\bar{P}, \varepsilon/8, m, k)+2m$).

(b) Every m-subword of xwy is in $F_m(\bar{P})$.

(c) xwy has good $\bar{P}-(\varepsilon/8, k, n)$ statistics, when $n = n(\bar{P}, \varepsilon/8, k, m)$.

Define the $Q-(t+i-2m)$-name of the point $T^m h$ to be w. The first and the last m-subwords of the $\bar{P}-(t+i)$-name of h have remained unchanged. Do so for all the points in $H^*_{t+i}\setminus A_t$ for all the i's $(0\leqq i\leqq t-1)$.

Because of (b) and the definition of Q, $F_m(Q)\subseteq F_m(\bar{P})$. According to (c) and the definition of Q, all the points in H^*_{t+i} have good $\bar{P}-(\varepsilon/8,k,n)$ statistics. Since $d(\bar{P},Q)<\bar{\delta}/3\leqq\varepsilon/24k$, all the points of H^*_{t+i} have good $Q-(\varepsilon/6,k,n)$ statistics. By choosing a sufficiently large $\bar{n}$ we can ensure that *all* the points of X will have good $Q-(\varepsilon/4,k,\bar{n})$ statistics.

Part II. Now we construct the partition R by changing Q on a set of measure less than $\bar{\delta}/3$. R will satisfy (v) of the proposition without spoiling Q in the sense that it satisfies properties (i) and (ii) of the proposition. Let $l=l(\bar{P},\varepsilon/8,m,k)$ and $\alpha=3l+4m$.

By Proposition 3.2 there are pairwise disjoint sets T^jA_β, $0\leqq j\leqq\beta-1$, $\beta=\alpha,\alpha+1,\dots$ of positive measure and with total measure less than $\bar{\delta}/3$. W.l.o.g all the points of A_β have the same $Q-\beta$-names for every $\beta\geqq\alpha$; otherwise take some positive measure set from $\{A_\beta\cap W\mid W\in\bigvee_{i=0}^{\beta-1}T^{-i}Q\}$.

In the description of the partition R we shall use the following notation. If x is a word and $0\leqq s\leqq t<|x|$ then $x(s,t)$ will be the $(t-s+1)$-subword of x which starts with its s'th index.

The partition R will be defined as follows:

(1) On $X\setminus\bigcup_{\beta=\alpha}^{\infty}\bigcup_{j=0}^{\beta-1}T^jA_\beta$, R is equal to Q.

(2) Let $|F_m(\bar{P})|=r$, $F_m(\bar{P})=\{x_1,x_2,\dots,x_r\}$. For each $\beta\geqq\alpha$ divide A_β into r^2+1 sets of positive measure, and call them B_{ij} $(1\leqq i,j\leqq r)$ and C^β. We shall only change the β-names of the points in B_{ij}. (In the meantime the β-names of the points in C^β will remain unchanged. These sets are reserved for the next part of the proof.)

Fix $\beta\geqq\alpha$, and let y be the $Q-\beta$-name of the points in A_β. By Proposition 3.1 for any $x_i,x_j\in F_m(\bar{P})$ there are words w_i, z_i and u_{ij} such that:

$$y(0,m-1)w_jx_j\in E=E(\bar{P},\varepsilon/8,m,k)\quad\text{and}\quad|w_j|=l=l(\bar{P},\varepsilon/8,m,k).$$

Also, $x_iz_iy(\beta-m,\beta-1)\in E$, $|z_i|=l$, $x_ju_{ji}x_i\in E$ and $|u_{ji}|=\beta-4m-2l$. There are such words because $\beta-4m-2l\geqq\alpha-4m-2l\geqq l$ (recall $\alpha=3l+4m$). Now define the $R-\beta$-names of each $h\in B_{ij}$ to be

$$v_{ij}=y(0,m-1)w_jx_ju_{ji}x_iz_iy(\beta-m,\beta-1).$$

The first and the last m-subwords of the $Q-\beta$-name of $h\in B_{ij}$ are not changed. In addition, all the m-subwords of v_{ij} are included in $F_m(\bar{P})$. After doing the same for every $\beta\geqq\alpha$, we will have, for every $x_i,x_j\in F_m(\bar{P})$, words in $E(R)$ with head x_j, tail x_i, and with any length exceeding $\bar{l}=l+2m-1$.

We have also $F_m(\bar{P})\subseteq F_m(R)\subseteq F_m(Q)\cup F_m(\bar{P})=F_m(\bar{P})$. Thus, $F_m(R)=F_m(\bar{P})$.

Part III. In this part we construct the final partition P that differs from R on a set of measure less than $\bar{\delta}/3$. The partition P will be constructed so as to be an $\bar{n}$-universal. P will differ from Q only in the β-names of points of a few of the C^β sets. This will ensure that the topological mixing achieved at Q will not be spoiled by P.

Take some sequence of mutually relatively-prime integers: $q_1 < q_2 < q_3 < \cdots$ and a word $x \in F_m(\bar{P})$. Since $\bar{P}$ is an m-universal partition, for every $q_i \geqq 2m + l + \bar{n}$ there is a word $u_i = xw_i$ such that $u_i x \in E(P)$ and $|xw_i| = q_i$. There are i and j $(j > i)$ that satisfy $u_i(0, \bar{n} - 1) = u_j(0, \bar{n} - 1)$, i.e. the first $\bar{n}$-subwords of u_i and of u_j are equal.

Let y be the $R - \gamma$-name of points in C^γ (where $\gamma = q_i + 2m + 2l + \bar{n}$) and let z be the $R - \sigma$-name of the points in C^σ (where $\sigma = q_j + 2m + 2l + \bar{n}$). By Proposition 3.1 there are words a_i, b_i, a_j, b_j each of which is of length l, so that the four words below are in $E(\bar{P}, \varepsilon/2, k, m)$:

$$y(0, m-1)a_i x, \qquad u_i(\bar{n} - m, \bar{n} - 1)b_i y(\gamma - m, \gamma - 1),$$

$$z(0, m-1)a_j x, \qquad u_j(\bar{n} - m, \bar{n} - 1)b_j z(\sigma - m, \sigma - 1).$$

P will be defined as follows:

(1) On $X \setminus (\bigcup_{t=0}^{\sigma-1} T^t C^\sigma \cup \bigcup_{t=0}^{\gamma-1} T^t C^\gamma)$, P will coincide with R.

(2) The $P - \gamma$-name of the points in C^γ will be defined as:

$$y(0, m-1)a_i x w_i u_i(0, \bar{n} - 1)b_i y(\gamma - m, \gamma - 1).$$

The $P - \sigma$-name of the points in C^σ will be defined as:

$$z(0, m-1)a_j x w_j u_j(0, \bar{n} - 1)b_j z(\sigma - m, \sigma - 1).$$

Notice that $xw_i u_i(0, \bar{n} - 1) = xw_i xw_i(0, \bar{n} - m - 1)$ and that $xw_j u_j(0, \bar{n} - 1) = xw_j xw_j(0, \bar{n} - m - 1)$.

Summary of the proof:

(a) For changes we use only words in $E(\bar{P}, \varepsilon/8, k, m)$, so that $F_m(P) \subseteq F_m(\bar{P}) \cup F_m(R) = F_m(\bar{P})$.

(b) Had $\bar{n}$ been asserted big enough, at the end of Part I, the good $Q - (\varepsilon/4, k, \bar{n})$ statistics of Part I would have degenerated into good $R - (\varepsilon/2, k, \bar{n})$ statistics after Part II (recall $d(Q, R) < \bar{\delta}/3 \leqq \varepsilon/24k$), and into good $P - (\varepsilon, k, \bar{n})$ statistics after Part III (recall $d(R, P) < \varepsilon/24k$ too).

(c) $d(P, \bar{P}) \leqq d(\bar{P}, Q) + d(Q, R) + d(R, P) < \bar{\delta}/3 + \bar{\delta}/3 + \bar{\delta}/3 \leqq \delta$.

(d) By (2) of Part III the words $u_i(0, \bar{n} - 1)u_i(\bar{n}, q_{i-1})u_i(0, \bar{n} - 1)$ and $u_j(0, \bar{n} - 1)u_j(\bar{n}, q_j - 1)u_j(0, \bar{n} - 1)$ are included in $F_{q_i + \bar{n}}(P)$ and in $F_{q_j + \bar{n}}(P)$ respectively. Since $u_j(0, \bar{n} - 1) = u_i(0, \bar{n} - 1)$, P is an $\bar{n}$-universal partition.

(e) In Part II we defined R so that it satisfies the following statement. For every $x_i, x_j \in F_m(R)$ and $l' > l + 2m$ there is a word w with length l' such that $x_i w x_j \in F_{l'+2m}(R)$. These words are the R-names of the points of $\bigcup_{\beta \geqq a} \bigcup_{1 \leqq i,j \leqq r} B_{ij}^{\beta}$. P too satisfies this statement since it coincides with R on $\bigcup_{\beta \geqq a} \bigcup_{t=0}^{\beta-1} \bigcup_{1 \leqq i,j \leqq r} T^t B_{ij}^{\beta}$. Q.E.D.

REMARK 3.3. The weak version of Proposition 3.1, as it was mentioned in Remark 3.2, enables us to establish the Jewett–Krieger Theorem, by using Parts I and III of the proof of Proposition 3.1 and the method of proof of the Theorem.

PROPOSITION 3.4. *If P is an m-universal partition, Q is any partition and $\varepsilon > 0$, then there is a partition Q^*, such that:*

(i) $d(Q, Q^*) < \varepsilon$,

(ii) *$P \vee Q^*$ is an m-universal partition.*

PROOF. Clear. Q.E.D.

The next propositions, 3.5 and 3.6, deal with the properties of the inverse limit.

PROPOSITION 3.5. *If (Y, U) is the inverse limit of the following uniquely ergodic systems:*

$$\cdots \to (Y_3, U_3) \xrightarrow{\Pi_2} (Y_2, U_2) \xrightarrow{\Pi_1} (Y_1, U_1),$$

then (Y, U) is uniquely ergodic.

PROOF. By the definition of the inverse limit, for each i there is a map $\bar{\Pi}_i$ such that the following diagram is commutative:

$$\begin{tikzcd} Y \arrow[r, "U"] \arrow[d, "\bar{\Pi}_i"'] & Y \arrow[d, "\bar{\Pi}_i"] \\ Y_i \arrow[r, "U_i"] & Y_i \end{tikzcd}$$

Furthermore, the σ-field of (Y, U) is the smallest one which includes all the sets of the following kind: $\bar{\Pi}_i^{-1}(A)$, A is in the σ-field of Y_i.

Any ergodic measure μ on (Y, U) induces the measure $\mu \circ \bar{\Pi}_i^{-1}$ on Y_i. But (Y_i, U_i) is uniquely ergodic, so that all the ergodic measures of (Y, U) are equal on the sets $\bar{\Pi}_i^{-1}(A)$, when A is a measurable set of Y_i, and thus are equal on the generator field, and therefore are equal. Q.E.D.

PROPOSITION 3.6. *If (Y, U) is the inverse limit of the following topologically mixing system:*

$$\cdots \xrightarrow{\Pi_2} (Y_2, U_2) \xrightarrow{\Pi_1} (Y_1, U_1),$$

then (Y, U) is topologically mixing.

PROOF. As in the previous proposition, we have for each i the map $\bar{\Pi}_i$. The topology on Y is generated by the sets $\bar{\Pi}_i^{-1}(A)$, with A as open set of Y_i. So, it is enough to show that for every $\bar{A} = \bar{\Pi}_i^{-1}(A)$ and $\bar{B} = \bar{\Pi}_j^{-1}(B)$ there is an l, such that if $l' > l$ then $\bar{A} \cap U^{l'}\bar{B} \neq \emptyset$, where A is an open set of Y_i and B is an open set of Y_j.

Assume that $j > i$. Then $\Pi_{j-1}^{-1} \circ \cdots \circ \Pi_{i+1}^{-1} \circ \Pi_i^{-1}(A) = D \subseteq Y_j$, and $\bar{\Pi}_j^{-1}(D) = \bar{A}$. But (Y_j, U_j) is topologically tmixing and so there is an l such that if $l' > l$ then $B \cap U_j^{l'} D \neq \emptyset$. Hence,

$$\emptyset \neq \bar{\Pi}_j^{-1}(B) \cap \bar{\Pi}_j^{-1}(U_j^{l'} D) = \bar{B} \cap U^{l'} \bar{\Pi}_j^{-1}(D) = \bar{B} \cap U^{l'} \bar{A}. \qquad \text{Q.E.D.}$$

4. Proof of the Theorem

In this section we will build the partitions $Q_1, Q_2, \ldots$ with the properties (p1)–(p4) as sketched in Section 2. The partition Q_i will be defined as the limit of the sequence $\{Q_i^j\}_{j \geq i}$. We shall build the partition

$$\begin{array}{lll} Q_1^1 & & \\ Q_1^2 & Q_2^2 & \\ Q_1^3 & Q_2^3 & Q_3^3 \\ \vdots & & \end{array}$$

inductively. At stage k we will construct the partitions $Q_1^k, Q_2^k, \ldots, Q_k^k$, and we will define two integers l_k and m_k and two reals δ_k and η_k.

Let $\{\varepsilon_i\}_{i=2}^{\infty}$ be a sequence of positive numbers that very quickly converges to zero (in particular, $\sum_{j>i} \varepsilon_j < \varepsilon_i$ for each i, and $i\varepsilon_i \to 0$). Let $\bar{Q}_1, \bar{Q}_2, \ldots$ be a sequences of partitions so that $\bigvee_{i=1}^{\infty} \bar{Q}_i = \mathcal{B}$. W.l.o.g.[†] it can be assumed that $\bar{Q}_1 \subseteq \bar{Q}_2$ and, furthermore, that $\bar{Q}_1$ and $\bar{Q}_2$ are both topologically independent. This means that if $l \geqq 1$ and $R, S \in \bar{Q}_i$ $(i = 1,2)$, then $\mu(T^l R \cap S) > 0$.

To begin the inductive process let $m_0 = 0$, $Q_1^1 = \bar{Q}_1$, and $l_1 = m_1 = \eta_1 = \delta_1 = \varepsilon_1 = 1$. Assume that after the k-th stage we have two finite sequences of integers, $\{l_i\}_{i=1}^k$ and $\{m_i\}_{i=0}^k$, where

$$l_{i+1} > l_i + 2m_{i-1} \qquad (i = 1, \ldots, k-1). \tag{1}$$

Assume also that we have sequences $\{\delta_i\}_{i=1}^k$ and $\{\eta_i\}_{i=1}^k$ so that

$$\delta_i < \delta_{i-1} \quad \text{and} \quad \delta_i \leqq \varepsilon_i \qquad (i \leqq k). \tag{2}$$

Finally, we have partitions:

$$\begin{array}{l} Q_1^1 \\ Q_1^2, Q_2^2 \\ \vdots \\ Q_1^k, Q_2^k, \ldots, Q_k^k \end{array}$$

[†] This can be achieved by using Part II of the proof of Proposition 3.3.

Assume that the following, (b1)–(b6), hold:

(b1) $F_{m_j}(Q_i^{j+1}) \subseteq F_{m_j}(Q_i^j)$, $i \leqq j < k$.

(b2) The words in $F_{m_j}(Q_i^j)$ have good $Q_i^j - (\varepsilon_j, j, m_j)$ statistics, $i \leqq j \leqq k$.

(b3) $d(Q_i^j, Q_i^{j+1}) < \delta_{j+1}$, $i \leqq j < k$.

(b4) Q_k^k is an m_k-universal parition.

(b5) For every $1 \leqq i \leqq j \leqq k$, $1 \leqq r < j$, $R, S \in Q_i^{j(r)}$ and $l_r + 2m_{r-1} \leqq u < l_{r+1} + 2m_r$ we have

$$\mu(T^u R \cap S) > \eta_{r+1} - \frac{\eta_{r+1}}{2^{r+1}} - \cdots - \frac{\eta_{r+1}}{2^j}.$$

Moreover, for every $1 \leqq i \leqq k$, $R, S \in Q_i^{k(k)}$ and $l_k + 2m_{k-1} \leqq u$ we have $\mu(T^u R \cap S) > 0$ (recall Notation 3.1).

(b6) $Q_i^j \subseteq Q_{i+1}^j$ $(1 \leqq j \leqq k)$.

The properties (b1)–(b5) correspond to (i)–(v) of Proposition 3.3, and the additional property (b6) is needed for the definition of the inverse limit (Y, U).

The only thing we have to check about $k = 1$ is (b5). However, Q_1^1 is topologically independent, so that it holds.

At the $(k+1)$-th stage we shall construct the partitions $Q_1^{k+1}, \ldots, Q_{k+1}^{k+1}$ and the numbers l_{k+1}, m_{k+1}, δ_{k+1}, η_{k+1} so that they satisfy (b1)–(b6) when the latter refer to $k+1$ instead of to k. Q_k^k is an m_k-universal partition. Therefore according to Proposition 3.4 there is a partition Q_{k+1}^*, such that $d(Q_{k+1}^*, \bar{Q}_{k+1}) < \varepsilon_{k+1}$ and that the partition $P_{k+1} = Q_k^k \vee Q_{k+1}^*$ is an m_k-universal partition. By Proposition 3.1 there is a set $E = E(P_{k+1}, \varepsilon_{k+1}, k+1, m_k)$ and an integer $l = l(P_{k+1}, \varepsilon_{k+1}, k+1, m_k)$ which satisfy that proposition. Define

$$l_{k+1} = \operatorname{Max}\{l, l_k + 2m_{k-1}\} + 1, \tag{3}$$

$$\eta_{k+1} = \operatorname{Min}\{\mu(T^j S \cap R) > 0 \mid j < l_{k+1} + 2m_k, R, S \in Q_k^{k(k)}\}, \tag{4}$$

$$\delta_{k+1} = \operatorname{Min}\{\eta_{k+1}, \varepsilon_{k+1}, \delta_k\}/((k+1) \cdot 2^{k+2}). \tag{5}$$

By Proposition 3.3 there exist a partition Q_{k+1}^{k+1}, and an integer $n = n(P_{k+1}, \varepsilon_{k+1}, \delta_{k+1}, k+1, m_k)$, which we will call m_{k+1}. The partition and the integer are such that the following five conditions, (c1)–(c5), hold:

(c1) $F_{m_k}(Q_{k+1}^{k+1}) \subseteq F_{m_k}(P_{k+1})$.

(c2) Each word in $F_{m_{k+1}}(Q_{k+1}^{k+1})$ has good $Q_{k+1}^{k+1} - (\varepsilon_{k+1}, k+1, m_{k+1})$ statistics.

(c3) $d(Q_{k+1}^{k+1}, P_{k+1}) < \delta_{k+1}$.

(c4) Q_{k+1}^{k+1} is an m_{k+1}-universal partition.

(c5) For every $R, S \in (Q_{k+1}^{k+1})^{(k+1)}$ and $u \geqq l_{k+1} + 2m_k$, $\mu(T^u R \cap S) > 0$.

Q_{k+1}^{k+1} improves P_{k+1} and $Q_k^k \subseteq P_{k+1}$; therefore the partition Q_k^{k+1} is defined naturally (for the precise definition see Appendix). Q_k^{k+1} improves Q_k^k and $Q_{k-1}^k \subseteq Q_k^k$; therefore Q_{k-1}^{k+1} is defined naturally. In the same way we can define all the partitions $Q_1^{k+1}, \dots, Q_{k+1}^{k+1}$. These partitions have the following properties (d1)–(d6) (compare with (b1)–(b6)).

(d1) By (c1), $F_{m_k}(Q_i^{k+1}) \subseteq F_{m_k}(Q_i^k)$, $i \leqq k$.

(d2) By (c2), each word in $F_{m_{k+1}}(Q_i^{k+1})$ has good $Q_i^{k+1} - (\varepsilon_{k+1}, k+1, m_{k+1})$ statistics $(1 \leqq i \leqq k+1)$.

(d3) By (c3) and because $Q_i^k \subseteq P_{k+1}$,

$$Q_i^k \underset{\delta_{k+1}}{\subseteq} Q_{k+1}^{k+1} \qquad (1 \leqq i \leqq k)$$

(recall Notation 3.10). Thus, by the definition of Q_i^{k+1},

$$d(Q_i^{k+1}, Q_i^k) < \delta_{k+1} \qquad (1 \leqq i \leqq k).$$

(d4) Q_{k+1}^{k+1} is an m_{k+1}-universal partition (as in (c4)).

(d5)

(i) If $1 \leqq i \leqq j \leqq k+1$, $r < j$, $R, S \in Q_i^{j(r)}$ and $l_r + 2m_{r-1} \leqq u < l_{r+1} + 2m_r$, then

$$\mu(T^u R \cap S) > \eta_{r+1} - \frac{\eta_{r+1}}{2^{r+1}} - \cdots - \frac{\eta_{r+1}}{2^k} - \frac{\eta_{r+1}}{2^{k+1}}\,.$$

(ii) If $l_{k+2} + 2m_{k+1} \leqq u$ and $R, S \in (Q_i^{k+1})^{(k+1)}$ then $\mu(T^u R \cap S) > 0$, $1 \leqq i \leqq k+1$.

(d5) needs a proof. We will first prove (ii). By the construction of Q_{k+1}^{k+1} it is ensured that if $l_{k+1} + 2m_{k+1} \leqq u$ and $R, S \in (Q_{k+1}^{k+1})^{(k+1)}$ then $\mu(T^u R \cap S) > 0$. Since $(Q_i^{k+1})^{(k+1)} \subseteq (Q_{k+1}^{k+1})^{(k+1)}$, (ii) holds.

In order to prove (i) assume first that $i = r = j = k$, and that $R, S \in Q_i^{j(r)}$. By (4), whenever $u \leqq l_{k+1} + 2m_k$ then $\mu(T^u R \cap S) \geqq \eta_{k+1}$, provided $\mu(T^u R \cap S) > 0$. However, according to (b5), if $u \geqq l_k + 2m_{k-1}$ then $\mu(T^u R \cap S) > 0$. Therefore

$$\mu(T^u R \cap S) \geqq \eta_{k+1} \qquad (l_k + 2m_{k-1} \leqq u < l_{k+1} + 2m_k). \tag{6}$$

Since $Q_i^{k(k)} \subseteq Q_k^{k(k)}$, (i) holds also for $i < k$ and $r = j = k$.

It is known by (b5) that if $r < j = k$ and $l_r + 2m_{r-1} \leqq u < l_{r+1} + 2m_r$ then $\mu(T^u R \cap S) > \eta_{r+1} - \eta_{r+1}/2^{r+1} - \cdots - \eta_{r+1}/2^k$, and by (6) it is known that if $r = j = k$ and $l_k + 2m_{k-1} \leqq u < l_{k+1} + 2m_k$ then $\mu(T^u R \cap S) > \eta_{k+1}$. By (d3) and (5), $d(Q_i^{k+1}, Q_i^k) < \eta_{k+1}/((k+1)2^{k+2})$. Therefore, for each atom $A \in Q_i^{k+1(r)}$ there is an atom $B \in Q_i^{k(r)}$ such that $\mu(A \,\triangle\, B) < \eta_{k+1}/2^{k+2} < \eta_{r+1}/2^{k+2}$. Thus, if $j = k+1$, $r \leqq k$ and $l_r + 2m_{r-1} \leqq u < l_{r+1} + 2m_r$ then for every $R, S \in Q_i^{k+1(r)}$ we have

$$\mu(T^u R \cap S) > \eta_{r+1} - \frac{\eta_{r+1}}{2^{r+1}} - \cdots - \frac{\eta_{r+1}}{2^k} - \frac{\eta_{r+1}}{2^{k+1}}\,.$$

This result and (b5) give the proof of (d5).

The last property is

(d6) By the definition of Q_i^{k+1}, $i = 1, \ldots, k+1$ and by (b6) it follows that $Q_i^j \subseteq Q_{i+1}^j$ $(i < j \leqq k+1)$.

After the inductive process is finished we have the partitions $\{Q_i^j\}_{i \in \mathbf{N}, j \geqq i}$. By (d3) the following holds:

$$\sum_{j'=j}^{\infty} d(Q_i^{j'}, Q_i^{j'+1}) < \sum_{j'=j+1}^{\infty} \delta_{j'} \leqq \delta_j.$$

Hence there is, for each i, a partition Q_i such that $Q_i^j \to Q_i$ as $j \to \infty$. Now we are able to proceed with the proof that the sequence $\{Q_i\}_{i=1}^{\infty}$ has the properties (p1)–(p4) mentioned in Section 2.

(p1) (d6) implies that $Q_i \subseteq Q_{i+1}$, $i = 1, 2, \ldots$.

(p2) We will first prove that $\bar{Q}_i \underset{3\varepsilon_i}{\subseteq} Q_i$. Then, it will follow that in the case where $\{\varepsilon_i\}$ shrinks very fast and because $\vee Q_i = \mathscr{B}$, $\vee Q_i = \mathscr{B}$ is ensured.

For each $1 \leqq i$, $d(Q_i^i, Q_i) < \delta_i < \varepsilon_i$. Moreover, $d(\bar{Q}_i, Q_i^*) < \delta_i < \varepsilon_i$, $Q_i^* \subseteq P_i$ and $d(P_i, Q_i^i) < \delta_i < \varepsilon_i$. Thus $\bar{Q}_i \underset{3\varepsilon_i}{\subseteq} Q_i$.

(p3) In order to prove that Q_i is a uniform partition for all $i \geqq 1$ use (d1). Fix Q_i and let $k \in \mathbf{N}$.

So, $F_{m_k}(Q_i) \subseteq F_{m_k}(Q_i^k)$. By (d2) all the words in $F_{m_k}(Q_i^k)$ have good $Q_i^k - (\varepsilon_k, k, m_k)$ statistics, and thus all the words of $F_{m_k}(Q_i)$ have good $Q_i - ((k+1)\varepsilon_k, k, m_k)$ statistics. Since $k \cdot \varepsilon_k \to 0$, Q_i is a uniform partition.

(p4) To prove that the symbolic system induced by (X, T, Q_i) is topologically mixing, take $R, S \in Q_i^{(r)}$ for some $r \in \mathbf{N}$, and take $u > l_r + 2m_{r-1}$. It will be shown that $\mu(T^u R \cap S) > 0$.

The sequence $\{l_k\}$ was defined in a way that $l_k > l_{k-1} + 2m_{k-2}$ for each $k \geqq 2$. Therefore there is an integer $s \geqq r$, so that $l_s + 2m_{s-1} \leqq u < l_{s+1} + 2m_s$.

As was shown $d(Q_i^s, Q_i) < \delta_s$, and thus there are two sets $R', S' \in Q_i^{s(r)}$ such that both $\mu(R' \triangle R)$ and $\mu(S' \triangle S)$ are less than $\delta_s \cdot r \leqq \delta_s \cdot s$. According to (d5) $\mu(T^u R' \cap S') > \eta_s/2$. Since $\delta_s < \eta_s/2^s$ it follows that

$$\begin{aligned}
\mu(T^u R \cap S) &\geqq \mu(T^u R' \cap S') - (T^u R' \triangle T^u R) - \mu(S' \triangle S) \\
&= \mu(T^u R' \cap S') - \mu(R' \triangle R) - \mu(S' \triangle S) \\
&\geqq \eta_s/2 - 2\delta_s \cdot s \\
&\geqq \eta_s/2 - s \cdot \eta_s/2^{s-1} \\
&> 0
\end{aligned}$$

Define (Y_i, U_i) to be the symbolic system induced by (X, T, Q_i). By (p3) and (p4), (Y_i, U_i) is uniquely ergodic and topologically mixing. By (p1) a natural mapping Π_i from (Y_{i+1}, U_{i+1}) to (Y_i, U_i) can be defined, $i = 1, 2, \ldots$.

Define (Y, U) to be the inverse limit of

$$\cdots \longrightarrow (Y_3, U_3) \xrightarrow{\Pi_2} (Y_2, U_2) \xrightarrow{\Pi_1} (Y_1, U_1).$$

Propositions 3.5 and 3.6 ensure that (Y, U) is uniquely ergodic and topologically mixing.

Let $\mathscr{C}$ be the σ-algebra generated by the open sets $\bar{\Pi}_i^{-1}(A)$, $i \in \mathbf{N}$, $A \in Q_i$ (recall the notation in Proposition 3.5). Let also $\nu(\bar{\Pi}_i^{-1}(A)) = \mu(A)$. By (p2) $(Y, U, \mathscr{C}, \nu)$ is isomorphic to $(X, T, \mathscr{B}, \mu)$. Q.E.D.

Appendix

Let V, U, U' be partitions of X, where $V \subseteq U$, and U' improves U (in particular $|U'| = |U|$); then U' induces a natural improvement V' of V.

Let $V = \{V_1, \ldots, V_a\}$, $U = \{U_1, \ldots, U_b\}$, and $U' = \{U'_1, \ldots, U'_b\}$. $V \subseteq U$ means that there is a function

$$\varphi: \{1, \ldots, a\} \rightarrow 2^{\{1, \ldots, b\}} \setminus \varnothing,$$

such that $\{\varphi(a')\}_{1 \leqq a' \leqq a}$ is a partition of $\{1, \ldots, b\}$, and for each $1 \leqq a' \leqq a$,

$$V_{a'} = \bigcup_{b' \in \varphi(a')} U_{b'}.$$

V' is defined to be $\{V'_1, \ldots, V'_a\}$, where

$$V'_{a'} = \bigcup_{b' \in \varphi(a')} U'_{b'} \qquad \text{for every } a' = 1, \ldots, a.$$

References

1. A. Bellow and H. Furstenberg, *An application of number theory to ergodic theory and the construction of uniquely ergodic models*, Isr. J. Math. **33** (1979), 231–240.

2. G. Hansel and J. P. Raoult, *Ergodicity, uniformity and unique ergodicity*, Indiana Univ. Math. J. **23** (1974), 221–237.

3. Robert I. Jewett, *The prevalence of uniquely ergodic systems*, J. Math. Mech. **19** (1970), 717–729.

4. W. Krieger, *On unique ergodicity*, Proc. Sixth Berkeley Symposium on Math., Statist. and Probability, June/July 1970, Vol. II, Univ. of California Press, 1972, pp. 327–345.