What Restrictions do Bayesian Games Impose on the Value of Information?

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Abstract

In a Bayesian game players play an unknown game. Before the game starts some players may receive a signal regarding the specific game actually played. Typically, information structures that determine different signals, induce different equilibrium payoffs. In zero-sum games the equilibrium payoff measures the value of the particular information structure which induces it. We pose a question as to what restrictions Bayesian games impose on the value of information. We provide answers in two kinds of information structures: symmetric, where both players are equally informed, and one-sided, where only one player is informed.

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1 Introduction

Markets or strategic interactions are typically not observable in full detail to the outside observer, may it be an econometrician or an analyst. Either the utilities of the agents or the actions available to them are unobservable. Frequently, only the outcome of the interaction is observable, if at all. The question arises as to what conditions the observable data should satisfy in order to be consistent with an underlying theoretical model. Stated differently, what restrictions on the outcomes of an interaction does the underlying model impose?

Afriat (1967) examined a situation where only finitely many observations of prices and consumption-bundles of an agent are available. Afriat's theorem (see also Varian, 1984) states that these observations may constitute a finite sample from a demand function induced by a continuous, concave and monotonic utility, if and only if a certain revealed preference condition is satisfied. Sonnenschein (1973), Debreu (1974) and Mantel (1974) examined functions that map prices to bundles. They questioned under what conditions such functions might convey the excess demand of a market with utility maximizing agents. It turns out that any function can be derived from rational individuals who maximize their utility.

This paper refers to strategic interactions and poses questions of a similar spirit. The exact specifications of the game played are unobservable to the outside observer. Only the payoffs received by the agents are knowable. In this case, what conditions should these payoffs satisfy in order to be consistent with the equilibrium paradigm of interactive models?

More specifically, consider a Bayesian game in which agents might receive information regarding the actual game played. As in Aumann (1974), we model the information structure in a Bayesian game by a partition of the state-space into disjoint cells: a player is informed of the cell containing the realized state. The information structure of the game affects the behavior of the agents; it determines the equilibrium payoffs. The data available to the economist about the game includes all possible information structures and the equilibrium payoffs associated with them. As in Afriat (1967) we look for conditions that data should satisfy in order to be consistent with the rational behavior of the agents in Bayesian games.

Another purpose of the paper is to study those properties essential to the functions that measure the value of information, as well as the role of information in Bayesian games and its effect on equilibrium payoffs. When the information structure changes, typically, the equilibrium payoffs also change. Specially interesting questions are: what is the extent to which information affects the outcome of the interaction; are there limitations on the way information affects the outcome; and should the contribution of additional information be related in any particular way to the information already available?

As a first step in studying the aforementioned questions, we restrict ourselves to zero-sum games, and to specific kinds of information structures : one sided and symmetric.

The main advantage of zero-sum games is that they have a unique equilibrium payoff, the value. This implies that any information structure is associated with a *unique* equilibrium payoff. Furthermore, in zero-sum games the effect of getting more information is always positive: the equilibrium payoff cannot decrease as a result of receiving more information.

The value-of-information function of a Bayesian zero-sum game maps each possible information structure to the corresponding equilibrium payoff. We characterize those real-valued functions defined over the (symmetric or one-sided) information structures that can be realized by an underlying Bayesian game, as value-of-information functions. That is, we specify the properties of functions over the state-space partitions that are necessary and sufficient for being value-of-information functions.

A Bayesian zero-sum game can be also perceived as a one-player decision problem under uncertainty when the decision-maker has a prior over her own payoff functions while she has no prior over the states nature may choose. Consider a decision-maker who takes a decision and then receives a payoff which depends also on the state nature chooses. Neither the payoff function nor the state of nature is known.

The payoff function reflects the decision-maker's own preferences, and therefore, she might have a prior over the possible payoff functions that may be relevant at the time the payoff is given. The state of nature, however, might be subject to complete ignorance: the decision-maker might have no assessment or hypothesis regarding the distribution of the states nature chooses. In such a situation a worst case analysis of nature's choice suggests that nature is malicious and it tries to minimize the decision-maker's payoff. Thus, in effect, the decision-maker plays a Bayesian zero-sum game against nature.

The issue of measuring the value of information has been previously addressed in the case of one decision-maker by Gilboa and Lehrer (1991). They characterized those functions that measure the value of information in optimization problems, where the decision-maker gets to know an equivalent class of states, rather than the realized state itself, and he has a prior on the set of states. In this paper we extend the model of Gilboa and Lehrer (1991) to zero-sum games and determine what kind of functions (of information) might measure the value of information. We answer this question in two polar cases: symmetric information in which the partitions of both players coincide and thus both obtain the same information about the state of nature; and one-sided information in which one player gets some information about the state of nature while the other does not.

In the case of symmetric information both players are equally informed. After being informed they actually play a Bayesian game restricted to the states within the informed cell. Therefore, the value of the original Bayesian game is the expected value of the Bayesian game played aposteriori. In other words, the value of the Bayesian game is a weighted sum of the values of the restricted Bayesian games played after the players have been informed. This implies, in particular, that a value-of-information function of a symmetric information game should be additively separable. It turns out that this very condition characterizes all possible value-of-information functions: any additively separable function over partitions is a value-of-information function.

When the information is one-sided, refining the partition of the informed player increases her equilibrium payoff. Thus, any value-of-information function must be monotonic (with respect to refinement). Our conclusion concerning one-sided information states that, unlike the case of one-player decision problems, no further condition beyond monotonicity is required to characterize the value-of-information functions.

To summarize, in both types of information structures − the symmetric and the one-sided − the Bayesian model is rich enough to allow for all value-of-information functions as long as they satisfy the obvious necessary conditions (i.e., additivity in the symmetric case and monotonicity in the one-sided case).

The paper is organized as follows. In Section 2 we present the model and the main issues treated by the paper. In Sections 3 and 4 we present the two main results: the characterizing of the value-of-information functions in symmetric and one-sided information structures. In Section 5 we prove these results. Section 6 reviews related literature and Section 7 is devoted to final comments.

2 The model

In this section we give a more formal content to the question asked in the introduction. We first define information structures and the corresponding Bayesian game. We then introduce the notion of the value of information in this context.

2.1 Information structures

We consider a game with incomplete information. A state of nature k is drawn from a finite set K according to a known probability p. None of the players observes k. The players, however, receive signals that depend on k through an information structure. This information structure is the main subject of this study and is to be distinguished from the uncertainty embeded in p.

In games with a general information structure, the players receive private signals selected randomly according to a distribution that depends on the realized state. In this paper we restrict ourselves to deterministic signals. Note that an information structure with deterministic signals induces a partition of K. Indeed, each signal can be identified with the subset of states that is consistent with it. Thus, we follow Aumann (1974) who defines an information structure as a partition of K into atoms: a player is informed of the atom containing the realized state.

Definition 1 *A* partitional information structure $I = (\mathcal{P}_1, \mathcal{P}_2)$ *consists of two* partitions of K. $\mathcal{P}_i$ *is the partition of player* i, $i = 1, 2$.

The signal player i receives about k is the atom of $\mathcal{P}_i$ that contains k.

In this paper we focus on two specific kinds of information structures: symmetric information in which both players receive the same signal and one-sided information in which only one player receives information.

Definition 2 *A partitional information structure* $I = (\mathcal{P}_1, \mathcal{P}_2)$ *is* symmetric *if* $\mathcal{P}_1 = \mathcal{P}_2$*.*

Definition 3 *A partitional information structure* $I = (\mathcal{P}_1, \mathcal{P}_2)$ *is* one-sided *if* $\mathcal{P}_2$ *contains only the set* K*.*

2.2 The game

The Bayesian game is defined by a finite state space K; a probability distribution p over K; a finite action set A_i for each player $i = 1, 2$; and a payoff function, g, defined on $K \times A_1 \times A_2$. As usual, the domain of g is extended to mixed strategies.

The zero-sum game associated with the information structure $I = (\mathcal{P}_1, \mathcal{P}_2)$ is played as follows. A state of nature is drawn from K according to the distribution p. Player i observes the cell of the partition $\mathcal{P}_i$ that contains the realized state. Then, both players simultaneously choose an action $a_i \in A_i$ and player 2 pays $g(k, a_1, a_2)$ to player 1.

A strategy of player i is a function, τ_i, that associates a probability distribution over A_i to each cell $B_i \in \mathcal{P}_i$. After being informed of $B_i \in \mathcal{P}_i$ player i chooses an action according to the distribution $\tau_i(B_i)$. Let $B_i(k)$ denote the cell of $\mathcal{P}_i$ that contains k. The payoff corresponding to a pair of strategies τ_1, τ_2 is $\sum_{k \in K} p(k) g(k, \tau_1(B_1(k)), \tau_2(B_2(k)))$. The game defined this way has a unique equilibrium payoff denoted by $v^I(p, g)$.

2.3 Measuring the contribution of information

We now define the value of information in Bayesian games. Consider a game with a state space K, a payoff function g and a distribution p over K. We define the value-of-information function of this game as $V(I) = v^I(p, g)$. That is, we fix the payoff function g and the distribution p and let the information structure vary. The value-of-information function reflects the impact of the information structure on the equilibrium payoff. For instance, player 1 would be willing to pay $V(\mathcal{P}_1', \mathcal{P}_2) - V(\mathcal{P}_1, \mathcal{P}_2)$ in order to exchange the structure $\mathcal{P}_1$ with $\mathcal{P}_1'$, given that player 2 is informed through $\mathcal{P}_2$.

The objective of this paper is to find what functions on information structures are inconsistent with the Bayesian model. In particular, we are interested in the restrictions imposed by the Bayesian model on the way information affects equilibrium outcomes. To this end, we study the value-of-information functions.

Formally, let V be a function over partitional information structures. The question is when this function is a value-of-information function of some Bayesian game.

In the case of one decision-maker this problem was analyzed by Gilboa and Lehrer (1991). They characterized the functions of partitions defined on the set of partitions that are the values of information of finite games. We need the following definition in order to present their result.

Definition 4 *Let V be a function defined over all the partitions of a finite set K. V is* separately additive *if there is a function v, defined over subsets of K, such that for any partition $\mathcal{P}$, $V(\mathcal{P}) = \sum_{B \in \mathcal{P}} v(B)$. In this case we say that V is separately additive with respect to v.*

Notation 1 *If $\emptyset \neq T \subseteq B \subseteq K$ and $(x_i)_{i \in B}$ is a vector, then $x(T)$ denotes $\sum_{i \in T} x_i$, and $x(\emptyset) = 0$.*

Definition 5 *For $B \subseteq K$, the B-anti-core of v is non-empty if there is a vector $(x_i)_{i \in B}$, such that $x(T) \leq v(T)$ for every $T \subseteq B$ and $x(B) = v(B)$.*

Gilboa and Lehrer (1991) showed that a function V defined over all the partitions of a finite set K is a value-of-information function of a one-player decision making problem with state space K if and only if there is a function v, defined over subsets of K, such that (i) $V(\mathcal{P}) = \sum_{B \in \mathcal{P}} v(B)$ (i.e., V is separately additive with respect to v); and (ii) for any $B \subset K$, the B-anti-core of v is non-empty. Moreover, there is no restriction on the underlying probability distribution over K, as long as the support is the entire K (i.e., any $k \in K$ is assigned a positive probability).

Condition (i) is necessary in a one-player decision problem for the following reason. Let $\mathcal{P}$ be a partition and $B \in \mathcal{P}$. Define $v(B)$ as $\max_{a \in A_1} \sum_{k \in B} p(k)g(k,a)$. $v(B)/p(B)$ is the optimal payoff that the decision-maker can achieve given that k is in B. The value of the decision problem $V(\mathcal{P})$ has to be $\sum_{B \in \mathcal{P}} v(B)$. Condition (i) will be extended to the case of two-player zero-sum games with symmetric information.

In the next two sections we provide similar characterizations of value-of-information functions in zero-sum games with two particular information structures: symmetric and one-sided. Note that the one-player case is a special case of a zero-sum game in which player 2 has only one action. Therefore, there are more zero-sum games than one-player decision problems. Thus, there are more value-of-information functions in zero-sum games than in one-player decision problems. It implies that the conditions that characterize value-of-information functions in zero-sum games are weaker than those characterizing value-of-information functions in one-player decision problems.

3 The value of symmetric partitional information

In this section we focus on zero-sum games with symmetric partitional information and state our first result.

Definition 6 *A function V defined over all the partitions of K is a* value-of-information function of a partitional symmetric information game *if there is a distribution p over K and a payoff function g such that for any partition $\mathcal{P}$ of K, $V(\mathcal{P}) = v^{(\mathcal{P},\mathcal{P})}(p, g)$.*

The following example illustrates this definition in a Bayesian game with two states.

Example 1

Let K be $\{1,2\}$. The payoff functions $g(1,\cdot)$ and $g(2,\cdot)$ are given by the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ respectively.}$$

Suppose that the probability of state $k = 1$ is p. If no player is informed of the state selected, the players actually play the game with matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1-p \end{pmatrix}.$$

The value of this game is $\frac{1-p}{2-p}$. On the other hand, if the players are informed of the game selected, then with probability p the value of the game played is 0 and with probability $1-p$ the value of the game played is $\frac{1}{2}$. Thus, the average of the Bayesian game is $\frac{1-p}{2}$.

To sum up, there are two possible partitional symmetric information structures: the trivial, $\mathcal{T}$, where no information about the state selected is given to the players, and the perfect one (that corresponds to the discrete partition), $\mathcal{D}$, where both players are fully informed of the state selected. The value-of-information function in this case is therefore given by $V(\mathcal{T}) = \frac{1-p}{2-p}$ and $V(\mathcal{D}) = \frac{1-p}{2}$. One can see that the additional information given by $\mathcal{D}$ is harmful to player 1.

We are now ready to characterize the functions over partitions that are values of information of zero-sum games with symmetric information.

Theorem 1 *Let V be a function defined over all the partitions of K. V is the value of information of a game with symmetric partitional information if and only if V is separately additive.*

Moreover, if V is separately additive, then for any probability distribution p on K that has a full support there is a payoff function g such that for any partition $\mathcal{P}$ over K, $V(\mathcal{P}) = v^{(\mathcal{P},\mathcal{P})}(p,g)$.

Note that as in the case of one-player decision problems, additivity is a necessary condition. Indeed, let $v(B)$ be the equilibrium payoff of the zero-sum matrix game with action sets A_1 and A_2, and payoff function $\sum_{k\in B} p(k)g(k,\cdot,\cdot)$. Then, for any partition $\mathcal{P}$, $V(\mathcal{P})$ is the sum of $v(B)$ over all atoms B of $\mathcal{P}$.

The main contribution of this theorem is to state that in a zero-sum game when both players receive the same information, no further condition, beyond additivity, is needed for a function to be the value of information. This means that in a game with symmetric information the impact of information can be literally unlimited (as long as additivity is preserved). Information may have a positive or negative contribution, and may alternate arbitrarily between having positive and negative effects, as the information increases. Furthermore, the marginal contribution of additional information may be arbitrarily small or large.

To sum up, the Bayesian model imposes only the obvious restriction on the impact of information – additivity. It means that when information is symmetric, Bayesianism cannot be rejected on the grounds that the effect of additional information is inconsistent with it.

The proof of Theorem 1 is postponed to Section 5.

4 One-sided information structures

In this section we discuss the case where one player, typically the maximizer, receives some information about the state selected, while the other player receives no information. Formally, player 1 will be informed of the cell of partition $\mathcal{P}$, while the other player's information structure is the trivial partition, $\mathcal{T}$.

Definition 7 *A function V defined over all the partitions of K is a* value-of-information function of a game with one-sided information *if there is a* *distribution p over K and a payoff function g such that for any partition $\mathcal{P}$ over K, $V(\mathcal{P}) = v^{(\mathcal{P};\mathcal{T})}(p, g)$.*

Definition 8 *A real-valued function V from the set of partitions of a finite set K is* monotonic *if $V(\mathcal{P}) \geq V(\mathcal{P}')$ whenever $\mathcal{P}$ refines $\mathcal{P}'$ (i.e., any atom of $\mathcal{P}'$ is a union of atoms of $\mathcal{P}$).*

When V is monotonic, additional information implies a higher payoff for player 1. Thus, monotonicity of V means that the value of information is positive for player 1.

The next example illustrates the above definitions.

Example 2

Recall Example 1 and consider one-sided partitional information. When the information is trivial, the equilibrium payoff, as in Example 1, is $\frac{1-p}{2-p}$. However, when player 1 is fully informed of the state and player 2 obtains no information, then the game actually played is

$$\begin{pmatrix} 1 & 0 \\ p & 1-p \\ 1-p & 0 \\ 0 & 1-p \end{pmatrix}.$$

The equilibrium payoff of this game is $\frac{1}{2}$ if $p \leq \frac{1}{2}$ and $1-p$ if $p > \frac{1}{2}$. Note that this game is the one-sided information corresponding to the discrete partition $\mathcal{D}$.

We conclude by writing down the value-of-information function of this one-sided information: $V(\mathcal{T}) = \frac{1-p}{2-p}$, and $V(\mathcal{D}) = \frac{1}{2}$ if $p \leq \frac{1}{2}$ and $V(\mathcal{D}) = 1-p$ if $p > \frac{1}{2}$. Note that V is monotonic, since $\mathcal{D}$ refines $\mathcal{T}$ and indeed, $\frac{1}{2} \geq \frac{1-p}{2-p}$ for $p \leq \frac{1}{2}$ and $1-p \geq \frac{1-p}{2-p}$ for $p > \frac{1}{2}$.

When only one player receives more information, his set of strategies expands while the set of the other players remains unchanged. In zero-sum games it implies that his equilibrium payoff increases. The reason is that in zero-sum games equilibrium strategies guarantee the equilibrium payoff against any strategy of the opponent and are not merely best response to the opponents' strategies. Thus, a greater set of strategies guarantees a higher payoff. Hence, the value-of-information functions of games with one-sided partitional information must be monotonic.

It turns out, as the following theorem states, that in games with one-sided information monotonicity is not only necessary but also sufficient for being a value-of-information function. As in the symmetric case, there is no restriction (as long as monotonicity is preserved) on the possible impacts of information. In other words, knowing the effect of information cannot help in accepting or rejecting the Bayesian model.

Theorem 2 *A function V from the set of partitions of a finite set K is a value-of-information function of a partitional one-sided information game if and only if it is monotonic.*

The proof of this theorem will be given in the next section.

5 Proofs of the theorems

5.1 The proof of Theorem 1.

We first prove that if V is a value-of-information function of a zero-sum game with symmetric information then it has to be additive. Recall that since each player knows the set of the partition $\mathcal{P}$ to which k belongs, the strategies τ_1 and τ_2 of player 1 and player 2 are functions from the atoms of the partition to probability distributions over A_1 and A_2, respectively. We will denote by $\tau_1(B)$ (resp. $\tau_2(B)$), $B \in \mathcal{P}$ the mixed action corresponding to the information B. Therefore

$$\begin{aligned} V(\mathcal{P},\mathcal{P}) &= v^{(\mathcal{P},\mathcal{P})}(p,g) \\ &= \max_{(\tau_1)} \min_{(\tau_2)} \sum_{B\in\mathcal{P}} \sum_{k\in B} p(k) g(k,\tau_1(B),\tau_2(B)) \\ &= \sum_{B\in\mathcal{P}} h(B), \end{aligned}$$

where $h(B) = \max_{\tau_1(B)} \min_{\tau_2(B)} \sum_{k\in B} p(k)g(k,\tau_1(B),\tau_2(B))$. Thus, V is additive.

Assume now that V is a separately additive function of partitions; we want to prove that it is a value-of-information function. In order to prove this result we will use the following proposition from Lehrer and Rosenberg (2003).

Denote by $\Delta(K)$ is the set of probability distributions over the set K.

Proposition 1 *Given a finite number of pairs* $(x_\ell, y_\ell) \in \Delta(K) \times I\!R$, $\ell = 1, ..., L$, *there exists a Bayesian zero-sum game with state space* K, *whose equilibrium payoff is* $f(p)$ *for any* $p \in \Delta(K$ *and* $f(x_\ell) = y_\ell$, $\ell = 1, ..., L$.

Let p be any probability distribution over K that has a full support. For any subset B of K we denote by p_B the conditional probability on B, namely $p_B(k) = p(k)/p(B)$ if $k \in B$ and 0 otherwise. Note that for any two different subsets of K, B and B', $p_B \neq p_{B'}$.

There are finitely many subsets of K. Thus, Proposition 1 ensures the existence a Bayesian zero-sum game with state space K, whose equilibrium payoff is $h(B)/p(B)$ whenever the distribution over K is p_B, $B \subseteq K$.

Let g denote the payoff function of this game. We have proven that

$$\begin{aligned} V(\mathcal{P},\mathcal{P}) &= \sum_{B\in\mathcal{P}} h(B) = \sum_{B\in\mathcal{P}} f(p_B)p(B) \\ &= \sum_{B\in\mathcal{P}} p(B) \max_{\tau_1(B)} \min_{\tau_2(B)} \sum_{k\in B} p_B(k) g(k,\tau_1(B),\tau_2(B)) \\ &= \sum_{B\in\mathcal{P}} \max_{\tau_1(B)} \min_{\tau_2(B)} \sum_{k\in B} p(k) g(k,\tau_1(B),\tau_2(B)), \end{aligned}$$

which is the desired result. ■

5.2 Proof of Theorem 2

The proof of Theorem 2 makes use of the following proposition.

Proposition 2 *A function V from the set of partitions of a finite set K is a value-of-information function of a one-sided information game if and only if it is a minimum of finitely many value-of-information functions of one-player decision making problems.*

Proof. Let V be the value-of-information function of the game G with state space K, a distribution p over K, action sets A_i, $i = 1, 2$, and a payoff function g. We prove that it is the minimum of finitely many value-of-information functions for one-player decision making problems.

Fix a partition $\mathcal{P}$. Denote by $G(\mathcal{P})$ the one-sided information game induced by the partition $\mathcal{P}$. Consider the following auxiliary multi-stage game, denoted $\bar{G}(\mathcal{P})$, that depends on $\mathcal{P}$. The action set of player 2 in $\bar{G}(\mathcal{P})$ is $\Delta(A_2)$, while the action set of player 1 in $\bar{G}(\mathcal{P})$ is A_1.

At the beginning of $\bar{G}(\mathcal{P})$ player 2 announces an action, say q; then a state in K is chosen with respect to the prior distribution p and player 1 is informed of the cell of $\mathcal{P}$ that contains it. Player 1 takes then an action, say a_1, and finally, a pure action in A_2 is selected according to the distribution q. The payoff of $\bar{G}(\mathcal{P})$ is defined as the expectation of g.

The equilibrium payoffs and the optimal strategies in the two games $\bar{G}(\mathcal{P})$ and $G(\mathcal{P})$ coincide.

For each action y of player 2 we define $D_y^{\mathcal{P}}$ as the one-player decision problem faced by player 1 in $\bar{G}(\mathcal{P})$, after player 2 announces y. The state space of $D_y^{\mathcal{P}}$ is K and its payoff function is $g(\cdot, y, \cdot)$. Denote by $U_y^{\mathcal{P}}$ the value of this problem.

Let $y_{\mathcal{P}}$ be an optimal strategy of player 2 in $G(\mathcal{P})$. Note that $V(\mathcal{P})$ coincides with $U_{y_{\mathcal{P}}}^{\mathcal{P}}$. Moreover, $U_{y_{\mathcal{P}}}^{\mathcal{P}} \leq U_{y_{\mathcal{P}}}^{\mathcal{Q}}$ for any partition $\mathcal{Q}$. Thus, for any $\mathcal{P}$, $V(\mathcal{P}) = \min_{\mathcal{Q}} U_{y_{\mathcal{P}}}^{\mathcal{Q}}$, which completes the necessity direction of the proof.

As for sufficiency, suppose that V is the minimum of finitely many values of one-player decision making problems, $D_1, .., D_n$. That is, for any partition $\mathcal{P}$, if $U_i(\mathcal{P})$ denotes the value of D_i when the information is induced by $\mathcal{P}$, then $V(\mathcal{P}) = \min_{1 \leq i \leq n} U_i(\mathcal{P})$. We need to show a zero-sum game whose value is V.

Theorem 4.4 in Gilboa and Lehrer (1991) implies that without loss of generality all problems $D_1, .., D_n$ share the same underlying probability p over K.

Consider the following multi-stage game, G. Player 2 chooses a whole number from $1, ..., n$, say r. Then a state k is drawn according to p, player 1 is informed of the cell containing this state, and finally, player 1 takes an action, say a. The payoff of player 1 is the payoff that corresponds to the action a and the state k in the decision problem D_r.

Note that for any partition $\mathcal{P}$, the value of G when the information is induced by $\mathcal{P}$ is $\min_{1 \leq i \leq n} U_i(\mathcal{P})$. Thus, the value of information of G coincides with V, as desired. ■

Definition 9 *Let $\mathcal{F}$ be an algebra of subsets of K. That is, $\mathcal{F}$ consists of subsets of K and is closed under unions and intersections. Let v be a real function defined over $\mathcal{F}$. We say that the* anti-core of $(v, \mathcal{F})$ is not empty, *if for every $A \in \mathcal{F}$ there is a vector $x_A \in I\!R^{|K|}$ such that $x_A(A) = v(A)$ and $x_A(B) \leq v(B)$ for every $B \subseteq A$ such that $B \in \mathcal{F}$.*

Remark 1 *Suppose that $\mathcal{F}$ is the set of all subsets of K. The anti-core of $(v, \mathcal{F})$ is not empty implies that the B-anti-core of v is not empty for every $B \subseteq K$.*

Lemma 1 *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two algebras of subsets of K such that $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Assume that the anti-core of $(v, \mathcal{F}_1)$ is not empty. Then, for every set of constants c_B, $B \in \mathcal{F}_2 \setminus \mathcal{F}_1$, there is u defined on $\mathcal{F}_2$ such that*

(a) *it coincides with v on $\mathcal{F}_1$;*

(b) *it satisfies $u(S) \geq c_S$ for every $S \in \mathcal{F}_2 \setminus \mathcal{F}_1$; and*

(c) *the anti-core of $(u, \mathcal{F}_2)$ is not empty.*

Proof. Suppose that the algebra $\mathcal{G}_2$ refines the algebra $\mathcal{G}_1$. We say that $\mathcal{G}_2$ is generated from $\mathcal{G}_1$ by splitting an atom of $\mathcal{G}_1$ into two subsets, if there is an atom A of $\mathcal{G}_1$, and a partition of A into two subsets B and B' that belong to $\mathcal{G}_2$, such that any set $C \in \mathcal{G}_2$ can be written as $C = C_1 \cup C_2$ with $C_1 \in \mathcal{G}_1$ and $C_2 \in \{B, B', \emptyset\}$.

Without loss of generality we can assume that $\mathcal{F}_2$ is generated from $\mathcal{F}_1$ by splitting an atom of $\mathcal{F}_1$ into two subsets. This is so because when $\mathcal{F}_2$ refines $\mathcal{F}_1$, any cell of $\mathcal{F}_1$ is a union of cells of $\mathcal{F}_2$. Thus, by finitely many successive splits of sets into two subsets one can generate $\mathcal{F}_2$ from $\mathcal{F}_1$. Therefore, if the lemma is proven for any two algebras such that the first is generated from the second by splitting an atom into two sets, one can apply it successively and obtain the desired result.

Let $B \in \mathcal{F}_2 \setminus \mathcal{F}_1$ be a set that does not contain any set from $\mathcal{F}_1$. That is, B is a proper subset of some $A \in \mathcal{F}_1$ (i.e., B is a result of splitting A into two subsets). Thus, the sets of $\mathcal{F}_2$ are of the type $D \cup E$, where $D \in \{B, A \setminus B, \emptyset\}$ and $E \in \mathcal{F}_1$.

Since by assumption the anti-core of $(v, \mathcal{F}_1)$ is not empty, for every $S \in \mathcal{F}_1$ there is a vector x_S that satisfies the conditions described in Definition 9. For $D = B, A \setminus B$, set $d_D = \max_{S;\ D \subseteq S \text{ and } S \in \mathcal{F}_1} x_S(D)$ and let b_D be a number greater than d_D. Define u as follows: u coincides with v on $\mathcal{F}_1$; $u(D) = b_D$ for $D = B, A \setminus B$; and finally, for $D \cup E$, where $D = B, A \setminus B$ and $E \in \mathcal{F}_1$, $u(D \cup E) = u(D) + u(E)$.

Note that if the numbers b_D, $D = B, A \setminus B$, are large enough, then $u(S) \geq c_S$ for every $S \in \mathcal{F}_2 \setminus \mathcal{F}_1$, as desired. It remains to show that the anti-core of $(u, \mathcal{F}_2)$ is not empty.

Fix $S \in \mathcal{F}_1$. If $C \subseteq S$ and $C \in \mathcal{F}_1$, then $u(C) = v(C) \geq x_S(C)$. If, however, $C \subseteq S$ and $C \notin \mathcal{F}_1$, then $C = D \cup E$, where $D = B, A \setminus B$ and $E \in \mathcal{F}_1$. By the definitions of u and d_D and since $b_D > d_D$, $u(C) = u(D) + u(E) > x_S(D) + x_S(E) = x_S(C)$.

Now fix $S \in \mathcal{F}_2 \setminus \mathcal{F}_1$. $S = D \cup E$, where $D = B, A \setminus B$ and $E \in \mathcal{F}_1$. Define the vector $x_S = (x_S(1), ..., x_S(|K|))$ as follows: if $k \in E$, then $x_S(k) = x_E(k)$, and for $k \in D$, the coordinates $x_S(k)$ are defined so as to satisfy $x_S(D) = u(D)$.

Let $D \cup C \subseteq S$, where $C \subseteq E$ is in $\mathcal{F}_1$. Then, $x_S(D \cup C) = x_S(D) + x_S(C) \geq u(D) + v(C) = u(D) + u(C) = u(D \cup C)$ which completes the proof that the anti-core of $(u, \mathcal{F}_2)$ is not empty. ■

Notation 2 *Denote by $\mathcal{A}(\mathcal{P})$ the algebra generated by a partition $\mathcal{P}$.*

Proof of Theorem 2. Let V be a monotonic function defined over the set of partitions of a set K. We prove that it is a value-of-information function. By Proposition 2 it is sufficient to show: (i) there are $v_1, ..., v_n$, where the B-anti-core of v_i is non-empty for any $B \subset K$, for $i = 1, ..., n$; and (ii) for any partition $\mathcal{P}$, $V(\mathcal{P}) = \min_i \sum_{A \in \mathcal{P}} v_i(A)$.

For any partition $\mathcal{P}$ we will find $v_{\mathcal{P}}$ that has three properties: (i) its B-anti-core is non empty for every $B \subseteq K$; (ii) $V(\mathcal{P}) = \sum_{A \in \mathcal{P}} v_{\mathcal{P}}(A)$; and (iii) $\sum_{A \in \mathcal{P}} v_{\mathcal{P}}(A) \leq \sum_{A \in \mathcal{P}} v_{\mathcal{Q}}(A)$ for any partition $\mathcal{Q}$. This will imply the result.

Fix a partition $\mathcal{P}$ and define $v_{\mathcal{P}}(A)$ for $A \in \mathcal{P}$ so that $\sum_{A \in \mathcal{P}} v_{\mathcal{P}}(A) = V(\mathcal{P})$. Thus, property (ii) is readily satisfied. Extend the definition of $v_{\mathcal{P}}$ to $\mathcal{A}(\mathcal{P})$ in a linear fashion. Note that this can be done in a unique way since any element of $\mathcal{A}(\mathcal{P})$ can be written uniquely as a union of cells of $\mathcal{P}$. Moreover, if $\mathcal{P}$ refines $\mathcal{Q}$, then $V(\mathcal{P}) = \sum_{A\in\mathcal{P}} v_{\mathcal{P}}(A) = \sum_{A\in\mathcal{Q}} v_{\mathcal{P}}(A)$. By monotonicity of V, $V(\mathcal{Q}) \le V(\mathcal{P}) = \sum_{A\in\mathcal{Q}} v_{\mathcal{P}}(A)$.

We claim that since $v_{\mathcal{P}}$ is linear on $\mathcal{A}(\mathcal{P})$, the anti-core of $(v_{\mathcal{P}}, \mathcal{A}(\mathcal{P}))$ is not empty. In order to justify this claim, fix $A \in \mathcal{P}$ and let x_A be any $|K|$ dimensional vector with two properties. First, the support of x_A is A (i.e., all the coordinates out of A are equal to zero); and second, $x_A(A) = v_{\mathcal{P}}(A)$. Define the vector $x = \sum_{A\in\mathcal{P}} x_A$. Note that for any $B \in \mathcal{A}(\mathcal{P})$, $x(B) = \sum_{k\in B}\sum_{A\in\mathcal{P}} x_A(k) = \sum_{A\in\mathcal{P}} x_A(A\cap B)$. Since $\mathcal{P}$ is a partition, $\sum_{A\in\mathcal{P}} x_A(A\cap B) = \sum_{A\in\mathcal{P} \text{ and } A\subseteq B} x_A(A)$ and therefore, $x(B) = \sum_{A\in\mathcal{P} \text{ and } A\subseteq B} x_A(A)$.

Since $x_A(A) = v_{\mathcal{P}}(A)$, we obtain $x(B) = \sum_{A\in\mathcal{P} \text{ and } A\subseteq B} v_{\mathcal{P}}(A)$. Due to the linearity of $v_{\mathcal{P}}$ on $\mathcal{A}(\mathcal{P})$, $\sum_{A\in\mathcal{P} \text{ and } A\subseteq B} v_{\mathcal{P}}(A) = v_{\mathcal{P}}(B)$. Thus, for any $B \in \mathcal{A}(\mathcal{P})$, x satisfies $x(B) = v_{\mathcal{P}}(B)$ and the anti-core of $(v_{\mathcal{P}}, \mathcal{A}(\mathcal{P}))$ is not empty.

We now extend $v_{\mathcal{P}}$ to all the subsets of K. The extension is defined so that on every partition $\mathcal{Q}$ it satisfies the linear inequality $\sum_{A\in\mathcal{Q}} v_{\mathcal{P}}(A) \ge V(\mathcal{Q})$. Let $c_A = v_{\mathcal{P}}(A)$ for every $A \in \mathcal{A}(\mathcal{P})$. Consider the following set of linear inequalities with the variables c_A, $A \subseteq K$ and $A \notin \mathcal{A}(\mathcal{P})$: $\sum_{A\in\mathcal{Q}} c_A \ge V(\mathcal{Q})$, for every partition $\mathcal{Q}$. Thus, every partition $\mathcal{Q}$ induces one inequality. This is a set of inequalities of the type "greater than or equal to". Moreover, the coefficients are either 0 or 1. Such a system has a solution. Furthermore, if $(c_A)_{A\subseteq K \text{ and } A\notin\mathcal{A}(\mathcal{P})}$ is a solution, then $(c_A + f_A)_{A\subseteq K \text{ and } A\notin\mathcal{A}(\mathcal{P})}$ is also a solution, whenever $f_A \ge 0$.

Now fix a solution (c_A) of the previous system of inequalities. Employing Lemma 1 with $\mathcal{F}_1 = \mathcal{A}(\mathcal{P})$, and $\mathcal{F}_2$ equals the set of all subsets of K, we obtain the extension of $v_{\mathcal{P}}$. This extension satisfies $v_{\mathcal{P}}(A) \ge c_A$ for every $A \subseteq K$. Therefore, $\sum_{A\in\mathcal{Q}} v_{\mathcal{P}}(A) \ge \sum_{A\in\mathcal{Q}} c_A \ge V(\mathcal{Q})$ for every partition $\mathcal{Q}$. Thus, $\sum_{A\in\mathcal{Q}} v_{\mathcal{P}}(A) \ge V(\mathcal{Q}) = \sum_{A\in\mathcal{Q}} v_{\mathcal{Q}}(A)$ for every $\mathcal{Q}$, which is requirement (iii). Finally, by Lemma 1, the anti-core of $(v_{\mathcal{P}}, \mathcal{F}_2)$ is non-empty, which is property (i). Thus, by Remark 1 it completes the proof that a monotonic function is a value-of-information function.

We prove now the inverse direction: if V is a value-of-information function it has to be monotonic. We claim that if $\mathcal{P}$ is a refinement of $\mathcal{Q}$, then in a one-sided information game induced by $\mathcal{P}$ player 1 has more strategies than in the game with information induced by $\mathcal{Q}$. For any strategy τ of player 1 in the game with information structure $\mathcal{Q}$ denote by $\tau(B)$ the mixed action prescribed by τ when the state chosen is in $B \in \mathcal{P}$. Define the strategy τ' of player 1 in the game with the information structure $\mathcal{P}$: when the state chosen is in $C \in \mathcal{P}$, where $C \subseteq B \in \mathcal{P}$, play $\tau(B)$. Since the set of strategies of player 2 is the same under both information structures, τ' guarantees at least $V(\mathcal{Q})$ in the game with information structure $\mathcal{P}$. This proves the monotonicity of V. ∎

6 Related literature

Most of the existing literature that relates to the role of information in interactive models compares different information structures. Blackwell (1951, 1953) initiated this trend. He dealt with one-player decision problems and information structures that provide a random signal whose distribution depends on the realized state of nature. Blackwell characterized when one information structure always provides at least as high a payoff as another information structure.

More information is always preferred by a rational decision-maker only when the underlying measures are countably additive. Kadaneet al. (1996) proved that this is not the case when the measures are finitely additive. Wakker (1988), Schlee (1990, 1991), Safra and Sulganik (1995), Chassagnon and Vergnaud (1999) showed that non expected utility maximizers may prefer not to be informed. In the case where the action set of the decision-maker depends on the state, Sulganik and Zilcha (1997) showed that more information does not always increase the expected payoff.

Lehmann (1988) compared information structures in decision problems with a restricted set of payoff functions and priors. The payoff functions he examined have the single crossing property and the priors have the monotone likelihood ratio property. Athey and Levin (1998) and Persico (2000) also analyzed restricted sets of decision problems.

In a non-zero-sum interactive context, players might prefer dropping payoff-relevant information. This might happen when the equilibrium payoffs of the better informed player are lower than her equilibrium payoffs before receiving the additional information. Hirshleifer (1971) showed that in economic situations additional information does not necessarily imply greater payoffs for the agent. In the games analyzed by Kamien et al. (1990a, 1990b) players receive signals from an outside agent. They showed that the impact of more informative messages on equilibrium payoffs is sometimes strictly negative. Bassan et al. (2003) introduced conditions that guarantee that getting more information always improves all players' payoffs. Neyman (1991) pointed out that a player might prefer not receiving information because other players would know that he was receiving this information.

Gossner and Mertens (2001) compared different information structures in zero-sum games, and Lehrer and Rosenberg (2003) compared them for in long-run repeated zero-sum games. Gossner (2003) showed that in the case where a player has more strategies in one game than in another can be interpreted as having more information.

7 Final remarks

7.1 Non-zero-sum games

In this paper we characterize the functions that are value-of-information functions for zero-sum games. In the non-zero-sum case, one could define the value-of-information correspondence that associates the set of corresponding Nash equilibrium payoffs with each information structure. Even the relatively easy problem of characterizing the set of value-of-information correspondences for symmetric or one-sided information structures is an open problem.

7.2 Games with two-sided information

In this work we focus on the two polar cases of symmetric and one-sided information. It would be interesting to characterize the functions V of pairs of partitions $(\mathcal{P}_1, \mathcal{P}_2)$ for which there is a p, sets of actions, and a payoff function g such that $V(\mathcal{P}_1, \mathcal{P}_2) = v^{(\mathcal{P}_1, \mathcal{P}_2)}(p, g)$.

As previously noted, when the information of one player becomes finer while the other's remains unchanged, the equilibrium payoff should increase. Therefore, $V(\mathcal{P}_1, \mathcal{P}_2)$ ought to be monotonically increasing in the first argument and monotonically decreasing in the second.

Define $CK(\mathcal{P}_1, \mathcal{P}_2)$ to be the coarsening of $\mathcal{P}_1$ and $\mathcal{P}_2$. That is, $CK(\mathcal{P}_1, \mathcal{P}_2)$ is the finest partition whose cells can be written as a union of cells of $\mathcal{P}_i$, $i = 1, 2$. Any cell of $\mathcal{P}_i$ is a subset of some cell of $CK(\mathcal{P}_1, \mathcal{P}_2)$. $CK(\mathcal{P}_1, \mathcal{P}_2)$ is the partition to common knowledge components (see Aumann, 1976).

We claim that $V(\mathcal{P}_1, \mathcal{P}_2)$ should be separably additive on the cells of $CK(\mathcal{P}_1, \mathcal{P}_2)$. Indeed, for any set $A \subseteq K$ and for any partition $\mathcal{P}$ denote by $\mathcal{P}|A$ the partition induced by $\mathcal{P}$ on A. When $k \in A \in CK(\mathcal{P}_1, \mathcal{P}_2)$, it is common knowledge among the players that a state in A has been realized. In this case the actual Bayesian game is the original Bayesian game restricted to A. In this game the information structure is $(\mathcal{P}_1|A, \mathcal{P}_2|A)$.

Denote the equilibrium payoff of this game by $v(A, \mathcal{P}_1|A, \mathcal{P}_2|A)$. With this notation $V(\mathcal{P}_1, \mathcal{P}_2) = \sum_{A \in CK(\mathcal{P}_1, \mathcal{P}_2)} p(A) v(A, \mathcal{P}_1|A, \mathcal{P}_2|A)$, where p is the prior over K. Thus, V is separably additive over the partition to common knowledge components.

Fully characterizing the value-of-information functions in the case of two-sided information is an open problem.

7.3 General information structures

In this paper we restricted ourselves to games in which the information structure is defined by a pair of partitions. One could more generally define the value of information as a function of general information structures (namely functions from K to probability distributions over a finite set of signals) and ask which functions are value-of-information functions.

7.4 On one-shot incomplete information zero-sum games

Mertens and Zamir (1971) showed that the values of one-shot incomplete information zero-sum games with state space K form a lattice and is closed under addition. It is therefore dense in the set of all continuous functions over $\Delta(K)$. This is not sufficient for our purposes. The proof of Theorem 1 requires that for any finite list of pairs $(x_\ell, y_\ell) \in \Delta(K) \times I\!R$, $\ell = 1, ..., L$, there exists a Bayesian zero-sum game whose equilibrium payoff when the prior is x_ℓ is **precisely** (not merely approximately) y_ℓ, $\ell = 1, ..., L$.

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