Evaluating information in zero-sum games with incomplete information on both sides

Ehud Lehrer* and Dinah Rosenberg†‡

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Abstract

In a Bayesian game some players might receive a signal regarding the specific game actually played before it starts. We study zero-sum games where each player receives a partial information about his own type and no information about that of the other player, and analyze the impact the signals have on the payoffs. It turns out that the functions that evaluate the value of information share *only* one property across all games: monotonicity. That is, the payoff increases with the amount of information the maximizing player obtains and decreases with the amount of information the minimizing player obtains.

*School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel; e-mail: lehrer@post.tau.ac.il; home page: www.math.tau.ac.il/∼lehrer

†LAGA Institut Galilée, Université Paris 13, avenue Jean Baptiste Clément, 93430 Villetaneuse, France, and Laboratoire d'Econométrie de l'Ecole Polytechnique, Paris France; e-mail: dinah@math.univ-paris13.fr

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1 Introduction

In strategic interaction some relevant aspects of the environment might be imperfectly known to players. Such situations are classically modeled as games with incomplete information. When the players are allowed to seek additional information or to buy it at some cost, the question arises as to what impact different information structures have on the result of the interaction. This paper contributes to the study of this question.

This problem is modeled by a game with incomplete information in which the payoffs depend not only on the players' actions but also on a random parameter. Before the game starts each player obtains a partial information about the realized parameter. This information is determined by an information structure that specifies how the signals that the players receive stochastically depend on the realized parameter.

In the most general framework there are several reasons why evaluating the impact of the information structure on the outcome of the game is not an easy task. First, the interpretation of 'outcome' depends on the solution concept applied. Second, for most solution concepts there are typically multiple outcomes (equilibria). Finally, players might get correlated information, which typically has a significant effect on the outcome. However, when restricting attention to zero-sum games, there is one natural solution concept, the value, and it is unique. We further reduce the difficulty of the problem by focusing on a case where the information the players get is independent and deterministic.

There are two main approaches to analyze the connection between information and payoffs. The first is to compare two information structures and find which is payoff-wise better than the other. This direction has been widely studied in the literature. Blackwell (1953) compared information structures in one-player decision problems. He proved that an information structure always yields a higher value of the problem than another one if and only if it is more informative in a certain sense. When restricted to deterministic information structures, namely to partitions of the state space[1], this property can be restated as follows: a partition $\mathcal{P}$ is more informative than

[1]A partition represents an information structure in the following sense: when a state k is chosen the agents gets to know only the cell of the partition to which k belongs. More generally if a state k is chosen an information structure sends to the agent a signal that may depend on k. Any

$\mathcal{Q}$ if $\mathcal{P}$ refines $\mathcal{Q}$. Blackwell's result was extended to zero-sum games (and random signals) by Gossner and Mertens (2001). It is well known that this property does not extend to zero-sum games and various attempts have been made to understand when it might be extended (see for instance Hirshleifer (1971), Bassan, et al. (2003), Kamien, et al. (1990), Neyman (1991), Lehrer, et al. (2006)).

The second approach is to study the impact of an information structure on the outcome of the interaction. A typical question in this line is whether the outcome depends in any particular and discernable way on the information. Since each specific game might have its own idiosyncrasies, an insight into the subject can be obtained only by looking at all possible interactions.

For a given game we define its value-of-information function that associates any information structure with the value of the corresponding Bayesian game. Such a function is called a value-of-information function. We study the properties that are common to *all* such functions and thereby, the connection between information and outcomes that is not specific to a particular game.

Gilboa and Lehrer (1991) treated deterministic information structures. They characterized those functions that are value-of-information functions of *one-player* decision problems. Blackwell's (1953) result implies that monotonicity is a necessary condition but it turns out not to be sufficient. The result of Gilboa and Lehrer has been extended to random information structures by Azrieli and Lehrer (2004).

Lehrer and Rosenberg (2006) studied the nature of value-of-information functions in two cases: (a) one-sided information: the game depends on a state of nature which is partially known only to player 1 (i.e., player 2 gets no information); and (b) symmetric information: the game depends on a state of nature which is known equally to both players. In case (a) the more refined the partition the higher the value. Lehrer and Rosenberg (2006) showed that *any* function defined over partitions which is increasing with respect to refinements is a value-of-information function of some game with incomplete information on one side.

This paper extends the discussions to the case where both players get partial information on the state of nature. We focus on zero-sum games with lack of information on both sides and independent deterministic information. The type k of player 1 and the type l of player 2 are drawn independently of each other. Each player obtains no information about the other player's type and just partial deterministic information

information structure in which the signal is a deterministic function of the state can be represented by a partition.

about his own: a player gets to know the set of types that contains his own type. This special case has been extensively studied in the context of repeated games with incomplete information (see Aumann and Maschler (1995), Mertens and Zamir (1971, 1980)).

Formally, the information structure is characterized by a pair of partitions: $\mathcal{P}$ a partition of K, and $\mathcal{Q}$ a partition of L. When k is selected to be the type of player 1, he is informed of the cell of $\mathcal{P}$ that contains k. Similarly, when l is selected to be the type of player 2, he is informed of the cell of $\mathcal{Q}$ that contains l. Then, a zero-sum game whose payoffs depend on the players' types is played. The value of this game depends on the information structure, $(\mathcal{P}, \mathcal{Q})$. The function that associates to every pair of partitions their corresponding value is called the value-of-information function of this game. The goal of this paper is to characterize those functions that are a value-of-information function of some game.

The result concerning games with one-sided information is certainly relevant to the current two-sided information case. A value-of-information function defined over pairs $(\mathcal{P}, \mathcal{Q})$ ought to be increasing with respect to refinement in $\mathcal{P}$ and decreasing with respect to refinement with $\mathcal{Q}$. The question arises as to whether this condition is sufficient for a function defined over pairs of partitions to be a value-of-information-function of some zero-sum game. We answer this question in the affirmative.

The implication of this result is that essentially no further condition beyond monotonicity is required to characterize the value-of-information functions. This means that the model of Bayesian game with varying information structures can be refuted only by observations that contradict monotonicity.

The main key of the proof is the duality between payoffs and information. This means that giving more information to a player amounts to giving him more payoffs in some sense (see Gossner (2006)). The proof method uses a duality technique inspired by Fenchel conjugate of convex functions (see, Rockafellar 1970). The duality technique has been introduced to games with incomplete information by De Meyer and widely investigated since then (see De Meyer (1996), De Meyer and Rosenberg (1999), De Meyer and Marino (2005), De Meyer and Moussa-Saley (2003), Laraki (2002), Sorin (2002)). For any game with incomplete information on one side, he defined a dual game for which the value is the Fenchel conjugate of the value of the initial game.

The paper is organized as follows. In Section 2 we present the model and value-of-information functions and the main result. Section 3 introduces the techniques of duality that are used in the proof, and in Section 4 we prove the result.

2 The model and the main result

2.1 Games with incomplete information on both sides

A state (k, l) is randomly drawn from a finite set $K \times L$ according to a product distribution $p \otimes q$. The players get a partial information about (k, l) through information structures. Player 1's information structure is given by a partition $\mathcal{P}$ of K and that of player 2 is given by a partition $\mathcal{Q}$ of L. When (k, l) is selected, player 1 is informed of the atom of $\mathcal{P}$ that contains k and player 2 is informed of the atom of $\mathcal{Q}$ that contains l. Note that player 1 gets no information about l and player 2 gets no information about k.

Upon getting the partial information about the state selected, the players take actions. Player 1's set of action is denoted by A and that of player 2 is denoted by B. The payoff function is $g : K \times L \times A \times B \to R$. Instead of $g(k, l, a, b)$ we denote by $g_{k,l}(a, b)$ the payoff when the state is (k, l), player 1 plays a and player 2 plays b. Player 1 tries to maximize the expected payoff and player 2 to minimize it.

A pure strategy of player 1 (resp. player 2) is a function from $\mathcal{P}$ (resp. $\mathcal{Q}$) to A (resp. B): players choose their action as a function of the information they get, i.e. the atom of the partition in which k (resp. l) is known to be.

This game is a finite game and has a value in mixed strategies denoted by $V^g_{p,q}(\mathcal{P}, \mathcal{Q})$. We will focus here on how $V^g_{p,q}(\mathcal{P}, \mathcal{Q})$ depends on $\mathcal{P}$ and $\mathcal{Q}$, and characterize the properties of this function that do not depend on a specific choice of g, p, q.

2.2 Value-of-information function

The set of partitions of K is denoted by $\mathcal{K}$ and the set of partitions on L is denoted by $\mathcal{L}$.

Definition 1 *A* value-of-information function *of games with incomplete information on both sides and independent information is a function $\boldsymbol{V}$ from $\mathcal{K} \times \mathcal{L}$ to $\mathbb{R}$ such that there exists a product distribution $p \otimes q$ over $K \times L$, action sets A and B and a payoff function g defined on $K \times L \times A \times B$, such that for every $\mathcal{P} \in \mathcal{K}$, $\mathcal{Q} \in \mathcal{L}$,*

$$V^g_{p,q}(\mathcal{P}, \mathcal{Q}) = \boldsymbol{V}(\mathcal{P}, \mathcal{Q}).$$

In this paper we characterize those functions defined over $\mathcal{K} \times \mathcal{L}$ that are value-of-information functions.

2.3 The main result

We first need the following definition.

Definition 2 *1. Partition $\mathcal{P}$ refines partition $\mathcal{P}'$ if any atom of $\mathcal{P}'$ is a union of atoms of $\mathcal{P}$.*
2. A function V from $\mathcal{K} \times \mathcal{L}$ to $\mathbb{R}$ is increasing (resp. decreasing) in $\mathcal{P}$ (resp. $\mathcal{Q}$) if for any $\mathcal{P}$, $\mathcal{P}'$ in $\mathcal{K}$, such that $\mathcal{P}$ refines $\mathcal{P}'$, and any $\mathcal{Q}$ in $\mathcal{L}$, $V(\mathcal{P}, \mathcal{Q}) \geq V(\mathcal{P}', \mathcal{Q})$ (resp. for any $\mathcal{P} \in \mathcal{K}$ and $\mathcal{Q}, \mathcal{Q}' \in \mathcal{L}$ such that $\mathcal{Q}$ refines $\mathcal{Q}'$, $V(\mathcal{P}, \mathcal{Q}) \leq V(\mathcal{P}, \mathcal{Q}')$).

Lehrer and Rosenberg (2006) proved the following result.

Proposition 1 *Assume that L is a singleton. A function V defined over $\mathcal{K}$ is a value-of-information function iff it is increasing in $\mathcal{P}$.*

The main result of the present paper is:

Theorem 1 *A function is a value-of-information function of a game with incomplete information on both sides and independent information iff it is increasing in $\mathcal{P}$ and decreasing in $\mathcal{Q}$.*

As in Lehrer and Rosenberg (2006) this result proves that no property but the well-known monotonicity property is common to all value of information functions. Therefore if one models a situation by a Bayesian game and gets to observe the result of the interaction for various information functions there is essentially no behavior of the value as a function of information that may disqualify the Baysian model.

By the result of Lehrer and Rosenberg (2006) monotonicity is clearly a necessary condition for a function to be a value of information function. Effort will be devoted in the sequel to proving it is also sufficient.

3 Duality

This section is devoted to duality which is the main tool in the proof of Theorem 1.

Definition 3 *A real function defined over $\mathcal{L}$ is a cost function if it is increasing. The set of cost functions is denoted by $\mathcal{C}$.*

For every function $V : \mathcal{K} \times \mathcal{L} \to \mathbb{R}$ which is increasing $\mathcal{P}$ and decreasing in $\mathcal{Q}$, we define a dual function W as follows. This definition is reminiscent of the Fenchel conjugation for concave functions.

For every partition $\mathcal{P}$ of K and every cost function c,

$$W(\mathcal{P}, c) = \min_{\mathcal{Q} \in \mathcal{L}} \left[V(\mathcal{P}, \mathcal{Q}) + c(\mathcal{Q}) \right].$$

Following the ideas of De Meyer (1996) we now define a game called the dual game for which the value is given by W. Suppose that V is a value-of-information function, and $G(\mathcal{P}, \mathcal{Q})$ is the game whose value is $V(\mathcal{P}, \mathcal{Q})$. Furthermore, suppose that c is a cost function. In the dual game there are two stages. First, player 2 can buy an information structure $\mathcal{Q}$ at the cost of $c(\mathcal{Q})$ and this choice is publicly observed. Then both players take part in the game $G(\mathcal{P}, \mathcal{Q})$ and get the corresponding payoffs. Therefore the overall payoff in the dual game is the sum of the payoff in the original game that has been played and the cost of the information structure $\mathcal{Q}$ that has been chosen. The idea is that on the one hand it is always better for player 2 to get more information but on the other hand since this information is costly, depending on the cost he might choose to buy a less informative structure.

W is the value of the dual game.

Remark 2 Since V is increasing in its first argument, so is W.

The following duality lemma is essential for the proof of Theorem 1.

Lemma 1 *If V is increasing in $\mathcal{P}$ and decreasing in $\mathcal{Q}$, then*

$$V(\mathcal{P}, \mathcal{Q}) = \max_{c \in \mathcal{C}} \left[W(\mathcal{P}, c) - c(\mathcal{Q}) \right]. \tag{1}$$

Proof. By definition of W it is clear that for all c and all $\mathcal{Q}$,

$$V(\mathcal{P}, \mathcal{Q}) \geq W(\mathcal{P}, c) - c(\mathcal{Q}).$$

Therefore, what needs to be proved is that for all $\mathcal{P}$, there exists an increasing cost function $c_{\mathcal{P}}$ such that

$$V(\mathcal{P}, \mathcal{Q}) \leq W(\mathcal{P}, c_{\mathcal{P}}) - c_{\mathcal{P}}(\mathcal{Q}).$$

Let $c_{\mathcal{P}}(\mathcal{Q}) := V(\mathcal{P},\mathcal{T}) - V(\mathcal{P},\mathcal{Q})$, where $\mathcal{T}$ is the trivial partition. This function is indeed increasing in $\mathcal{Q}$. Moreover

$$\begin{aligned}
W(\mathcal{P},c_{\mathcal{P}}) - c_{\mathcal{P}}(\mathcal{Q}) &= \min_{\mathcal{Q}'\in\mathcal{L}} V(\mathcal{P},\mathcal{Q}') + c_{\mathcal{P}}(\mathcal{Q}') - c_{\mathcal{P}}(\mathcal{Q}) \\
&= V(\mathcal{P},\mathcal{Q}^*) + V(\mathcal{P},\mathcal{T}) - V(\mathcal{P},\mathcal{Q}^*) - V(\mathcal{P},\mathcal{T}) + V(\mathcal{P},\mathcal{Q}) \\
&= V(\mathcal{P},\mathcal{Q}),
\end{aligned}$$

where $\mathcal{Q}^*$ is the partition that attains the minimum. This is the desired result. ■

Notation: For every pair $\mathcal{P},\mathcal{Q}$ there is a cost function $c_{\mathcal{P},\mathcal{Q}}$ that achieves the maximum on the right-hand side of eq. (2). For a fixed function V, set $\mathcal{C}_V = \{c_{\mathcal{P},\mathcal{Q}};\ \mathcal{P},\mathcal{Q} \in \mathcal{K}\times\mathcal{L}\}$. That is, $\mathcal{C}_V$ is the set that contains all the cost functions $c_{\mathcal{P},\mathcal{Q}}$. Note that $\mathcal{C}_V$ is finite and that Lemma 1 implies that

$$V(\mathcal{P},\mathcal{Q}) = \max_{c\in\mathcal{C}_V} \left[W(\mathcal{P},c) - c(\mathcal{Q})\right].$$

4 Proof of Theorem 1

The result of Lehrer and Rosenberg (2006) implies that since $\mathcal{P} \mapsto W(\mathcal{P},c)$ is increasing, it is a value-of-information function of a game, denoted G_c, with lack of information on one side, state space K in which player 1 is informed. Similarly, $-c(\cdot)$ is a value-of-information function of a game, denoted Γ_c, with lack of information on one side, state space L, in which player 2 is informed.

The main idea of Theorem 1's proof is to define a bidual game $\mathcal{G}$ for which the value will be given by V. This game is played in two stages. Before the game starts both players are informed about the selected state through their respective partitions. Then player 1 first chooses a cost function $c \in \mathcal{C}_V$, this choice is publicly observed. Finally both G_c and Γ_c are played simultaneously. The payoff is the sum of the payoffs obtained in the two games.

If player 1 chooses a cost function c independently of his information, then by observing the selected c player 2 gets no information about the state k. In such a case player 1 guarantees the payoff $W(\mathcal{P},c)$ in G_c and the payoff $-c(\mathcal{Q})$ in Γ_c. The sum is $W(\mathcal{P},c) - c(\mathcal{Q})$. If player 1 plays optimally he maximizes $W(\mathcal{P},c) - c(\mathcal{Q})$ and by Lemma 1 the value is at least $V(\mathcal{P},\mathcal{Q})$.

We will show that an optimal c is indeed independent of player 1's information. In other words, although player 1 can choose the cost function in a way that may depend on his information, he will choose c independently of it. This will prove that the value of the game $\mathcal{G}$ is exactly $V(\mathcal{P}, \mathcal{Q})$.

If player 1 uses his information when choosing c, the conditional probability over k given the observed choice of c will be some π_c, typically different from p. Therefore, in G_c, the value is no longer $W(\mathcal{P}, c)$. Rather, the value is that of the game G_c in which k is initially drawn according to π_c which can be denoted by $W^{\pi_c}(\mathcal{P}, c)$. All these possibilities should be taken into account when analyzing the choice of player 1.

4.1 One sided information

The following lemma shows that in one-sided information games the probability p can be taken to be the uniform probability.

Lemma 2 *Let U be a value-of-information function of a game G in which k is drawn with probability π. Then, there is a game G' whose value-of-information function coincides with V and, moreover, k is drawn according to the uniform distribution.*

Proof. Let A, B and g be the action sets and payoff function of G. Define G' with the same action sets and payoff $g'_{kl}(a, b) = \frac{g_{kl}(a,c)\pi(k)}{p_k}$. For any information partitions and any strategies of the players, the expected payoff in G where k, l are drawn according to probability $\pi \otimes q$ is the same as the expected payoff in G' where k, l are drawn according to $p \otimes q$. ∎

4.2 Auxiliary games

Fix V which is increasing in the first argument and decreasing in the second. We employ Lehrer and Rosenberg (2006) and define some auxiliary games which will be needed for the proof of Theorem 1.

Theorem 2 in Lehrer and Rosenberg (2006) ensures that for each $\mathcal{Q}$ there is a $p_{\mathcal{Q}}$ and a game $\mathcal{G}_{\mathcal{Q}}$ such that $V(\mathcal{P}, \mathcal{Q})$ is the value of $\mathcal{G}_{\mathcal{Q}}$ where player 1 is informed about the chosen k through $\mathcal{P}$, and player 2 gets no further information. Lemma 2 ensures that the probability $p_{\mathcal{Q}}$ with respect to which the state k is chosen in $\mathcal{G}_{\mathcal{Q}}$ can be taken to be the uniform distribution over K.

Note that in this game player 2 gets no information at all and $\mathcal{Q}$ is just an abstract parameter of the game, it does not model an information structure.

In what follows we need other distributions than the uniform one. Fix a distribution π over K. Let $V^\pi(\mathcal{P},\mathcal{Q})$ be the value of the game $\mathcal{G}^\pi_\mathcal{Q}$ which coincides with $\mathcal{G}_\mathcal{Q}$ in all respects except for one, the initial distribution: the state k is drawn in $\mathcal{G}^\pi_\mathcal{Q}$ according to π.

Define $W^\pi(\mathcal{P},c)$ to be the conjugate function of $V^\pi(\mathcal{P},\mathcal{Q})$:

$$W^\pi(\mathcal{P},c) = \min_{\mathcal{Q}\in\mathcal{L}} \left[V^\pi(\mathcal{P},\mathcal{Q}) + c(\mathcal{Q})\right]. \tag{2}$$

For a cost function c consider the following game, denoted $\mathcal{G}^*_c(\pi,\mathcal{P})$. A state k is drawn according to the distribution π, player 1 is informed according to the partition $\mathcal{P}$ and player 2 chooses a partition $\mathcal{Q}$. Then, $\mathcal{G}^\pi_\mathcal{Q}$ is played and $c(\mathcal{Q})$ is added to its payoff.

Plainly, for *any* initial probability π and *any* information partition $\mathcal{P}$, the value of this game is $W^\pi(\mathcal{P},c)$. Denote by A_c , B_c and g_c the action sets and the payoff function of $\mathcal{G}^*_c(\pi,\mathcal{P})$.

We turn to the definition of a second auxiliary game. Suppose now that player 2 is the informed player in a one-sided information game with state space L. For each increasing cost function c, Theorem 2 in Lehrer and Rosenberg (2006) ensures that $-c$ is a value-of-information function. Thus, there is a one-sided information game Γ_c, with a state space L, whose value is $-c(\mathcal{Q})$ for any information structure of player 2, $\mathcal{Q}$. Denote by A'_c, B'_c and γ_c its action sets and payoff function (the proofs are omitted).

Both games Γ_c and $\mathcal{G}^*_c$ will be used in the definition of a game with value V.

4.3 Concavity of value functions

We say that π is a *stochastic map from $\mathcal{P}$ to $\mathcal{C}$ with finite support* if for every cell $B \in \mathcal{P}$, $\pi(\cdot|B)$ is a distribution over $\mathcal{C}$ with a finite support. For stochastic map from $\mathcal{P}$ to $\mathcal{C}$ with finite support π_c denotes the probability over K given by $\pi_c(k) = \frac{p(k)\pi(c|B(k))}{\sum_{B\in\mathcal{P}} p(B)\pi(c|B)}$, where $B(k)$ is the cell of $\mathcal{P}$ that contains k. The following lemma phrases the concavity condition in terms of stochastic maps.

Lemma 3 *1. Let h be a function from $\Delta(K)\times\mathcal{K}$ to $\mathbb{R}$. Fix a partition $\mathcal{P}$. $h(\cdot,\mathcal{P})$ is concave iff for every stochastic map from $\mathcal{P}$ to $\mathcal{C}$ with finite support, and every* *probability $\bar{p}$ over K,*

$$\sum_{c\in\mathcal{C}}\left(\sum_{B\in\mathcal{P}}\bar{p}(B)\pi(c|B)\right)h(\pi_c,\mathcal{P})\leq h(\bar{p},\mathcal{P}).$$

2. Let h be a function from $\Delta(K)\times\mathcal{C}\times\mathcal{K}$ to $\mathbb{R}$. $h(\cdot,\cdot,\mathcal{P})$ is concave iff for every stochastic map from $\mathcal{P}$ to $\mathcal{C}$ with finite support, and every probability $\bar{p}$ over K,

$$\sum_{c\in\mathcal{C}}\left(\sum_{B\in\mathcal{P}}\bar{p}(B)\pi(c|B)\right)h(\pi_c,\mathcal{P},c)\leq h(\bar{p},\mathcal{P},\bar{c}),$$

where

$$\bar{c}=\sum_{c\in\mathcal{C}}\sum_{B\in\mathcal{P}}\bar{p}(B)\pi(c|B)c.$$

Let $v(\bar{p},\mathcal{P})$ be the value of a game G in which k is drawn according to probability $\bar{p}$ and player 1 is informed through $\mathcal{P}$.

Lemma 4 *For every $\mathcal{P}$, $v(\cdot,\mathcal{P})$ is concave.*

Proof. $v(\bar{p},\mathcal{P})$ is actually the value of the game with incomplete information whose state space is $\mathcal{P}$ (i.e., the states are the cells of $\mathcal{P}$), player 1 knows the state and player 2 does not. The payoff corresponding to the actions (a,b) and the state $B\in\mathcal{P}$ is $\sum_{k\in B}\frac{\bar{p}(k)}{\bar{p}(B)}g_k(a,b)$.

It is well known that the value of such a game is a concave function of the initial probability. ■

Lemma 5 *Define $h(\pi,c,\mathcal{P})=W^{\pi}(\mathcal{P},c)$. Then, $h(\cdot,\cdot,\mathcal{P})$ is concave.*

Proof. Let π be a stochastic map with a finite support from $\mathcal{P}$ to $\mathcal{C}$. Then, for every partition $\mathcal{Q}$,

$$\begin{aligned}\textstyle\sum_{c\in\mathcal{C}}\sum_{B\in\mathcal{P}}p(B)\pi(c|B)W^{\pi_c}(\mathcal{P},c) &\leq \textstyle\sum_{c\in\mathcal{C}}\sum_{B\in\mathcal{P}}p(B)\pi(c|B)(V^{\pi_c}(\mathcal{P},\mathcal{Q})+c(\mathcal{Q}))\\ &\leq V^{p}(\mathcal{P},\mathcal{Q})+\bar{c}(\mathcal{Q}).\end{aligned}$$

The first inequality is due to eq. (2). The second inequality is due to Lemma 4 and to the definition of $\bar{c}$. Since the last inequality holds for every $\mathcal{Q}$, by eq. (2),

$$\textstyle\sum_{c\in\mathcal{C}}\sum_{B\in\mathcal{P}}p(B)\pi(c|B)W^{\pi_c}(\mathcal{P},c)\quad\leq\quad W^{p}(\mathcal{P},\bar{c}).$$

■

4.4 Definition of the game

Step 0: k and l are drawn with respect to the probability $p \otimes q$. Player 1 is informed about k through $\mathcal{P}$ and player 2 about l through $\mathcal{Q}$.
Step 1: Player 1 chooses a cost function $c \in \mathcal{C}_V$ (possibly as a function of his information). This choice is observed by player 2.
Step 2: The games $\mathcal{G}_c^*$ and Γ_c are played simultaneously.

The payoff is the sum $\gamma_c + g_c$.

Lemma 6 *The value of this game is* $V(\mathcal{P}, \mathcal{Q})$

Proof. (i) This game is a finite game and therefore has a value in mixed strategies.
(ii) Player 1 can guarantee $V(\mathcal{P}, \mathcal{Q})$ in this game. Indeed, by choosing c independent of his information and then playing optimally in both $\mathcal{G}_c^*$ and Γ_c. In $\mathcal{G}^*$, player 1 guarantees $W^p(\mathcal{P}, c)$. In Γ_c he guarantees $-c(\mathcal{Q})$, so that by choosing c optimally he can guarantee $\max_{c \in \mathcal{C}} W^p(\mathcal{P}, c) - c(\mathcal{Q}) = V(\mathcal{P}, \mathcal{Q})$.
(iii) We show now that player 2 can defend $V(\mathcal{P}, \mathcal{Q})$. Fix a (behavioral) strategy of player 1. Assume that in stage 1 he chooses c with probability $\pi(c|B)$ when his information is that the selected k is in $B \in \mathcal{P}$.

Denote by $\pi(c)$ the total probability that player 1 chooses c. π_c denotes the conditional probability on K given the choice c. Knowing the strategy of player 1, player 2 can compute π_c for each observed c and the continuation strategy of player 1 both in $\mathcal{G}^*$ and in Γ_c. He then plays in each of these games a best reply to the continuation strategy of player 1. Thus, knowing the chosen c, player 2 defends $W^{\pi_c}(\mathcal{P}, c)$ in $\mathcal{G}^*$ and $-c(\mathcal{Q})$ in Γ_c. Note that the information about the distribution over K is irrelevant to Γ_c and the change of the conditional probability on K does not affect the value. Therefore, the maximal payoff given by this strategy of player 1 is bounded by $\sum_c \pi(c)(W^{\pi_c}(\mathcal{P}, c) - c(\mathcal{Q}))$.

Denote by $\bar{c} := \sum_c \pi(c)c$ and note that $p = \sum_c \pi(c)\pi_c$. By Lemma 5,

$$\sum_c \pi(c)(W^{\pi_c}(\mathcal{P}, c) - c(\mathcal{Q})) \leq W^p(\mathcal{P}, \bar{c}) - \bar{c}(\mathcal{Q}).$$

By the duality lemma, we conclude that the best player 1 can guarantee is at most

$$\max_{\bar{c} \in \mathcal{C}} W^p(\mathcal{P}, \bar{c}) - \bar{c}(\mathcal{Q}) = V(\mathcal{P}, \mathcal{Q}).$$

∎

Theorem 1 is a consequence of Lemma 6.

Remark: We make use of the independence between k and l in assuming that the choice of c by player 1 does not affect the value of Γ_c. This is so because knowing k is irrelevant to this game. More importantly, since the information that player 2 receives is irrelevant to $\mathcal{G}^*$, we could separate the games $\mathcal{G}^*$ and Γ_c.

5 Final remarks

5.1 Non-independent probabilities

Let $r \in R$ be a parameter drawn with some probability ρ. Let $\mathcal{P}$ and $\mathcal{Q}$ be partitions of R through which player 1 and player 2 get information on the realized r. The value-of-information function is defined in a similar way it was defined above. We were unable to extend the characterization of value-of-information functions to this case.

It is clear that value-of-information functionsare increasing in $\mathcal{P}$ and decreasing in $\mathcal{Q}$. We proved here that in the special case where $R = K \times L$, ρ is the product of independent probabilities $\rho = p \otimes q$ and $\mathcal{P}$ is a partition on K and $\mathcal{Q}$ on L, then these conditions are sufficient. Lehrer and Rosenberg (2006) characterized value-of-information functions in the symmetric case, where $\mathcal{P} = \mathcal{Q}$. The general characterization problem remains open.

5.2 Random signals

Our main result refers to deterministic signaling structure. Assume however that the state is a couple (k, l) where k and l are drawn independently, but player 1 and 2 get random signals s and t that stochastically depend on k and l, respectively. We conjecture that any function on such information structures, that is increasing when the signal of player 1 becomes more informative and decreasing when the signal of player 2 becomes less informative, is a value-of-information function.

6 References

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