Partially-specified probabilities and related topics

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Talk map

- Partially-specified probabilities (PSP): motivation and examples
- PSP: examples
- PSP: definition
- Growing awareness and extension rules (with Roee Teper)
- Games with PSP

Motivation

- Growing awareness - partitional information.
- Nash equilibrium - how players get into playing in concert? Learning with imperfect monitoring, off equilibrium path.

Advantages

- PSP: a model (strictly) between full Bayesian and ambiguity (set of priors - as in Levi (1980), Giron and Rios (1980) - first axiomatic treatment (?), Bewely (80's) and Gardenfors and Sahlin (1982) - introduced MMEU Gilboa-Schmeidler (1989) - axiomatization of MMEU).
- Advantages: tractability, enables close formulas, naturally arises in various contexts, endogenizing ambiguity.

Examples

- Noisy signals
- Ellsberg urn - static and dynamic
- Subjective/Self-confirming/cojectural equilibrium

Partially-specified probability: Definition

Definition

A partially-specified probability consists of a list $\mathcal{Y}$ of random variables and their expectations w.r.t. to a probability distribution.

- The probability in the definition is referred to as the REAL or OBJECTIVE probability.

The examples above and more

- A regular probability as PSP.
- Null information as PSP.
- Growing awareness – partitional information.
- Noisy channel and Blackwell informational monotonicity.
- Dynamic Ellsberg urn again.
- PSP in interactive settings

How to take a Decision under PSP: some examples

- Lower expectation - two ways: unanimity (Giron and Rios, 1980) and worst case (concave approximation: Levi, 1980, Gardenfors and Sahlin, 1982 and Gilboa-Schmeidler, 1989)
- Upper expectation - two ways: Best case (convex approximation) and Justifiable preferences(Lehrer-Teper, 2011 and cojectural equilibrium, 1987 and subjective/self-confirming, 1991)
- Principle of indifference or insufficient reason: uniform and in general PSP domain – maximal entropy

Growing awareness, incomplete order

Setup:

- S be a finite non–empty set of *states of nature*.
- $\mathcal{F} = L^S$ – acts: functions from states of nature to lotteries.
- for $\alpha \in [0,1]$, denote by $\alpha f + (1-\alpha)g$ the point-wise mixture.
- $\mathcal{H}$ - a subset of acts that contains the constants $\mathcal{F}_c$.
- $\succeq$ is incomplete over $\mathcal{F}$, but complete over $\mathcal{H}$.

Example

- $S = \{R, B, W\}$, $\mathcal{H}$ – acts of the form (a, b, b), $a, b \in [0, 1]$
- $(a, b, b) \succeq (a', b', b')$ iff $\frac{(a-a')}{3} + \frac{2(b-b')}{3} \geq 0$.
- How to extend this to $g = (.5, .4, .2)$ and $h = (.7, .2, .1)$?

Extension of incomplete order: The prudent rule

Denote by $\succeq^S$ the dominance relation.

Definition. Fix two acts $g, h \in \mathcal{F}$. Two $\succeq$-comparable acts, f_1 and f_2, *testify that g is preferred over h* if there exists $\alpha \in (0,1)$ such that $\alpha g + (1-\alpha) f_1 \succeq^S \alpha h + (1-\alpha) f_2$ and $f_2 \succeq f_1$.

The prudent rule (PR) $g \succeq_P h$ if and only if there are $\succeq$-comparable acts testifying that g is preferred over h.

Example - cont.

- consider $f_1 = (-0.2, 0.1, 0.1)$ and $f_2 = (0, 0, 0)$ (both in $\mathcal{H}$).
- Recall, $g = (.5, .4, .2)$ and $h = (.7, .2, .1)$.
- Mix g with f_2 and h with f_1 and get,
- $\frac{1}{2}g + \frac{1}{2}f_2 = (0.25, 0.2, 0.1)$ and $\frac{1}{2}h + \frac{1}{2}f_1 = (0.25, 0.15, 0.1)$.
- The former dominates the latter.
- f_1, f_2 testify for the preference of g over h.

After using the extension rule

$\succeq$ is induced by PSP: $\succeq = \succeq_{(u,\mathcal{H},p)}$

Theorem

Suppose that $\succeq$ is induced by PSP relation $\succeq_{(u,\mathcal{H},p)}$. The following are equivalent: (1) **PR** *is applied. That is, $\succeq_P$ extends $\succeq_{(u,\mathcal{H},p)}$; (2) For every $g, h \in \mathcal{F}$,*

$$g \succeq_P h \quad \textit{if and only if} \quad \mathbb{E}_q(u(g)) \geq \mathbb{E}_q(u(h)) \ \forall q \in \mathcal{P}(u, \mathcal{H}, p).$$

The extension of PSP with the prudent rule results in Knightian preferences (Bewley, 2002)

Games and PSP

- PSP about the strategies played by others.
- Games with signals induce PSP, but not the other way around.
- Advantage 1: endogenizing ambiguity. No need for additional consistency requirement.
- Advantage 2: enables application to games with incomplete information – PSP about the state (of nature) space.
- Advantage 3: allows non-independent distributions – correlated eq.

In equilibrium

- Distributions consistent with observations. Prob. Dist. that are consistent with PSP.
- All consistent distribution: correlated vs. independent (not convex).
- Not always a convex set.
- Full information: Nash

Example with lower expectation

	L	M	R
T	3,3	0,0	0,0
C	0,0	2,2	0,0
B	0,0	0,0	1,1

There are three equilibria with **lower expectation – worst case** (i.e., maxmin):
a. $p = (\frac{2}{5}, \frac{3}{5}, 0)$, $q = (\frac{2}{5}, \frac{3}{5}, 0)$;
b. $p = (0, 0, 1)$, $q = (0, 0, 1)$; and
c. $p = (\frac{2}{11}, \frac{3}{11}, \frac{6}{11})$, $q = (\frac{2}{11}, \frac{3}{11}, \frac{6}{11})$.

- Note: in this example all equilibria are also regular (fully-specified strategies) NE.

Example using uniforms dist.

	L	M	R
T	3,3	0,0	0,0
C	0,0	2,2	0,0
B	0,0	0,0	1,1

There are three equilibria with **uniforms dist.** (i.e., insufficient reason):

a. $p = (1, 0, 0)$, $q = (1, 0, 0)$;

b. $p = (0, 0, 1)$, $q = (0, 0, 1)$; and

c. $p = (\frac{2}{5}, 0, \frac{3}{5})$, $q = (\frac{2}{5}, 0, \frac{3}{5})$.

- Note: last equilibrium is not a regular (fully-specified strategies) NE.

Learning

- PSP-Correlated equilibrium.
- Regret minimization: definition and existence.
- Convergence to PSP-correlated equilibrium.

What players best respond to?

Issues related to a definition of equilibrium:

- Players best respond to a strategy consistent with information.
- Response correspondence: R_i defined over PSP's about others' strategies.
- independence of other players: issue beyond linear constraints over distributions.

Example

Consider the following game with a player having no information about the column player's behavior

	L	M	R
T	3	0	0
B	0	2	2

Upper expectation: Best case: $R_1(\Delta_2) = \{(1,0)\}$

Lower expectation: Worst case: $R_1(\Delta_2) = \{(\frac{2}{5}, \frac{3}{5})\}$

Maximum entropy: $R_1(\Delta_2) = \{(0,1)\}$

Additional related issues

- Risk measures induced by PSP: axiomatization.
- Existence of response equilibrium.

In short, PSP. Points mentioned

- PSP as an information-based model of ambiguity.
- I discussed only games with complete information.
- Incorporating PSP in strategic competitions.
- One can apply PSP to games with incomplete information.
- Solution concepts - Response equilibrium.
- Nash vs. correlated eq.
- Learning procedure.