CATEGORIZATION GENERATED BY PROTOTYPES – AN AXIOMATIC APPROACH

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Outline

- Categorization and Prototypes - short general background.
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- Categorization system – the model.
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- Categorization system – the model.
- Categorization systems generated by prototypes – Axiomatization.
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- Proof sketch.
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- Proof sketch.
- Examples.
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- Categorization systems generated by prototypes – Axiomatization.
- Voronoi diagrams.
- Proof sketch.
- Examples.
- Discussion.
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- Voronoi diagrams.
- Proof sketch.
- Examples.
- Discussion.
- Remarks, open questions,...
General background

Why do we categorize?
General background

Why do we categorize?
If you have never seen this particular creature before, why are you afraid of it?
General background

It seems that for decision under stress we use a prototype-oriented decision process.
General background

- It seems that for decision under stress we use a prototype-oriented decision process.
- A team trained to recognize a prototype from a set of prototypical scenarios and react accordingly, do not need a coordinator to act in harmony. This is crucial when the team might need to react instantaneously with no time for coordination.
General background

The classical view:

A category as a list of features

“If ‘man’ has one meaning, let this be ‘two-footed animal’. By ‘has one meaning’ I mean this: if X means ‘man’, then if anything is a man, it’s humanity will consist in being X.”

General background

The classical view:

Categories are defined in terms of a conjunction of necessary and sufficient features.
General background

The classical view:

- Categories are defined in terms of a conjunction of necessary and sufficient features.
- Features are binary.
General background

The classical view:

- Categories are defined in terms of a conjunction of necessary and sufficient features.
- Features are binary.
- All members of a category have equal status.
General background

The prototypes approach:

“Another way to achieve separateness and clarity of actually continuous categories is by conceiving of each category in terms of its clear cases rather than its boundaries. As Wittgenstein (1953) has pointed out, categorical judgments become a problem only if one is concerned with boundaries – in the normal course of life, two neighbors know on whose property they are standing without exact demarcation of the boundary line.”

General background

The prototypes approach:

Certain members of a category are more typical than others.
General background

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- Categories have radial structure. They form around the prototypical cases.
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- Certain members of a category are more typical than others.
- Categories have radial structure. They form around the prototypical cases.
- Category membership is a matter of degree.
- People categorize more typical members faster than less typical ones.
A finite set of categories $L = \{1, 2, \ldots, l\}$
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Entities and Categories

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- Typically, the $d$ attributes represent just a partial list of attributes of the entity under consideration.
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The set \( \mathbb{R}^d_+ \) can be replaced by the entire Euclidean space, or the simplex.
Definition:
An open partition of $\mathbb{R}_+^d$ is a collection of non-empty, pairwise disjoint open sets, say, $A_1, A_2, \ldots$ such that $\overline{\bigcup A_i} = \mathbb{R}_+^d$. 
Categorization system

**Definition:**
An *open partition* of $\mathbb{R}^d_+$ is a collection of non-empty, pairwise disjoint open sets, say, $A_1, A_2, \ldots$ such that $\text{cl} \cup A_i = \mathbb{R}^d_+$.

**Definition:**
1. A *categorization system* is a collection of open partitions $P_A = (P_A(i))_{i \in A}$ of $\mathbb{R}^d_+$, $A \subseteq L$ ($|A| \geq 2$).
2. When $x \in P_A(i)$ we say that $x$ is *categorized as* $i$ when $A$ is considered.
Axioms

Convexity: For every $A \subseteq L$ and for each $i \in A$, $P_A(i)$ is a convex set.
**Axioms**

- **Convexity**: For every $A \subseteq L$ and for each $i \in A$, $P_A(i)$ is a convex set.

- **Hierarchic Consistency**: For every $A \subseteq L$ and for each $i \in A$, $P_A(i) = \bigcap_{B \subseteq A, i \in B} P_B(i)$. 

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- **Non-Redundancy**: For every three categories $\{i, j, k\} \subseteq L$, $P_{\{i, j\}}(i) \not\subseteq P_{\{i, k\}}(i)$. 
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- **Hierarchic Consistency**: For every $A \subseteq L$ and for each $i \in A$, $P_A(i) = \bigcap_{B \subseteq A, i \in B} P_B(i)$.

- **Non-Redundancy**: For every three categories $\{i, j, k\} \subseteq L$, $P_{\{i,j\}}(i) \notin P_{\{i,k\}}(i)$.

- **Variety**: For every four distinct categories $\{i, j, k, m\} \subseteq L$,

\[
\text{cl}(P_{\{i,j,k\}}(i)) \cap \text{cl}(P_{\{i,j,k\}}(j)) \cap \text{cl}(P_{\{i,j,k\}}(k)) \neq \text{cl}(P_{\{m,j,k\}}(m)) \cap \text{cl}(P_{\{m,j,k\}}(j)) \cap \text{cl}(P_{\{m,j,k\}}(k))
\]
Definition: A categorization system, \( P_A = (P_A(i))_{i \in A} \), \( A \subseteq L \), is *generated by extended prototypes*, if there exist \( \ell \) points \( x_1, x_2, \ldots, x_\ell \) in \( \mathbb{R}^{d+1}_+ \), such that for any \( A \subseteq L \) and any \( i \in A \), \( P_A(i) = \{ y \in \mathbb{R}^d_+ : d_i(y) < d_j(y) \text{ for every } j \in A, j \neq i \} \), where \( d_i(y) = \| (y, 0) - x_i \|^2 \) (\( i = 1, \ldots, \ell \)) and \((y, 0)\) is the vector in \( \mathbb{R}^{d+1}_+ \) whose first \( d \) coordinates coincide with \( y \) and the last coincides with \( 0 \).
Extended prototypes

**Definition**: A categorization system, \( P_A = (P_A(i))_{i \in A} \), \( A \subseteq L \), is *generated by extended prototypes*, if there exist \( \ell \) points \( x_1, x_2, \ldots, x_\ell \) in \( \mathbb{R}^{d+1}_+ \), such that for any \( A \subseteq L \) and any \( i \in A \), \( P_A(i) = \{ y \in \mathbb{R}^d_+ : d_i(y) < d_j(y) \text{ for every } j \in A, j \neq i \} \), where \( d_i(y) = \| (y, 0) - x_i \|^2 (i = 1, \ldots, \ell) \) and \( (y, 0) \) is the vector in \( \mathbb{R}^{d+1}_+ \) whose first \( d \) coordinates coincide with \( y \) and the last coincides with 0.

**Interpretation**: The extended prototype is in \( \mathbb{R}^{d+1}_+ \). Recall that the \( d \) attribute may represent only a partial information about the entity. The hidden attribute may me interpreted as an imaginary completion of the unknown attributes.
Partial information

Suppose that you encounter this:
Partial information

Suppose that you encounter this:
Partial information

Suppose that you encounter this: What do you do?
Axiomatization

The main result:

If \( \{P_A\}_{A \subseteq L} \) is a categorization system generated by extended prototypes, then it satisfies **Convexity** and **Hierarchic Consistency**.
Axiomatization

The main result:

- If \( \{P_A\}_{A \subseteq L} \) is a categorization system generated by extended prototypes, then it satisfies **Convexity** and **Hierarchic Consistency**.

- If a categorization system \( \{P_A\}_{A \subseteq L} \) satisfies **Convexity** and **Hierarchic Consistency**, and in addition satisfies **Non-Redundancy** and **Variety**, then it is generated by extended prototypes.
Remarks

- **Hierarchic Consistency** is a necessary condition for a categorization system to be generated by prototypes regardless of the metric (e.g., maximum norm, $\ell_1$ etc.).
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- The Euclidean metric is the result of the **Convexity** axiom.
Voronoi Diagrams

Let $x_1, x_2, \ldots, x_n$ be some $n$ distinct points in $\mathbb{R}^d$.

For every $i = 1, 2, \ldots, n$, let

$$A_i = \{ y \in \mathbb{R}^d : \| y - x_i \| < \| y - x_j \| \text{ for every } j = 1, 2, \ldots, n \ j \neq i \}$$

where $\| . \|$ stands for the Euclidian norm.

**Definition:** The *Voronoi diagram* induced by $x_1, x_2, \ldots, x_n$ is the collection of sets $A_1, A_2, \ldots, A_n$. 
Voronoi Diagrams
Voronoi Diagrams

Simple observations:

For every two points $x_i, x_j$, the set of points in $\mathbb{R}^d$ which are in equal distance from $x_i$ and $x_j$ is the hyperplane which is the perpendicular bisector of the segment between $x_i$ and $x_j$. 
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- For every \( i = 1, \ldots, n \), \( A_i \) is the intersection of a (finite) number of half spaces. Therefore, it is a convex polyhedral set.
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- For every $i = 1, \ldots, n$, $A_i$ is the intersection of a (finite) number of half spaces. Therefore, it is a convex polyhedral set.

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- For every $i = 1, \ldots, n$, $A_i$ is the intersection of a (finite) number of half spaces. Therefore, it is a convex polyhedral set.

- $x_i \in A_i$. Therefore $A_i$ is not empty.

- The closure of the union of the sets $A_i$ ($1 \leq i \leq n$) is the entire space.
Voronoi Diagrams

Generalization: Power diagrams

Every point $x_i$ has a non-negative “weight” $w_i$. For a point $y \in \mathbb{R}^d$, let $d_i(y) = \|y - x_i\|^2 + w_i$.

Similarly to the former case, define for $i = 1, 2, ..., n$

$$B_i = \{ y \in \mathbb{R}^d : d_i(y) < d_j(y) \text{ for every } j = 1, 2, ..., n \ j \neq i \}$$

**Definition:** The *Power diagram* induced by $(x_1, w_1), ..., (x_n, w_n)$ is the collection of sets $B_1, B_2, ..., B_n$. 
For every two points $x_i, x_j$, the set of points in $\mathbb{R}^d$ which are in equal \textbf{weighted} distance from $x_i$ and $x_j$ is a hyperplane perpendicular to the segment between $x_i$ and $x_j$ (not necessarily the perpendicular bisector).
Voronoi Diagrams

Simple observations:

- For every two points $x_i, x_j$, the set of points in $\mathbb{R}^d$ which are in equal **weighted** distance from $x_i$ and $x_j$ is a hyperplane perpendicular to the segment between $x_i$ and $x_j$ (not necessarily the perpendicular bisector).

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- For every \( i = 1, \ldots, n \), \( B_i \) is the intersection of a (finite) number of half spaces. Therefore, it is a convex polyhedral set.

- \( B_i \) can be empty.
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- For every $i = 1, ..., n$, $B_i$ is the intersection of a (finite) number of half spaces. Therefore, it is a convex polyhedral set.

- $B_i$ can be empty.

- The closure of the union of the sets $B_i$ ($1 \leq i \leq n$) is the entire space.
Proof sketch

**Part 1 of the main theorem:**
If \( \{P_A\}_{A \subseteq L} \) is a categorization system generated by extended prototypes, then it satisfies **Convexity** and **Hierarchic Consistency**.
Proof sketch

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  If \( \{P_A\}_{A \subseteq L} \) is a categorization system generated by extended prototypes, then it satisfies **Convexity** and Hierarchic Consistency.

- **Lemma:**
  Let \( \{P_A\}_{A \subseteq L} \) be a categorization system. Then \( \{P_A\}_{A \subseteq L} \) satisfies **Hierarchic Consistency** if and only if for every \( A \subseteq L \ (|A| \geq 2) \) and for each \( i \in A \),
  \[
P_A(i) = \bigcap_{j \in A \setminus \{i\}} P_{\{i,j\}}(i).
  \]
Proof sketch

Part 2 of the main theorem:
If a categorization system \( \{P_A\}_{A \subseteq L} \) satisfies **Convexity** and **Hierarchic Consistency**, and in addition satisfies **Non-Redundancy** and **Variety**, then it is generated by extended prototypes.
Proof sketch

**Lemma:** If \( \{P_A\}_{A \subseteq L} \) satisfies **Convexity** then for every two categories \( i, j \in L \) there exists a hyperplane \( H_{i,j} \), such that \( P_{\{i,j\}}(i) \) is the open half space on one side of it and \( P_{\{i,j\}}(j) \) is the complementary open half.
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**Lemma:** If \( \{ P_A \}_{A \subseteq L} \) satisfies **Convexity** then for every two categories \( i, j \in L \) there exists a hyperplane \( H_{i,j} \), such that \( P_{\{i,j\}}(i) \) is the open half space on one side of it and \( P_{\{i,j\}}(j) \) is the complementary open half.

**Lemma:** **Non-Redundancy** implies that for any three distinct categories \( i, j, k \) the hyperplanes \( H_{i,j} \) and \( H_{i,k} \) are not parallel.
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**Lemma:** **Non-Redundancy** implies that for any three distinct categories \( i, j, k \) the hyperplanes \( H_{i,j} \) and \( H_{i,k} \) are not parallel.

**Lemma:** Let \( \{P_A\}_{A \subseteq L} \) be a categorization system which satisfies **Convexity**, **Hierarchic Consistency** and **Non-Redundancy**. For any three distinct categories \( i, j \) and \( k \)

\[
H_{i,j} \cap H_{i,k} = H_{i,j} \cap H_{j,k} = H_{j,k} \cap H_{i,k}
\]
Proof sketch

**Proposition:** If a categorization system \( \{P_A\}_{A \subseteq L} \) satisfies Convexity, Hierarchic Consistency, Non-Redundancy and Variety, then there are \( \ell \) points \( x'_1, \ldots, x'_\ell \in \mathbb{R}^d_+ \) such that:

1. \( x'_i - x'_j \) is perpendicular to the hyperplane \( H_{i,j} \), for every \( i, j \in L \); and
2. The direction from \( x'_i \) to \( x'_j \) is the same as the direction from \( P_{\{i,j\}}(i) \) to \( P_{\{i,j\}}(j) \) (we call such points ‘well oriented’).
Proof sketch
Proof sketch

\[ P_L(1) \]

\[ P_L(2) \]

\[ P_L(3) \]

\[ P_L(4) \]
Proof sketch

\[ P_L(1) \]

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**Proposition:** Let \( \{x'_1, \ldots, x'_\ell\} \subseteq \mathbb{R}^d_+ \) be points that satisfy the previous proposition. There exist a set of numbers \( w_1, \ldots, w_\ell \), such that \( P_{i,j}(i) = \{y \in \mathbb{R}^d_+; d_i(y) < d_j(y)\} \) for every \( i, j \in L \), where \( d_i(y) = \| (y, 0) - (x'_i, w_i) \|^2 \).
**Proof sketch**

**Conclusion of the main result’s proof:**
It is possible to find points \( x_1 = (x'_1, w_1), \ldots, x_\ell = (x'_\ell, w_\ell) \) in \( \mathbb{R}^{d+1}_+ \) such that for every two categories \( i, j \in L \),
\[
P_{\{i,j\}}(i) = \{ y \in \mathbb{R}^d_+; \| (y, 0) - x_i \|_2^2 < \| (y, 0) - x_j \|_2^2 \}.
\]
Therefore, for any \( A \subseteq L \) and any \( i \in A \),
\[
\forall \{ y \in \mathbb{R}^d_+; \| (y, 0) - x_i \|_2^2 < \| (y, 0) - x_j \|_2^2 \} = \bigcap_{j \in A \setminus \{i\}} \{ y \in \mathbb{R}^d_+; \| (y, 0) - x_i \|_2^2 < \| (y, 0) - x_j \|_2^2 \}
\]
\[
\{ y \in \mathbb{R}^d_+; \| (y, 0) - x_i \|_2^2 < \| (y, 0) - x_j \|_2^2 \text{ for every } j \in A, j \neq i \}
\]
Thus, the categorization system is generated by extended prototypes.
Examples

Categorization which is not generated by extended prototypes

\[ L = \{i, j, k\} \text{ (3 categories), } d = 2 \text{ (2 attributes).} \]

\[
P_{\{i,j\}}(i) = \{(x_1, x_2) \in \mathbb{R}_+^2; x_1 < 1\}
\]
\[
P_{\{i,k\}}(i) = \{(x_1, x_2) \in \mathbb{R}_+^2; x_2 < 1\}
\]
\[
P_{\{j,k\}}(j) = \{(x_1, x_2) \in \mathbb{R}_+^2; x_2 < 2 - x_1\}
\]

By Hierarchic Consistency, when \( L \) is considered:

\[
P_L(i) = \{(x_1, x_2) \in \mathbb{R}_+^2; x_1 < 1, x_2 < 1\}
\]
\[
P_L(j) = \{(x_1, x_2) \in \mathbb{R}_+^2; x_1 > 1, x_2 < 2 - x_1\}
\]
\[
P_L(k) = \{(x_1, x_2) \in \mathbb{R}_+^2; x_2 > 1, x_2 > 2 - x_1\}
\]
Examples

Categorization which is not generated by extended prototypes

\[ P_L(j) \]

\[ P_L(i) \]

\[ P_L(k) \]

\[ x_2 \]

\[ x_1 \]
Examples

The hidden attribute

$L = \{i, j, k, m\}$ (4 categories), $d = 2$ (2 attributes).

\[
P_{\{i,j\}}(i) = \{(x_1, x_2) \in \mathbb{R}^2_+; \ 2x_2 > 4 - x_1\}
\]

\[
P_{\{i,k\}}(i) = \{(x_1, x_2) \in \mathbb{R}^2_+; \ x_2 > 1\}
\]

\[
P_{\{i,m\}}(i) = \{(x_1, x_2) \in \mathbb{R}^2_+; \ 2x_2 > x_1 - 1\}
\]

\[
P_{\{j,k\}}(j) = \{(x_1, x_2) \in \mathbb{R}^2_+; \ 2x_2 > x_1\}
\]

\[
P_{\{j,m\}}(j) = \{(x_1, x_2) \in \mathbb{R}^2_+; \ 2x_1 < 5\}
\]

\[
P_{\{k,m\}}(k) = \{(x_1, x_2) \in \mathbb{R}^2_+; \ 2x_2 < 5 - x_1\}
\]

When more than 2 categories are considered, the partitions are determined by Hierarchic Consistency.
Examples

The hidden attribute

\[ P_L(j), P_L(k), P_L(m) \]

\[ x_1 \]

\[ x_2 \]
Examples

Convexity and Hierarchic Consistency are insufficient

$L = \{i, j, k\}$ (3 categories), $d = 2$ (2 attributes).

\[ P_{\{i,j\}}(i) = \{(x_1, x_2) \in \mathbb{R}^2_+; x_1 < 1\} \]
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By Hierarchic Consistency, when $L$ is considered:

\[ P_L(i) = \{(x_1, x_2) \in \mathbb{R}^2_+; x_1 < 1\} \]
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\[ P_L(k) = \{(x_1, x_2) \in \mathbb{R}^2_+; x_1 > 1, x_2 < x_1\} \]
Examples

Convexity and Hierarchic Consistency are insufficient
Examples

Variety is not satisfied

\[ L = \{i, j, k, m\} \text{ (4 categories), } d = 2 \text{ (2 attributes).} \]

\[
\begin{align*}
P_{\{i,j\}}(i) &= \{(x_1, x_2) \in \mathbb{R}^2_+; x_1 < 1\} \\
P_{\{i,k\}}(i) &= \{(x_1, x_2) \in \mathbb{R}^2_+; x_2 > 1\} \\
P_{\{i,m\}}(i) &= \{(x_1, x_2) \in \mathbb{R}^2_+; x_2 > 2x_1 - 1\} \\
P_{\{j,k\}}(j) &= \{(x_1, x_2) \in \mathbb{R}^2_+; 2x_2 > 3 - x_1\} \\
P_{\{j,m\}}(j) &= \{(x_1, x_2) \in \mathbb{R}^2_+; x_2 > x_1\} \\
P_{\{k,m\}}(k) &= \{(x_1, x_2) \in \mathbb{R}^2_+; x_2 < 2 - x_1\}
\]

When more than 2 categories are considered, the partitions are determined by Hierarchic Consistency.
Examples
Variety is not satisfied

\[ P_{\{i,j,k\}}(i) \]
\[ P_{\{i,j,k\}}(j) \]
\[ P_{\{i,j,k\}}(k) \]

\[ P_{\{m,j,k\}}(j) \]
\[ P_{\{m,j,k\}}(m) \]
\[ P_{\{m,j,k\}}(k) \]
Discussion

Limitations of the axioms:

- More than one prototypical example resulting in non-convex categories.
Discussion

Limitations of the axioms:

- More than one prototypical example resulting in non-convex categories.
- The effect of the set of categories under consideration on the prototypes – Violation of Hierarchic Consistency.
Discussion

Decision theory and categorization

Categorization of decision problems according to their best response satisfies Hierarchic Consistency.
Categorization of decision problems according to their best response satisfies Hierarchic Consistency.

If a decision problem is defined by a distribution $P$ over some state space (say $\Omega$), and $u(P, a)$ is the expected utility when choosing the action $a$, then the categorization of decision problems (distributions) according to their best response forms convex categories.
Remarks & Open questions

Categorizations generated by (non-extended) prototypes.
Remarks & Open questions

- Categorizations generated by (non-extended) prototypes.
- Other domains.
Remarks & Open questions

- Categorizations generated by (non-extended) prototypes.
- Other domains.
- Other metrics.
Remarks & Open questions

- Categorizations generated by (non-extended) prototypes.
- Other domains.
- Other metrics.
- Prototypical sets.
Remarks & Open questions

- Categorizations generated by (non-extended) prototypes.
- Other domains.
- Other metrics.
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- Another version of the Non-redundancy axiom which is also necessary:

**Non-Redundancy**: For every three categories \( \{i, j, k\} \subseteq L \), if \( P_{\{i,j\}}(i) = P_{\{i,k\}}(i) \) then either 
\[
P_{\{i,j,k\}}(i) \subseteq P_{\{j,k\}}(j) \quad \text{or} \quad P_{\{i,j,k\}}(i) \subseteq P_{\{j,k\}}(k).
\]
Remarks & Open questions

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- Is the theorem still true without assuming **Variety**?