Reward Schemes

David Lagziel & Ehud Lehrer

Tel-Aviv University

Maastricht University
January 2016

An overview

- A combined Mechanism Design and Game Theory problem.
- A decision maker (DM) uses investment firms to invest.
- Every year she collects the profits and redistributes the funds.
- She chooses firms according to publicly-known results.
- These allocation rules are called *Reward Schemes*.

An overview

- A combined Mechanism Design and Game Theory problem.
- A decision maker (DM) uses investment firms to invest.
- Every year she collects the profits and redistributes the funds.
- She chooses firms according to publicly-known results.
- These allocation rules are called *Reward Schemes*.

An overview

- A combined Mechanism Design and Game Theory problem.
- A decision maker (DM) uses investment firms to invest.
- Every year she collects the profits and redistributes the funds.
- She chooses firms according to publicly-known results.
- These allocation rules are called *Reward Schemes*.

An overview

- A combined Mechanism Design and Game Theory problem.
- A decision maker (DM) uses investment firms to invest.
- Every year she collects the profits and redistributes the funds.
- She chooses firms according to publicly-known results.
- These allocation rules are called *Reward Schemes*.

An overview

- A combined Mechanism Design and Game Theory problem.
- A decision maker (DM) uses investment firms to invest.
- Every year she collects the profits and redistributes the funds.
- She chooses firms according to publicly-known results.
- These allocation rules are called *Reward Schemes*.
- Her objective is to maximize the total expected earnings.
- Each firm wishes to maximize its share of managed funds.

An overview

- A combined Mechanism Design and Game Theory problem.
- A decision maker (DM) uses investment firms to invest.
- Every year she collects the profits and redistributes the funds.
- She chooses firms according to publicly-known results.
- These allocation rules are called *Reward Schemes*.
- Her objective is to maximize the total expected earnings.
- Each firm wishes to maximize its share of managed funds.

Main Goal

For every market, find an optimal reward scheme such that the firms are motivated to act according to the interests of the DM.

An overview

- A motivating example.
- The model.
- Positive result: *Every market has an optimal reward scheme*.
- Negative result: *A universal reward scheme does not exist*.
- Concluding remarks.

A motivating example

The market

- A 2-firms market with two bonds, X_1 and X_2, such that X_1 gives 5% and X_2 gives 5.1% w.p. 0.6 and 0% w.p. 0.4.

$$X_1 = 1.05 \text{ per year w.p. } 1, \qquad X_2 = \begin{cases} 1.051, & \text{per year w.p. } \frac{3}{5}, \\ 1.0, & \text{per year w.p. } \frac{2}{5}. \end{cases}$$

- Clearly, X_1 is better than X_2 in terms of expected payoff and risk.
- However, X_2 presents higher results than X_1 w.p. 0.6.

A motivating example

The market

- A 2-firms market with two bonds, X_1 and X_2, such that X_1 gives 5% and X_2 gives 5.1% w.p. 0.6 and 0% w.p. 0.4.

$$X_1 = 1.05 \text{ per year w.p. } 1, \qquad X_2 = \begin{cases} 1.051, & \text{per year w.p. } \frac{3}{5}, \\ 1.0, & \text{per year w.p. } \frac{2}{5}. \end{cases}$$

- Clearly, X_1 is better than X_2 in terms of expected payoff and risk.
- However, X_2 presents higher results than X_1 w.p. 0.6.

A reward scheme

Winner takes all. The DM decides to allocate the entire amount to highest-earnings firm, with a symmetric tie-breaking rule.

A motivating example

Utility functions

- The portfolio Y_i of firm i is based on either X_1 or X_2 or a mixture.
- Fix $\lambda \in (0, 1)$.
- The goal function of firm 1 is a λ-weighted average of the earnings and the (normalized) redistributed funds:

$$U_1(Y_1, Y_2) = \lambda Y_1 + (1 - \lambda) \left[\mathbf{1}_{\{Y_1 > Y_2\}} + \frac{\mathbf{1}_{\{Y_1 = Y_2\}}}{2} \right].$$

A motivating example

Utility functions

- The portfolio Y_i of firm i is based on either X_1 or X_2 or a mixture.
- Fix $\lambda \in (0, 1)$.
- The goal function of firm 1 is a λ-weighted average of the earnings and the (normalized) redistributed funds:

$$U_1(Y_1, Y_2) = \lambda Y_1 + (1 - \lambda) \left[\mathbf{1}_{\{Y_1 > Y_2\}} + \frac{\mathbf{1}_{\{Y_1 = Y_2\}}}{2} \right].$$

Equilibrium result

If $0 \leq \lambda \leq \frac{1}{1.194} \approx 0.83$, the only equilibrium is (X_2, X_2).

A motivating example

Proof.

There are 4 possible *pure profiles* and the payoff from each is:

$$\begin{aligned}
E[U_1(X_1, X_1)] &= 0.5 + 0.55\lambda; \\
E[U_1(X_2, X_1)] &= 0.6 + 0.4306\lambda; \\
E[U_1(X_1, X_2)] &= 0.4 + 0.65\lambda; \\
E[U_1(X_2, X_2)] &= 0.5 + 0.5306\lambda.
\end{aligned}$$

Inserting the expected gain of the two firms to a 2-player game yields:

	X_1	X_2
X_1	$0.5 + 0.55\lambda$, *	$0.4 + 0.65\lambda$, *
X_2	$0.6 + 0.4306\lambda$, *	$0.5 + 0.5306\lambda$, *

If $\lambda \in \left[0, \frac{1}{1.194}\right]$, then for every firm $i \in \{1, 2\}$, action X_2 strongly dominates action X_1.

A motivating example

Proof.

If the portfolios $Y_i = \alpha_i X_1 + (1-\alpha_i)X_2$ are diversified, then:

$$E[U_1(Y_1, Y_2)] = \lambda(1.0306 + 0.0194\alpha_1) + (1-\lambda) \cdot \begin{cases} 3/5, & \text{if } \alpha_1 < \alpha_2, \\ 1/2, & \text{if } \alpha_1 = \alpha_2, \\ 2/5, & \text{if } \alpha_1 > \alpha_2. \end{cases}$$

A motivating example

Proof.

If the portfolios $Y_i = \alpha_i X_1 + (1-\alpha_i)X_2$ are diversified, then:

$$E[U_1(Y_1, Y_2)] = \lambda(1.0306 + 0.0194\alpha_1) + (1-\lambda)\cdot \begin{cases} 3/5, & \text{if } \alpha_1 < \alpha_2, \\ 1/2, & \text{if } \alpha_1 = \alpha_2, \\ 2/5, & \text{if } \alpha_1 > \alpha_2. \end{cases}$$

- A profile of strategies in which $\alpha_1 < \alpha_2$ cannot be an equilibrium.
- If $\alpha_1 = \alpha_2 > 0$, then any firm can deviate to $\alpha_i - \epsilon$.
- Thus, we are left with (X_2, X_2) and the previous analysis.

A motivating example

Proof.

If the portfolios $Y_i = \alpha_i X_1 + (1-\alpha_i)X_2$ are diversified, then:

$$E[U_1(Y_1, Y_2)] = \lambda(1.0306 + 0.0194\alpha_1) + (1-\lambda) \cdot \begin{cases} 3/5, & \text{if } \alpha_1 < \alpha_2, \\ 1/2, & \text{if } \alpha_1 = \alpha_2, \\ 2/5, & \text{if } \alpha_1 > \alpha_2. \end{cases}$$

- A profile of strategies in which $\alpha_1 < \alpha_2$ cannot be an equilibrium.
- If $\alpha_1 = \alpha_2 > 0$, then any firm can deviate to $\alpha_i - \epsilon$.
- Thus, we are left with (X_2, X_2) and the previous analysis.

Note that for every $\lambda \in \left(\frac{1}{1.194}, 1\right)$, there is *no equilibrium*!

The Model

- $A = \{X_1, \ldots, X_n\}$ is a finite set of pure actions of the players (firms).
- Every X_j has a finite expectation.
- A strategy, or *diversified action*, q is a diversified portfolio $q = \sum_{j=1}^{n} q_j X_j$ when $(q_1, \ldots, q_n)$ is a probability distribution over A.
- For simplicity, assume $E[X_1] > E[X_j]$, for every $j = 2, \ldots, n$.

Definition

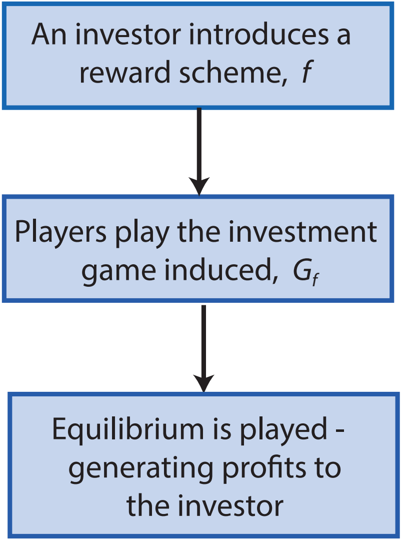
Fix a natural $k \geq 2$. A *reward scheme* (RS) of dimension k is a function $f : \mathbb{R}^k \to \mathbb{R}^k$ such that $\sum_{i=1}^{k} f_i(r) = 1$ and $f(r) \in [0,1]^k$, for every $r \in \mathbb{R}^k$.

The model

A k-player *investment game* evolves as follows:

- The investor publicly commits to a reward scheme f.
- The RS defines a k-player investment game G_f.
- In the investment game every player i chooses a strategy: a composition of financial assets. Denote it by σ_i and $\sigma = (\sigma_1, \ldots, \sigma_k)$.
- Then, a random state $\omega \in \Omega$ is chosen, and
  (i) Player i receives $f_i(\sigma(\omega))$.
  (ii) The investor receives $\sum_{i=1}^{k} \sigma_i(\omega)$.

The model - flowchart



Optimal reward scheme

Definition

A RS f is *optimal*, if every equilibrium σ in the induced investment game G_f satisfies the following *optimality condition*:

$$E\left[\sum_{i=1}^{k} \sigma_i\right] = k \max_{i \in N} E[X_i].$$

When a RS is optimal, any equilibrium played by the investment firms serves best the interests of the investor.

A positive result

Theorem

For every finite A, there is an optimal reward scheme.

A positive result – bounded assets

- Suppose that all X_j's are bounded between $-M$ and M .

Theorem

The following Linear Reward Scheme *is optimal:*

$$f_i(r) \quad = \quad \frac{1}{k} + \frac{1}{2M(k-1)}\left[r_i - \frac{1}{k}\sum_{\ell=1}^{k} r_\ell\right].$$

A positive result – bounded assets

- Suppose that all X_j's are bounded between $-M$ and M .

Theorem

The following Linear Reward Scheme *is optimal:*

$$f_i(r) \quad = \quad \frac{1}{k} + \frac{1}{2M(k-1)}\left[r_i - \frac{1}{k}\sum_{\ell=1}^{k} r_\ell\right].$$

- Note that in order to define the RS, the set A need not be known to the designer.
- However, the value of M is determined according to A.

The Linear Reward Scheme & previous example

- Before proving the theorem, we observe how The Linear Reward Scheme solves the problem of the motivating example.
- Since $k = 2$ and taking $M = 2$,

$$U_1(Y_1, Y_2) \quad = \quad \lambda Y_1 + (1 - \lambda)\left[\frac{1}{2} + \frac{Y_1 - Y_2}{8}\right]$$

- The Linear Reward Scheme induces a 2-player game, where the utilities are linear w.r.t. the profits.
- For every $\lambda \in [0, 1]$, the *dominant-strategy* equilibrium is (X_1, X_1) .

A positive result – bounded assets cont.

Proof.

Fix strategies $\sigma_2, \ldots, \sigma_k$ of players $2, \ldots, k$ respectively, and consider any strategy $\sigma_1 \neq X_1$ of Player 1.

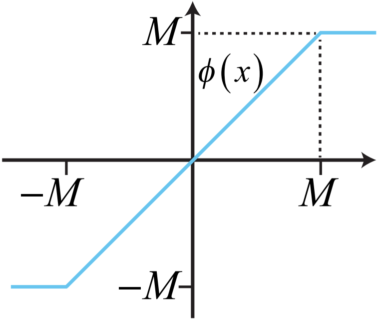
$$
\begin{aligned}
E\left[f_1\left(\sigma_1, \sigma_2, \ldots, \sigma_k\right)\right] &= E\left[\frac{1}{2M(k-1)}\left(\sigma_i - \frac{1}{k}\sum_{\ell=1}^{k}\sigma_\ell\right) + \frac{1}{k}\right] \\
&= E\left[\frac{(k-1)\sigma_1 - \sum_{\ell=2}^{k}\sigma_\ell}{2k(k-1)M} + \frac{1}{k}\right] \\
&< E\left[\frac{(k-1)X_1 - \sum_{\ell=2}^{k}\sigma_\ell}{2k(k-1)M} + \frac{1}{k}\right] \\
&= E\left[f_i\left(X_1, \sigma_2, \ldots, \sigma_k\right)\right],
\end{aligned}
$$

when the inequality follows from the fact that $E[\sigma_1] < E[X_1]$. □

A positive result – unbounded assets

Define the real-valued function $\phi : \mathbb{R} \to \mathbb{R}$ as

$$\phi(x) = \begin{cases} -M, & \text{if } x < -M, \\ x, & \text{if } -M \leq x \leq M, \\ M, & \text{if } x > M. \end{cases}$$



A positive result – unbounded assets

Define the real-valued function $\phi : \mathbb{R} \to \mathbb{R}$ as

$$\phi(x) = \begin{cases} -M, & \text{if } x < -M, \\ x, & \text{if } -M \leq x \leq M, \\ M, & \text{if } x > M. \end{cases}$$

Theorem

For every finite set A, there exists $M > 0$ such that The following General Reward Scheme f *is optimal:*

$$f_i(r) \quad = \quad \frac{1}{k} + \frac{1}{2M(k-1)} \left[\phi(r_i) - \frac{1}{k} \sum_{\ell=1}^{k} \phi(r_\ell) \right].$$

Extensions - General

- *Not all eggs in one basket*. The share of each firm is bounded between 0 and $\frac{2}{k}$.
- The RS remains optimal even when firms *share the profits*.

Extensions - General

- *Not all eggs in one basket*. The share of each firm is bounded between 0 and $\frac{2}{k}$.
- The RS remains optimal even when firms *share the profits*.

Dynamics

- *Conjecture* - Generalizing the same model to a dynamic environment will still produce optimality.
- Specifically, if all sides wish to maximize a discounted sum of single-round payoffs, and if the firms can update their strategy in every period, the Linear Reward Scheme remains optimal.

Extensions - Combining risk via utility function

- If the investor is an expected utility maximizer with utility function u. That is, the investor wishes to maximize

$$E\left[\sum_{i=1}^{k} u\left(\sigma_i(\omega)\right)\right].$$

- Use the same RS w.r.t. $u(r_i)$ instead of r_i,

$$f_i(r) \quad = \quad \frac{1}{k} + \frac{1}{2M(k-1)}\left[u(r_i) - \frac{1}{k}\sum_{\ell=1}^{k} u(r_\ell)\right].$$

- This RS solves the *moral hazard problem*, while a constant RS does not. E.g., a risk-averse investor and firms with goal functions as in the motivating example.

Extensions - Uniqueness

- The optimal RS is not unique since it remains optimal with different normalization factors.
- However, the form of the RS is unique in the sense that linearity is crucial.

Theorem

Let f be a RS such that for every finite set of bounded actions A, the investment game G_f has an optimal dominant-strategy equilibrium. Then, $f_i(r)$ is linear in r_i.

The negative result

Universal reward scheme (definition)

A RS f is *universal* if for every finite set of actions A, there exists an optimal equilibrium.

The negative result

Universal reward scheme (definition)

A RS f is *universal* if for every finite set of actions A, there exists an optimal equilibrium.

Theorem

If f is a universal reward scheme and there are only two players, then every profile of actions is an equilibrium.

In other words, the only 2-player RS that always (i.e., in every market) generates at least one optimal equilibrium is constant.

The negative result

Strongly-universal reward scheme (definition)

A RS f is *strongly-universal* if for every finite set of actions A, every optimal profile of actions is an equilibrium.

The negative result

Strongly-universal reward scheme (definition)

A RS f is *strongly-universal* if for every finite set of actions A, every optimal profile of actions is an equilibrium.

Theorem

If f is a strongly universal reward scheme, then every profile of actions is an equilibrium.

That is, in a k-player investment game, if the RS is strongly universal, then any profile is an equilibrium.

Non-existence of a universal reward scheme

Intuition behind the proof.

- Assume that a non-constant universal RS f exists.
- To ensure that firms prefer higher payoffs, f needs to be monotonic.
- Since $f \in [0,1]^2$ is bounded, $f_1(x, y)$ tends to concavity as x increases.
- Fix a market with $A = \{X_1, X_2\}$ where $E[X_1] > E[X_2]$ and:
  - (i) X_1 is very risky. Very high values with small probabilities.
  - (ii) Firms prefer X_2. Sufficiently high values with high probabilities.

A continuous investment game with no equilibrium

The no-equilibrium investment game

- Fix a large $M > 0$ and consider the reward scheme f defined by

$$f_i(r) = \frac{1}{k} + \frac{\sum_{\ell=1}^{k} \phi(r_i - r_\ell)}{2k(k-1)M}.$$

Proposition

There is a set A such that the game G_f, induced by f, has no equilibrium.

A continuous investment game with no equilibrium

Proof - an intuitive sketch.

- Assume there are only two players.
- Fix $A = \{X, Y\}$ where $X \equiv 0$ and choose Y s.t. $E[Y] = 0$, and for every $M > 0$, there exists $n_+, n_- > M$ where

$$\Pr(Y > n_\pm) = \Pr(Y < -n_\pm),$$

and

$$\pm E[Y \mathbf{1}_{\{|Y| \leq n_\pm\}}] > 0.$$

- Given $\sigma_i = \alpha_i X + (1 - \alpha_i) Y$, we get

$$f_1(\sigma_1, \sigma_2) = \frac{1}{2} + \frac{\phi((\alpha_2 - \alpha_1) Y)}{4M}.$$

Summary & future research

- An investor incentivizes funds via a reward scheme.
- It induces a competition (an investment game) among funds.
- An optimal reward scheme incentivizes funds to invest in the assets that serve best the interests of the investor.

Summary & future research

- An investor incentivizes funds via a reward scheme.
- It induces a competition (an investment game) among funds.
- An optimal reward scheme incentivizes funds to invest in the assets that serve best the interests of the investor.
- When the assets are bounded, an optimal reward scheme exists.
- When there is a known set of assets, an optimal reward scheme exists.
- If the assets are unbounded and the set is unknown to the investor, there is no satisfactory reward scheme.

Summary & future research

- An investor incentivizes funds via a reward scheme.
- It induces a competition (an investment game) among funds.
- An optimal reward scheme incentivizes funds to invest in the assets that serve best the interests of the investor.
- When the assets are bounded, an optimal reward scheme exists.
- When there is a known set of assets, an optimal reward scheme exists.
- If the assets are unbounded and the set is unknown to the investor, there is no satisfactory reward scheme.

What's next?

- Different utility functions (e.g., combine risk, general preferences).
- Different information structures.
- Heterogeneous firms.
- Dynamics.