INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL
Int. J. Robust. Nonlinear Control 0000; **00**:1–31
Published online in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/rnc

Filtering Homogeneous Observers in Control of Integrator Chains

Adam Jbara[a], Arie Levant[b], Avi Hanan[c]

a,b,c - School of Mathematical Sciences, Tel-Aviv University, Tel Aviv, Israel,
a,b,c - adamjbara@mail.tau.ac.il, levant@tauex.tau.ac.il, avih@mail.tau.ac.il

SUMMARY

Any homogeneous controller asymptotically stabilizing disturbed integrator chains can be coupled with the proposed robust homogeneous filtering observer. The type of convergence (in finite-time, in fixed-time to any ball or exponential) is determined by the sign of the system homogeneity degree (HD). Filtering high-gain observers and sliding-mode-based observers/differentiators are important particular cases corresponding to zero and negative HDs respectively. Stabilization accuracy is estimated in the presence of possibly unbounded sampling noises featuring locally-bounded iterated integrals. Accurate stabilization and differentiation are demonstrated for extremely large noises.

Received ...

KEY WORDS: Nonlinear output-feedback control, filtering, sliding mode control, robustness.

1. INTRODUCTION

Output-regulation problem is the classical problem of control theory. In the case of well defined relative degrees [18] it can be reduced to the stabilization of interacting integrator chains of the form $\sigma^{(n)} = h + gu$, $g \neq 0$, where σ is the output, and uncertain functions h and g depend on the time and the system state. Substituting the unknown functions h, g with some *concrete* sets depending on $\sigma, ..., \sigma^{(n-1)}$, produces a separate differential inclusion (DI) to be stabilized. The length of the chain determines the number of the real-time output derivatives to be estimated for the output-feedback control.

The corresponding control design is often based on the homogeneous extension or domination [7, 37]. The homogeneity degrees (HDs) are defined up to proportionality. In the following we fix the output HD $\deg \sigma = 1$.

The system HD is defined as the constant difference $q = \deg \sigma^{(i)} - \deg \sigma^{(i-1)}$, $i = 1, ..., n$. In particular, if h, g admit the representation $h = c_0\sigma + ... + c_{n-1}\sigma^{(n-1)}$, $c_i \in [\underline{c}_i, \overline{c}_i]$, $g \in [\underline{g}, \overline{g}]$, $\underline{g} > 0$, the chain is described by a homogeneous DI of the zero HD, $q = 0$.

The sliding-mode (SM) control (SMC) case corresponds to the system HD $q = -1/n$, and bounded h, g, $h \in [\underline{h}, \overline{h}]$, $g \in [\underline{g}, \overline{g}]$, $\underline{g} > 0$. Some solutions of that SMC problem are available in [6, 9, 12, 13, 15, 16, 22, 23, 25]. Disturbed integrator chains for $q > -1/n$ are stabilized by homogeneous output feedbacks in such papers as [17, 36, 37, 21].

The problem is solved in finite time (FT) for the negative system HDs, or asymptotically for non-negative HDs [27]. The convergence time to any ball around the origin is uniformly bounded for positive HDs, $q > 0$, (fixed time (FxT) convergence) [1, 2, 35].

*Correspondence to: levant@tauex.tau.ac.il

Being very sensitive to sampling time periods and very large already at moderate distances from the origin, FxT control is often difficult to apply. Nevertheless, it becomes the best choice in the control of explosive systems capable of finite-time escape to infinity. In the homogeneity theory such escaping corresponds to positive HDs. Contrary to this, closed systems of non-positive HDs are always robust to small sampling noises and delays [23, 27].

Consider the cases $q = 0$ and $q = -1/n$. High-gain observers (HGOs) are known to be universally feedback applicable for $q = 0$ providing for the global exponential stabilization [3, 19]. In their turn SMC methods employ robust exact SM-based observers/differentiators for the FT global system stabilization [22, 41] if $q = -1/n$. Compared with HGOs the SM-based differentiators do not contain high gains, feature no peaking effect, but slowly converge from large initial errors. In both cases the *separation principle* [3] holds, i.e. the observer is designed independently of the control.

Note that in spite of being exact, SM-based differentiators [22] are only semiglobally applicable for HDs $q > -1/n$. The incompatible observer homogeneity in that case also destroys the accuracy asymptotics in the presence of small sampling noises.

The HGOs [3] and the SM-based differentiators [22] are known to share the same structure for $q = 0$ and $q = -1/n$ respectively [34]. In the case of sufficiently small $|q|$, $q < 0$, their output-feedback FT-stabilization application was established for perfectly known systems [34]. In the general case $q \neq 0, -1/n$ such differentiators/observers are known to be only exact on polynomials of degrees $n - 1$ and lower [10], and the corresponding output-feedback application has not been established.

This paper closes the above gap. We prove that for any HD $q \geq -1/n$ the corresponding observer/differentiator can be coupled with any homogeneous locally-bounded stabilizing controller providing for the global output-feedback asymptotic stabilization of disturbed homogeneous integrator chains. Thus, we establish the separation principle for homogeneous observers/differentiators [10, 34] of *any HDs* including the case $q > 0$.

We establish the recursion procedure for choosing the observer parameters which starts with the trivial observer for the relative degree $n = 1$ and adds one parameter each time the relative degree is increased by one. The procedure generalizes the recursion of SM-based observers [22].

The observation problem is aggravated in the presence of large sampling noises. Thus, we equip the proposed observers with the filtering capabilities previously being only available for $q = -1/n$ [32, 30]. These extensions preserve the system exactness and ensure good practical stabilization even for very large or *unbounded* noises having small iterated integrals. In the case $q = 0$ modified HGOs are produced which are naturally called filtering HGOs.

Extensive simulation demonstrates successful differentiation and output-feedback stabilization in the presence of **very large** measurement noises.

Notation. A binary operation $\diamond$ of two sets is defined as $A \diamond B = \{a \diamond b |\ a \in A, b \in B\}$; $a \diamond B = \{a\} \diamond B$. A function of a set is the set of function values on this set; the norm $||x||$ stays for the standard Euclidian norm of x, $B_\varepsilon = \{x \mid ||x|| \leq \varepsilon\}$; $||x||_h$ is a homogeneous norm; $\lfloor a \rceil^b = |a|^b \operatorname{sign} a$, $\lfloor a \rceil^0 = \operatorname{sign} a$; $\vec{s}_k = (s_0, s_1, ..., s_k)^T$ for any finite sequence $\{s_0, ..., s_k\}$, in the corresponding context *the same notation* $\vec{\phi}_k(t) = (\phi(t), \dot{\phi}(t), ..., \phi^{(k)}(t))^T$ also denotes the sequence of derivatives of the orders $0, ..., k$.

2. COORDINATE HOMOGENEITY BASICS

Recall that solutions of the differential inclusion (DI)

$$\dot{x} \in F(x),\ F(x) \subset T_x\mathbb{R}^{n_x}, \tag{1}$$

are defined as locally absolutely continuous functions $x(t)$, satisfying the DI for almost all t. Here $T_x\mathbb{R}^{n_x}$ denotes the tangent space to $\mathbb{R}^{n_x}$ at $x \in \mathbb{R}^{n_x}$.

DOI: 10.1002/rnc

We call the DI (1) *Filippov DI*, if the vector-set field $F(x) \subset T_x\mathbb{R}^{n_x}$ is non-empty, compact and convex for any x, and F is an upper-semicontinuous set function. The latter means that the maximal distance of the points of $F(x)$ from the set $F(y)$ tends to zero, as $x \to y$.

Filippov DIs feature existence, and extendability of solutions, but not their uniqueness [14]. The Filippov definition replaces a discontinuous vector field $f(x)$ with a Filippov DI (1), $F = K_F[f]$, where

$$K_F[f](x) = \bigcap_{\mu_L N=0} \bigcap_{\delta>0} \overline{co}\, f((x + B_\delta)\backslash N). \tag{2}$$

Here $\overline{co}$, μ_L stand for the convex closure and the Lebesgue measure respectively, (2) defines the celebrated Filippov procedure [14]. Filippov solutions are proved to appear in real systems as the limit motions corresponding to various forms of high-frequency system switching.

Introduce the homogeneous weights $m_1, ..., m_{n_x} > 0$ of the coordinates $x_1, ..., x_{n_x}$ in $\mathbb{R}^{n_x}$, $\deg x_i = m_i$. The dilation [4] is introduced for any $\kappa \geq 0$ as the linear transformation

$$d_\kappa : (x_1, x_2, ..., x_{n_x})^T \mapsto (\kappa^{m_1}x_1, \kappa^{m_2}x_2, ..., \kappa^{m_{n_x}}x_{n_x})^T.$$

Recall [4] that a vector function $g : \mathbb{R}^{n_x} \to \mathbb{R}^k$ is said to have the homogeneity degree (HD) (weight) $q \in \mathbb{R}$, $\deg g = q$, if the identity $g(x) = \kappa^{-q}g(d_\kappa x)$ holds for any $x \in \mathbb{R}^{n_x}$ and $\kappa > 0$. Similarly we say that a *set function* $G : \mathbb{R}^{n_x} \to 2^{\mathbb{R}^k}$, $G(x) \subset \mathbb{R}^k$, is homogeneous of the HD q if $G(x) = \kappa^{-q}G(d_\kappa x)$.

Consider the combined time-coordinate transformation

$$(t, x) \mapsto (\kappa^{-q}t, d_\kappa x), \quad \kappa > 0, \tag{3}$$

where the number $-q \in \mathbb{R}$ might naturally be interpreted as the weight of t, $\deg t = -q$. *The DI* $\dot{x} \in F(x), x \in \mathbb{R}^{n_x}$, and *the vector-set field* $F(x) \subset T_x\mathbb{R}^{n_x}$ are called homogeneous of the HD q, if the identity $F(x) = \kappa^{-q}d_\kappa^{-1}F(d_\kappa x)$ holds for any x and $\kappa > 0$ [23]. It implies that the DI (1) is invariant with respect to (3), i.e. $\dot{x} \in F(x) \Leftrightarrow \frac{d(d_\kappa x)}{d(\kappa^{-q}t)} \in F(d_\kappa x)$.

Note that we distinguish a *vector function* $f : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$, $f : x \mapsto f(x) \in R^{n_x}$, and a *vector field* $f : \mathbb{R}^{n_x} \to T\mathbb{R}^{n_x}$, $f : x \mapsto f(x) \in T_x\mathbb{R}^{n_x}$. In the considered fixed coordinates the difference is in the transformation of the function/field induced by transformation (3).

A system of differential equations (DEs) $\dot{x}_i = f_i(x)$, $f_i : \mathbb{R}^{n_x} \to \mathbb{R}$, $i = 1, ..., n_x$, can be considered as a particular case of DI, when the set $F(x)$ has only one element $f(x) = (f_1(x), ..., f_{n_x}(x))^T \in T_x\mathbb{R}^{n_x}$. Then the above homogeneity definition of DI and of the vector field $f : \mathbb{R}^{n_x} \to T\mathbb{R}^{n_x}$ turns into the classical definition $\deg \dot{x}_i = \deg x_i - \deg t = m_i + q = \deg f_i$ [4].

The inequalities $\deg x_i + q \geq 0$, $i = 1, ..., n_x$, hold for homogeneous Filippov DIs and DEs [28]. If the homogeneous vector field $f : \mathbb{R}^{n_x} \to T\mathbb{R}^{n_x}$ is discontinuous, the homogeneous DE $\dot{x} = f(x)$ is equivalent to the corresponding homogeneous Filippov DI (1), $F = K_F[f]$ [23].

A function $g : \mathbb{R}^{n_x} \to \mathbb{R}^k$ is called *quasi-continuous* [24], if it is continuous everywhere except, possibly, at $x = 0$.

Note that if $g : \mathbb{R}^{n_x} \to \mathbb{R}^k$ is continuous and $\deg g > 0$, then $g(0) = 0$. If g is not identical zero, and $\deg g \leq 0$ then it can be quasi-continuous, but it is never continuous at $x = 0$ for $\deg g < 0$, and is only continuous for $\deg g = 0$ if g is a constant.

Any continuous positive-definite function of the HD 1 is called a homogeneous norm. We denote it $||x||_h$. In particular, denote $||x||_{h\infty} = \max_i |x_i|^{1/m_i}$. The quotient $||x||_{h1}/||x||_{h2}$ of any two homogeneous norms is uniformly bounded and separated from zero for $x \neq 0$.

Note that the weights/degrees q, $m_1, ..., m_{n_x}$ are defined up to the proportionality $\tilde{q} = \gamma q, \tilde{m}_i = \gamma m_i$ for any $\gamma > 0$. Such proportional change of weights certainly does not preserve homogeneous norms.

Lemma 1. *Let g be differentiable and homogeneous, then its partial derivatives are also homogeneous, and* $\deg \frac{\partial}{\partial x_i} g = \deg g - m_i$, *whereas its total time derivative $\dot{g}(x(t))$ along any solution $x(t)$ of DI (1) satisfies $\dot{g} \in G(x) = \frac{\partial}{\partial x} g(x)F(x) \subset \mathbb{R}^k$. Also $G(x)$ is a homogeneous vector-set function, and* $\deg G = \deg g - \deg t = \deg g + q$,

DOI: 10.1002/rnc

Proof. The calculation shows that
$\frac{\partial g}{\partial x_i}(d_\kappa x) = \lim_{\Delta x_i \to 0} \frac{g(d_\kappa x + (0,...,\kappa^{m_i}\Delta x_i,...,0)) - g(d_\kappa x)}{\kappa^{m_i}\Delta x_i} = \kappa^{\deg g - m_i}\frac{\partial g}{\partial x_i}(x)$;
$G(d_\kappa x) = \frac{\partial g}{\partial x}(d_\kappa x)F(d_\kappa x) =$
$\frac{\partial g}{\partial x}(x)\,\mathrm{diag}(\kappa^{\deg g - m_1}, ..., \kappa^{\deg g - m_{n_x}})\,\mathrm{diag}(\kappa^{m_1+q}, ..., \kappa^{m_{n_x}+q})F(x) = \kappa^{\deg g + q}\frac{\partial}{\partial x}g(x)F(x)$.
Here $\mathrm{diag}(\cdot)$ denotes a diagonal matrix, $\mathrm{diag}(\kappa^{m_1}, ..., \kappa^{m_{n_x}}) = d_\kappa$. □

It follows from Lemma 1 that any homogeneous function has only finite number of continuous partial derivatives which are not identical zero.

Any asymptotically stable homogeneous Filippov DI (1) has a proper *global* C^∞ Lyapunov function [8]. Moreover, it has a proper *homogeneous* C^k Lyapunov function $V(x)$, $\deg V = a$, for any a, k satisfying $a > \max[q, k \max m_i]$ [39, 7]. The following simple lemma is often used in the homogeneous backstepping [17, 33, 37].

Lemma 2. *Consider the homogeneous DI* (1)*, differentiable homogeneous functions* $v_1, v_2 : R^{n_x} \to \mathbb{R}$*,* $\deg v_1(x) = \deg v_2(x) = m_v$*, and a continuous homogeneous function* $\theta(x,s)$*,* $\theta : \mathbb{R}^{n_x+1} \to \mathbb{R}$*,* $\deg s = m_v > 0$*,* $\deg \theta = m_\theta \geq 0$*. Let the function* $\theta(x,s)$ *be strictly increasing in* s *for any fixed* x*,* $\theta(x, v_1(x)) \equiv 0$*. Define*

$$\Theta(x) = \int_{v_1(x)}^{v_2(x)} \theta(x,s)ds.$$

Then the function $\Theta(x)$ *is non-negative and homogeneous,* $\{x \in \mathbb{R}^{n_x} | \Theta(x) = 0\} = \{x \in \mathbb{R}^{n_x} | v_1(x) = v_2(x)\}$*,* $\deg \Theta = m_\theta + m_v$*, and*

$$\dot{\Theta} \in \theta(x, v_2(x))\frac{\partial}{\partial x}v_2(x)F(x) + \int_{v_1(x)}^{v_2(x)} \frac{\partial}{\partial x}\theta(x,s)ds\; F(x),$$

where the right-hand side is a homogeneous set-valued function of the degree $m_\theta + m_v + q$*. If* $\dot{\Theta}$ *is a single-valued function, then also* $\deg \dot{\Theta} = m_\theta + m_v + q$*.*

The proof is straightforward.

Lemma 3 ([1])**.** *Let two homogeneous quasi-continuous functions* $v_1, v_2 : R^{n_x} \to \mathbb{R}$*,* $\deg v_1(x) = \deg v_2(x)$*, feature the property that for* $x \neq 0$ *the function* v_1 *is positive on the set* $v_2 = 0$*, whereas* v_2 *is non-negative. Then for sufficiently large* $\gamma > 0$ *the combination* $v_1(x) + \gamma v_2(x)$ *is positive definite.*

Proof. It is enough to consider the functions on the homogeneous sphere $S_1 = \{x|\, ||x||_h = 1\}$. Due to the continuity v_1 remains positive in some open vicinity Ω of the set $v_2 = 0$ on S_1. The function v_2 is strictly positive and separated from zero on the compact set $S_1 \backslash \Omega$. □

Contractivity principle. It is proved in [23, 28, 27] that Filippov homogeneous DI (1) is asymptotically stable iff there exist numbers $T, R_M, R_m > 0$, $R_M > R_m$, such that all solutions starting in B_{R_M} at $t = 0$ concentrate in B_{R_m} at $t = T$. Moreover, if $q > 0$ then asymptotic stability (AS) implies FxT convergence to any ball around 0, AS is exponential for $q = 0$, and if $q < 0$ then AS implies FT stability.

Consider (1) in the presence of the maximal delay $\tau \geq 0$ and noises of the magnitudes $\varepsilon_i \geq 0$, $i = 1, 2, ..., n_x$,

$$\dot{x} \in F(x(t - \tau[0,1]) + ([-\varepsilon_1, \varepsilon_1], ..., [-\varepsilon_{n_x}, \varepsilon_{n_x}])^T). \tag{4}$$

For simplicity suppose that solutions depend on the initial condition $x(t_0)$, but do not depend on the values of $x(t)$ for $t < t_0$. In particular, that condition holds for sampling-based zero-hold control and Euler approximations. The general case is considered in [27, 28].

Accuracy. All extendable-in-time solutions of the disturbed DI (4) starting from some time satisfy the inequalities $|x_i| \leq \mu_i \rho^{m_i}$ for some $\mu_i > 0$. Here $\rho = \max[||\varepsilon||_{h\infty}, \tau^{-1/q}]$ for $q < 0$; $\rho = ||\varepsilon||_{h\infty}$ for $q = 0$ and sufficiently small τ; $\rho = ||\varepsilon||_{h\infty}$ for $q > 0$. In the latter case, in order to avoid the possible solution escape to infinity, the initial value $x(t_0)$ is to be sufficiently small for each fixed τ, or τ is to be sufficiently small for each fixed $R > 0$, $||x(t_0)||_{h\infty} \leq R$.

3. HOMOGENEOUS OBSERVATION AND FILTERING

In this section we introduce homogeneous filtering differentiators which generalize the SM-based filtering differentiators [32], [30] to any HD and extend some well-known results on SM-based exact differentiation from [22]. New differentiators do not contain discontinuities, and under usual circumstances are only capable of exactly differentiating polynomials. Alternatively, their application can be based on the high-gain technique [3, 34]. Nevertheless, further in Section 4 we establish their direct application for the output-feedback stabilization of disturbed homogeneous integrator chains.

3.1. Homogeneous differentiation

Let the measured input $f(t) = f_0(t) + \eta(t)$ consist of a Lebesgue-measurable noise $\eta(t)$ and an unknown basic signal $f_0(t)$ with the locally Lipschitzian nth derivative. Restrictions on the noise and f_0 are introduced later. The problem is to evaluate the derivatives $f_0^{(i)}(t)$, $i = 0, 1, ..., n_d$, in real time.

Introduce the weights $\deg z_i = 1 + iq,\ i = 0, 1, ..., n_d$. The homogeneous filtering differentiator of the differentiation order $n_d \geq 0$, the HD $q > -1/(n_d + 1)$, with the input $f(t)$ and the parameter $L > 0$ has the form

$$\begin{aligned} \dot{z}_0 &= -\tilde{\lambda}_{n_d} L^{|q|} \lfloor z_0 - f(t) \rceil^{1+q} + z_1, \\ \dot{z}_1 &= -\tilde{\lambda}_{n_d-1} L^{2|q|} \lfloor z_0 - f(t) \rceil^{1+2q} + z_2, \\ &\dots \\ \dot{z}_{n_d} &= -\tilde{\lambda}_0 L^{(n_d+1)|q|} \lfloor z_0 - f(t) \rceil^{1+(n_d+1)q}. \end{aligned} \tag{5}$$

Here $\tilde{\lambda}_j > 0$, $j = 0, ..., n_d$. The outputs $z_i(t)$ are to estimate the derivatives $f_0^{(i)}(t)$. System (5) is understood in the Filippov sense and is homogeneous for $f(t) \equiv 0$.

Here and further we assume that non-existing variables are replaced with zero. In particular, get the differentiator $\dot{z}_0 = -\tilde{\lambda}_0 L^{|q|} \lfloor z_0 - f(t) \rceil^{1+q}$ for $n_d = 0$.

The introduced differentiator coincides with the "continuous differentiator" [40] for $-1/(n_d + 1) < q < 0$, $L = 1$. A stable linear filter [3] is produced for $q = 0$, if the polynomial $s^{n_d+1} + \tilde{\lambda}_{n_d} s^{n_d} + ... + \tilde{\lambda}_0$ is Hurwitz. It is called high gain observer (HGO [3]), if the eigenvalues are large. The nonlinear filter is stable, provided the above polynomial is Hurwitz, and $|q|$ is small enough. Such filter for small $q < 0$, $L = 1$ is considered in [34], whereas small $q > 0$ is considered in [2]. The well-known SM-based differentiator [22] is produced for $q = -1/(n_d + 1)$. The general case $q > 0$ has never been previously considered.

Theorem 1. *Fix any $\gamma_L \geq 0$ and let $\eta = 0$. Then there exist such $\tilde{\lambda}_0, ..., \tilde{\lambda}_{n_d} > 0$ that differentiator* (5) *produces asymptotically exact estimations z_i of $f_0^{(i)}(t)$ provided $L \geq 1$ and*

$$|f_0^{(n_d+1)}(t)| \leq \gamma_L \max_{i=0,...,n_d} \left[L^{\frac{(n_d+1-i)|q|}{1+iq}} |z_i(t) - f_0^{(i)}(t)|^{\frac{1+(n_d+1)q}{1+iq}} \right] \tag{6}$$

holds for all $t > 0$. The convergence is in FT for $q < 0$, in FxT to any ball of errors $z_i - f_0^{(i)}$ for $q > 0$ and exponential for $q = 0$.

It is formally assumed here that $A^0 = 1$ for any $A \geq 0$. *Here and further theorem proofs are placed in the Appendix.*

In the particular case $q = -1/(n_d + 1)$ convergence condition (6) reduces to the simple inequality $|f_0^{(n_d+1)}(t)| \leq \gamma_L L$. In that case differentiator (5) in FT provides for the exact derivatives of f_0 for $\eta = 0, \gamma_L = 1$ and $L > 0$ [22].

In the case $\gamma_L = 0$ condition (6) means exactness on polynomials of degrees not exceeding n_d. Naturally, if $q > -1/(n_d + 1)$, in stand-alone (signal processing) applications that convergence condition practically only holds for such polynomials. Nevertheless, this differentiator is further proved to be applicable for the global output-feedback stabilization of homogeneous systems (Section 4).

DOI: 10.1002/rnc

3.2. Recursion of differentiators and their parameters

It is not easy to find n_d+1 coefficients of an n_dth-order differentiator for $n_d>1$. Finding them using Lyapunov functions is possible, but is numerically difficult and still seems to always include some numeric procedure. In particular, the recent important paper [11] only succeeds to find parameters of the SM-based differentiators (i.e. for $q=-1/(n_d+1)$) for $n_d=1,2,3$. A bit later we present the coefficients till $n_d=12$ (Section 3.3).

Introduce the short notation for (5):

$$\dot z = D_{n_d,q}(z_0-f,z,L,\vec{\tilde\lambda}_{n_d}). \tag{7}$$

The following recursion theorem establishes the step-by-step way of choosing the coefficients $\vec{\tilde\lambda}_{n_d+1}$ of the (n_d+1)th-order differentiator, provided coefficients of the n_dth-order differentiator (7) of *the same HD* q are available, and $q\ge -1/(n_d+2)$, $\gamma_L\ge 0$ are *fixed*.

Theorem 2. *Any* $\tilde\lambda_0>\gamma_L$, $\gamma_L\ge 0$, $q\ge -1$ *constitute a proper parametric choice for Theorem 1 if* $n_d=0$. *Let* $n_d>0$, *and* $q\ge -1/(n_d+2)$, $\tilde\lambda_0,...,\tilde\lambda_{n_d}>0$ *be properly chosen according to Theorem 1. Consider the* (n_d+1)*th-order differentiator*

$$\begin{array}{l} \dot z_0 = -\tilde\lambda_{n_d+1}L^{|q|}\lfloor z_0-f(t)\rceil^{1+q}+z_1,\\ \vec z_{1,n_d+1}=(z_1,...,z_{n_d+1})^T,\\ \dot{\vec z}_{1,n_d+1}=D_{n_d,q}(z_1-\dot z_0,\vec z_{1,n_d+1},L,\vec{\tilde\lambda}_{n_d}). \end{array} \tag{8}$$

It is reduced to form (5) *by substituting* $\dot z_0$ *from the first equation into the last line and the obvious algebraic simplification. Then for sufficiently large* $\tilde\lambda_{n_d+1}\ge 1$ *the resulting coefficients constitute a proper set of coefficients for the differentiator* (5) *of the order* n_d+1.

Theorem 2 builds a sequence of valid differentiators for each fixed q,γ_L starting from an initial differentiator of a lower order. Only one number $\tilde\lambda_{n_d+1}$ is to be assigned to increase the differentiation order n_d by one. The proof is based on the Lyapunov approach, and a numeric procedure similar to [11] can be established.

Note that the coefficient $\tilde\lambda_{n_d+1}$ in (8) is preserved by the transformation of (8) to the standard form (5) corresponding to the differentiation order n_d+1.

Due to the restriction $q>-1/(n_d+1)$ the length of a differentiators' recursion sequence does not exceed the integer part of $1/|q|$ for each fixed $q<0$. It constitutes a serious obstacle for the recursive parametric choice in that important case. In the case $q=0$ the sequence is infinitely extendable, but classic linear methods are probably more effective. In the case $q>0$ Theorem 2 allows sequences of any length, but only a finite number of steps are available for each initial differentiator.

Let $q\ne 0$, and $s_q=\operatorname{sign} q=\pm 1$. Change all weights proportionally so that the HD be $q/|q|=s_q$. The corresponding weights are $\deg t=-s_q$, $\deg \dot z_{n_d}=d=(1+(n_d+1)q)/|q|\ge 0$, $\deg z_i=d-(n_d+1-i)s_q$, $i=0,1,...,n_d$, and the convergence condition (6) is rewritten as

$$\begin{array}{l} d=1/|q|+(n_d+1)s_q,\ s_q=\operatorname{sign} q,\ q\ne 0,\ L\ge 1,\\ |f_0^{(n_d+1)}(t)|\le \gamma_L \max_{i=0,...,n_d}\left[L^{\frac{n_d+1-i}{d-(n_d+1-i)s_q}}|z_i(t)-f_0^{(i)}(t)|^{\frac{d}{d-(n_d+1-i)s_q}}\right]. \end{array} \tag{9}$$

Correspondingly, differentiator (5) is *identically* rewritten as

$$\begin{array}{l} \dot z_0 = -\tilde\lambda_{n_d}L^{\frac{1}{d-(n_d+1)s_q}}\lfloor z_0-f(t)\rceil^{\frac{d-n_ds_q}{d-(n_d+1)s_q}}+z_1,\\ \dot z_1 = -\tilde\lambda_{n_d-1}L^{\frac{2}{d-(n_d+1)s_q}}\lfloor z_0-f(t)\rceil^{\frac{d-(n_d-1)s_q}{d-(n_d+1)s_q}}+z_2,\\ \quad \dots\\ \dot z_{n_d-1} = -\tilde\lambda_1 L^{\frac{n_d}{d-(n_d+1)s_q}}\lfloor z_0-f(t)\rceil^{\frac{d-s_q}{d-(n_d+1)s_q}}+z_{n_d},\\ \dot z_{n_d} = -\tilde\lambda_0 L^{\frac{n_d+1}{d-(n_d+1)s_q}}\lfloor z_0-f(t)\rceil^{\frac{d}{d-(n_d+1)s_q}}. \end{array} \tag{10}$$

DOI: 10.1002/rnc

Note that if $q \neq 0$ then the condition $q \geq -1/(n_d+1)$ is equivalent to $d \geq 0$ and implies $d-(n_d+1)s_q > 0$. The following alternative recursion theorem is now formulated.

Theorem 3. *Let $n_d > 0$, $s_q = \operatorname{sign}\tilde{q} \neq 0$, $\gamma_L \geq 0$, $\tilde{\lambda}_0, ..., \tilde{\lambda}_{n_d} > 0$ be properly chosen according to Theorem 1 for some HD $\tilde{q} \neq 0$. Let $d = 1/|\tilde{q}| + (n_d+1)s_q \geq 0$, $d-(n_d+2)s_q > 0$. Consider the following differentiator scheme of the differentiation order n_d+1,*

$$\begin{aligned} \dot{z}_0 &= -\tilde{\lambda}_{n_d+1} L^{\frac{1}{d-(n_d+2)s_q}} \lfloor z_0 - f(t) \rceil^{\frac{d-(n_d+1)s_q}{d-(n_d+2)s_q}} + z_1, \\ \vec{z}_{1,n_d+1} &= (z_1, ..., z_{n_d+1})^T, \\ \dot{\vec{z}}_{1,n_d+1} &= D_{n_d,\tilde{q}}(z_1 - \dot{z}_0, \vec{z}_{1,n_d+1}, L, \vec{\tilde{\lambda}}_{n_d}). \end{aligned} \tag{11}$$

It is reducible to form (10) *(and also to* (5)*) by substituting $\dot{z}_0$ from the first equation into the dynamics of $\vec{z}_{1,n_d+1}$ and algebraic simplification. Then for sufficiently large $\tilde{\lambda}_{n_d+1} \geq 1$ the resulting coefficients constitute a proper set of coefficients for the differentiator* (10) *(and* (5)*) of the order n_d+1 with the same parameter d, but of the new HD $q = s_q/(d-(n_d+2)s_q)$.*

Contrary to Theorem 2, Theorem 3 builds infinite recursion sequences for negative HDs, since $s_q = -1$ implies $d-(n_d+2)s_q > 0$. In the case $q > 0$ the recursion sequences stay finite.

Therefore, Theorem 3 establishes a step-by-step algorithm of finding proper coefficients for any differentiator with the HD $q \neq 0$ by calculating the corresponding parameter $d = 1/|q| + (n_d+1)\operatorname{sign} q$ and building the sequence from $n_d = 0$.

In the important particular case $\tilde{q} = -1/(n_d+1)$ get $d = 0$ and the HD q of the differentiator (11) of the order n_d+1 is $q = -1/(n_d+2)$. In that concrete case the convergence condition turns to be $|f_0^{(n_d+1)}| \leq \gamma_L L$, and the differentiator equations are valid for any $L > 0$ (i.e. the requirement $L \geq 1$ is removed [22]). The corresponding SM-based differentiators are FT exact [22] and asymptotically optimal in the presence of bounded sampling noises [31].

Theorem 4. *Let $\tilde{\lambda}_0, ..., \tilde{\lambda}_{n_d} > 0$ be properly chosen according to Theorem 1 for some $n_d \geq 0$, $\gamma_L \geq 0$, $q \geq -1/(n_d+1)$. Then any other sufficiently close choice of parameters $\vec{\tilde{\lambda}}_{n_d}, \gamma_L, q$ keeping $\gamma_L \geq 0$ and $q \geq -1/(n_d+1)$ is also valid.*

3.3. Filtering homogeneous observation

The homogeneous filtering differentiator of the differentiation order $n_d \geq 0$, the filtering order $n_f \geq 0$ and the HD q, $-1/(n_d+1) \leq q < 1/n_f$, with the input $f(t)$ and the parameter $L > 0$ has the form

$$\begin{aligned} \dot{w}_1 &= -\tilde{\lambda}_{n_d+n_f} L^{\frac{|q|}{1-n_f q}} \lfloor w_1 \rceil^{\frac{1-(n_f-1)q}{1-n_f q}} + w_2, \\ \dot{w}_2 &= -\tilde{\lambda}_{n_d+n_f-1} L^{\frac{2|q|}{1-n_f q}} \lfloor w_1 \rceil^{\frac{1-(n_f-2)q}{1-n_f q}} + w_3, \\ &\ \ \dots \\ \dot{w}_{n_f-1} &= -\tilde{\lambda}_{n_d+2} L^{\frac{(n_f-1)|q|}{1-n_f q}} \lfloor w_1 \rceil^{\frac{1-q}{1-n_f q}} + w_{n_f}, \\ \dot{w}_{n_f} &= -\tilde{\lambda}_{n_d+1} L^{\frac{n_f|q|}{1-n_f q}} \lfloor w_1 \rceil^{\frac{1}{1-n_f q}} + w_{n_f+1}, \ w_{n_f+1} = z_0 - f(t), \end{aligned} \tag{12}$$

$$\begin{aligned} \dot{z}_0 &= -\tilde{\lambda}_{n_d} L^{\frac{(1+n_f)|q|}{1-n_f q}} \lfloor w_1 \rceil^{\frac{1+q}{1-n_f q}} + z_1, \\ \dot{z}_1 &= -\tilde{\lambda}_{n_d-1} L^{\frac{(2+n_f)|q|}{1-n_f q}} \lfloor w_1 \rceil^{\frac{1+2q}{1-n_f q}} + z_2, \\ &\ \ \dots \\ \dot{z}_{n_d} &= -\tilde{\lambda}_0 L^{\frac{(n_d+1+n_f)|q|}{1-n_f q}} \lfloor w_1 \rceil^{\frac{1+(n_d+1)q}{1-n_f q}}. \end{aligned} \tag{13}$$

In the case $f(t) \equiv 0$ system (12), (13) becomes homogeneous with the weights $\deg t = -q$, $\deg w_i = 1-(n_f+1-i)q$, $i = 1, ..., n_f+1$, $\deg z_i = 1+iq$, $i = 0, 1, ..., n_d$.

The parameters $\tilde{\lambda}_j > 0$, $j = 0, ..., n_d+n_f$, are assumed to constitute a valid parametric set for the differentiator of the form (5) corresponding to the differentiation order n_d+n_f.

Note that in the case $n_f=0$ the DEs in (12) disappear, $w_1=z_0-f(t)$, and (12), (13) are reduced to (5). That arrangement turns differentiator (5) into the particular case of (12), (13) for $n_f=0$.

In the case $q=-1/(n_d+1)$ the SM-based filtering differentiator [30] is obtained.

Theorem 5. *Fix any $\gamma_L\geq 0$. Let $L\geq 1$, $\eta=0$, and (6) hold. Then the parameters $\tilde\lambda_0,...,\tilde\lambda_{n_d+n_f}>0$ chosen as in Theorem 1, but applied in filter (12), (13) provide for the asymptotic convergence of the outputs z_i to $f_0^{(i)}(t)$. The convergence is in FT for $q<0$, in FxT to any ball of errors $z_i-f_0^{(i)}$ for $q>0$, and exponential for $q=0$.*

Finding proper parameters is facilitated by the identical recursive form of the filter

$$\begin{aligned}
\dot w_1 &= -\lambda_{n_d+n_f}L^{\frac{|q|}{1-n_f q}}\lfloor w_1\rceil^{\frac{1-(n_f-1)q}{1-n_f q}}+w_2,\\
\dot w_2 &= -\lambda_{n_d+n_f-1}L^{\frac{|q|}{1-(n_f-1)q}}\lfloor w_2-\dot w_1\rceil^{\frac{1-(n_f-2)q}{1-(n_f-1)q}}+w_3,\\
&\dots\\
\dot w_{n_f-1} &= -\lambda_{n_d+2}L^{\frac{|q|}{1-2q}}\lfloor w_{n_f-1}-\dot w_{n_f-2}\rceil^{\frac{1-q}{1-2q}}+w_{n_f},\\
\dot w_{n_f} &= -\lambda_{n_d+1}L^{\frac{|q|}{1-q}}\lfloor w_{n_f}-\dot w_{n_f-1}\rceil^{\frac{1}{1-q}}+w_{n_f+1},\\
w_{n_f+1} &= z_0-f(t),
\end{aligned}\tag{14}$$

$$\begin{aligned}
\dot z_0 &= -\lambda_{n_d}L^{\frac{|q|}{1}}\lfloor w_{n_f+1}-\dot w_{n_f}\rceil^{\frac{1+q}{1}}+z_1,\\
\dot z_1 &= -\lambda_{n_d-1}L^{\frac{|q|}{1+q}}\lfloor z_1-\dot z_0\rceil^{\frac{1+2q}{1+q}}+z_2,\\
&\dots\\
\dot z_{n_d} &= -\lambda_0 L^{\frac{|q|}{1+n_d q}}\lfloor z_{n_d}-\dot z_{n_d-1}\rceil^{\frac{1+(n_d+1)q}{1+n_d q}}.
\end{aligned}\tag{15}$$

If $n_f=0$ the value $\dot w_{n_f}=0$ is taken in (15). If $n_d=0$ then (15) only contains the first equation, and z_1 is replaced with 0. It is easy to see that

$$\begin{aligned}
\tilde\lambda_{n_d+n_f} &= \lambda_{n_d+n_f},\\
\tilde\lambda_{n_d+n_f-i} &= \lambda_{n_d+n_f-i}\cdot\tilde\lambda_{n_d+n_f-i+1}^{\frac{1-(n_f-i-1)q}{1-(n_f-i)q}},\ i=1,2,...,n_f,\\
\tilde\lambda_{n_d-i} &= \lambda_{n_d-i}\cdot\tilde\lambda_{n_d-i+1}^{\frac{1+(i+1)q}{1+iq}},\ i=1,2,...,n_d.
\end{aligned}\tag{16}$$

Let $q\neq 0$. Recall that $s_q=\operatorname{sign} q$. Then the recursive form (14), (15) is rewritten as

$$\begin{aligned}
\dot w_1 &= -\lambda_{n_d+n_f}L^{\frac{1}{d-(n_d+n_f+1)s_q}}\lfloor w_1\rceil^{\frac{d-(n_d+n_f)s_q}{d-(n_d+n_f+1)s_q}}+w_2,\\
\dot w_2 &= -\lambda_{n_d+n_f-1}L^{\frac{1}{d-(n_d+n_f)s_q}}\lfloor w_2-\dot w_1\rceil^{\frac{d-(n_d+n_f-1)s_q}{d-(n_d+n_f)s_q}}+w_3,\\
&\dots\\
\dot w_{n_f} &= -\lambda_{n_d+1}L^{\frac{1}{d-(n_d+2)s_q}}\lfloor w_{n_f}-\dot w_{n_f-1}\rceil^{\frac{d-(n_d+1)s_q}{d-(n_d+2)s_q}}+w_{n_f+1},\\
w_{n_f+1} &= z_0-f(t),\ s_q=\operatorname{sign} q,
\end{aligned}\tag{17}$$

$$\begin{aligned}
\dot z_0 &= -\lambda_{n_d}L^{\frac{1}{d-(n_d+1)s_q}}\lfloor w_{n_f+1}-\dot w_{n_f}\rceil^{\frac{d-n_d s_q}{d-(n_d+1)s_q}}+z_1,\\
\dot z_1 &= -\lambda_{n_d-1}L^{\frac{1}{d-n_d s_q}}\lfloor z_1-\dot z_0\rceil^{\frac{d-(n_d-1)s_q}{d-n_d s_q}}+z_2,\\
&\dots\\
\dot z_{n_d} &= -\lambda_0 L^{\frac{1}{d-s_q}}\lfloor z_{n_d}-\dot z_{n_d-1}\rceil^{\frac{d}{d-s_q}},
\end{aligned}\tag{18}$$

and (16) gets the simpler form

$$\begin{aligned}
\tilde\lambda_k &= \lambda_k,\ k=n_d+n_f;\\
\tilde\lambda_{k-i} &= \lambda_{k-i}\cdot\tilde\lambda_{k-i+1}^{\frac{d-(k-i)s_q}{d-(k-i+1)s_q}},\ i=1,2,...,k.
\end{aligned}\tag{19}$$

The following theorem establishes the recursive choice of parameters λ_i, $i=0,1,...$, for $q=0$, or, in the case $q\neq 0$, for each fixed $d\geq 0$. It is the direct corollary of Theorems 2, 3.

DOI: 10.1002/rnc

Theorem 6. *Let $\lambda_0 > \gamma_L \geq 0$. Then the parameters λ_i of the filter* (14), (15) *can be taken from parametric sequences $\vec{\lambda} = \{\lambda_0, \lambda_1, ...\}$ which are valid for any admissible $n_f, n_d \geq 0$ (i.e. for $-1/(n_d+1) \leq q < 1/n_f$) and are specified below. The estimations $z_i(t)$ asymptotically converge to the exact derivatives $f^{(i)}(t)$ for any input $f(t)$ satisfying* (6)*. Parameters $\lambda_i \geq 1$ are chosen one-by-one sufficiently large starting from λ_1. Concretely, the following holds:*

- *Let $q < 0$ and $d \geq 0$. Then the sequence $\vec{\lambda}$ is separately build for each d and is **infinite**. The convergence of the filter* (12), (13) *(and* (17), (18)*) is in FT.*
- *Let $q > 0$ and $d \geq 0$. Then the sequence $\vec{\lambda}$ is separately build for each d and is **finite**, $n_f + n_d \leq \max\{k \in \mathbb{Z} \mid d - k - 1 > 0\}$. The convergence of the filter* (12), (13) *(and* (17), (18)*) is slower then exponential, but it features FxT convergence to any ball around the origin of the errors' space.*
- *Let $q = 0$. Then L disappears both from the filter* (14), (15) *(and* (12), (13)*) and the convergence condition* (6)*. The corresponding sequence $\vec{\lambda}$ is **infinite**. The convergence of the resulting filter* (14), (15) *is exponential.*

Note that both recursion Theorems 2, 3 at each step directly provide for the next element of the sequence $\vec{\lambda}$, since $\tilde{\lambda}_{n_d+n_f} = \lambda_{n_d+n_f}$. All three forms (12), (13); (14), (15); (17), (18) are *completely identical* for $q \neq 0$.

The particular case $d = 0$ implies the SMC case $q = -1/(n_d+1)$. That is the important case of the exact SM-based filtering differentiation [30]. In that concrete case the convergence condition turns to be $|f_0^{(n_d+1)}| \leq \gamma_L L$, and the differentiator equations are valid for any $L > 0$ (i.e. the requirement $L \geq 1$ is removed).

The simulation-based step-by-step assignment of λ_i, $i = 1, 2, ...$, turns out to be surprisingly easy. Let $\lambda_0 = 1.1$, $\gamma_L = 1$, $d = 0$. The following is the corresponding parametric set well validated up to the order $n_d + n_f = 12$:

$$q = -\tfrac{1}{n_d+1},\ d = 0,\ 0 \leq n_d + n_f \leq 12,\ \vec{\lambda}_{12} = (1.1, 1.5, 2, 3, 5, 7, 10, 12, 14, 17, 20, 26, 32). \quad (20)$$

Parameters $\vec{\tilde{\lambda}}_{n_d}$ are calculated according to (19) for $d = 0$ (also (16) produces the same numbers). The authors prefer to directly apply the recursive form (17), (18) (or (14), (15)).

Note that parameters (20) extend another known set valid up to the order $n_d + n_f = 7$ [29, 30]. The first known set [22] was valid for $n_d + n_f \leq 5$. It is not difficult to further extend this sequence basing on Theorem 3. The corresponding filtering SM-based differentiators are FT exact in the absence of noises and asymptotically optimal in the presence of bounded sampling noises [30, 32].

It follows from Theorem 6 that one can choose a sequence $\{\lambda_i\}$ valid simultaneously for any predefined finite set of HDs q_ω and $\gamma_{L\omega} \geq 0$. In particular, the sequence (20) is also valid for $q = 0$ with $\gamma_L = 0$, i.e. the corresponding linear filters produced for $q = 0$, $n_d + n_f \leq 12$, are Hurwitz. There exists a hypothesis that it is always true in the SM case $d = 0$ [38].

Introduce the short notation for (12), (13):

$$\begin{aligned} \dot{w} &= \Omega_{n_d,n_f,q}(w, z_0 - f, L, \vec{\lambda}_{n_d+n_f}), \qquad && w \in \mathbb{R}^{n_f}, \\ \dot{z} &= D_{n_d,n_f,q}(w_1, z, L, \vec{\lambda}_{n_d+n_f}), && z \in \mathbb{R}^{n_d+1}, \end{aligned} \quad (21)$$

where $\vec{\lambda}_{n_d+n_f} = \{\lambda_0, ..., \lambda_{n_d+n_f}\}$ are provided by Theorem 6.

Note that the parameter $d = 1/|q| + (n_d+1)\,\mathrm{sign}\, q$ introduced in (9) tends to infinity as q approaches zero. Correspondingly the formal substitution $d = \infty$ implies the convergence condition $|f_0^{(n_d+1)}(t)| \leq \gamma_L \max_i |z_i(t) - f_0^{(i)}(t)|$, L disappears from (17), (18) and all powers turn into 1. The same condition is obtained from (6) by the direct substitution of $q = 0$. That case is considered in Section 3.4 below.

3.4. The linear case $q = 0$

In the important particular case $q = 0$ filter (12), (13) turns into the modified linear filter [3] of the form

$$\begin{aligned} \dot{w}_1 &= -\tilde{\lambda}_{n_d+n_f}\, w_1 + w_2, \\ \dot{w}_2 &= -\tilde{\lambda}_{n_d+n_f-1}\, w_1 + w_3, \\ &\dots \\ \dot{w}_{n_f-1} &= -\tilde{\lambda}_{n_d+2}\, w_1 + w_{n_f}, \\ \dot{w}_{n_f} &= -\tilde{\lambda}_{n_d+1}\, w_1 + w_{n_f+1},\ w_{n_f+1} = z_0 - f(t), \end{aligned} \tag{22}$$

$$\begin{aligned} \dot{z}_0 &= -\tilde{\lambda}_{n_d}\, w_1 + z_1, \\ \dot{z}_1 &= -\tilde{\lambda}_{n_d-1}\, w_1 + z_2, \\ &\dots \\ \dot{z}_{n_d} &= -\tilde{\lambda}_0\, w_1, \end{aligned} \tag{23}$$

with the corresponding convergence condition $|f_0^{(n_d+1)}(t)| \le \gamma_L \max_i |z_i(t) - f_0^{(i)}(t)|$.

Asymptotic stability of filter (22) is completely determined by its characteristic polynomial $p(s) = s^{n_d+n_f+1} + \tilde{\lambda}_{n_d+n_f} s^{n_d+n_f} + \dots + \tilde{\lambda}_0$. The classical HGO corresponds to $n_f = 0$ and large eigenvalues. It is suggested in [42] to choose the multiple eigenvalue $\hat{\varepsilon}^{-1}$, $0 < \hat{\varepsilon} << 1$. In that case $p(s) = (s + \hat{\varepsilon}^{-1})^{n_d+n_f+1}$ and

$$\tilde{\lambda}_j = \begin{pmatrix} n_d + n_f + 1 \\ j \end{pmatrix} \hat{\varepsilon}^{-(n_d+n_f+1-j)},\ \tilde{\lambda}_0 = \hat{\varepsilon}^{-(n_d+n_f+1)}. \tag{24}$$

In the case $n_f = 0$ the transfer function from f to z_i takes the form $\frac{s^i + \hat{\varepsilon}\hat{p}_i(s,\hat{\varepsilon})}{1+\hat{\varepsilon}\hat{p}(s,\hat{\varepsilon})}$, where $\hat{p}_i, \hat{p}$ are some polynomial functions [42].

Filter (22) is to be naturally called a *filtering* HGO for $n_f > 0$. In that case it gets new filtering capabilities (Section 5) due to its transfer function from f to z_i having the form $(\tilde{\lambda}_0 s^i + \tilde{\lambda}_1 s^{i+1} + \dots + \tilde{\lambda}_{n_d-i} s^{n_d})/p(s)$.

In the case $q = 0$ the recursive equations (14), (15) take the form

$$\begin{aligned} \dot{w}_1 &= -\lambda_{n_d+n_f} w_1 + w_2, \\ \dot{w}_2 &= -\lambda_{n_d+n_f-1}(w_2 - \dot{w}_1) + w_3, \\ &\dots \\ \dot{w}_{n_f} &= -\lambda_{n_d+1}(w_{n_f} - \dot{w}_{n_f-1}) + w_{n_f+1},\ w_{n_f+1} = z_0 - f(t), \end{aligned} \tag{25}$$

$$\begin{aligned} \dot{z}_0 &= -\lambda_{n_d}(w_{n_f+1} - \dot{w}_{n_f}) + z_1, \\ \dot{z}_1 &= -\lambda_{n_d-1}(z_1 - \dot{z}_0) + z_2, \\ &\dots \\ \dot{z}_{n_d} &= -\lambda_0(z_{n_d} - \dot{z}_{n_d-1}), \end{aligned} \tag{26}$$

which is formally obtained from (17), (18) for $L = 1,\ d = 1,\ s_q = 0$. While Theorem 6 is also true for $q = 0$ the corresponding standard linear technique (24) [42] of choosing the parameters is obviously more adequate.

4. HOMOGENEOUS STABILIZATION

4.1. The problem statement

Let $\sigma \in \mathbb{R}$, introduce the homogeneous weights $\deg \sigma^{(i)} = 1 + iq,\ i = 0, 1, \dots, n$. Denote $\vec{\sigma}_k = (\sigma, \dot{\sigma}, \dots, \sigma^{(k)})^T$, $k \in \mathbb{N} \cup \{0\}$. Consider the DI

$$\begin{aligned} &\sigma^{(n)} \in [-C, C]||\vec{\sigma}_{n-1}||_h^{1+nq} + [K_m, K_M]u, \\ &C \ge 0,\ 0 < K_m \le K_M, \end{aligned} \tag{27}$$

where $u \in \mathbb{R}$ is the control, $||\cdot||_h$ is some homogeneous norm. The problem is to globally asymptotically stabilize the system while only using the real-time measurements of σ. The solution is to be robust with respect to *unaccounted for* sampling noises.

A number of homogeneous output-feedback stabilizers are known for $q > -1/n$ [36, 37, 21]. The robustness of homogeneous systems to sampling noises is established in [27].

In the case $q = 0$ the problem is universally solved by the HGO feedback application of linear control [3, 19]. The problem also has a well-known robust output-feedback SMC solution for $q = -1/n$ [22, 23]. The standard differentiators with constant [22] or variable [29] gains are only locally applicable for $q \neq -1/n$, since $||\vec{\sigma}_{n-1}||_h$ is not available and possibly features higher than exponential growth.

Assuming that a locally-bounded homogeneous stabilizing controller is available, below we present a robust homogeneous observer applicable with any such stabilizer. That separation-principle problem remains unsolved for $q \neq 0, -1/n$.

4.2. Output feedback stabilization and separation principle

In practice condition (6) is not natural for $q \neq -1/(n_d+1)$, and, generally speaking, differentiators (12), (13) require $f^{(n_d+1)} \equiv 0$, i.e. only differentiate polynomials of the order n_d and less. Nevertheless, they still can be used as observers in a homogeneous feedback.

Theorem 7. *Let $q \neq 0$, $\gamma_L > 0$, and assume that a locally-bounded homogeneous control $u = U(\vec{\sigma}_{n-1})$, $\deg U = 1 + nq$, asymptotically stabilizes system (27). Then the output-feedback control*

$$\begin{aligned} &u = U(z), \\ &\dot{w} = \Omega_{n-1,n_f,q}(w, z_0 - \sigma, L, \vec{\lambda}_{n+n_f-1}), \\ &\dot{z} = D_{n-1,n_f,q}(w_1, z, L, \vec{\lambda}_{n+n_f-1}) \end{aligned} \tag{28}$$

globally asymptotically stabilizes system (27) for any $n_f \geq 0$ and sufficiently large L. The convergence is in FT for $q < 0$ and in FxT to any ball around the origin for $q > 0$.

In the case $q = 0$ choosing $\vec{\lambda}_{n-1}$ corresponding to sufficiently large γ_L (Theorem 6) renders the closed-loop system globally exponentially stable.

Note that in the case $q = 0$ it is reasonable to simply assign a large multiple eigenvalue $\hat{\varepsilon}^{-1}$ taking the coefficients (24) [42].

5. PERFORMANCE IN THE PRESENCE OF NOISE

Let us first explain the idea of the filtering extension. Consider the case $n_f = 1$, $f(t) = f_0(t) + \eta_0(t) + \eta_1(t)$, and let $|\eta_0(t)| \leq \varepsilon_0$, $\xi(t) = \int_0^t \eta_1(s)ds$, $|\xi(t)| \leq \varepsilon_1$ hold for any $t \geq 0$. Consider the dynamics of w_1 taken from (12):

$$\dot{w}_1 = -\tilde{\lambda}_{n_d+1} L^{\frac{|q|}{1-q}} \lfloor w_1 \rceil^{\frac{1-(n_f-1)q}{1-n_f q}} + z_0 - f_0 - \eta_0(t) - \eta_1(t).$$

It can be rewritten as

$$\tfrac{d}{dt}(w_1 + \xi(t)) = -\tilde{\lambda}_{n_d+1} L^{\frac{|q|}{1-q}} \lfloor w_1 + \xi(t) - \xi(t) \rceil^{\frac{1-(n_f-1)q}{1-n_f q}} + z_0 - f_0 - \eta_0(t).$$

Now denoting $\omega_1 = w_1 + \xi(t)$, $\zeta_i = z_i - f_0^{(i)}$, $i = 0, 1, ..., n_d$, obtain

$$\dot{\omega}_1 \in -\tilde{\lambda}_{n_d+1} L^{\frac{|q|}{1-q}} \lfloor \omega_1 + [-\varepsilon_1, \varepsilon_1] \rceil^{\frac{1-(n_f-1)q}{1-n_f q}} + \zeta_0 + [-\varepsilon_0, \varepsilon_0],$$

and the overall dynamics of ω_1, z coinciding with the disturbed asymptotically stable error dynamics of the differentiator (5), but of the order $n_d + 1$.

In spite of η_1 possibly being a very large signal, its integral can be small, $|\xi| \leq \varepsilon_1$, in which case also the disturbance of the error dynamics is small for $\varepsilon_0, \varepsilon_1 << 1$. In its turn it implies small differentiation errors. The following is the corresponding notion from [30].

Definition 1 ([30]). *A signal $\nu(t)$, $\nu:[0,\infty)\to\mathbb{R}$, is called globally filterable, or a signal of the (global) filtering order $k\geq 0$, if it is a locally integrable Lebesgue-measurable function, and there exists a globally bounded Caratheodory solution $\xi(t)$, $\xi:[0,\infty)\to\mathbb{R}$, of the equation $\xi^{(k)}=\nu$. Correspondingly $\xi^{(k-1)}(t)$ is locally absolutely-continuous if $k>0$, and the signal $\nu(t)$ has the filtering order $k=0$, if ν is essentially bounded. Any number exceeding* $\operatorname{ess\,sup}|\xi(t)|$ *is called a kth-order (global) integral magnitude of ν.*

Let the input $f_0(t)$ of the differentiator (12), (13) be sampled as $f(t)=f_0(t)+\eta(t)$ with the noise $\eta(t)=\eta_0(t)+\eta_1(t)+...+\eta_{n_f}(t)$, where each η_k, $k=0,...,n_f$, is a signal of the global filtering order k and the kth-order integral magnitude $\varepsilon_k\geq 0$. Correspondingly, the noise components $\eta_1,...,\eta_{n_f}$ are possibly unbounded, whereas $\operatorname{ess\,sup}|\eta_0|\leq\varepsilon_0$.

Theorem 8. *Let $-1/(n_d+1)\leq q<1/n_f$, $\gamma_L\geq 0$, coefficients of the filtering differentiator* (12), (13) *be properly chosen, and* (6) *hold. Then after some transient the differentiator provides for the accuracy*

$$|z_i-f_0^{(i)}|\leq\mu_i L^{-s_q}\rho^{1+iq},\ i=0,...,n_d;$$
$$\rho=\max[(L^{s_q}\varepsilon_0)^{\frac{1}{1}},(L^{s_q}\varepsilon_1)^{\frac{1}{1-q}},...,(L^{s_q}\varepsilon_{n_f})^{\frac{1}{1-n_f q}}]. \tag{29}$$

Here μ_i only depend on the differentiator parameters $\vec{\lambda}$, and $s_q=0$ is taken for $q=0$.

Note that in spite of the complicated dependence on L the upper steady-state error estimations (29) monotonously grow for growing L and $q\neq 0$. One can also rewrite (29) as

$$|z_i-f_0^{(i)}|\leq\mu_i L^{-s_q}\max_{k=1,...,n_f}(L^{s_q}\varepsilon_k)^{\frac{\deg z_i}{\deg w_{n_f-k}}},\ i=0,...,n_d, \tag{30}$$

which shows that the estimation does not depend on rescaling the weights.

Example 1. *The noise $\eta=\gamma\cos(\omega_* t)$ features any global filtering order $k\geq 0$ with the integral magnitude γ for $k=0$ and $2\gamma/\omega_*{}^k$ for $k>0$. It means that it has at least n_f+1 trivial expansions of the form $\eta=\eta_k$ for any $k=0,1,...,n_f$ and other components being equal 0. Therefore, Theorem 8 implies that the estimation* (30) *holds for each possible representation of η, i.e.*

$$|z_i-f_0^{(i)}|\leq\mu_i L^{-s_q}\min_{k=1,...,n_f}(L^{s_q}\gamma\omega_*^{-k})^{\frac{d-(n_d+1-i)s_q}{d-(n_d+k+1)s_q}},\ i=0,...,n_d, \tag{31}$$

for $q\neq 0$. Note that $q=0$ corresponds to the limit case $d\to\infty$ both for $q>0$ and $q<0$.

Continue that consideration. From (31) taking $\eta=\eta_{n_f}$ one gets the conservative upper estimation of the steady-state accuracy

$$|z_i-f_0^{(i)}|\leq\mu_i\begin{cases}L^{\frac{n_f+i}{d-n_d-n_f-1}}\gamma^{\frac{d-n_d-1+i}{d-n_d-n_f-1}}\omega_*{}^{-n_f\frac{d-n_d-1+i}{d-n_d-n_f-1}},\ q>0\\ L^{\frac{n_f+i}{d+n_d+n_f+1}}\gamma^{\frac{d+n_d+1-i}{d+n_d+n_f+1}}\omega_*{}^{-n_f\frac{d+n_d+1-i}{d+n_d+n_f+1}},\ q<0\\ \gamma\omega_*{}^{-n_f},\ q=0,\end{cases} \tag{32}$$

which holds for **any** $\omega_*>0$.

Suppose that $\omega_*>1$ and is large, then indefinitely increasing n_f is probably beneficial for $q\leq 0$. In particular, *the case* $q=0$ in (32) is then adequately described by the filter transfer function.

In the case $q<0$ one gets for large n_f from (32) that $|z_i-f_0^{(i)}|=O(L\omega_*^{-(d+n_d+1-i)})$, $i=0,...,n_d$. The coefficient γ is "killed" for sufficiently large n_f! That sudden conclusion is further confirmed by simulation (Section 7.1).

If $q>0$ the filtering order n_f is bounded for each fixed d, and for some values of d the power denominator $d-n_d-n_f-1$ can turn out to be very small. In such a case the representation $\eta=\eta_{n_f}$ is obviously not optimal.

Other simple examples of the signals of global filtering orders are arbitrary-order derivatives of periodic functions and their linear combinations.

DOI: 10.1002/rnc

Example 2. *The signal* $\eta = \gamma \frac{d^k}{dt^k} \lfloor \cos(\omega_* t) \rceil^{k-0.5}$, $k \in \mathbb{N}$, $\omega_* > 0$, *is unbounded and features the global filtering order* k *with the integral magnitude* γ.

It is not obvious that globally filterable noises are widespread in practice, but that condition can be significantly relaxed. The following definition extends the corresponding definition from [30].

Definition 2. *A locally integrable Lebesgue-measurable function* $\nu(t)$, $\nu : [0, \infty) \to \mathbb{R}$, *is called locally* T*-filterable signal of the filtering order* $k > 0$ *and the integral magnitudes* $a_0, a_1, ..., a_{k-1} \geq 0$, *if there exists an infinite sequence* $\hat{t}_0, \hat{t}_1, ...,$ $\hat{t}_0 \geq 0$, $\hat{t}_{s+1} - \hat{t}_s \geq T > 0$, $s = 0, 1, ...,$ *such that for each* s *there exists a Caratheodory solution* $\xi(t)$, $t \in [\hat{t}_s, \hat{t}_{s+1}]$, *of the equation* $\xi^{(k)}(t) = \nu(t)$ *which satisfies* $|\xi^{(l)}(t)| \leq a_l$ *for* $l = 0, 1, ..., k-1$. *The number* a_l *is called the local* $(k-l)$*th-order integral magnitude of* ν. *Signals of local filtering order 0 are trivially defined as uniformly essentially bounded Lebesgue-measurable signals of the magnitude* a_0, $\operatorname{ess\,sup}_{t \geq 0} |\nu(t)| \leq a_0$.

In particular, locally filterable noises can be concatenated producing a new locally filterable noise (see Example 3 below). The following lemma shows that filtering differentiators can be applied when the noises are only locally filterable.

Lemma 4 ([30])**.** *Any signal* $\nu(t)$ *of the* ***local*** T*-filtering order* $k \geq 0$ *can be represented as* $\nu = \eta_0 + \eta_1 + \eta_k$, *where* η_0, η_1, η_k *are signals of the (global) filtering orders* $0, 1, k$ *respectively. Their magnitudes continuously depend on* $\vec{a}_{k-1}$ *and* T.

In particular, in the important case $k = 1$ *get* $\nu = \eta_0 + \eta_1$, *where* $|\eta_0| \leq a_0/T$, *and the first-order integral magnitude of* η_1 *is* $2a_0$. *In the general case* $k > 1$ *fix any number* $\hat{\rho}_0 > 0$. *Then, provided* $\hat{\rho} \leq \hat{\rho}_0$ *holds for* $\hat{\rho} = \max[a_0^{1/k}, a_1^{1/(k-1)}, ..., a_{k-1}]$, *the integral magnitudes of the signals* η_0, η_1, η_k *are calculated as* $\gamma_0 \hat{\rho}/T$, $\gamma_1 \hat{\rho}$, $\gamma_k \hat{\rho}^k$ *respectively, where the constants* $\gamma_0, \gamma_1, \gamma_k > 0$ *only depend on* k *and* $\hat{\rho}_0$.

Example 3. *Construct a new combined signal* $\nu(t)$ *from the signals of the form* $\cos \omega_l t$, $\omega_l \geq \omega_*$, $l = 1, 2, ...,$ *by applying them over the consecutive time intervals exceeding* T *in their length. The new signal is a locally* T*-filterable signal of any order* k *and the magnitudes* $1, 2\omega_*^{-1}, ..., 2\omega_*^{-k}$. *According to Lemma (4) one can represent this signal as a sum of globally filterable discrete signals,* $\nu = \eta_0 + \eta_1 + \eta_k$, *of the sampling filtering orders* $0, 1$ *and* k. *The bounded component* η_0 *is small provided* T *is taken large.*

5.1. Output-feedback stabilization in the presence of noise

Assumption 1. *The output* $\sigma(t)$ *of system* (27) *is sampled with the noise* $\eta(t) = \eta_0(t) + \eta_1(t) + ... + \eta_{n_f}(t)$, *where each* η_k, $k = 0, ..., n_f$, *is a signal of the global filtering order* k *and the* k*th-order integral magnitude* $\varepsilon_k \geq 0$.

Theorem 9. *Let* $q \neq 0$, $-1/n \leq q < 1/n_f$, $\gamma_L > 0$. *Consider the proposed output-feedback controls. Then all solutions stabilize in the set*

$$|\sigma^{(i)}| \leq \tilde{\mu}_i \rho^{1+iq}, \; i = 0, ..., n-1; \tag{33}$$

$$\rho = \max[\varepsilon_0^{\frac{1}{1}}, \varepsilon_1^{\frac{1}{1-q}}, ..., \varepsilon_{n_f}^{\frac{1}{1-n_f q}}], \tag{34}$$

where $\tilde{\mu}_i > 0$ *only depend on the system parameters* C, K_m, K_M, $\vec{\lambda}$, q, *the chosen controller and* L. *In the case* $q = 0$ *any dependence on* L *disappears.*

Remark 1. *Obtained accuracy estimations hold for any possible expansion of the sampling noise* $\eta(t) = \eta_0(t) + \eta_1(t) + ... + \eta_{n_f}(t)$. *Thus the real accuracy observed in the practice inevitably satisfies such estimation obtained for the "optimal" noise expansion often unknown to the "user" (see* (44) *and Fig. 2, Section 7.1).*

6. DISCRETIZATION ISSUES

In reality a filter is a discrete dynamic system obtaining discretely sampled input $f(t)$. Since in fact f is an output of a continuous-time dynamic system, the resulting closed-loop system is a hybrid one [5, 20]. Some dicretization strategy is needed to preserve the features of the continuous-time systems studied in Sections 3, 4.

Let the sampling take place at the times $t_0, t_1, ..., t_0 = 0$, $0 < t_{j+1} - t_j = \tau_j$, $\lim_{j\to\infty} t_j = \infty$. The sampling steps are assumed bounded, $\tau_j \le \tau$, though τ itself can be unknown.

It follows from the Nyquist-Shannon sampling rate principle that not all sampling-time sequences are admissible, since noises small in average under one sequence can be large under another. Therefore, we assume that the set of admissible sampling-time sequences contains sequences for any $\tau > 0$. In particular, the number of sequences is infinite.

Notation. Denote $\delta_j \phi = \phi(t_{j+1}) - \phi(t_j)$ for any sampled vector signal $\phi(t_j)$.

Definition 3. *A discretely sampled signal $\nu : \mathbb{R}_+ \to \mathbb{R}$ is said to be a signal of the global sampling filtering order $k \ge 0$ and the global kth order integral sampling magnitude $a \ge 0$ if for each admissible sequence t_j there exists a discrete vector signal $\xi(t_j) = (\xi_0(t_j), ..., \xi_k(t_j))^T \in \mathbb{R}^{k+1}$, $j = 0, 1, ...,$ which satisfies the relations*

$$\begin{array}{l} \delta_j \xi_i = \xi_{i+1}(t_j)\tau_j, \; i = 0, 1, ..., k-1, \\ \xi_k(t_j) = \nu(t_j), \; |\xi_0(t_j)| \le a. \end{array}$$

Example 4 ([30])**.** *Choose any $\tau, a > 0$, $0 < \tau_j = t_{j+1} - t_j \le \tau$, $t_0 = 0$, and let $\xi_0(t_j) = (-1)^j a$. Introduce $\xi_l(t_j)$ by the equalities $\xi_l(0) = 0$, $\delta_j \xi_{l-1} = \xi_l(t_j)\tau_j$, i.e. $\xi_l(t_{j+1}) = \xi_l(t_j) + \delta_j \xi_{l-1}/\tau_j$, $j = 1, 2, ...,$ for $l = 1, 2, ..., k$. Now $\nu(t_j) = \xi_k(t_j)$ is a signal of the global sampling filtering order k. Even if a is very small, ν turns to be very large for large k, provided $\tau < 1$.*

The following assumption on noises is naturally replacing Assumption 1.

Assumption 2. *The sampled noise signal is comprised of $n_f + 1$ components, $\eta(t_j) = \eta_0(t_j) + \eta_1(t_j) + ... + \eta_{n_f}(t_j)$. The discretely sampled signals $\eta_l(t_j)$ are of the global sampling filtering order l and integral magnitude ε_l, $l = 0, 1, ..., n_f$. Components $\eta_1, ..., \eta_{n_f}$ possibly are unbounded.*

That assumption is proved to hold for the steady-state SMC and $n_f = 1$, if the deviation of the SMC from the equivalent-control u_{eq} is considered the noise. That allows direct extraction of $u_{eq}, ..., u_{eq}^{(n)}$ by the filtering differentiator [32]. The assumption also often holds due to the statistical features of the noise (see [26]).

The proposed homogeneous discretization of the filtering differentiator (21) has the form

$$\begin{array}{rl} \delta_j w = & \Omega_{n_d,n_f,q}(w(t_j), z_0(t_j) - f(t_j), L, \vec{\lambda}_{n_d+n_f})\tau_j, \\ \delta_j z = & D_{n_d,n_f,q}(w_1(t_j), z(t_j), L, \vec{\lambda}_{n_d+n_f})\tau_j + T_{n_d}(z(t_j), \tau_j), \end{array} \tag{35}$$

where the Taylor-like term $T_{n_d} \in \mathbb{R}^{n_d+1}$ is defined as

$$\begin{array}{l} T_{n_d,0} = \frac{1}{2!} z_2(t_j)\tau_j^2 + ... + \frac{1}{n_d!} z_{n_d}(t_j)\tau_j^{n_d}, \\ T_{n_d,1} = \frac{1}{2!} z_3(t_j)\tau_j^2 + ... + \frac{1}{(n_d-1)!} z_{n_d}(t_j)\tau_j^{n_d-1}, \\ ... \\ T_{n_d,n_d-2} = \frac{1}{2!} z_{n_d}(t_j)\tau_j^2, \\ T_{n_d,n_d-1} = 0, \; T_{n_d,n_d} = 0. \end{array} \tag{36}$$

That discretization is proposed for the case $q = -(n_d + 1)^{-1}$ in [30] for the stand-alone signal-processing applications. In the case $q = 0$ the alternative discretization can be the exact one, based on the calculation of the matrix exponent. In both cases the error asymptotics (29) is preserved [30].

DOI: 10.1002/rnc

Note that (35) does not contain Taylor-like terms in the dynamics of w. Insertion of such terms would interfere with filtering out inherently discontinuous signals (Example 4, Fig. 2g,h,i, Section 7.1).

In the case $q > -1/(n_d+1)$ differentiator (12), (13) is practically only exact on the polynomials of the degrees not exceeding n_d. Therefore, we are mainly interested in its output-feedback application.

Theorem 10. *Under the conditions of Theorem 7 and Assumption 2 let system* (27)

$$\begin{array}{l} \sigma^{(n)} \in [-C,C]||\vec{\sigma}_{n-1}||_h^{1+nq} + [K_m, K_M]u, \\ C \geq 0,\ 0 < K_m \leq K_M, \end{array}$$

be closed by the feedback

$$\begin{array}{rl} u(t) = & U(z(t_j)),\ t \in [t_j, t_j+1), \\ \delta_j w = & \Omega_{n_d,n_f,q}(w, z_0(t_j) - \sigma(t_j) - \eta(t_j), L, \vec{\lambda}_{n_d+n_f})\tau_j, \\ \delta_j z = & D_{n_d,n_f,q}(w_1(t_j), z(t_j), L, \vec{\lambda}_{n_d+n_f})\tau_j. \end{array} \tag{37}$$

based on the zero-hold and the explicit Euler integration methods. Then the same accuracy (33) *is obtained in some FT T_c, but for different value of ρ. In particular,*

- *for $q<0$ obtain $|\sigma^{(i)}| \leq \mu_i \rho^{1-i|q|}$, $i = 0,...,n-1$, $\rho = \max[\tau^{\frac{1}{|q|}}, \max_{0\leq l\leq n_f} \varepsilon_l^{\frac{1}{1+l|q|}}]$;*
- *$|\sigma^{(i)}| \leq \mu_i \max_{0\leq l\leq n_f} \varepsilon_l$ for $q=0$ and sufficiently small τ, also $T_c \to \infty$ as $\vec{\varepsilon}_{n_f} \to 0$;*
- *for $q>0$ obtain $|\sigma^{(i)}| \leq \mu_i \rho^{1+iq}$, $\rho = \max_{0\leq l\leq n_f} \varepsilon_l^{\frac{1}{1-lq}}$, also $T_c \to \infty$ as $\vec{\varepsilon}_{n_f} \to 0$. In order to avoid the possible solution escape to infinity, the initial values $z(t_0), w(t_0)$ are to be sufficiently small for each fixed τ, or τ is to be sufficiently small for each fixed $||z(t_0), w(t_0)||_{h\infty}$.*

Here $\mu_i > 0$ only depend on the controller U and parameters $q, K_m, K_M, C, L, \vec{\lambda}_{n+n_f-1}$ of the system and the observer.

The theorem formally remains true also for $\vec{\varepsilon}_{n_f} = 0$ and $\tau = 0$ if $T_c = \infty$ is taken for $q \geq 0$. It is natural to always take $w(t_0) = 0$. If the noise is known to be small one can choose the initial sampled value $z_0(t_0) = \sigma(t_0) + \eta(t_0)$, and determine other initial values using finite differences in order to shorten the transient. If the noise is large, it is reasonable to assign $z(t_0) = 0$.

The following definition slightly extends the similar definition from [30].

Definition 4. *A discretely sampled signal $\nu(t_j)$ is said to be locally T-filterable of the local sampling filtering order $k > 0$ and the integral magnitudes $a_0, a_1, ..., a_{k-1} \geq 0$, if there exists an infinite sequence $\hat{t}_0, \hat{t}_1, ...,$ $\hat{t}_0 \geq 0$, $\hat{t}_{l+1} - \hat{t}_l \geq T > 0$, $l = 0, 1, ...,$ such that for any sufficiently small τ, admissible sequence $\{t_j\}$, and any $l \geq 0$ there exists a discrete vector signal*

$$\xi(t_j) = (\xi_0(t_j), ..., \xi_k(t_j))^T \in \mathbb{R}^{k+1},\ j = j_0, j_0+1, ..., j_1,\ t_{j_0} \in [\hat{t}_l, \hat{t}_l + \tau),\ t_{j_1} \in (\hat{t}_{l+1} - \tau, \hat{t}_{l+1}]$$

which satisfies the relations

$$\begin{array}{l} \delta_j \xi_i = \xi_{i+1}(t_j)\tau_j,\ i = 0, 1, ..., k-1, \\ \xi_k(t_j) = \nu(t_j), \\ |\xi_i(t_j)| \leq a_i,\ i = 0, 1, ..., k-1. \end{array} \tag{38}$$

Numbers a_i are called the local $(k-i)$th-order sampling integral magnitudes of ν. Signals of local sampling filtering order 0 by definition are just bounded signals of the magnitude a_0.

Similarly to the continuous-time case, one can concatenate locally filterable signals. The following lemma is analogous to Lemma 4 and allows application of Theorem 10 in the case of locally filterable sampled noises.

DOI: 10.1002/rnc

Lemma 5 ([30]). *Let all admissible sampling time sequences satisfy the condition* $\sup \tau_j / \inf \tau_j \le c_\tau$ *for some* $c_\tau > 0$. *Then any discretely sampled signal* $\nu(t_j)$ *of the* ***local*** *sampling* T*-filtering order* $k \ge 0$ *can be represented as* $\nu = \eta_0 + \eta_1 + \eta_k$, *where* η_0, η_1, η_k *are signals of the global sampling filtering orders* $0, 1, k$.

In particular, if $k = 1$ *get* $\nu = \eta_0 + \eta_1$, *where* $|\eta_0| \le a_0/T$, *and the first-order integral sampling magnitude of* η_1 *is* $2a_0$. *If* $k > 1$ *fix any number* $\rho_0 > 0$. *Then, provided* $\rho = \max[a_0^{1/k}, a_1^{1/(k-1)}, ..., a_{k-1}] \le \rho_0$ *the sampling integral magnitudes of the signals* η_0, η_1, η_k *are calculated as* $\gamma_0 \rho/T$, $\gamma_1 \rho$, $\gamma_k \rho^k$ *respectively, where the constants* $\gamma_0, \gamma_1, \gamma_k > 0$ *only depend on* k *and* ρ_0.

Lemma 6. *Let* $\nu(t)$ *be bounded and locally* T*-filterable of the order* k *with the integral magnitudes* $a_0, a_1, ..., a_{k-1} \ge 0$ *and the bounded sequence* $\hat{t}_{s+1} - \hat{t}_s \ge T$ *from Definition 2. Let it also be uniformly equicontinuous (in particular, absolutely continuous) over the intervals* $(\hat{t}_s, \hat{t}_{s+1})$. *Then the sampled signal* $\nu(t_j)$ *is a locally* T*-filterable signal of the sampling filtering order* k *and some integral magnitudes* $\tilde{a}_0, \tilde{a}_1, ..., \tilde{a}_{k-1} \ge 0$. *Moreover,* $\vec{\tilde{a}}_{k-1} \to \vec{a}_{k-1}$ *as* $\tau \to 0$.

By uniform equicontinuity we mean that $\forall \epsilon_1 > 0\ \exists \epsilon_2 > 0\ \forall s \in \mathbb{Z}, s \ge 0,\ \forall t_1, t_2 \in (\hat{t}_s, \hat{t}_{s+1}) : |t_1 - t_2| < \epsilon_2 \Rightarrow |\nu(t_1) - \nu(t_2)| < \epsilon_1$. The lemma follows from the uniform convergence of the Euler approximations to the unique DE solution (see the Appendix).

Similarly to Remark 1 also here one might avoid checking Assumption 2 (Fig. 2g,h,i).

Remark 2. *Discretely sampling a globally filterable signal of the order* k *can produce a discrete signal of the global sampling filtering order* k. *Indeed, an iterated numeric integral of the signal from Example 2 can be itself unbounded and even locally not filterable. It completely depends on the concrete sampling sequence* t_j.

On the other hand, if the signal is large and fast, but bounded and Lipschitzian (for example, see Example 1), then for sufficiently small τ *its* ***local*** *iterated numeric integrals approach their continuous-time counterparts (Lemma 6). A discrete locally* T*-filterable signal is produced for any fixed* $T > 0$. *Such signals can be processed due to Lemma 5. Similarly the saturated sampled signal of Example 2 is locally filterable for any (even large) saturation value.*

7. NUMERIC EXPERIMENTS

7.1. Numeric differentiation

It is well-known that filtering differentiators (12), (13) of the HD $q = -1/(n_d + 1)$ converge in FT and are exact under condition (6). Contrary to this, filtering differentiators of the HD $q > -1/(n_d + 1)$ are only capable to asymptotically exactly differentiate polynomials of the degrees $0, 1, ..., n_d$. Nevertheless, they still can be used for the high-gain observation and differentiation [3, 19, 42].

In the case $q \ne 0$ high-gain differentiation is obtained by taking $L >> 1$. In the case $q = 0$ the filtering HGO (FHGO) (22), (23) is produced, and high-gain differentiation is obtained by taking $\tilde{\lambda}_i = \tilde{\lambda}_{0i} \hat{\varepsilon}^{-(n_d+n_f+1-i)}$, $0 < \hat{\varepsilon} << 1$, for any characteristic Hurwitz polynomial $p_{n_d+n_f+1}(s) = s^{n_d+n_f+1} + \tilde{\lambda}_{0,n_d+n_f} s^{n_d+n_f} + ... + \tilde{\lambda}_{0,0}$. Recall that the HGO [3] is also described by (22), (23), but corresponds to $n_f = 0$.

It is easy to see that in the absence of noises the error dynamics of the HGO of the differentiation order $n_d + n_f$ and of the FHGO of the differentiation and filtering orders n_d, n_f, coincide (see the proof of Theorem 5 in the Appendix). It implies the accuracies

$$\begin{array}{l} |z_i - f_0^{(i)}| \le \gamma_{n_d+n_f,i} L \hat{\varepsilon}^{(n_f+n_d+1-i)} \text{ for HGO of the order } n_d + n_f,\ i = 0, ..., n_d + n_f, \\ |z_i - f_0^{(i)}| \le \gamma_{n_d+n_f,n_f+i} L \hat{\varepsilon}^{(n_d+1-i)} \text{ for FHGO of the orders } n_d, n_f,\ i = 0, ..., n_d, \end{array}$$

for some constants $\gamma_{n_d+n_f,j} > 0$, $j = 0, ..., n_d + n_f$, only depending on choice of $\vec{\tilde{\lambda}}_{0,n_d+n_f}$ [3, 42]. Note that due to the *different differentiation orders* the corresponding conditions on the inputs are $|f_0^{(n_d+n_f+1)}| \le L$ for the HGO and $|f_0^{(n_d+1)}| \le L$ for the FHGO.

DOI: 10.1002/rnc

In the following numeric experiment we compare the FHGO of the orders $n_d = 2$, $n_f = 8$ with the HGO of the order $n_d = 2$, correspondingly providing for the accuracies

$$|z_i - f_0^{(i)}| \leq \gamma_{10,i} L \hat{\varepsilon}^{(3-i)} \text{ and } |z_i - f_0^{(i)}| \leq \gamma_{2,i} L \hat{\varepsilon}^{(3-i)},\ i = 0, 1, 2,\ |\dddot{f}_0| \leq L. \tag{39}$$

Thus, in the absence of sampling noises (and for the ideal continuous sampling) the accuracies of HGO and FHGO are of the same order.

Concretely, consider the FHGO and the HGO with the characteristic polynomials $p_{11}(s) = (s+1)^{11}$ and $p_3(s) = (s+1)^3$ respectively and $\hat{\varepsilon} = 0.01$. For both of them we take $z(0) = 0$, also $w(0) = 0$ is taken for the FHGO.

Consider the noisy input signal

$$f(t) = f_0(t) + \eta(t),\ f_0(t) = 0.8 \cos t - \sin(0.6t),\ n_d = 2, \tag{40}$$

where $|\dddot{f}_0(t)| \leq 1$, $\eta(t)$ is a noise, $L = 1$.

For the comparison we also take the filtering SM-based differentiator (12), (13) rewritten in the equivalent recursive form (14), (15) for $n_d = 2$, $n_f = 8$, $\vec{\lambda}_{10} = (1.1, 1.5, 2, 3, 5, 7, 10, 12, 14, 17, 20)$, $q = -(n_d + 1)^{-1} = -1/3$. We once more take $z(0) = 0$, $w(0) = 0$.

Due to the Shannon sampling-rate principle, performance of any filter in the presence of large high-frequency noises is very sensitive to the sampling/integration step. Thus we take equal sampling steps of the length $\tau = 10^{-6}$ to reliably demonstrate the filter features. The discrete form (35), (36) is applied.

Performance in the absence of noises of the HGO and the filtering HGO, corresponding to $n_f = 0$ and $n_f = 8$ respectively, is shown in Fig. 1. Both linear filters demonstrate fast convergence, good precision and some peaking phenomena ($\max |z_2| \approx 2000$ for HGO, $\max |z_2| \approx 250$ for FHGO). The corresponding steady-state accuracies of all three filters are provided by the following component-wise inequalities:

$$\begin{array}{l}(|z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|) \leq (8.1 \cdot 10^{-7}, 2.8 \cdot 10^{-4}, 2.4 \cdot 10^{-2}),\ n_f = 0,\ q = 0; \\ (|z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|) \leq (1.3 \cdot 10^{-4}, 4.4 \cdot 10^{-3}, 8.9 \cdot 10^{-2}),\ n_f = 8,\ q = 0; \\ (|z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|) \leq (6.5 \cdot 10^{-12}, 7.5 \cdot 10^{-5}, 2.9 \cdot 10^{-4}),\ n_f = 8,\ q = -\frac{1}{3}.\end{array} \tag{41}$$

Comparing (41) with the theoretical accuracy (39), we see that probably $\gamma_{2,i} < \gamma_{10,8+i}$, $i = 0, 1, 2$, hold for our *specific choices* of $\vec{\tilde{\lambda}}_{0,10}$ and $\vec{\tilde{\lambda}}_{0,2}$.

Note that homogeneous SM-based filtering differentiators remain practically stable for any bounded noise and any $\tau > 0$ [30]. The above accuracies do not significantly improve for smaller values of τ due to the computer double-precision round-up errors [29, 5].

Introduce now the sampling noise $\eta(t) = \eta_G(t) + \cos(11111t)$, $\eta_G(t) \in N(0, 0.5^2)$, where η_G is the Gaussian noise of the standard deviation 0.5 (Fig. 1). Values sampled at different sampling times are independent. The corresponding steady-state accuracies are

$$\begin{array}{l}(|z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|) \leq (5.6 \cdot 10^{-2}, 5.2, 167),\ n_f = 0,\ q = 0; \\ (|z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|) \leq (2.8 \cdot 10^{-2}, 0.53, 4.9),\ n_f = 8,\ q = 0; \\ (|z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|) \leq (8.1 \cdot 10^{-3}, 7.0 \cdot 10^{-2}, 0.36),\ n_f = 8,\ q = -\frac{1}{3}.\end{array} \tag{42}$$

The HGO does not evaluate derivatives, whereas the FHGO provides some first-derivative approximation, and only the SM-based differentiator remains reasonably accurate even in the second derivative.

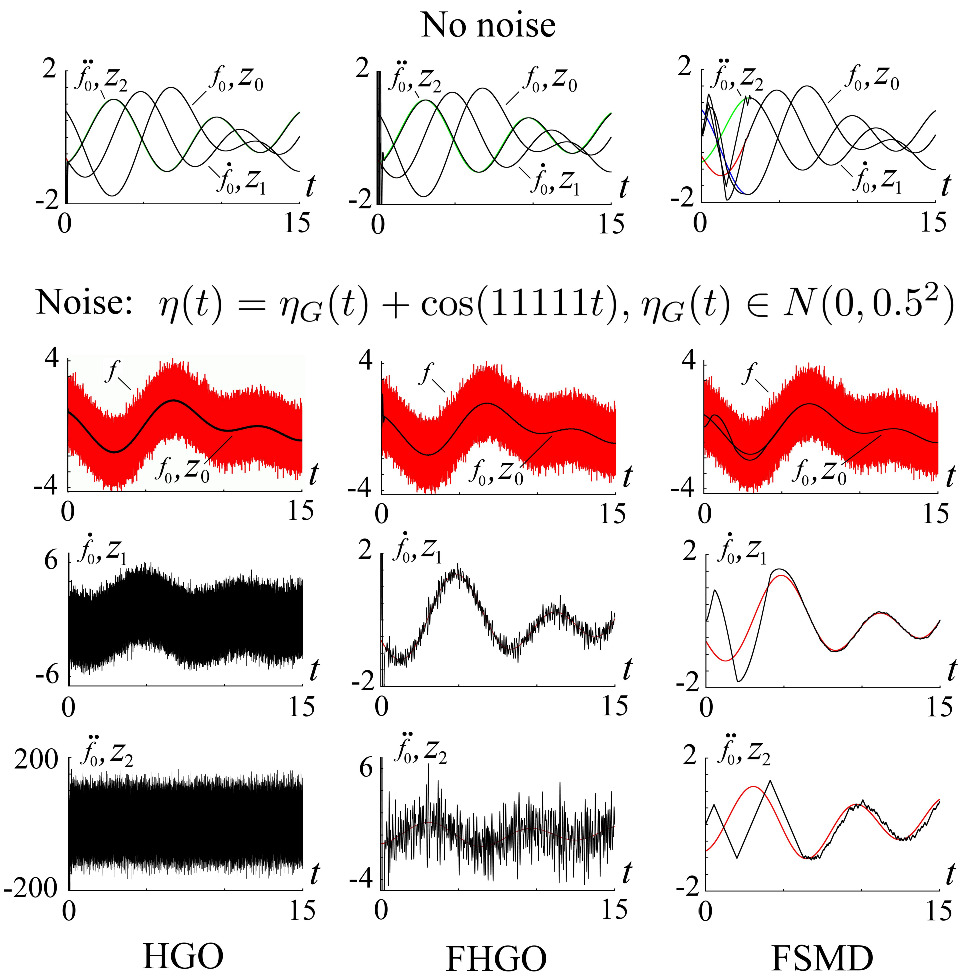

Figure 1: Performance of the HGO for $n_d = 2, n_f = 0$, $q = 0$, the filtering HGO (FHGO) for $n_d = 2, n_f = 8$, $q = 0$, with the high-gain parameter $\hat{\varepsilon}^{-1} = 100$ and of the filtering SM-based differentiator (FSMD) for $n_d = 2, n_f = 8, L = 1$, $q = -1/3$, without noise and for the sampling noise $\eta = \eta_G + \cos(11111t)$ with the Gaussian component $\eta_G \in N(0, 0.5^2)$. The HGO is the most sensitive to noise. Graphs of z_1, z_2 for HGO, FHGO are cut from above and from below.

Both linear filters are linear also in noise, which means that proportionally increasing noises completely destroys both HGO and FHGO performance. Check the performance of the SM-based filter for different noises, including **very large** ones:

$$\begin{aligned}
&n_d = 2,\ n_f = 8,\ q = -\tfrac{1}{3},\ \tau = 10^{-6}, \eta_G(t) \in N(0, 0.5^2),\\
&(|z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|) \le (7.1 \cdot 10^{-6}, 7.3 \cdot 10^{-4}, 0.038) \quad \text{for } \eta = \cos(11111t),\\
&(|z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|) \le (5.3 \cdot 10^{-4}, 1.3 \cdot 10^{-2}, 0.16) \text{ for } \eta = 10^7 \cos(11111t),\\
&(|z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|) \le (5.4 \cdot 10^{-4}, 1.3 \cdot 10^{-2}, 0.16)\\
&\qquad\qquad \text{for } \eta = 10^7 \cos(11111t) - 2 \cdot 10^7 \cos(22222t),\\
&(|z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|) \le (7.0 \cdot 10^{-3}, 6.4 \cdot 10^{-2}, 0.34) \text{ for } \eta = \eta_G,\\
&(|z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|) \le (8.1 \cdot 10^{-3}, 7.0 \cdot 10^{-2}, 0.36)\\
&\qquad\qquad \text{for } \eta = \eta_G + 10^7 \cos(11111t).
\end{aligned} \tag{43}$$

One sees that the superposition principle does not work here. Performance corresponding to the third line (the noise $\eta = 10^7 \cos(11111t) - 2 \cdot 10^7 \cos(22222t)$) is shown in Fig. 2a-f. Obviously w_7, w_8 absorb the main part of the noise, in particular, $|w_8| \le 1584$ is kept.

Comparing the last lines of (42) and (43) we find out that practically identical accuracies are obtained for $\eta = \eta_G + \cos(11111t)$ and $\eta = \eta_G + 10^7 \cos(11111t)$. The difference is that whereas in the first case $|w_8| \leq 7 \cdot 10^{-4}$ is kept, in the second case $|w_8| \leq 900$ holds (not shown). Other components of w stay small.

The normally distributed noise component η_G in practice has the local sampling filtering order 1 [26], and is the most difficult to deal with (still simulation in [30] shows similar or possibly better performance than the Kalman filter).

The filter also suppresses noises for which Assumption 2 is at least not obvious. Introduce very large noises of a complicated structure:

$$\begin{array}{l}(|z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|) \leq (4.4 \cdot 10^{-5}, 2.1 \cdot 10^{-3}, 6.3 \cdot 10^{-2}), \\ \qquad\qquad\qquad\qquad \text{for } \eta = 10^7 \cos(10^5 t + 3\pi \sin(10^3 t)); \\ (|z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|) \leq (1.2 \cdot 10^{-2}, 8.8 \cdot 10^{-2}, 0.38), \\ \qquad\qquad\qquad\qquad \text{for } \eta = 10^7 \cos(10^8 t + 3\pi \sin(10^4 t)).\end{array} \tag{44}$$

Note that the noise $\eta = 10^7 \cos(10^8 t + 3\pi \sin(10^4 t))$ from the second line is practically random due to the "large" sampling step $\tau = 10^{-6}$. The corresponding performance is shown in Fig. 2g,h,i. Also here w_8 absorbs the main part of the noise. Other components of w look similar to Fig. 2b. Changing 10^4 to 10^5 causes explosion-like divergence of the estimations.

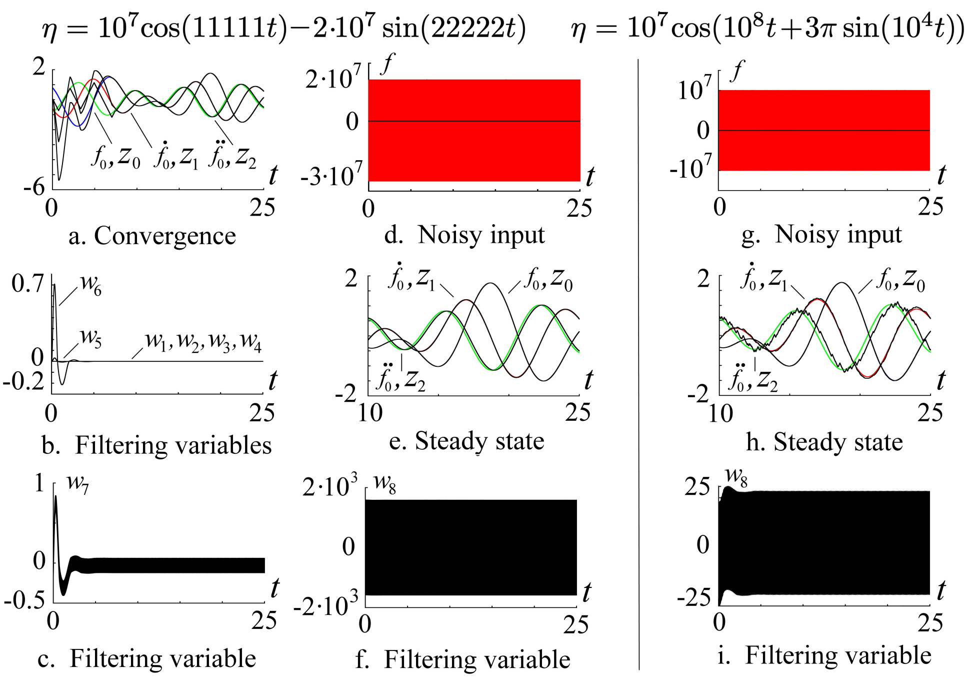

Figure 2: $q = -1/(n_d + 1) = -1/3$. Performance of the SM-based filtering differentiator for $n_d = 2, n_f = 8, L = 1$, $\tau = 10^{-6}$, in the presence of the noise $\eta = 10^7 \cos(11111t) - 2 \cdot 10^7 \sin(22222t)$ (a-f) and the noise $\eta = 10^7 \cos(10^8 t + 3\pi \sin(10^4 t))$ (g-i).

7.2. Output-feedback stabilization

Consider the discontinuous nonlinear dynamic system

$$\dddot{\sigma} = \cos(120t + \text{sat}_{100}(\frac{\ddot{\sigma}}{\sigma}))|\dot{\sigma}|^{\frac{1+3q}{1+q}} + (2 + \cos(\text{sat}_{100}(\frac{\dot{\sigma}}{\sigma})))u \tag{45}$$

where the function $\text{sat}_{\varpi}(s) = \max(-\varpi, \min(\varpi, s))$ saturates at $\pm\varpi$. The control can be discontinuous, therefore system (45) is understood in the Filippov sense [14]. Obviously its

DOI: 10.1002/rnc

solutions satisfy the DI

$$\dddot{\sigma} \in [-1,1]||\vec{\sigma}_2||_{h\infty}^{1+3q} + [1,3]u, \tag{46}$$

where $||\vec{\sigma}_2||_{h\infty} = \max_i |\sigma^{(i)}|^{1/(1+iq)}$ is the homogeneous norm, $\deg \sigma^{(i)} = 1 + iq,\ i = 0,1,2$. DI (46) is a Filippov homogeneous DI of the HD q, provided $u = U(\vec{\sigma}_2)$ is homogeneous, $\deg U = 1 + 3q$.

Choose a homogeneous stabilizing control template [25] of the form

$$u = -\alpha||\vec{\sigma}_2||_{h\infty}^{\frac{1}{2}+3q}[\lfloor\ddot{\sigma}\rceil^{\frac{1}{2(1+2q)}} + \beta_1\lfloor\dot{\sigma}\rceil^{\frac{1}{2(1+q)}} + \beta_0\lfloor\sigma\rceil^{\frac{1}{2}}], \tag{47}$$

which is valid for the relative degree 3 and any $q \in \mathbb{R}$, provided $\beta_0, \beta_1 > 0$ are properly chosen and α is sufficiently large. Note that this control is quasi-continuous, but not smooth, and it does not allow developing a Lyapunov function in the standard way [9].

The value $q = -1/3$ corresponds to homogeneous SMC and is well studied. Thus only consider $q = 0, -0.2, 0.2$. The same control parameters $(\beta_0, \beta_1, \alpha) = (1, 2, 5)$ have been found accidentally valid for all these values of q.

The authors have taken $\gamma_L = 1$ and found parameters $\vec{\lambda}$ one by one by simulation, using the recursive form (17), (18) for the corresponding $d, q,\ n_d = 2,\ n_f = 3$. The initial values $w(0) = 0,\ z(0) = (10, -10, 10),\ \vec{\sigma}_2(0) = (50, -50, 50)$ are taken to show the transient. In the case $q = 0$ the filtering HGO of the form (22), (23) is applied with the multiple eigenvalue 100. The integration is performed by the Euler method with the integration step $\tau = 10^{-6}$.

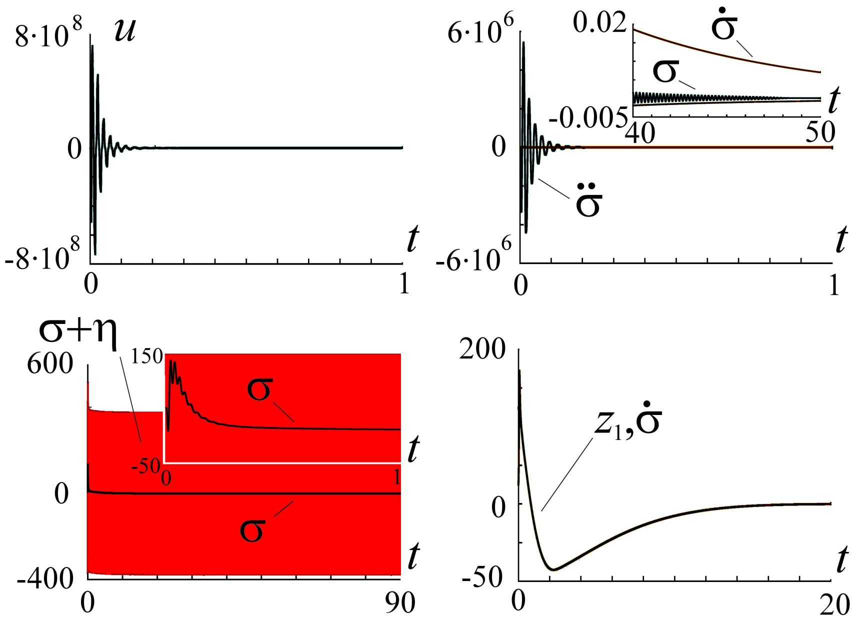

Figure 3: The case $q = 0.2$. Asymptotic stability is got for the exact sampling. The system remains practically exact in spite of the very large sampling noise η.

1. Let $q = 0.2$. The filtering differentiator, $n_d = 2,\ n_f = 3,\ d = 8$, is employed in the feedback. Its parameters are $\vec{\lambda} = \{1.1, 2, 3, 9, 30, 200\},\ L = 5 \cdot 10^9$. Note that actual coefficients are not so large, since form (17), (18) involves $L^{1/7}, ..., L^{1/2}$.

In the absence of noises the accuracy is described by the component-wise inequality

$$(|\sigma|, |\dot{\sigma}|, |\ddot{\sigma}|) \le (3.5 \cdot 10^{-3}, 1.5 \cdot 10^{-4}, 8 \cdot 10^{-5}) \text{ for } t \ge 70.$$

Introduce the sampling noise $\eta(t) = 100\cos(10000t) - 200\cos(50000t) + 100\sin(70000t)$. In spite of the large noise the system converges into the region $(|\sigma|, |\dot{\sigma}|, |\ddot{\sigma}|) \le (5 \cdot 10^{-3}, 4 \cdot 10^{-4}, 4 \cdot$

10^{-4}) (Fig. 3). Note that the accuracies are practically the same in the presence and the absence of noises. It is the result of the positive HD. Such systems are very sensitive to large initial values and the sampling period, but are less sensitive to noises ([27] and Section 2).

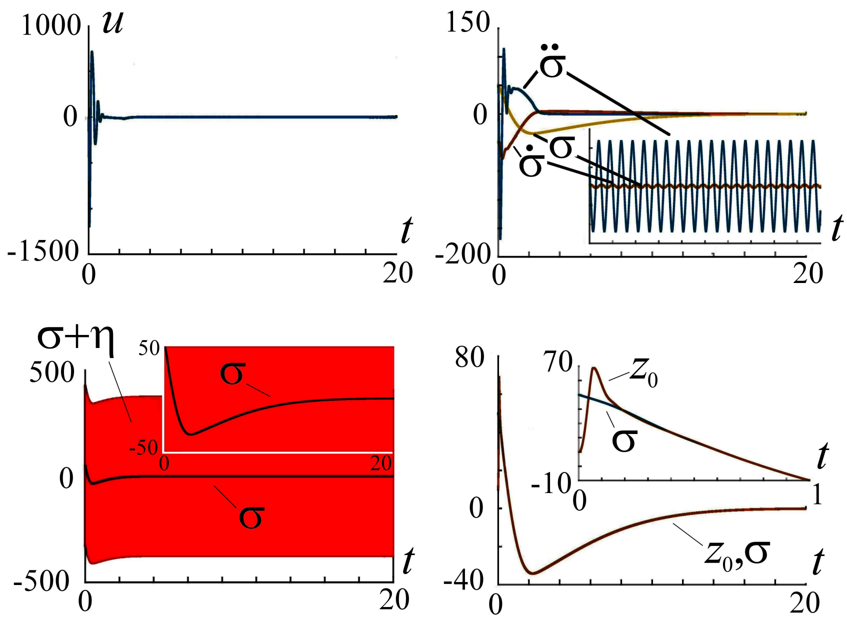

Figure 4: The case $q = -0.2$. FT stability is got for the exact sampling. The system remains practically stable in spite of the very large sampling noise $\eta(t) = 100\cos(10000t) - 200\cos(50000t) + 100\sin(70000t)$

2. Let $q = -0.2$. Once more apply the filtering differentiator with $n_d = 2, n_f = 3$. This time $d = 2$, and the parameters are $\vec{\lambda}_5 = \{1.1, 1.4, 2.4, 5, 6, 8\}$, $L = 5 \cdot 10^5$. Also here the actual coefficients are much smaller.

The system stabilizes in FT at $t = 26$. In the absence of noises the accuracy is described by the component-wise inequality

$$(|\sigma|, |\dot{\sigma}|, |\ddot{\sigma}|) \leq (7 \cdot 10^{-19}, 2 \cdot 10^{-14}, 10^{-10}) \text{ for } t \geq 30.$$

Introduce the same large noise (Fig. 4). In spite of it the system converges into the region $(|\sigma|, |\dot{\sigma}|, |\ddot{\sigma}|) \leq (5 \cdot 10^{-3}, 6 \cdot 10^{-2}, 1.5)$.

3. Let $q = 0$. Apply the filtering HGO (22), (23) with $n_d = 2$, $n_f = 3$ and the parameters corresponding to the multiple eigenvalue 100, i.e. $\vec{\lambda}_5 = \{100^6, 6 \cdot 100^5, 15 \cdot 100^4, 20 \cdot 100^3, 15 \cdot 100^2, 6 \cdot 100\}$. In the absence of noises the accuracy is described by the component-wise inequality

$$(|\sigma|, |\dot{\sigma}|, |\ddot{\sigma}|) \leq (6 \cdot 10^{-9}, 2 \cdot 10^{-8}, 2 \cdot 10^{-6}) \text{ for } t \geq 25.$$

Introduce the noise $\eta(t) = 10\cos(11111t)$. In spite of this noise the filtering HGO provides for practically the same accuracy. Also the graphs are indistinguishable (Fig. 5).

Also the HGO performs quite well in that framework providing for the very high accuracy in the absence of noises,

$$(|\sigma|, |\dot{\sigma}|, |\ddot{\sigma}|) \leq (8 \cdot 10^{-8}, 6 \cdot 10^{-8}, 5 \cdot 10^{-8}) \text{ for } t \geq 25.$$

In the presence of noises the steady-state accuracy is significantly lower, but still acceptable,

$$(|\sigma|, |\dot{\sigma}|, |\ddot{\sigma}|) \leq (0.02, 0.05, 3) \text{ for } t \geq 14.$$

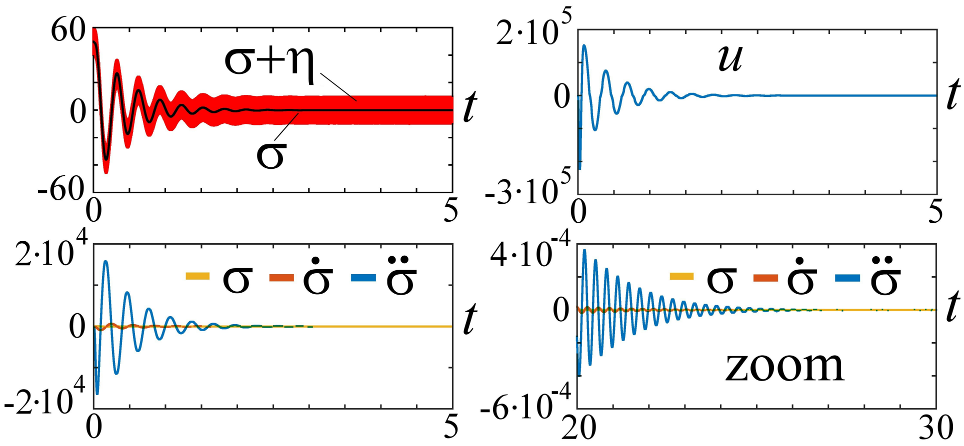

Figure 5: The case $q = 0$, filtering HGO with the multiple eigenvalue 100 is applied. The system remains practically exact in spite of the sampling noise $\eta(t) = 10\cos(11111t)$.

8. CONCLUSION

The robust homogeneous output-feedback stabilization of disturbed integrator chains has been obtained for any homogeneous degree. The corresponding separation principle is formulated.

The proposed new homogeneous observers feature strong filtering properties and ensure system robustness for any homogeneity degree in the presence of very large noises featuring locally bounded iterated integrals. No knowledge on the presence of noises or their nature is required.

The recursive method of choosing the coefficients has been established for any homogeneity degree, extending and strengthening the known method for the SM-based differentiators [22].

The proposed filtering high-gain observers can be applied instead of the high-gain observers to suppress large sampling noises.

The authors believe that the proposed extremely robust observers with positive homogeneity degrees can prove themselves in observation and control of potentially "explosive" systems like nuclear and thermonuclear reactors.

9. APPENDIX: PROOFS

The stability proofs in this paper are based on the Lyapunov approach inspired by [17, 9, 37, 33].

Proof of Theorem 1. Obtain the error dynamics equations for (5). For this end subtract $f_0^{(i+1)}$ from the both sides of the equation for $\dot{z}_i$. If $q > 0$ multiply both sides by L, and denote $\zeta_i = L(z_i - f_0^{(i)}),\ i = 0, 1, ..., n_d,\ \hat{f}_0(t) = Lf_0(t)$. If $q < 0$ divide both sides by L and denote $\zeta_i = (z_i - f_0^{(i)})/L,\ \hat{f}_0(t) = f_0(t)/L$. In the case $q = 0$ just define $\zeta_i = z_i - f_0^{(i)},\ \hat{f}_0(t) = f_0(t)$.

In the new coordinates condition (6) obtains the form

$$|\hat{f}_0{}^{(n_d+1)}(t)| \leq \gamma_L ||\vec{\zeta}_{n_d}||_{h\infty}^{1+(n_d+1)q} = \gamma_L \max_i |\zeta_i|^{\frac{1+(n_d+1)q}{1+iq}}. \quad (48)$$

Correspondingly (5) is transformed into

$$\begin{aligned} \dot{\zeta}_0 &= -\tilde{\lambda}_{n_d} \lfloor \zeta_0 \rceil^{1+q} + \zeta_1, \\ \dot{\zeta}_1 &= -\tilde{\lambda}_{n_d-1} \lfloor \zeta_0 \rceil^{1+2q} + \zeta_2, \\ &\dots \\ \dot{\zeta}_{n_d} &\in -\tilde{\lambda}_0 \lfloor \zeta_0 \rceil^{1+(n_d+1)q} + \gamma_L[-1, 1]||\vec{\zeta}_{n_d}||_{h\infty}^{1+(n_d+1)q}. \end{aligned} \quad (49)$$

DOI: 10.1002/rnc

Strictly speaking the last line of (49) can contain discontinuity, and should have the form

$$\dot{\zeta}_{n_d} \in -\tilde{\lambda}_0 L^{(n_d+1)|q|} K_F \left[\lfloor \cdot \rceil^{1+(n_d+1)q} \right] (\zeta_0) + \gamma_L[-1,1] ||\vec{\zeta}_{n_d}||_{h\infty}^{1+(n_d+1)q},$$

producing a Filippov DI in the case $q = -1/(n_d+1)$. Here and further in the similar cases we omit the Filippov procedure for the brevity.

In fact Theorem 5 is a corollary of Theorem 2. That is why the following is only a sketch of the independent proof.

Due to the robustness of the AS of homogeneous systems [27] it is enough to prove the existence of parameters for $\gamma_L = 0$. After a suitable recursive change of coefficients the reduced errors' dynamics get the form

$$\begin{aligned} \dot{\zeta}_0 &= c_0(\zeta_1 - \lfloor \zeta_0 \rceil^{1+q}), \\ \dot{\zeta}_1 &= c_1(\zeta_2 - \lfloor \zeta_0 \rceil^{1+2q}), \\ &\dots \\ \dot{\zeta}_{n_d} &= -c_{n_d} \lfloor \zeta_0 \rceil^{1+(n_d+1)q}. \end{aligned} \quad (50)$$

A proper choice of $c_i > 0$ is to asymptotically stabilize (50). Fix any $a > 3\max[1, 1+(n_d+1)q]$. The following Lyapunov function (LF) candidate is inspired by [17, 11]:

$$\begin{aligned} V = \int_{\lfloor \zeta_1 \rceil^{\frac{1}{1+q}}}^{\zeta_0} (\lfloor s \rceil^{\frac{a-1}{1}} - \lfloor \zeta_1 \rceil^{\frac{a-1}{1+q}}) ds + \dots + \\ \int_{\lfloor \zeta_{n_d} \rceil^{\frac{1+(n_d-1)q}{1+n_d q}}}^{\zeta_{n_d-1}} (\lfloor s \rceil^{\frac{a-1-(n_d-1)q}{1+(n_d-1)q}} - \lfloor \zeta_{n_d} \rceil^{\frac{a-1-(n_d-1)q}{1+n_d q}}) ds \\ + \int_0^{\zeta_{n_d}} \lfloor s \rceil^{\frac{a-1-n_d q}{1+n_d q}} ds. \end{aligned} \quad (51)$$

It is easily checked that V is positive definite due to Lemma 2. According to the same lemma its derivative has the form

$$\begin{aligned} \dot{V} = -[c_0 W_{\dot{\zeta}_0 \neq 0}(z) + c_1 H_{\dot{\zeta}_0 = 0}(z)] - \dots \\ - [c_i W_{\dot{\zeta}_{i-1}=0, \dot{\zeta}_i \neq 0}(z) + c_{i+1} H_{\dot{\zeta}_{i-1}, \dot{\zeta}_i = 0}(z)] - \dots \\ - c_{n_d} W_{\dot{\zeta}_{n_d-1}=0, \dot{\zeta}_{n_d} \neq 0}(z), \end{aligned}$$

where all components are continuous homogeneous functions, $\deg \dot{V} = a + q > 0$, $W_{\dot{\zeta}_0 \neq 0}$ is positive whenever $\dot{\zeta}_0 \neq 0$, $W_{\dot{\zeta}_{i-1}=0, \dot{\zeta}_i \neq 0}$ is positive whenever $\dot{\zeta}_{i-1} = 0$ and $\dot{\zeta}_i \neq 0$, $i = 1, \dots, n_d$. The function $H_{\dot{\zeta}_{i-1}, \dot{\zeta}_i = 0}$ vanishes whenever $\dot{\zeta}_{i-1} = \dot{\zeta}_i = 0$, and $H_{\dot{\zeta}_0 = 0} = 0$ if $\dot{\zeta}_0 = 0$.

Now one takes any $c_{n_d} > 0$, then chooses c_i sufficiently large to provide for the negative-definiteness of $\dot{V}$ on the set $\dot{\zeta}_0 = \dots = \dot{\zeta}_{i-1} = 0$, $i = n_d - 1, \dots, 1$ (Lemma 3). At last step c_0 is chosen sufficiently large to provide for the global negative-definiteness of $\dot{V}$. □

Proof of Theorems 2, 3. The case $q \neq 0$. In the case of Theorem 2 perform the proportional change of the homogeneity weights, so that the new HD be $s_q = \operatorname{sign} q$. Once more define $\zeta_i = (z_i - f_0^{(i)}) L^{s_q}$. Thus, the recursive error dynamics (14), (15) for $n_f = 0$ corresponding to (8) and (11) take the same form

$$\begin{aligned} \dot{\zeta}_0 &= -\lambda_{n_d} \lfloor \zeta_0 \rceil^{\frac{d - n_d s_q}{d-(n_d+1)s_q}} + \zeta_1, \\ \dot{\zeta}_1 &= -\lambda_{n_d-1} \lfloor \zeta_1 - \dot{\zeta}_0 \rceil^{\frac{d-(n_d-1)s_q}{d - n_d s_q}} + \zeta_2, \\ &\dots \\ \dot{\zeta}_{n_d-1} &= -\lambda_1 \lfloor \zeta_{n_d-1} - \dot{\zeta}_{n_d-2} \rceil^{\frac{d-s_q}{d-2s_q}} + \zeta_{n_d}, \\ \dot{\zeta}_{n_d} &\in -\lambda_0 \lfloor \zeta_{n_d} - \dot{\zeta}_{n_d-1} \rceil^{\frac{d}{d-s_q}} + \gamma_L[-1,1] ||\vec{\zeta}_{n_d}||_{h\infty}^{d}. \end{aligned} \quad (52)$$

DOI: 10.1002/rnc

Recall that in the case of Theorem 2 the HD q is fixed, and $d = 1/|q| + (n_d+2)s_q$, $d \geq 0$ hold. Under the conditions of Theorem 3 $d \geq 0$ is fixed in advance, and q actually depends on n_d according to the equations

$$1/|q(n_d+1)| + (n_d+2)s_q = d,\ 1/|q(n_d)| + (n_d+1)s_q = d.$$

In both cases s_q remains fixed.

The coefficients λ_k satisfy the recursive formulas (16) for $n_f = 0$. Note that $\lambda_{n_d} = \tilde{\lambda}_{n_d}$, $\lambda_0 = \tilde{\lambda}_0$. The homogeneity degree of the system (52) is $s_q = \operatorname{sign} q$, and does not depend on the differentiation order. Moreover, adding one to the order n_d simply leads to the addition of one more equation from above and renumbering the variables. Thus, provided $d - (k+1)s_q > 0$, the weight $\deg \zeta_{n_d-k} = d - (k+1)s_q$ only depends on k and d.

Using (52) rewrite the error dynamics of (8) as

$$\begin{aligned}
&\dot{\tilde{\zeta}}_0 = -c\left(\lfloor\tilde{\zeta}_0\rceil^{\frac{d-(n_d+1)s_q}{d-(n_d+2)s_q}} - \zeta_1\right),\\
&\dot{\zeta}_1 = -\lambda_{n_d}\left\lfloor\zeta_1 - \tfrac{1}{c}\dot{\tilde{\zeta}}_0\right\rceil^{\frac{d-n_d s_q}{d-(n_d+1)s_q}} + \zeta_2,\\
&\quad\dots\\
&\dot{\zeta}_{n_d} = -\lambda_1\lfloor\zeta_{n_d} - \dot{\zeta}_{n_d-1}\rceil^{\frac{d-s_q}{d-2s_q}} + \zeta_{n_d+1},\\
&\dot{\zeta}_{n_d+1} \in -\lambda_0\lfloor\zeta_{n_d+1} - \dot{\zeta}_{n_d}\rceil^{\frac{d}{d-s_q}} + \gamma_L[-1,1]||\vec{\zeta}_{n_d+1}||_{h\infty}^{d}.
\end{aligned} \tag{53}$$

Here $c = \lambda_{n_d+1}^{\frac{d-(n_d+2)s_q}{d-(n_d+1)s_q}}$, $\tilde{\zeta}_0 = c\zeta_0$, $\lambda_{n_d+1} = \tilde{\lambda}_{n_d+1}$. It is easy to check that the homogeneous functions $\tilde{\zeta}_0, \dot{\tilde{\zeta}}_0, \dot{\zeta}_1, \dots, \dot{\zeta}_{n_d}$, as well as $\dot{\tilde{\zeta}}_0, \dot{\zeta}_1, \dots, \dot{\zeta}_{n_d}, \zeta_{n_d+1}$, form a set of alternative *algebraic* coordinates for the space $\vec{\zeta}_{n_d+1}$.

In the case $n_d = 0$ get $c = 1, \tilde{\zeta}_0 = \zeta_0$. Then the Lyapunov function of the homogeneity degree $a > d - s_q$ for (52) is defined as

$$V_0 = \int_0^{\zeta_0} \lfloor s\rceil^{\frac{a-d+s_q}{d-s_q}} ds.$$

One easily checks that

$$\dot{V}_0 \in \lfloor\zeta_0\rceil^{\frac{a-d+s_q}{d-s_q}}\left(-\lambda_0\lfloor\zeta_0\rceil^{\frac{d}{d-s_q}} + [-\gamma_L, \gamma_L]\cdot|\zeta_0|^{\frac{d}{d-s_q}}\right)$$

and $\dot{V}_0 \leq -(\lambda_0 - \gamma_L)|\zeta_0|^{\frac{d}{d-s_q}}$ for $\lambda_0 > \gamma_L$.

Now prove that provided the coefficients $\lambda_0, \lambda_1, \dots, \lambda_{n_d}$ are properly chosen, any sufficiently large $c \geq 1$ (and $\lambda_{n_d+1} > 1$) provides a proper choice of the coefficients for the differentiator of the order $n_d + 1$. Construct a homogeneous Lyapunov function V_{n_d+1} by induction.

Induction step. Choose any a, $a > 2\max[d - (n_d+2)s_q, d] + 1$, and let $V_{n_d}(\vec{\zeta}_{n_d})$ be a homogeneous Lyapunov function of the homogeneity degree a for system (52) of the order n_d, $V_{n_d} \in C^1$. Such function always exists [7].

In order to simplify the further calculations, consider the lower subsystem of (53). After renumbering the variables $i \mapsto i+1$, it is identical to (52) if $\dot{\tilde{\zeta}}_0 = \dot{\zeta}_0 = \zeta_0 = 0$. The time derivative of $V_{n_d}(\zeta_1, \dots, \zeta_{n_d+1})$ with respect to that subsystem is negative definite, which means that

$$\begin{aligned}
&\dot{V}_{n_d}(\zeta_1, \dots, \zeta_{n_d+1}) \leq\\
&\qquad \dot{V}_{n_d}|_{\gamma_L=0} + \gamma_L\left|\frac{\partial V_{n_d}}{\partial \zeta_{n_d+1}}\right|\,||(0, \zeta_1, \dots, \zeta_{n_d+1})||_{h\infty}^{d} < 0 \text{ for } (\zeta_1, \dots, \zeta_{n_d+1}) \neq 0.
\end{aligned} \tag{54}$$

Let the Lyapunov function for (53) have the form

$$\begin{aligned}
&V_{n_d+1} = W_{n_d+1}(\tilde{\zeta}_0, \zeta_1) + V_{n_d}(\zeta_1, \dots, \zeta_{n_d+1}),\\
&W_{n_d+1} = \int_{\lfloor\zeta_1\rceil^{\frac{d-(n_d+2)s_q}{d-(n_d+1)s_q}}}^{\tilde{\zeta}_0} \left(\lfloor s\rceil^{\frac{a-d+(n_d+2)s_q}{d-(n_d+2)s_q}} - \lfloor\zeta_1\rceil^{\frac{a-d+(n_d+2)s_q}{d-(n_d+1)s_q}}\right) ds
\end{aligned} \tag{55}$$

According to Lemma 2 the function W_{n_d+1} is non-negative, $\deg W_{n_d+1} = a$, and it only vanishes for $\dot{\tilde{\zeta}}_0 = 0$. Thus, V_{n_d+1} only vanishes for $\dot{\tilde{\zeta}}_0 = 0$, $\zeta_1 = ... = \zeta_{n_d+1} = 0$, in which case (53) implies that $\vec{\zeta}_{n_d+1} = 0$. Therefore, V_{n_d+1} is positive definite.

Denote $r_i = \deg \zeta_i = d - (n_d + 2 - i)s_q$. Obviously,

$$W_{n_d+1} = \int_{\lfloor \zeta_1 \rceil^{\frac{r_0}{r_1}}}^{\tilde{\zeta}_0} (\lfloor s \rceil^{\frac{a-r_0}{r_0}} - \lfloor \zeta_1 \rceil^{\frac{a-r_0}{r_1}}) ds$$

and

$$\dot{W}_{n_d+1} = (\lfloor \tilde{\zeta}_0 \rceil^{\frac{a-r_0}{r_0}} - \lfloor \zeta_1 \rceil^{\frac{a-r_0}{r_1}}) \dot{\tilde{\zeta}}_0 - \tfrac{a-r_0}{r_1} \int_{\lfloor \zeta_1 \rceil^{\frac{r_0}{r_1}}}^{\tilde{\zeta}_0} (|\zeta_1|^{\frac{a-r_0-r_1}{r_1}}) ds \, \dot{\zeta}_1 =$$
$$- c(\lfloor \tilde{\zeta}_0 \rceil^{\frac{a-r_0}{r_0}} - \lfloor \zeta_1 \rceil^{\frac{a-r_0}{r_1}}) (\lfloor \tilde{\zeta}_0 \rceil^{\frac{r_1}{r_0}} - \zeta_1)$$
$$+ \tfrac{a-r_0}{r_1} (\tilde{\zeta}_0 - \lfloor \zeta_1 \rceil^{\frac{r_0}{r_1}}) \, |\zeta_1|^{\frac{a-r_0-r_1}{r_1}} \, (\lambda_{n_d} \left\lfloor \zeta_1 - \tfrac{1}{c} \dot{\tilde{\zeta}}_0 \right\rceil^{\frac{r_2}{r_1}} - \zeta_2). \quad (56)$$

Recall that for any $A, B \geq 0$, $(A + B)^p \leq 2^{p-1}(A^p + B^p)$ holds for $p \geq 1$, and $(A + B)^p \leq (A^p + B^p)$ holds for $0 < p \leq 1$. Taking into account that $0 < r_2/r_1 < 1$ for $q < 0$, and $r_2/r_1 > 1$ for $q > 0$ get

$$|\zeta_1 - \tfrac{1}{c}\dot{\tilde{\zeta}}_0|^{\frac{r_2}{r_1}} \leq |\zeta_1|^{\frac{r_2}{r_1}} + |\tfrac{1}{c}\dot{\tilde{\zeta}}_0|^{\frac{r_2}{r_1}} = |\zeta_1|^{\frac{r_2}{r_1}} + |\lfloor \tilde{\zeta}_0 \rceil^{\frac{r_1}{r_0}} - \zeta_1|^{\frac{r_2}{r_1}}, \; q > 0;$$
$$|\zeta_1 - \tfrac{1}{c}\dot{\tilde{\zeta}}_0|^{\frac{r_2}{r_1}} \leq 2^{\frac{r_2}{r_1}-1}(|\zeta_1|^{\frac{r_2}{r_1}} + |\lfloor \tilde{\zeta}_0 \rceil^{\frac{r_1}{r_0}} - \zeta_1|^{\frac{r_2}{r_1}}), \; q < 0.$$

Thus, substituting the value of c obtain from (56) that

$$\begin{aligned}
\dot{W}_{n_d+1} \leq \; & -\lambda_{n_d}^{\frac{r_0}{r_1}} \hat{W}_{n_d+1,1} + \hat{W}_{n_d+1,2}, \\
\hat{W}_{n_d+1,1} = \; & (\lfloor \tilde{\zeta}_0 \rceil^{\frac{a-r_0}{r_0}} - \lfloor \zeta_1 \rceil^{\frac{a-r_0}{r_1}}) (\lfloor \tilde{\zeta}_0 \rceil^{\frac{r_1}{r_0}} - \zeta_1), \\
\hat{W}_{n_d+1,2} = \; & \lambda_{n_d} \hat{\gamma} |\zeta_1|^{\frac{a-r_0-r_1}{r_1}} |\tilde{\zeta}_0 - \lfloor \zeta_1 \rceil^{\frac{r_0}{r_1}}| \, |\lfloor \tilde{\zeta}_0 \rceil^{\frac{r_1}{r_0}} - \zeta_1|^{\frac{r_2}{r_1}} \\
& + \hat{\gamma} |\zeta_1|^{\frac{a-r_0-r_1}{r_1}} |\tilde{\zeta}_0 - \lfloor \zeta_1 \rceil^{\frac{r_0}{r_1}}| \, |\zeta_2|, \\
\hat{\gamma} = \; & \begin{cases} \frac{a-r_0}{r_1} & \text{for } q < 0, \\ \frac{a-r_0}{r_1} 2^{\frac{r_2}{r_1}-1} & \text{for } q > 0. \end{cases}
\end{aligned} \quad (57)$$

Also note that

$$\begin{aligned}
& (\lfloor \tilde{\zeta}_0 \rceil^{\frac{a-r_0}{r_0}} - \lfloor \zeta_1 \rceil^{\frac{a-r_0}{r_1}}) (\lfloor \tilde{\zeta}_0 \rceil^{\frac{r_1}{r_0}} - \zeta_1) > 0 \text{ for } \dot{\zeta}_0 \neq 0, \\
& (\lfloor \tilde{\zeta}_0 \rceil^{\frac{a-r_0}{r_0}} - \lfloor \zeta_1 \rceil^{\frac{a-r_0}{r_1}}) (\lfloor \tilde{\zeta}_0 \rceil^{\frac{r_1}{r_0}} - \zeta_1) = 0 \text{ for } \dot{\zeta}_0 = 0, \\
& \tilde{\zeta}_0 - \lfloor \zeta_1 \rceil^{\frac{r_0}{r_1}} = 0 \text{ for } \dot{\zeta}_0 = 0, \\
& ||\vec{\zeta}_{n_d+1}||_{h\infty} = ||(0, \zeta_1, ..., \zeta_{n_d+1})||_{h\infty} \text{ for } \dot{\zeta}_0 = 0.
\end{aligned} \quad (58)$$

The last equality is due to $\lambda_{n_d+1} \geq 1$.

Now combining (55), (54), (57) obtain

$$\dot{V}_{n_d+1} \leq -\lambda_{n_d+1}^{\frac{r_0}{r_1}} \hat{W}_{n_d+1,1} + \hat{W}_{n_d+1,2} + \dot{V}_{n_d}|_{\gamma_L=0} + \gamma_L |\tfrac{\partial V_{n_d}}{\partial \zeta_{n_d+1}}| \, ||\vec{\zeta}_{n_d+1}||_{h\infty}^{d}. \quad (59)$$

Obviously, due to (58), (54) and Lemma 3, the right-hand side of (59) is negative definite for sufficiently large λ_{n_d+1}. It finishes the proof for $q \neq 0$.

The case $q = 0$ (Theorem 2) formally corresponds to $s_q = 0$, $d = 1$ in (53), correspondingly $r_i = 1$, $i = 0, 1, ..., n_d + 1$. The further proof is exactly the same. □

Proof of Theorem 4. The theorem immediately follows from the contractivity principle ([22], Section2) and the continuous dependence of the DI solutions on the right-hand-side

DOI: 10.1002/rnc

graph [14]. □

Proof of Theorem 5. Consider (12), (13). Subtract $f_0^{(i+1)}$ from the both sides of the equation for $\dot{z}_i$, $i = 0, ..., n_d$. Define $\zeta_j = w_{j+1}L^{s_q}$, for $j = 0, n_f - 1$, and $\zeta_j = (z_{j-n_f} - f_0^{(j-n_f)})L^{s_q}$ for $j = n_f, ..., n_f + n_d$. The resulting dynamics of errors coincide with the dynamics (49) of the pure differentiator (5), but of the order $n_d + n_f$. Correspondingly the choice of coefficients is reduced to the case of the pure differentiator (5), i.e. to Theorem 1. □

Proof of Theorem 6. The error dynamics of the filtering differentiator coincides with those of differentiator (5), but of the order $n_d + n_f$. Thus, the theorem is reduced to differentiator (5), and Theorem 2 for $q = 0$, and Theorem 3 for $q \neq 0$. □

Proof of Theorem 7. Define $\zeta_i = z_i - \sigma^{(i)}$, $\deg \zeta_i = \deg z_i = \deg \sigma^{(i)} = 1 + iq$, $\deg w_i = 1 - (n_f - i + 1)q$. The closed loop system can be rewritten as the homogeneous DI of the HD q,

$$\begin{array}{l} \sigma^{(n)} \in [-C, C]||\vec{\sigma}_{n-1}||_h^{1+nq} \\ \qquad\qquad +[K_m, K_M]U(\vec{\sigma}_{n-1} + \zeta), \\ \dot{w} = \Omega_{n-1,n_f,q}(w, \zeta_0, L, \vec{\lambda}_{n+n_f-1}), \\ \dot{\zeta} = D_{n-1,n_f,q}(w_1, \zeta, L, \vec{\lambda}_{n+n_f-1}). \end{array} \qquad (60)$$

Under the exact measurements of $\vec{\sigma}_{n-1}$ get

$$|\sigma^{(n)}| \leq C||\vec{\sigma}_{n-1}||_h^{1+nq} + K_M|U(\vec{\sigma}_{n-1})| \leq L_0||\vec{\sigma}_{n-1}||_{h\infty}^{1+nq}$$

for some $L_0 > 0$. It is easy to see that (6) is satisfied for $n_d = n - 1$, provided $|f_0^{(n)}| \leq \gamma_L L^\delta ||z_0 - f_0, ..., z_{n-1} - f_0^{(n-1)}||_{h\infty}^{1+nq}$, where $\delta = |q|$ for $q < 0$, and $\delta = \frac{q}{1+(n-1)q}$ for $q > 0$.

Show that the statement of Theorem 5 follows from the AS of the homogeneous DI

$$\begin{array}{ll} \dot{w} = & \Omega_{n-1,n_f,q}(w, \zeta_0, L, \vec{\lambda}_{n+n_f-1}), \\ \dot{\zeta} \in & D_{n-1,n_f,q}(w_1, \zeta, L, \vec{\lambda}_{n+n_f-1}) + (0, ..., 0, \gamma_L L^\delta ||\zeta||_{h\infty}^{1+nq})^T. \end{array} \qquad (61)$$

Choose $L > 1$ such that $\gamma_L L^\delta > L_0$, and let $V_u(\vec{\sigma}_{n-1})$ and $V_f(w, \zeta)$ be the homogeneous C^1 LFs for (27) closed by $u = U(\vec{\sigma}_{n-1})$ and (61) respectively, $\deg V_u = \deg V_f = a > -q$. Such functions always exist [7].

Note that $\dot{V}_u(\vec{\sigma}_{n-1}) = \{\nabla V_u \dot{\vec{\sigma}}_{n-1}\}$ and similarly $\dot{V}_f(w, \zeta)$ are compact numeric sets. Concequently $\sup \dot{V}_u(\vec{\sigma}_{n-1}) \leq -W_u(\vec{\sigma}_{n-1})$, $\sup \dot{V}_f(w, \zeta) \leq -W_f(w, \zeta)$ where W_u, W_f are positive-definite functions of their arguments, $\deg W_u = \deg W_f = a + q > 0$.

Search for the LF in the form $V(\vec{\sigma}_{n-1}, w, \zeta) = V_u(\vec{\sigma}_{n-1}) + \mu V_f(w, \zeta)$, $\mu > 0$. Then

$$\dot{V} = \dot{V}_u(\vec{\sigma}_{n-1}) + \tfrac{\partial}{\partial \sigma^{(n-1)}} V_u [K_m, K_M](U(\vec{\sigma}_{n-1} + \zeta) - U(\vec{\sigma}_{n-1})) + \mu \dot{V}_f(w, \zeta).$$
$$\sup \dot{V} \leq -W_u(\vec{\sigma}_{n-1}) + W_1(\vec{\sigma}_{n-1}, \zeta) - \mu W_f(w, \zeta),$$

where the continuous homogeneous function $W_1(\vec{\sigma}_{n-1}, \zeta)$ vanishes for $\zeta = 0$. It follows now from Lemma 3 that the right hand side is negative definite for μ large enough. □

Proof of Theorem 8. According to the filtering-order definition introduce the functions $\xi_k(t)$, $|\xi_k| \leq \varepsilon_k$, $\xi_k^{(k)}(t) = \eta_k(t)$, $k = 1, ..., n_f$. Let

$$\begin{array}{l} \tilde{\omega}_1 = w_1 + \xi_{n_f}, \tilde{\omega}_2 = w_2 + \dot{\xi}_{n_f} + \xi_{n_f-1}, ..., \\ \tilde{\omega}_{n_f} = w_{n_f} + \xi_{n_f}^{(n_f-1)} + ... + \dot{\xi}_2 + \xi_1; \\ \tilde{\zeta}_i = z_i - f_0^i, \; i = 0, ..., n. \end{array} \qquad (62)$$

Then $f = f_0 + \eta_0 + \dot{\xi}_1 + ... + \xi_{n_f}^{(n_f)}$, and one can rewrite (12), (13) in the form

$$\begin{array}{l}\dot{\tilde{\omega}}_1 = -\tilde{\lambda}_{n_d+n_f} L^{\frac{|q|}{1-n_f q}} \lfloor \tilde{\omega}_1 - \xi_{n_f} \rceil^{\frac{1-(n_f-1)q}{1-n_f q}} + \tilde{\omega}_2 - \xi_{n_f-1},\\ \dot{\tilde{\omega}}_2 = -\tilde{\lambda}_{n_d+n_f-1} L^{\frac{2|q|}{1-n_f q}} \lfloor \tilde{\omega}_1 - \xi_{n_f} \rceil^{\frac{1-(n_f-2)q}{1-n_f q}} + \tilde{\omega}_3 - \xi_{n_f-2},\\ \quad ...\\ \dot{\tilde{\omega}}_{n_f-1} = -\tilde{\lambda}_{n_d+2} L^{\frac{(n_f-1)|q|}{1-n_f q}} \lfloor \tilde{\omega}_1 - \xi_{n_f} \rceil^{\frac{1-q}{1-n_f q}} + \tilde{\omega}_{n_f} - \xi_2,\\ \dot{\tilde{\omega}}_{n_f} = -\tilde{\lambda}_{n_d+1} L^{\frac{n_f|q|}{1-n_f q}} \lfloor \tilde{\omega}_1 - \xi_{n_f} \rceil^{\frac{1}{1-n_f q}} + \tilde{\zeta}_0 + \eta_0,\end{array} \tag{63}$$

$$\begin{array}{l}\dot{\tilde{\zeta}}_0 = -\tilde{\lambda}_{n_d} L^{\frac{(1+n_f)|q|}{1-n_f q}} \lfloor \tilde{\omega}_1 - \xi_{n_f} \rceil^{\frac{1+q}{1-n_f q}} + \tilde{\zeta}_1,\\ \dot{\tilde{\zeta}}_1 = -\tilde{\lambda}_{n_d-1} L^{\frac{(2+n_f)|q|}{1-n_f q}} \lfloor \tilde{\omega}_1 - \xi_{n_f} \rceil^{\frac{1+2q}{1-n_f q}} + \tilde{\zeta}_2,\\ \quad ...\\ \dot{\tilde{\zeta}}_{n_d} = -\tilde{\lambda}_0 L^{\frac{(n_d+1+n_f)|q|}{1-n_f q}} \lfloor \tilde{\omega}_1 - \xi_{n_f} \rceil^{\frac{1+(n_d+1)q}{1-n_f q}}.\end{array} \tag{64}$$

Multiply by L for $q > 0$, or divide by L for $q < 0$. Denote $\omega_k = \tilde{\omega}_k L^{s_q}$, $\zeta_k = \tilde{\zeta}_k L^{s_q}$. In the case $q = 0$ simply let $\omega_k = \tilde{\omega}_k$, $\zeta_k = \tilde{\zeta}_k$. Rewrite (63), (64) as the inclusion

$$\begin{array}{l}\dot{\omega}_1 \in -\lambda_{n_d+n_f} \lfloor \omega_1 + \rho^{1-n_f q}[-1,1] \rceil^{\frac{1-(n_f-1)q}{1-n_f q}} + \omega_2 + \rho^{1-(n_f-1)q}[-1,1],\\ \dot{\omega}_2 \in -\lambda_{n_d+n_f-1} \lfloor \omega_1 + \rho^{1-n_f q}[-1,1] \rceil^{\frac{1-(n_f-2)q}{1-n_f q}} + \omega_3 + \rho^{1-(n_f-2)q}[-1,1],\\ \quad ...\\ \dot{\omega}_{n_f-1} \in -\lambda_{n_d+2} \lfloor \omega_1 + \rho^{1-n_f q}[-1,1] \rceil^{\frac{1-q}{1-n_f q}} + \omega_{n_f} + \rho^{1-q}[-1,1],\\ \dot{\omega}_{n_f} \in -\lambda_{n_d+1} \lfloor \omega_1 + \rho^{1-n_f q}[-1,1] \rceil^{\frac{1}{1-n_f q}} + \zeta_0 + \rho[-1,1],\end{array} \tag{65}$$

$$\begin{array}{l}\dot{\zeta}_0 \in -\lambda_{n_d} \lfloor \omega_1 + \rho^{1-n_f q}[-1,1] \rceil^{\frac{1+q}{1-n_f q}} + \zeta_1,\\ \dot{\zeta}_1 \in -\lambda_{n_d-1} \lfloor \omega_1 + \rho^{1-n_f q}[-1,1] \rceil^{\frac{1+2q}{1-n_f q}} + \zeta_2,\\ \quad ...\\ \dot{\zeta}_{n_d} \in -\lambda_0 \lfloor \omega_1 + \rho^{1-n_f q}[-1,1] \rceil^{\frac{1+(n_d+1)q}{1-n_f q}} + \gamma_L ||\vec{\zeta}_{n_d}||_{h\infty}^{1+(n_d+1)q}.\end{array} \tag{66}$$

which actually is the perturbation of the FT stable homogeneous error dynamics (49) of the $(n_d + n_f)$th-order differentiator (5) obtained by substituting $n_d + n_f$ for n_d and renaming the first n_f variables to ω_k, $k = 1, ..., n_f$, and the rest to ζ_i, $i = 0, ..., n_d$. Obviously, $\deg \omega_k = 1 - (n_f + 1 - k)q$, $\deg \zeta_i = 1 + iq$, $\deg t = -q$, the system HD is q, $\deg \rho = 1$ is assigned.

It follows from [27] (also see Section 2) that $\sup |\zeta_i| \le \mu_i \rho^{1-iq}$, $\sup |\omega_k| \le \hat{\mu}_{wk} \rho^{1-(n_f+1-k)q}$ for some $\mu_i, \hat{\mu}_{wk} > 0$. Now the accuracies of z_i and w_1 are directly obtained from these relations. Estimations of w_k, $k > 1$, can be similarly obtained from (62) under additional assumptions on the noises. □

Proof of Theorem 9. The closed-loop system gets the form

$$\begin{array}{l}\sigma^{(n)} \in [-C, C] ||\vec{\sigma}_{n-1}||_h^{1+nq} + [K_m, K_M] U(\vec{\sigma}_{n-1} + \vec{z}_{n-1} - \vec{\sigma}_{n-1}),\\ \dot{w} = \Omega_{n-1,n_f,q}(w, z_0 - \sigma - \eta, L, \vec{\lambda}_{n+n_f-1}),\\ \dot{\vec{z}}_{n-1} = D_{n-1,n_f,q}(w_1, \vec{z}_{n-1}, L, \vec{\lambda}_{n+n_f-1}).\end{array} \tag{67}$$

Using Theorem 8 obtain that after the differentiator transient the dynamics of σ satisfy the DI

$$\begin{array}{l}\sigma^{(n)} \in [-C, C] ||\vec{\sigma}_{n-1}||_h^{1+nq} + [K_m, K_M] U(\vec{\sigma}_{n-1} + h),\\ h = \nu(\rho^1, \rho^{1+q}, ..., \rho^{1+(n-1)q})^T [-1, 1]\end{array} \tag{68}$$

for some $\nu > 0$. The DI is asymptotically stable and homogeneous of the HD q. Assigning the weight $\deg \rho = 1$ obtain a homogeneously disturbed DI. The resulting accuracy now follows

DOI: 10.1002/rnc

from [27] (also see Section 2). □

Proof of Theorem 10. First let $q \neq 0$. The closed-loop system gets the form

$$\begin{array}{l} \sigma^{(n)} \in [-C,C]||\vec{\sigma}_{n-1}||_h^{1+nq} + [K_m, K_M]U(\vec{z}_{n-1}(t_j)),\ t \in [t_j, t_{j+1}), \\ \delta_j w = \Omega_{n-1,n_f,q}(w(t_j), z_0(t_j) - \sigma(t_j) - \eta(t_j), L, \vec{\lambda}_{n+n_f-1})\tau_j, \\ \delta_j \vec{z}_{n-1} = D_{n-1,n_f,q}(w_1(t_j), \vec{z}_{n-1}(t_j), L, \vec{\lambda}_{n+n_f-1})\tau_j. \end{array} \quad (69)$$

Introduce $\xi_{l,k}(t_j)$ satisfying $\delta_j \xi_{l,k} = \xi_{l,k+1}\tau_j$ for $k = 0, 1, ..., l-1$, $\xi_{l,l}(t_j) = \eta_l(t_j)$, $|\xi_{l,0}(t_j)| \leq \varepsilon_l$, $l = 1, 2, ..., n_f$. Define

$$\begin{array}{ll} \omega_1(t_j) = & w_1(t_j) + \xi_{n_f,0}(t_j), \\ \omega_2(t_j) = & w_2(t_j) + \xi_{n_f,1}(t_j) + \xi_{n_f-1,0}(t_j), \\ & ..., \\ \omega_{n_f}(t_j) = & w_{n_f}(t_j) + \xi_{n_f,n_f-1}(t_j) + ... + \xi_{2,1}(t_j) + \xi_{1,0}(t_j), \end{array}$$

Then system (67) gets the form

$$\begin{array}{l} \sigma^{(n)} \in [-C,C]||\vec{\sigma}_{n-1}||_h^{1+nq} + [K_m, K_M]U(\vec{z}_{n-1}(t_j)),\ t \in [t_j, t_{j+1}), \\ \delta_j \omega = \tau_j \Omega_{n-1,n_f,q}(\omega(t_j) - d_f(t_j), \sigma(t_j) + \eta_0(t_j), L, \vec{\lambda}_{n+n_f-1}), \\ \delta_j \vec{z}_{n-1} = \tau_j D_{n-1,n_f,q}(\omega_1(t_j) - \xi_{n_f,0}(t_j), \vec{z}_{n-1}(t_j), L, \vec{\lambda}_{n+n_f-1}), \\ d_f(t_j) = (\xi_{n_f,0}(t_j), \xi_{n_f-1,0}(t_j), ..., \xi_{1,0}(t_j))^T. \end{array}$$

Discrete solutions $(\omega(t_j), z(t_j))$ can be considered as the node points of the piece-wise linear Euler solutions $(\omega(t), z(t))$. The corresponding continuous time solutions satisfy the DI

$$\begin{array}{l} \sigma^{(n)} \in [-C,C]||\vec{\tilde{\sigma}}_{n-1}||_h^{1+nq} + [K_m, K_M]U(\vec{z}_{n-1}(t - [0,\tau])), \\ \dot{\omega} \in \Omega_{n-1,n_f,q}(\omega(t - [0,\tau]) + h_w, \tilde{\sigma}(t_j) + \rho[-1,1], L, \vec{\lambda}_{n+n_f-1}), \\ \dot{\vec{z}}_{n-1} \in D_{n-1,n_f,q}(\omega_1(t - [0,\tau]) + \mu_w \rho^{1-n_f q}[-1,1], \vec{z}_{n-1}(t - [0,\tau]), L, \vec{\lambda}_{n+n_f-1}), \\ h_w = \mu_w(\rho^{1-n_f q}, ..., \rho^{1-q})^T[-1,1], \end{array}$$

for some $\mu_w > 0$. This DI is asymptotically stable and homogeneous for $\rho = 0$. Assigning the weight $\deg \rho = 1$ to the disturbance get the desired accuracy from [27] (also see Section 2). □

Proof of Lemma 6. Introducing virtual division points if needed, without loss of generality assume that $T \leq \hat{t}_{s+1} - \hat{t}_s \leq 2T$, $s = 1, 2, ...$. Let t_{j_s} and $t_{j_{s+1}}$ be the closest approximations from above of $\hat{t}_s$ and $\hat{t}_{s+1}$ by the time division t_j, $j = 0, 1, ...$, $t_{j+1} - t_j = \tau_j \leq \tau$. Let $|\nu(t)| \leq M_\nu$. Let τ be small enough to provide for $|\nu(t_{j+1}) - \nu(t_j)| \leq \delta_\tau$ due to the equicontinuity, $\delta_\tau \to 0$ as $\tau \to 0$.

Approximate the DEs

$$\dot{\xi}_i = \xi_{i+1},\ i = 1, ..., k-1,\ \dot{\xi}_k = \nu$$

by the Euler approximation

$$\dot{\hat{\xi}}_i(t) = \xi_{i+1}(t_j),\ i = 1, ..., k-1,\ \dot{\hat{\xi}}_k(t) = \nu(t_j),\ t \in [t_j, t_{j+1}).$$

Let $\Delta\xi_i = \hat{\xi}_i - \xi_i$. Then $\frac{d}{dt}\Delta\xi_k \in \delta_\tau[-1,1]$ holds. Correspondingly, taking into account two sampling intervals added at the ends of the segment $[\hat{t}_s, \hat{t}_{s+1}]$ get that $||\Delta\xi|| = O(\delta_\tau + M_\nu \tau)$ holds over the intervals $[t_{j_s}, t_{j_{s+1}}]$ uniformly in s. □

REFERENCES

[1] V. Andrieu, L. Praly, and A. Astolfi. Homogeneous approximation, recursive observer design, and output feedback. *SIAM Journal on Control and Optimization*, 47(4):1814–1850, 2008.

[2] M.T. Angulo, J.A. Moreno, and L.M. Fridman. Robust exact uniformly convergent arbitrary order differentiator. *Automatica*, 49(8):2489–2495, 2013.

[3] A.N. Atassi and H.K. Khalil. Separation results for the stabilization of nonlinear systems using different high-gain observer designs. *Systems & Control Letters*, 39(3):183–191, 2000.

[4] A. Bacciotti and L. Rosier. *Liapunov Functions and Stability in Control Theory.* Springer Verlag, London, 2005.

[5] J.-P. Barbot, A. Levant, M. Livne, and D. Lunz. Discrete differentiators based on sliding modes. *Automatica*, 112:available online, 2020.

[6] G. Bartolini, A. Pisano, E. Punta, and E. Usai. A survey of applications of second-order sliding mode control to mechanical systems. *International Journal of Control*, 76(9/10):875–892, 2003.

[7] E. Bernuau, D. Efimov, W. Perruquetti, and A. Polyakov. On an extension of homogeneity notion for differential inclusions. In *Proc. of 12th European Control Conference ECC, 17-19 July, 2013, Zurich*, pages 2204–2209, 2013.

[8] F.H. Clarke, Y.S. Ledayev, and R.J. Stern. Asymptotic stability and smooth Lyapunov functions. *Journal of Differential Equations*, 149(1):69–114, 1998.

[9] E. Cruz-Zavala and J.A. Moreno. Lyapunov approach to higher-order sliding mode design. In *Recent Trends in Sliding Mode Control*, pages 3–28. Institution of Engineering and Technology IET, 2016.

[10] E. Cruz-Zavala and J.A. Moreno. Lyapunov functions for continuous and discontinuous differentiators. *IFAC-PapersOnLine*, 49(18):660–665, 2016.

[11] E. Cruz-Zavala and J.A. Moreno. Levant's arbitrary order exact differentiator: a Lyapunov approach. *IEEE Transactions on Automatic Control*, 64(7):3034–39, 2019.

[12] J. Davila, L. Fridman, A. Pisano, and E. Usai. Finite-time state observation for non-linear uncertain systems via higher-order sliding modes. *International Journal of Control*, 82(8):1564–1574, 2009.

[13] C. Edwards and S.K. Spurgeon. *Sliding Mode Control: Theory And Applications.* Taylor & Francis, London, 1998.

[14] A.F. Filippov. *Differential Equations with Discontinuous Right-Hand Sides.* Kluwer Academic Publishers, Dordrecht, 1988.

[15] T. Floquet, J.P. Barbot, and W. Perruquetti. Higher-order sliding mode stabilization for a class of nonholonomic perturbed systems. *Automatica*, 39(6):1077–1083, 2003.

[16] M. Harmouche, S. Laghrouche, Y. Chitour, and M. Hamerlain. Stabilisation of perturbed chains of integrators using Lyapunov-based homogeneous controllers. *International Journal of Control*, 90(12):2631–2640, 2017.

[17] Y.G. Hong. Finite-time stabilization and stabilizability of a class of controllable systems. *Systems & Control Letters*, 46(4):231–236, 2002.

[18] A. Isidori. *Nonlinear Control Systems, Second edition.* Springer Verlag, New York, 1989.

[19] H.K. Khalil and L. Praly. High-gain observers in nonlinear feedback control. *International Journal of Robust and Nonlinear Control*, 24(6):993–1015, 2014.

[20] S. Koch, M. Reichhartinger, M. Horn, and L. Fridman. Discrete-time implementation of homogeneous differentiators. *IEEE Transactions on Automatic Control*, 65(2):757–762, 2020.

[21] H. Lei and W. Lin. Robust control of uncertain systems with polynomial nonlinearity by output feedback. *International Journal of Robust and Nonlinear Control*, 19(6):692–723, 2009.

[22] A. Levant. Higher order sliding modes, differentiation and output-feedback control. *International J. Control*, 76(9/10):924–941, 2003.

[23] A. Levant. Homogeneity approach to high-order sliding mode design. *Automatica*, 41(5):823–830, 2005.

[24] A. Levant. Quasi-continuous high-order sliding-mode controllers. *IEEE Trans. Aut. Control*, 50(11):1812–1816, 2005.

[25] A. Levant. Non-Lyapunov homogeneous SISO control design. In *56th Annual IEEE Conference on Decision and Control (CDC) 2017*, pages 6652–6657, 2017.

[26] A. Levant. Filtering differentiators and observers. In *15th International Workshop on Variable Structure Systems (VSS), 2018*, pages 174–179, 2018.

[27] A. Levant, D. Efimov, A. Polyakov, and W. Perruquetti. Stability and robustness of homogeneous differential inclusions. In *Proc. of the 55th IEEE Conference on Decision and Control, Las-Vegas, December 12-14, 2016*, 2016.

[28] A. Levant and M. Livne. Weighted homogeneity and robustness of sliding mode control. *Automatica*, 72(10):186–193, 2016.

[29] A. Levant and M. Livne. Globally convergent differentiators with variable gains. *International Journal of Control*, 91(9):1994–2008, 2018.

[30] A. Levant and M. Livne. Robust exact filtering differentiators. *European Journal of Control*, available online, 2019.

[31] A. Levant, M. Livne, and X. Yu. Sliding-mode-based differentiation and its application. *IFAC-PapersOnLine*, 50(1):1699–1704, 2017.

[32] A. Levant and X. Yu. Sliding-mode-based differentiation and filtering. *IEEE Transactions on Automatic Control*, 63(9):3061–3067, 2018.

[33] W. Lin and C. Qian. Adding one power integrator: a tool for global stabilization of high-order lower-triangular systems. *Systems & Control Letters*, 39(5):339–351, 2000.

[34] W. Perruquetti, T. Floquet, and E. Moulay. Finite-time observers: application to secure communication. *Automatic Control, IEEE Transactions on*, 53(1):356–360, 2008.

[35] A. Polyakov. Nonlinear feedback design for fixed-time stabilization of linear control systems. *IEEE Transactions on Automatic Control*, 57(8):2106–2110, 2012.

[36] C. Qian. A homogeneous domination approach for global output feedback stabilization of a class of nonlinear systems. In *Proceedings of the IEEE American Control Conference 2005*, pages 4708–4715, 2005.

[37] C. Qian and J. Li. Global output feedback stabilization of upper-triangular nonlinear systems using a homogeneous domination approach. *International Journal of Robust and Nonlinear Control*, 16(9):441–463, 2006.

[38] M. Reichhartinger, S.K. Spurgeon, M. Forstinger, and M. Wipfler. A robust exact differentiator toolbox for Matlab/Simulink. *IFAC-PapersOnLine*, 50(1):1711–1716, 2017.

[39] L. Rosier. Homogeneous Lyapunov function for homogeneous continuous vector field. *Systems & Control Letters*, 19(6):467–473, 1992.

[40] T. Sanchez, E. Cruz-Zavala, and J.A. Moreno. An SOS method for the design of continuous and discontinuous differentiators. *International Journal of Control*, 91(11):2597–2614, 2018.

[41] Y. Shtessel, C. Edwards, L. Fridman, and A. Levant. *Sliding mode control and observation.* Birkhauser, 2014.

[42] L.K. Vasiljevic and H.K. Khalil. Error bounds in differentiation of noisy signals by high-gain observers. *Systems & Control Letters*, 57(10):856–862, 2008.

DOI: 10.1002/rnc