GROWTH GAP VERSUS SMOOTHNESS FOR DIFFEOMORPHISMS OF THE INTERVAL

LEV BUHOVSKI AND ROMAN MURAVIEV
SCHOOL OF MATHEMATICAL SCIENCES
TEL-AVIV UNIVERSITY

Abstract. Given a diffeomorphism of the interval, consider the uniform norm of the derivative of its n-th iteration. We get a sequence of real numbers called the growth sequence. Its asymptotic behavior is an invariant which naturally appears both in smooth dynamics and in geometry of the diffeomorphisms groups. We find sharp estimates for the growth sequence of a given diffeomorphism in terms of the modulus of continuity of its derivative. These estimates extend previous results of Polterovich-Sodin and Borichev.

1. Introduction and main results

Denote by $\mathrm{Diff}_0[0,1]$ the group of all C^1-smooth diffeomorphisms of the interval $[0,1]$ fixing the end points 0 and 1. For any $f \in \mathrm{Diff}_0[0,1]$, we define the growth sequence of f by

$$\Gamma_n(f) = \max\{\|(f^n)'(x)\|_\infty, \|(f^{-n})'(x)\|_\infty\},$$

for all $n \in \mathbb{N}$, where $\|.\|_\infty$ stands for the uniform norm.

We say that a subgroup $G \subseteq \mathrm{Diff}_0[0,1]$ admits a *growth gap* if there exists a sequence of positive numbers $\gamma_n(G)$ that grows sub-exponentially to $+\infty$, such that for any $f \in G$, either $\Gamma_n(f)$ tends exponentially to $+\infty$, or $\Gamma_n(f) \leq C(f) \cdot \gamma_n(G)$, for all $n \in \mathbb{N}$.

From a viewpoint of dynamics, growth sequence of an element reflects how the length changes asymptotically under iterations. At the same time, geometrically, growth sequence indicates how an element is distorted with respect to the multiplicative norm. In $[DG]$, D'Ambra and Gromov suggested to study growth sequences of various classes of diffeomorphisms.

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The growth sequence is always submultiplicative:

$$\Gamma_{m+n}(f) \leq \Gamma_m(f) \cdot \Gamma_n(f),$$

for all $m, n \in \mathbb{N}$. Therefore, the limit

$$\gamma(f) = \lim_{n \to \infty} \sqrt[n]{\Gamma_n(f)}$$

always exists. Using standard arguments of ergodic theory, one can check that

$$\gamma(f) = 1 \ \text{ if and only if } \ f'(\xi) = 1 \text{ for every } \xi \in \mathrm{Fix}(f),$$

(see $[PS]$, page 199). The following theorem shows that the whole group $\mathrm{Diff}_0[0,1]$ does not admit a growth gap (see $[B]$),

Theorem 1. *Given any monotone decreasing sequence of positive numbers $\{\alpha_n\}_{n=1}^{\infty}$ tending to 0, there exists $f \in$Diff$_0[0,1]$ such that $Fix(f) = \{0,1\}$, $\gamma(f) = 1$ and*

$$\Gamma_n(f) \geq e^{\alpha_n \cdot n}$$

for all $n \in \mathbb{N}$.

As it is shown in Theorem 1, weakening of smoothness assumptions leaves more room for exponential growth, i.e., the growth sequence $\Gamma_n(f)$ becomes bigger, or in other words, "the growth gap" is smaller. Therefore, smaller subgroups of $\mathrm{Diff}_0[0,1]$ should be considered in order to discover a growth gap. In [PS] a growth gap was found for the subgroup of C^2-diffeomorphisms of $\mathrm{Diff}_0[0,1]$. Namely,

Theorem 2. *Let $f \in$Diff$_0[0,1]$ be a C^2-diffeomorphism with $\gamma(f) = 1$. Then*

$$\Gamma_n(f) \leq C(f) \cdot n^2,$$

for all $n \in \mathbb{N}$.

This result leads to a natural question on the growth gap for subgroups of $\mathrm{Diff}_0[0,1]$ with intermediate smoothness rate between C^1 and C^2. A partial answer is provided in [B]. To introduce this result, we consider the following subgroup of $\mathrm{Diff}_0[0,1]$ which is associated with the Hölder condition, $\mathrm{H}_\alpha[0,1] = \{f \in \mathrm{Diff}_0[0,1] : |f'(x) - f'(y)| \leq C(f) \cdot |x-y|^\alpha\}$, for $0 < \alpha < 1$.

Theorem 3. *If $f \in H_\alpha[0,1]$ with $\gamma(f) = 1$, then*

$$\log \Gamma_n(f) \leq C(f, \alpha) \cdot n^{1-\alpha},$$

for all $n \in \mathbb{N}$.

In the present work we obtain a growth gap for the following intermediate subgroups of diffeomorphisms:

Case (a): Subgroups between $C^2[0,1] \bigcap \mathrm{Diff}_0[0,1]$ and $\cap_{0<\alpha<1} H_\alpha[0,1]$.
Case (b): Subgroups between $\mathrm{Diff}_0[0,1]$ and $\cup_{0<\alpha<1} H_\alpha[0,1]$.

To describe subgroups of smoothness between C^1 and C^2, we use the terminology of moduli of continuity, i.e., non-decreasing continuous functions $\omega : [0,1] \to \mathbb{R}$ satisfying $\omega(0) = 0$ and $\omega(\delta_1 + \delta_2) \le \omega(\delta_1) + \omega(\delta_2)$. Given a modulus of continuity $\omega : [0,1] \to \mathbb{R}_+$, we consider the subgroup

$$\mathrm{Diff}_0^\omega[0,1] = \{f \in \mathrm{Diff}_0[0,1] : \omega_{f'}(\delta) \le C(f) \cdot \omega(\delta)\},$$

where $\omega_{f'}(\delta) = \max_{|x-y|\le\delta} |f'(x) - f'(y)|$.

It is not hard to check that $\mathrm{Diff}_0^\omega[0,1]$ is a non-empty subgroup. Indeed, the identity map is an element of $\mathrm{Diff}_0^\omega[0,1]$. Furthermore, for any two $f, g \in \mathrm{Diff}_0^\omega[0,1]$,

$$\begin{gathered} |(f\circ g)'(x) - (f\circ g)'(y)| \\ \le |f'(g(x))g'(x) - f'(g(x))g'(y)| + |f'(g(x))g'(y) - f'(g(y))g'(y)| \\ \le A(f)\cdot|g'(x)-g'(y)| + B(g)\cdot|f'(g(x)) - f'(g(y))| \le C(f,g)\cdot\omega(|x-y|). \end{gathered}$$

$$\begin{gathered} |(f^{-1})'(x) - (f^{-1})'(y)| = |\frac{f'(f^{-1}(x)) - f'(f^{-1}(y))}{f'(f^{-1}(x))\cdot f'(f^{-1}(y))}| \\ \le |\frac{f'(f^{-1}(x)) - f'(f^{-1}(y))}{a(f)^2}| \le B(f)\cdot\omega(|f^{-1}(x) - f^{-1}(y)|) \\ \le C(f)\cdot\omega(|x-y|). \end{gathered}$$

Our first result generalizes Theorems 2 and 3 and provides a growth gap for *case (a)*.

Theorem 4. *Let $\omega(x) : [0,1] \to \mathbb{R}_+$ be a strictly increasing modulus of continuity. Then, for each $f \in \mathrm{Diff}_0^\omega[0,1]$, such that $\gamma(f) = 1$, we have*

$$(*) \quad \log\Gamma_n(f) \le \log\frac{n}{\omega^{-1}(\frac{2}{n})} + C(f)n\omega(\frac{1}{n}).$$

Here we denote by ω^{-1} the inverse function to ω.

One can substitute $\omega(\delta) = \delta$ and $\omega(\delta) = \delta^\alpha$ into Theorem 4 for achieving Theorems 2 and 3. In the following corollary, we consider two toy models related to *case (a)* in order to test how Theorem 4 provides a growth gap. In the case when the modulus of continuity $\omega(\delta)$ is close to the identity, the second term on the right hand side of $(*)$ can be absorbed into the first one. Namely,

Corollary 1. *(1) If*

$$\limsup_{x\to 0} \frac{\omega(x)}{x \cdot \log \frac{e}{x}} < +\infty,$$

then

$$\log \Gamma_n(f) \leq C(f,\omega) \cdot \log \frac{n}{\omega^{-1}(\frac{2}{n})}.$$

(2) If

$$\lim_{x\to 0} \frac{\omega(x)}{x \cdot \log \frac{e}{x}} = 0,$$

then

$$\log \Gamma_n(f) \leq (1 + o(1)) \cdot \log \frac{n}{\omega^{-1}(\frac{2}{n})}.$$

The proofs easily follow by substituting the relevant assumptions into Theorem 4.

The drawback of Theorem 4 is that it does not provide a growth gap for *case (b)*. For instance, if we consider a diffeomorphism $f(x)$ from *case (b)* with $\omega_{f'}(\delta) \leq \frac{1}{\log \frac{e}{\delta}}$, then an attempt to apply Theorem 4 for this diffeomorphism yields only a trivial estimate

$$\log \Gamma_n(f) \leq C(f) \cdot n.$$

Our second theorem mends this disadvantage. It shows that in *case (b)* (under additional regularity assumption imposed on ω) one can discard the first term on the right hand side of $(*)$:

Theorem 5. *Let $\omega(x) : [0,1] \to \mathbb{R}_+$ be a modulus of continuity such that for some $0 < \alpha < 1$, $\frac{\omega(x)}{x^\alpha}$ is a decreasing function on $(0, a(\alpha))$, where $0 < a(\alpha) < 1$. Then for $f \in \mathrm{Diff}_0^\omega[0,1]$,* such that $\gamma(f) = 1$, we have

$$\log \Gamma_n(f) \leq C(f) \cdot n\omega(\frac{1}{n}).$$

The next set of theorems present a sufficient sharpness for the estimates of the bounds in Theorems 4 and 5 respectively.

Theorem 6. *Suppose that for each* $0 < \alpha < 1$ *there exists* $0 < a(\alpha) < 1$ *such that the function* $\frac{\omega(x)}{x^\alpha}$ *increases for all* $x \in [0, a(\alpha)]$ *and suppose that*

$$\lim_{x\to 0} \frac{\omega(x)}{x\cdot\log(\frac{\varepsilon}{x})} = 0.$$

Then, there exists a diffeomorphism $f \in \mathrm{Diff}_0^\omega[0,1]$ *with* $\gamma(f) = 1$ *such that for any* $\varepsilon > 0$,

$$\log \Gamma_n(f) \geq (1-\varepsilon)\cdot \log \frac{n}{\omega^{-1}(\frac{c(f)}{n})}\ , \qquad n \to \infty\,.$$

Theorem 7. *Suppose that the modulus of continuity* ω *satisfies assumptions of Theorem 5. Then there exists a diffeomorphism* $f \in \mathrm{Diff}_0^\omega[0,1]$ *with* $\gamma(f) = 1$, *such that for each* $\varepsilon > 0$,

$$\log \Gamma_n(f) \geq c(\varepsilon) n^{1-\varepsilon}\omega\left(\frac{1}{n}\right)\ , \qquad n \to \infty\,.$$

The proofs of Theorems 4-7 use ideas and techniques introduced in [L, chapterII] and especially in [B].

2. Growth gap: Proofs of theorems 4 and 5

The following lemma (see [EF, Dz]) states that every modulus of continuity admits an equivalent concave modulus of continuity:

Lemma 1. *For any modulus of continuity* ω *there exists a concave modulus of continuity* ω^* *such that* $\omega \leq \omega^* \leq 2\omega$ *everywhere on* $[0,1]$.

Due to this lemma, we assume in the proofs of Theorems 4 and 5 that ω is a concave modulus of continuity.

Proof of Theorem 4. First, we will introduce several notations and definitions: $\phi(x) := f(x) - x$; $x_n = f(x_{n-1})$; $A = \max_{x\in[0,1]} f'(x)$, $a = \min_{x\in[0,1]} f'(x)$. Choose a sufficiently small $\varepsilon > 0$, such that we will have $\omega(\varepsilon) < 1$. WLOG, we assume that $\phi(x)$ is positive. Consider a function $x \mapsto x\cdot\omega(x)$ which maps $[0,\varepsilon]$ on $[0, \varepsilon\cdot\omega(\varepsilon)]$, and denote by $\Omega(x) : [0, \varepsilon\cdot\omega(\varepsilon)] \to [0,\varepsilon]$ its inverse. Now pick a positive $\delta < \varepsilon$, so that the following requirement will be satisfied:

For all $x \in [0,\delta]$ we have $\phi(x) \in [0, \varepsilon\cdot\omega(\varepsilon)]$ and

$$J_x := [x, f(x)] \subseteq I_x := [x - \Omega(\phi(x)), x + \Omega(\phi(x))] \subseteq [0,\varepsilon].$$

Let us explain why it is possible. It is obviously possible to require that $\phi(x) \in [0, \varepsilon \cdot \omega(\varepsilon)]$ and $x + \Omega(\phi(x)) \leq \varepsilon$ for all $x \in [0, \delta]$, due to continuity. The inequality

$$0 \leq x - \Omega(\phi(x))$$

is equivalent to that

$$\phi(x) \leq x \cdot \omega(x)$$

which is satisfied for all $x \in [0, \delta]$, since $|\phi'(x)| \leq \omega(x)$.

We will present now a sequence of technical claims, which will be used later in the proof of Theorem 4.

Claim 1. *(a) For any $x \in [0, 1]$ and $y \in [x, f(x)]$,*

$$\frac{1}{A} \leq \frac{\phi(x)}{\phi(y)} \leq \frac{1}{a}.$$

(b) For any $x_1 \in [0, 1]$ and $n \in \mathbb{N}$,

$$\frac{1}{A} \cdot n \leq \int_{x_1}^{x_{n+1}} \frac{dt}{\phi(t)} \leq \frac{1}{a} \cdot n.$$

Proof of Claim 1. For any $y \in [x, f(x)]$, there exists $0 \leq \theta \leq 1$ such that $y = x + \theta \cdot \phi(x)$ and $0 \leq \theta_1 \leq 1$, such that

$$\frac{\phi(y)}{\phi(x)} = \frac{\phi(x) + \theta\phi(x)\phi'(x + \theta_1\theta\phi(x))}{\phi(x)} \leq 1 + \max_{x \in [0,1]} \phi'(x) \leq A.$$

In the same way,

$$\frac{\phi(x)}{\phi(y)} = \frac{\phi(x)}{\phi(x) + \theta\phi(x)\phi'(x + \theta_1\theta\phi(x))} \leq \frac{1}{1 + \phi'(x + \theta_1\theta\phi(x))} \leq \frac{1}{a}.$$

Therefore, for all $k \in \mathbb{N}$:

$$\frac{1}{A} \leq \int_{x_k}^{x_{k+1}} \frac{dt}{\phi(t)} \leq \frac{1}{a}.$$

By summing the integrals we obtain the desirable inequality. $\square$

Claim 2. *For all $x \in [0, \delta]$ and $y \in I_x$, we have*

$$(a) \quad |\phi'(y)| \leq 3\omega(\Omega(\phi(x))) = 3\frac{\phi(x)}{\Omega(\phi(x))}.$$

$$(b) \quad \frac{\phi(y)}{\phi(x)} \leq 2.5.$$

Remark: In particular, we obtain that for all $x \in [0, \delta]$, $|\phi'(x)| \leq 3\omega(\Omega(\phi(x)))$.

Proof of Claim 2. (a) Suppose that there exists $y_0 \in I_x$ such that $\phi'(y_0) > 3 \cdot \omega(\Omega(\phi(x)))$. Note that the following inequalities are satisfied for all $y \in I_x$:

$$\phi'(y) \geq \phi'(y_0) - \omega(|y - y_0|) > 3 \cdot \omega(\Omega(\phi(x))) - \omega(2\Omega(\phi(x)))$$
$$\geq (3-2) \cdot \omega(\Omega(\phi(x))) = \omega(\Omega(\phi(x))).$$

Therefore,

$$\phi(x) - \phi(x - \Omega(\phi(x))) = \int_{x-\Omega(\phi(x))}^{x} \phi'(t)dt > \omega(\Omega(\phi(x))) \cdot \Omega(\phi(x)) = \phi(x).$$

It follows that $\phi(x - \Omega(\phi(x))) < 0$, what contradicts our assumptions. Now, assume that there exists a point $y_0 \in I_x$ such that $\phi'(y_0) < -3 \cdot \omega(\Omega(\phi(x)))$. Then for all $y \in I_x$,

$$\phi'(y) \leq \phi'(y_0) + \omega(|y - y_0|) < -3 \cdot \omega(\Omega(\phi(x))) + \omega(2\Omega(\phi(x)))$$
$$\leq (-3+2)\omega(\Omega(\phi(x))) \leq -\omega(\Omega(\phi(x))).$$

Therefore,

$$\phi(x + \Omega\phi(x)) - \phi(x) = \int_{x}^{x+\Omega(\phi(x))} \phi'(t)dt < -\omega(\Omega(\phi(x))) \cdot \Omega(\phi(x)) = -\phi(x),$$

whence $\phi(x + \Omega\phi(x)) < 0$, this is a contradiction.
(b) Using (a) we obtain for some $0 < \theta_1 < 1$,

$$\frac{\phi(y)}{\phi(x)} = \frac{\phi(x) + (y-x)\phi'(x + \theta_1(y-x))}{\phi(x)} \leq 1 + \max_{y \in I_x} |\phi'(y)| \cdot \frac{|I_x|}{2\phi(x)} \leq$$
$$\leq 1 + \frac{3}{2} \cdot \omega(\Omega(\phi(x))) \frac{\Omega(\phi(x))}{\phi(x)} = 2.5.$$

□

Claim 3. *Let $z \in [0, \delta]$ and $n \in \mathbb{N}$ be such that*

$$n \geq c(f) \cdot \frac{\Omega(z)}{z}.$$

Then,

$$\frac{1}{z} \leq C(f) \cdot \frac{n}{\omega^{-1}(\frac{1}{n})}.$$

Proof of Claim 3. Denote $s = \Omega(z)$, and notice that $s \cdot \omega(s) = z$. Thus,

$$\omega(s) = \frac{z}{\Omega(z)} \geq \frac{c(f)}{n}$$

$$s \geq \omega^{-1}(\frac{c(f)}{n}) \geq C(f) \cdot \omega^{-1}(\frac{1}{n}),$$

therefore,

$$z \geq s \cdot \frac{c(f)}{n} \geq c(f) \cdot C(f) \frac{\omega^{-1}(\frac{1}{n})}{n},$$

and we are done. $\square$

We turn now to the following two lemmas, on which the proof of Theorem 4 will be based.

Lemma 2. *Suppose that* $x_1, ..., x_{n+1} \in (0, \delta)$. *Then,*

$$|\log(\frac{\phi(x_{n+1})}{\phi(x_1)})| \leq \log \frac{n}{\omega^{-1}(\frac{1}{n})} + C(f, \omega).$$

Proof of Lemma 2. We split the proof into 2 cases.

Case 1: a. $x_{n+1} \in I_{x_1}$ and $\phi(x_1) < \phi(x_{n+1})$. In this case,

$$|\log \frac{\phi(x_1)}{\phi(x_{n+1})}| = \log \frac{\phi(x_{n+1})}{\phi(x_1)} < \log 2.5,$$

due to Claim 2
b. $x_{n+1} \in I_{x_1}$ and $\phi(x_1) > \phi(x_{n+1})$. We have two possibilities:
(i) $x_1 + \Omega(\phi(x_{n+1})) > x_{n+1}$ and by Claim 2 $\frac{\phi(x_1)}{\phi(x_{n+1})} < 2.5$.
(ii) $x_1 + \Omega(\phi(x_{n+1})) \leq x_{n+1}$, then :

$$n \geq a \cdot \int_{x_1}^{x_{n+1}} \frac{dt}{\phi(t)} \geq a \cdot \int_{x_{n+1} - \Omega(\phi(x_{n+1}))}^{x_{n+1}} \frac{dt}{\phi(t)}$$
$$\geq 2.5 \cdot a \cdot \frac{\Omega(\phi(x_{n+1}))}{\phi(x_{n+1})}.$$

Hence, this case is completed due to Claim 3

Case 2: a. $x_{n+1} \notin I_{x_1}$ and $\phi(x_1) < \phi(x_{n+1})$.

$$n \geq a \cdot \int_{x_1}^{x_{n+1}} \frac{dt}{\phi(t)} \geq a \cdot \int_{x_1}^{x_1 + \Omega(\phi(x_1))} \frac{dt}{\phi(t)} \geq \frac{a}{2.5 \cdot \phi(x_1)} \cdot \Omega(\phi(x_1)),$$

in the last inequality we have used Claim 1, hence we are done due to Claim 3.
b. $x_{n+1} \notin I_{x_1}$ and $\phi(x_1) > \phi(x_{n+1})$.

$$n \geq \int_{x_1}^{x_{n+1}} \frac{dt}{\phi(t)} \geq a \int_{x_{n+1} - \Omega(\phi(x_1))}^{x_{n+1}} \frac{dt}{\phi(t)} \geq a \int_{x_{n+1} - \Omega(\phi(x_{n+1}))}^{x_{n+1}} \frac{dt}{\phi(t)}$$
$$\geq \frac{a}{2.5} \cdot \frac{\Omega(\phi(x_{n+1}))}{\phi(x_{n+1})}$$

In the last inequality we have used Claim 1, hence we are done due to Claim 3. $\square$

Lemma 3. *Suppose that $x_1, ..., x_{n+1} \in (0,\delta)$. Then,*

$$|\log{(f^n)'(x_1)} - \log(\frac{\phi(x_{n+1})}{\phi(x_1)})| \leq C(f)\cdot n\cdot \omega(\frac{1}{n}).$$

Proof of Lemma 3. We have

$$|\log{(f^n)'(x_1)} - \log\frac{\phi(x_{n+1})}{\phi(x_1)}| = |\sum_{k=1}^{n}(\log(1+\phi'(x_k)) - \log\frac{\phi(x_{k+1})}{\phi(x_k)})|$$

$$\leq \sum_{k=1}^{n}|\int_{x_k}^{x_{k+1}}\frac{\phi'(t)}{\phi(t)}dt - \log(1+\phi'(x_k))|.$$

The inequality $-\frac{y^2}{1+y} \leq \log(1+y) - y < 0$, which is valid for all $y > -1$, implies that $|\log(1+y) - y| \leq \frac{y^2}{1+y}$. In our context, we may use both inequalities, since $\min_{x\in[0,1]}\phi'(x) > -1$.

$$\sum_{k=1}^{n}|\int_{x_k}^{x_{k+1}}\frac{\phi'(t)}{\phi(t)}dt - \log(1+\phi'(x_k))| \leq$$

$$\sum_{k=1}^{n}|\int_{x_k}^{x_{k+1}}\frac{\phi'(t)}{\phi(t)}dt - \phi'(x_k)| + \sum_{k=1}^{n}|\log(1+\phi'(x_k)) - \phi'(x_k)|$$

$$\leq \sum_{k=1}^{n}|\int_{x_k}^{x_{k+1}}\frac{\phi'(t)}{\phi(t)}dt - \phi'(x_k)| + \sum_{k=1}^{n}\frac{[\phi'(x_k)]^2}{1+\phi'(x_k)}$$

$$\leq \sum_{k=1}^{n}|\int_{x_k}^{x_{k+1}}\frac{\phi'(t)}{\phi(t)}dt - \phi'(x_k)| + \frac{1}{a}\cdot\sum_{k=1}^{n}[\phi'(x_k)]^2.$$

Now we are going to estimate these sums. For any $x \in [0,\delta]$, there exists $0 < \theta < 1$ such that

$$\int_{x}^{x+\phi(x)}\frac{\phi'(t)}{\phi(t)}dt - \phi'(x) = \frac{\phi'(x+\theta\cdot\phi(x))}{\phi(x+\theta\cdot\phi(x))}\cdot\phi(x) - \phi'(x) =$$

$$= [\phi'(x+\theta\phi(x)) - \phi'(x)]\cdot\frac{\phi(x)}{\phi(x+\theta\phi(x))} + \phi'(x)[\frac{\phi(x)}{\phi(x+\theta\phi(x))} - 1].$$

By Claim 1,

$$|[\phi'(x+\theta\phi(x)) - \phi'(x)]\cdot\frac{\phi(x)}{\phi(x+\theta\phi(x))}|$$

$$\leq (|\phi'(x+\theta\phi(x)) - \phi'(x)|)\cdot\max_{y\in J_x}\frac{\phi(x)}{\phi(y)}$$

$$\leq A\cdot\omega(\theta\cdot\phi(x)) \leq A\cdot\omega(\phi(x)).$$

Then, there exists some $0 < \theta_1 < 1$, such that: $\phi(x+\theta\phi(x)) - \phi(x) = \theta\phi(x) \cdot \phi'(x + \theta \cdot \theta_1 \phi(x))$. Using it together with Claim 1, we get

$$|\frac{\phi(x)}{\phi(x+\theta\phi(x))} - 1| = |\frac{\phi(x)}{\phi(x) + \theta\phi(x)\phi'(x+\theta_1\theta\phi(x))} - 1| =$$

$$= |\frac{1}{1+\theta\phi'(x+\theta_1\theta\phi(x))} - 1| = |\frac{\theta\phi'(x+\theta_1\theta\phi(x))}{1+\theta\phi'(x+\theta_1\theta\phi(x))}|$$

$$\leq \frac{1}{a} \cdot |\phi'(x+\theta_1\theta\phi(x))| \leq \frac{3}{a} \cdot \omega(\Omega(\phi(x))).$$

Therefore

$$|\phi'(x) \cdot [\frac{\phi(x)}{\phi(x+\theta\phi(x)} - 1]| \leq \frac{3}{a} \cdot |\phi'(x)| \cdot \omega(\Omega(\phi(x)))$$

$$\leq \frac{9}{a} \cdot \omega^2(\Omega(\phi(x))).$$

Since $\Omega(x) \geq x$, it follows that $\Omega(\phi(x)) \geq \phi(x)$. Additionally, $\frac{\omega(x)}{x}$ is decreasing, thus

$$\frac{\omega(\Omega(\phi(x)))}{\Omega(\phi(x))} \leq \frac{\Omega(\phi(x))}{\phi(x)}.$$

The substitution of it yields the following:

$$\omega^2(\Omega(\phi(x))) = \phi(x) \cdot \frac{\omega(\Omega(\phi(x)))}{\Omega(\phi(x))} \leq \phi(x) \cdot \frac{\omega(\phi(x))}{\phi(x)} = \omega(\phi(x)).$$

Adding those results together, we have the following estimate:

$$|\int_x^{x+\phi(x)} \frac{\phi'(t)}{\phi(t)} dt - \phi'(x)| \leq (A + \frac{9}{a}) \cdot \omega(\phi(x)).$$

Using the previous estimate, we have also:

$$|\phi'(x)|^2 \leq 9 \cdot \omega^2(\Omega(\phi(x))) \leq 9 \cdot \omega(\phi(x)).$$

Let us apply the above estimates for bounding our initial expressions:

$$\sum_{k=1}^{n} |\int_{x_k}^{x_{k+1}} \frac{\phi'(t)}{\phi(t)} dt - \phi'(x_k))| + C \cdot \sum_{k=1}^{n} [\phi'(x_k)]^2$$

$$\leq C(f) \cdot \sum_{k=1}^{n} \omega(\phi(x_k)),$$

with $C(f) = 10 + A + \frac{9}{a}$. By Jensen's inequality

$$\sum_{k=1}^{n} \omega(\phi(x_k)) = \sum_{k=1}^{n} \omega(x_{k+1} - x_k) \leq n \cdot \omega(\frac{1}{n}),$$

completing the proof of Lemma 3. $\square$

Combining Lemmas 2 and 3, we get

Corollary 2. *Suppose that* $x_1, ..., x_{n+1} \in (0, \delta)$. *Then,*

$$(**) \quad |\log (f^n)'(x_1)| \leq \log \frac{n}{\omega^{-1}(\frac{1}{n})} + C(f) \cdot n \cdot \omega(\frac{1}{n}).$$

At last, we turn to the details of the proof of Theorem 4, we shall show that estimate $(**)$ holds for each $x \in (0,1)$. Consider the decomposition of the interval into a union of open intervals $[0,1]\backslash \mathrm{Fix}(f) = \cup_{i \in I}(a_i, b_i)$. Let $x \in (0,1)$ be an arbitrary point, then $x \in (a_i, b_i)$ for some $i \in I$. If $|b_i - a_i| \leq \delta$, then the proof is complete by Corollary 2. There are only finitely many intervals such that $|b_i - a_i| > \delta$. We take one of them and divide it into 3 subintervals:

$$[a_i, b_i] = [a_i, a_i + \delta_0] \cup [a_i + \delta_0, b_i - \delta_0] \cup [b_i - \delta_0, b_i],$$

when $\delta_0 \leq \delta$ and $\Omega(\phi(x)) \in [a_i, b_i - \delta_0]$ for all $x \in [a_i, a_i + \delta_0]$. We denote by n_1, n_2, n_3 the length of the trajectory of the sequence (x_n) in each of the 3 subintervals respectively.

It is evident that n_2 is bounded by some constant $N(f)$. If $n_3 = 0$ or $n_1 = 0$, then we are done due to Corollary 2. Otherwise, $n = n_1 + n_2 + n_3$,

$$|\log (f^n)'(x_1)| \leq |\log (f^{n_2})'(x_{n_1+1})| + |\log (f^{n_1})'(x_1) + \log (f^{n_3})'(x_{n_1+n_2+1})|,$$

we continue using Lemma 2,

$$\leq N(f) \cdot C(f) + |\log \frac{\phi(x_{n_1}) \cdot \phi(x_n)}{\phi(x_1) \cdot \phi(x_{n_1+n_2+1})}| + C(f) n_1 \omega(\frac{1}{n_1}) + C(f) n_3 \omega(\frac{1}{n_3}).$$

Note that

$$C(f) \cdot n_1 \omega(\frac{1}{n_1}) + C(f) \cdot n_3 \omega(\frac{1}{n_3}) \leq 2C(f) n \cdot \omega(\frac{1}{n}).$$

Moreover, we have the following estimate:

$$|\log \frac{\phi(x_{n_1})}{\phi(x_{n_1+n_2+1})}| \leq c_i = \max_{z \in [f^{-1}(a_i+\delta_0), a_i+\delta_0], w \in [f^{-1}(b_i-\delta_0), b_i-\delta_0]} |\log \frac{\phi(z)}{\phi(w)}|.$$

Now we are going to find an upper bound for $|\log \frac{\phi(x_n)}{\phi(x_1)}|$. As before, we split into two cases:

a. $\phi(x_n) > \phi(x_1)$. By using Claim 1 and the choice of δ_0, we have

$$n \geq a \cdot \int_{x_1}^{x_{n_1+n_2}} \frac{dt}{\phi(t)}$$

$$\geq a \cdot \int_{x_1}^{x_1 + \Omega(\phi(x_1))} \frac{dt}{\phi(t)} \geq \frac{a}{2.5} \cdot \frac{\Omega(\phi(x_1))}{\phi(x_1)},$$

the last inequality is due to Claim 2. Thus by Claim 3, we have

$$|\log \frac{\phi(x_n)}{\phi(x_1)}| \leq \log \frac{1}{\phi(x_1)} \leq C(f) \cdot \frac{n}{\omega^{-1}(\frac{1}{n})}.$$

b. $\phi(x_n) < \phi(x_1)$. Then,

$$n \geq n_2 + n_3 \geq a \cdot \int_{x_{n_1}}^{x_n} \frac{dt}{\phi(t)}$$

$$\geq a \cdot \int_{x_n - \Omega(\phi(x_n))}^{x_n} \frac{dt}{\phi(t)} \geq \frac{a}{2.5} \cdot \frac{\Omega(\phi(x_n))}{\phi(x_n)}.$$

In the same way, by Claim 3 it follows that

$$|\log \frac{\phi(x_n)}{\phi(x_1)}| \leq \log \frac{1}{\phi(x_n)} \leq C(f) \cdot \frac{n}{\omega^{-1}(\frac{1}{n})}.$$

□

Proof of Theorem 5. Without limiting the generality, we assume that $\omega(x)$ is a C^1 smooth concave function. We assume that by $f(x) = x - \phi(x) > 0$. By Lemma 2,

$$\log(f^n)'(x_1) \leq \log \frac{\phi(x_1)}{\phi(x_n)} + Cn\omega(\frac{1}{n}).$$

Therefore, it is sufficient to show that there exists a constant $C > 0$, such that for every $n \geqslant 1$ we have $\phi(x_n) \geqslant e^{-Cn\omega(\frac{1}{n})}$.
The proof is by induction. We shall determine the value of $C > 0$ during the proof. Take big enough n, and suppose that we have $\phi(x_{n-1}) \geqslant e^{-C(n-1)\omega(\frac{1}{n-1})}$. We wish to prove that $\phi(x_n) \geqslant e^{-Cn\omega(\frac{1}{n})}$. Assume in a counter that

$$\phi(x_n) < e^{-Cn\omega(\frac{1}{n})}.$$

Let us show that in this case we must have

$$\phi'(t) \leqslant 3 \cdot \omega(\frac{1}{n}),$$

for any $t \in [x_n, x_{n-1}]$.

Assume in a counter that we have

$$\phi'(t) > 3\omega(\frac{1}{n}),$$

for some $t \in [x_n, x_{n-1}]$. Note that,

$$\phi(x_{n-1}) \leqslant \phi(x_n) + (x_{n-1} - x_n) \max_{s \in [x_n, x_{n-1}]} \phi'(s)$$

$$\leqslant \phi(x_n) + (x_{n-1} - x_n)\omega(x_{n-1} - x_n) = \phi(x_n) + \phi(x_{n-1})\omega(\phi(x_{n-1})),$$

hence $\phi(x_{n-1}) \leqslant \frac{\phi(x_n)}{1-\omega(\phi(x_{n-1}))}$. Therefore, for big enough n we have,

$$\phi(x_{n-1}) \leqslant 2\phi(x_n) < 2e^{-Cn\omega(\frac{1}{n})} < \frac{1}{n},$$

the last inequality is satisfied since $\frac{\omega(x)}{x^\alpha}$ is decreasing for small x and $0 < \alpha < 1$. Thus, for big enough n, we have $\phi(x_{n-1}) = x_{n-1} - x_n < \frac{1}{n}$. We have $\phi'(t) > 3\omega(\frac{1}{n})$, hence

$$\begin{aligned}\phi'(x_n) &\geqslant \phi'(t) - \omega(t - x_n) \geqslant \phi'(t) - \omega(x_{n-1} - x_n)\\ &\geqslant \phi'(t) - \omega(\frac{1}{n}) > 2\omega(\frac{1}{n}).\end{aligned}$$

In particular, $\omega(x_n) \geqslant \phi'(x_n) > 2\omega(\frac{1}{n}) > \omega(\frac{1}{n})$, hence $x_n > \frac{1}{n}$. For any $s \in [x_n - \frac{1}{n}, x_n]$, we have

$$\phi'(s) \geqslant \phi'(x_n) - \omega(\frac{1}{n}) > \omega(\frac{1}{n}).$$

Therefore, by the mean value theorem

$$\phi(x_n) \geqslant \phi(x_n - \frac{1}{n}) + \frac{1}{n}\omega(\frac{1}{n}) \geqslant \frac{1}{n}\omega(\frac{1}{n}).$$

On the other hand, we have assumed that $\phi(x_n) < e^{-Cn\omega(\frac{1}{n})}$ and thus

$$\frac{1}{n} \cdot \omega(\frac{1}{n}) < e^{-Cn\omega(\frac{1}{n})}.$$

Now, for any $0 < \alpha < 1$ and big n, observe that

$$(C_1)^{n^\alpha} \cdot \omega(\frac{1}{n}) \leq n$$

where the constant C_1 is an increasing function of C. Now, for any $0 < \beta < 1$ and big enough n, we have

$$C_2 \cdot (C_1)^{n^\alpha} < n^{1+\beta},$$

that is a contradiction. Therefore, we have proved that $\phi'(t) \leqslant 3\omega(\frac{1}{n})$, for any $t \in [x_n, x_{n-1}]$.

Notice that, $n^\alpha\omega(\frac{1}{n}) \geqslant (n-1)^\alpha\omega(\frac{1}{n-1})$. Choose some $\alpha < \beta < 1$. For big enough n, we have

$$\begin{aligned}(1 + \frac{\beta}{n-1})\omega(\frac{1}{n}) &> (1 + \frac{1}{n-1})^\alpha\omega(\frac{1}{n})\\ &= \frac{1}{(n-1)^\alpha}n^\alpha\omega(\frac{1}{n}) \geqslant \omega(\frac{1}{n-1}),\end{aligned}$$

hence,

$$n\omega(\frac{1}{n}) - (n-1)\omega(\frac{1}{n-1}) \geqslant (1-\beta)\omega(\frac{1}{n}).$$

Therefore, we conclude that

$$e^{Cn\omega(\frac{1}{n})-C(n-1)\omega(\frac{1}{n-1})} > 1 + Cn\omega(\frac{1}{n}) - C(n-1)\omega(\frac{1}{n-1})$$
$$> 1 + (1-\beta)C\omega(\frac{1}{n}),$$

that is,

$$e^{-C(n-1)\omega(\frac{1}{n-1})} - e^{-Cn\omega(\frac{1}{n})} > (1-\beta)C\omega(\frac{1}{n})e^{-Cn\omega(\frac{1}{n})}.$$

Finally, recall that for big n, $\phi(x_n) \geq 2\phi(x_{n-1})$, and by the initial assumption that $\phi(x_n) < e^{-Cn\omega(\frac{1}{n})}$ and $\phi(x_{n-1}) \geq e^{-C(n-1)\omega(\frac{1}{n-1})}$ we obtain

$$(1-\beta)C\omega(\frac{1}{n})e^{-Cn\omega(\frac{1}{n})} < e^{-C(n-1)\omega(\frac{1}{n-1})} - e^{-Cn\omega(\frac{1}{n})} < \phi(x_{n-1}) - \phi(x_n)$$
$$\leqslant (x_{n-1}-x_n) \max_{s\in[x_n,x_{n-1}]} \phi'(s) \leqslant 3\phi(x_{n-1})\omega(\frac{1}{n}) \leqslant 6\phi(x_n)\omega(\frac{1}{n}) <$$
$$6e^{-Cn\omega(\frac{1}{n})}\omega(\frac{1}{n}).$$

This inequality is surely false for big enough C.

$\square$

3. Sharpness: proofs of theorems 6 and 7

Proof of Theorem 6. Define $\phi(x) = \int_0^x \omega(t)dt$ and $f(x) = x - \phi(x)$, in some interval $[0,\varepsilon]$. Extend $f(x)$ arbitrarily C^∞-smoothly to the whole interval $[0,1]$ in such way that $f(1) = f'(1) = 1$. We work in the interval $[0,\varepsilon]$. By Lemma 3,

$$(\triangle) \quad \log(\frac{1}{\phi(x_n)}) - C\cdot n\cdot\omega(\frac{1}{n}) \leq \log(f^n)'(x_1).$$

Now, we estimate from below the left hand side of $(\triangle)$.

Claim 4.

$$\omega(x_n) \leq \frac{C}{n}$$

Proof. We shall do it by induction. Assume that we have proved the claim for $n-1$, namely,

$$x_{n-1} \leq \omega^{-1}(\frac{C}{n-1}).$$

Since f is monotonic, it suffices to verify

$$\omega^{-1}(\frac{C}{n}) \geq f(\omega^{-1}(\frac{C}{n-1})) \geq f(x_{n-1}) = x_n.$$

The last inequality is equivalent to

$$\omega^{-1}(\frac{C}{n}) + \phi(\omega^{-1}(\frac{C}{n-1})) \geq \omega^{-1}(\frac{C}{n-1}),$$

By the Mean Value theorem and the assumptions that $\omega(x)$ is concave and monotonic and $\phi(x)$ is monotonic, it is enough to show that

$$\phi(\omega^{-1}(\frac{C}{n})) \geq \frac{C}{n^2 \cdot \omega'(\omega^{-1}(\frac{C}{n})},$$

denote $x = \omega^{-1}(\frac{C}{n})$ and observe

$$\phi(x) \geq \frac{\omega^2(x)}{C \cdot \omega'(x)}.$$

Now, recall that $\phi(x) \geq \frac{x}{4} \cdot \omega(x)$ and consider

$$\omega'(x) \geq \frac{C}{4} \cdot \frac{\omega(x)}{x},$$

this inequality is equivalent to

$$\big(\ln \frac{\omega(x)}{x^{\frac{4}{C}}}\big)' \geq 0$$

this holds since we know that $\frac{\omega(x)}{x^\alpha}$ increases for all $0 < \alpha < 1$ on the corresponding intervals $(0, a(\alpha))$.

□

Recall that $\phi(x) \leq x\omega(x)$. Due to Claim 9 and the monotonicity of ω^{-1}, we have

$$\phi(x_n) \leq x_n\omega(x_n) \leq \frac{a}{n} \cdot \omega^{-1}(\frac{a}{n}).$$

Therefore,

$$\frac{n}{a \cdot \omega^{-1}(\frac{a}{n})} \leq \frac{1}{\phi(x_n)}.$$

Substituting it into $(\triangle)$, we have

$$\log \frac{n}{a \cdot \omega^{-1}(\frac{a}{n})} - C \cdot n \cdot \omega(\frac{1}{n}) \leq \log(f^n)'(x_1).$$

Consider any $\varepsilon > 0$, let us check that

$$(1-\varepsilon) \cdot \log \frac{n}{\omega^{-1}(\frac{c}{n})} \leq \log(\frac{n}{a \cdot \omega^{-1}(\frac{a}{n})}) - C \cdot n \cdot \omega(\frac{1}{n}),$$

when $n \to \infty$. That is equivalent to

$$\frac{Cn\omega(\frac{1}{n})}{\log \frac{n}{a\omega^{-1}(\frac{a}{n})}} \leq \varepsilon,$$

as $n \to \infty$. Indeed,

$$\frac{Cn\omega(\frac{1}{n})}{\log \frac{n}{a\omega^{-1}(\frac{a}{n})}} \le \frac{Cn\omega(\frac{1}{n})}{\log \frac{n^2}{a^2}} \le \frac{C}{2} \cdot \frac{\omega(\frac{1}{n})}{\frac{\log n}{n}} \to 0,$$

here we used $\omega^{-1}(x) \le x$ and that $\lim_{x\to 0} \frac{\omega(x)}{x \log \frac{1}{x}} = 0$. It completes the proof of Theorem 6. □

Proof of Theorem 7. The proof is based on the construction presented in [B]. Let $0 < \varepsilon < 1$ be an arbitrary number, define

$$\phi_\varepsilon(x) = x - (1 + \frac{1}{x})^{-1} - x^{2+\varepsilon} \cdot \omega(x) \cdot \sin(\frac{2\pi}{x})$$

$$f_\varepsilon(x) = x - \phi_\varepsilon(x)$$

on some interval $[0, a(\varepsilon)]$. Note that $f_\varepsilon(0) = 0, f'_\varepsilon(0) = 1$ and for $0 < k^{-1} < a(\varepsilon)$, $f_\varepsilon(k^{-1}) = (k+1)^{-1}$. It is possible to choose $a(\varepsilon)$ in a way that

1. $f'_\varepsilon(x) > 0$ for all $x \in [0, a(\varepsilon)]$.
2. $f_\varepsilon(x)$ does not admit any fixed points in $(0, a(\varepsilon)]$.
3. The following inequality is satisfied

$$|f'_\varepsilon(x) - f'_\varepsilon(y)| \le C \cdot \omega(|x - y|),$$

for all $x, y \in [0, a(\varepsilon)]$, where C is an absolute constant, which does not depend on ε.

Then, for $0 < k^{-1} < a(\varepsilon)$,

$$\log(f_\varepsilon^N)'(k^{-1}) = \sum_{j=0}^{N-1} \log f'_\varepsilon(\frac{1}{k+j})$$

$$\ge \sum_{j=0}^{N-1} \log(((k+j)^{-2} + 1)^{-2} + (k+j)^{-\varepsilon} \cdot \omega((k+j)^{-1}))$$

$$\ge c'(\varepsilon) \cdot N \cdot (k + N - 1)^{-\varepsilon} \cdot \omega((k + N - 1)^{-1}))$$

$$\ge c(\varepsilon) \cdot N^{1-\varepsilon} \cdot \omega(N^{-1}),$$

as $N \to \infty$.

We are going to construct a diffeomorphism $f \in \mathrm{Diff}_0^\omega[0,1]$, which will be composed of a suitable pasting of the frame functions f_ε.

Let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ be an arbitrary monotonically decreasing sequence of real numbers which tends to 0. Pick two sequences $\{a_k\}_{k\in\mathbb{N}}$, $\{b_k\}_{k\in\mathbb{N}}$ monotonically decreasing sequences of real numbers which tend to 0, such that $a_k > b_{k+1}$, for all $k \in \mathbb{N}$. Define now

$$\widetilde{f}_{\varepsilon_k}(x) = \begin{cases} f_{\varepsilon_k}(x - a_k), & x \in [a_k, a_k + a(\varepsilon_k)] \\ \Psi_k(x), & x \in [a_k + a(\varepsilon_k), b_k] \end{cases}$$

where $\Psi_k(x)$ is a monotonic C^∞-continuation of $f_{\varepsilon_k}(x-a_k)$ to the whole interval $[a_k, b_k]$, without fixed points on the interval $[a_k+a(\varepsilon_k), b_k]$ with the property $\Psi_k(b_k) = b_k$, $\Psi_k'(b_k) = 1$, and with bounded second derivative $|\Psi_k''(x)| < 1$. Define

$$f(x) = \begin{cases} \widetilde{f}_{\varepsilon_k}(x), & x \in [a_k, b_k] \\ x, & x \in [0,1] \setminus \cup_{k\in\mathbb{N}}[a_k, b_k] \end{cases}$$

Since $\Psi_k(x)$ is C^∞ with second bounded derivative, it is not hard see that $|f'(x) - f'(y)| \le C(f)\omega(|x-y|)$, for all $x, y \in [0,1]$.
Now, choose an arbitrary $\varepsilon > 0$, there exists $\varepsilon_k < \varepsilon$. Pick any $m^{-1} < a(\varepsilon_k)$. Thus, we have

$$\begin{aligned} \log \Gamma_N(f) &\ge \log{(f^N)'}(a_k + m^{-1}) = \log{(f^N_{\varepsilon_k})'}(m^{-1}) \\ &\ge c(\varepsilon_k) \cdot N^{1-\varepsilon_k}\omega(N^{-1}) \ge c(\varepsilon) \cdot N^{1-\varepsilon}\omega(N^{-1}), \end{aligned}$$

for large N. Theorem 7 is proved. □

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School of Mathematical Sciences, Tel-Aviv University, Ramat-Aviv, Tel-Aviv 69978, Israel.

E-mail address: `levbuh@post.tau.ac.il, moraviev@post.tau.ac.il.`