

# HOMOLOGY OF LAGRANGIAN SUBMANIFOLDS IN COTANGENT BUNDLES

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## 1. INTRODUCTION AND MAIN RESULTS

The present note deals with topological restrictions on Lagrangian submanifolds in cotangent bundles. This subject has attracted a great deal of attention since the beginning of symplectic topology (see e.g. [Gr, LS, P, V]). In this note we concentrate on cotangent bundles of spheres and of Lens spaces. Denote by  $T^*(S^{2k+1})$  the cotangent bundle to the  $(2k + 1)$  - dimensional sphere, endowed with its canonical symplectic structure. Henceforth all Lagrangian submanifolds are assumed to be *compact, connected and with no boundary*. Our first result is:

**Theorem 1.** *Let  $L \subset T^*(S^{2k+1})$  be a Lagrangian submanifold with  $H_1(L, \mathbb{Z}) = 0$ . Then the cohomology of  $L$ , with  $\mathbb{Z}_2$  - coefficients, equals the cohomology of the zero-section, namely  $H^*(L, \mathbb{Z}_2) \cong H^*(S^{2k+1}, \mathbb{Z}_2)$ .*

A few remarks are in order before we continue. Clearly, Theorem 1 is not empty, since we have the zero-section (and its images under symplectic diffeomorphisms) as Lagrangian submanifolds of  $T^*(S^{2k+1})$ . However, beyond these we do not know any other examples of Lagrangians  $L \subset T^*(S^{2k+1})$  with  $H_1(L, \mathbb{Z}) = 0$ . In fact, a long standing (folkloric) conjecture asserts that the only exact Lagrangian submanifolds in cotangent bundles are isotopic to the zero-section. Theorem 1 can be viewed as some supporting evidence to this conjecture.

Theorem 1 admits the following generalization to Lens spaces. Denote by  $Lens_m^{2k+1} = S^{2k+1}/\mathbb{Z}_m$  the  $(2k + 1)$ -dimensional Lens space.

**Theorem 2.** *Let  $L \subset T^*(Lens_m^{2k+1})$  be a Lagrangian submanifold with  $\pi_1(L) = \mathbb{Z}_m$ . Then  $H^*(L, \mathbb{Z}_2) = H^*(Lens_m^{2k+1}, \mathbb{Z}_2)$ .*

The phenomenon arising from Theorems 1 and 2 can be described as some kind of "homological rigidity" (or uniqueness). Namely, low-dimensional topological invariants of Lagrangian submanifolds determine their *entire homology*. First examples of this phenomenon were discovered by Seidel [Se-1] and then by Biran [B-1, B-2, B-3].

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Very recently<sup>1</sup> Siedel [Se-2] has generalized, using other methods, Theorem 1. His result deals with compact connected Lagrangian submanifold of  $L \subset M = T^*(S^n)$  (where  $n$  can be also even) with  $H^1(L, \mathbb{Z}) = 0$  and vanishing second Whitney class  $w_2(L)$ . Under these assumptions, he proves (among other results) uniqueness of cohomology with coefficients in the field  $\mathbb{C}$ , that is,  $H^*(L, \mathbb{C}) \cong H^*(S^n, \mathbb{C})$ . The proof of this results uses different tools than ours.

In comparison to Siedel's approach, our methods apply only to odd-dimensional spheres, since we use in a crucial way existence of a fixed point free circle action on the sphere. Also note that our results are concerned with homology with coefficients in  $\mathbb{Z}_2$ . However, we believe that essentially the same proof should imply, with some corrections, homological uniqueness with coefficients in  $\mathbb{Z}$ .

The rest of the paper is devoted to proving Theorems 1 and 2. The proofs are based on techniques of symplectic topology, the main ingredient being computations in Floer homology. In Section 2 below we outline these computations and in Section 3 we apply them in order to prove our theorems.

## 2. COMPUTATION IN FLOER HOMOLOGY

Before we prove our main theorems in Section 3 we need to recall some important facts from Floer theory that will be used in our proof. Most of the theory in this section is due to Oh. We refer the reader to [Oh-2] for more details (see also [Se-1], and [B-2, B-3]).

Let  $(M, \omega)$  be a tame symplectic manifold (see [A-L-P] for the definition), and let  $L \subset (M, \omega)$  be a monotone Lagrangian submanifold with minimal Maslov number  $N_L \geq 2$  (see [Oh-1, Oh-2]). For any Hamiltonially isotopic copy of  $L$ , say  $L'$ , we denote by  $HF(L, L')$  the Floer homology of the pair  $(L, L')$ . Since this invariant is independent of the choice of  $L'$  we shall sometimes denote it also by  $HF(L)$ . We now describe Oh's [Oh-2] approach for computing  $HF(L)$  using a spectral sequence whose first step is the singular cohomology  $H^*(L; \mathbb{Z}_2)$  of  $L$ . Throughout this paper we work with Floer and Morse homologies with  $\mathbb{Z}_2$ -coefficients.

Let  $L_\epsilon \subset (M, \omega)$  be a small Hamiltonian perturbation of  $L$  defined using a Morse function  $f : L \rightarrow \mathbb{R}$  and a Weinstein tubular neighborhood  $\mathcal{U}$  of  $L$ . Then the total Floer complex  $CF(L, L_\epsilon)$  can be graded by the Morse indices of  $f$ . We shall denote this grading by  $CF^*(L, L_\epsilon)$ . Then, the Floer differential  $d_F : CF(L, L_\epsilon) \rightarrow CF(L, L_\epsilon)$  can be written as  $d_F = \partial_0 + \partial'$  where  $\partial_0$  comes from counting Floer

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<sup>1</sup>After the first draft of this paper has been written.

trajectories lying entirely inside  $\mathcal{U}$  while  $\partial'$  comes from counting those trajectories going out of  $\mathcal{U}$ . By Floer's work [F-1, F-2],  $\partial_0$  can be identified with the differential of Morse homology of the function  $f$ , hence  $\partial_0 : CF^*(L, L_\epsilon) \rightarrow CF^{*+1}(L, L_\epsilon)$  (i.e.  $\partial_0$  raises grading by 1) and  $H^*(\partial_0) \cong H^*(L; \mathbb{Z}_2)$ . (Note however that the operator  $\partial'$  is not a differential.) It is shown in [Oh-2] that the operator  $\partial'$  splits as  $\partial' = \partial_1 + \cdots + \partial_\nu$ , where  $\nu = \lceil \frac{\dim L + 1}{N_L} \rceil$ , and the behavior of each  $\partial_i$  in terms of grading is  $\partial_i : CF^*(L, L_\epsilon) \rightarrow CF^{*+1-iN_L}(L, L_\epsilon)$ . By a straightforward algebraic computation (see [Oh-2]) it follows that  $\partial_1$  descends to a homomorphism  $[\partial_1] : H^*(\partial_0) \rightarrow H^{*+1-N_L}(\partial_0)$  with  $[\partial_1]^2 = 0$ . Thus  $[\partial_1]$  is a differential on the total homology  $\bigoplus_{k=0}^{\dim L} H^k(\partial_0)$ . (Note that  $[\partial_1]$  is not compatible with the grading of  $H^*(\partial_0)$  in the sense that it does not raise grading by 1.) Denote by  $H^*([\partial_1])$  the homology of  $[\partial_1]$ , namely

$$H^k([\partial_1]) = \frac{\ker([\partial_1] : H^i(\partial_0) \rightarrow H^{k+1-N_L}(\partial_0))}{\text{im}([\partial_1] : H^{k-1+N_L}(\partial_0) \rightarrow H^k(\partial_0))} \quad \text{for every } k.$$

Here for the grading of  $H^*([\partial_1])$  we use the one induced from that of  $H^*(\partial_0)$ .

Continuing by induction we get that  $\partial_i$  descends to a homomorphism  $[\partial_i] : H^*([\partial_{i-1}]) \rightarrow H^{*+1-iN_L}([\partial_{i-1}])$  with  $[\partial_i]^2 = 0$ . We denote by  $H^*([\partial_i])$  the corresponding homology (with grading induced from  $H^*([\partial_{i-1}])$ ), namely

$$H^k([\partial_i]) = \frac{\ker([\partial_i] : H^k([\partial_{i-1}]) \rightarrow H^{k+1-iN_L}([\partial_{i-1}]))}{\text{im}([\partial_i] : H^{k-1+iN_L}([\partial_{i-1}]) \rightarrow H^k([\partial_{i-1}]))} \quad \text{for every } k.$$

Finally, it is shown in [Oh-2] that

$$\bigoplus_{k=0}^{\dim L} H^k([\partial_\nu]) \cong \ker d_F / \text{im } d_F = HF(L, L_\epsilon) \cong HF(L).$$

### 3. PROOFS OF THEOREMS

*Proof of Theorem 1.* Let us first describe the main ideas of the proof. The first observation is that a small neighborhood of the zero-section of  $T^*(S^{2k+1})$  can be symplectically embedded into  $\mathbb{C}P^k \times \mathbb{C}^{k+1}$  (such an embedding was first described by Audin, Lalonde and Polterovich [A-L-P]). Therefore, if  $L \subset T^*(S^{2k+1})$  is a Lagrangian submanifold we can Lagrangianly embed it also into  $\mathbb{C}P^k \times \mathbb{C}^{k+1}$ . The benefit of this embedding is that in  $\mathbb{C}P^k \times \mathbb{C}^{k+1}$ , due to the  $\mathbb{C}^{k+1}$ -factor, any compact subset can be disjoint from itself via a Hamiltonian isotopy (e.g. by a linear translation). In particular, viewing  $L$  as a Lagrangian

in  $\mathbb{C}P^k \times \mathbb{C}^{k+1}$  we have  $HF(L, L) = 0$ . Comparing this vanishing of Floer homology with the spectral sequence computation from Section 2, we shall derive our restrictions on the first step of this sequence which is  $H^*(L, \mathbb{Z}_2)$ .

Let us turn now to the details of the proof. We first use a Lagrangian embedding  $S^{2k+1} \hookrightarrow \mathbb{C}P^k \times \mathbb{C}^{k+1}$  due to Audin, Lalonde and Polterovich [A-L-P]. To describe this embedding denote by  $i : S^{2k+1} \subset \mathbb{C}^{k+1}$  the standard inclusion as the unit sphere and by  $\pi : S^{2k+1} \rightarrow \mathbb{C}P^k$  the Hopf fibration. Next, we endow  $\mathbb{C}P^k$  with its standard symplectic structure, normalized so that the symplectic area of a projective line is  $\pi$ . A simple computation shows that

$$S^{2k+1} \hookrightarrow \mathbb{C}P^k \times \mathbb{C}^{k+1}, \quad z \mapsto (\pi(z), \overline{i(z)}).$$

is a Lagrangian embedding (here,  $\overline{(\cdot)}$  stands for usual complex conjugation).

By Darboux-Weinstein theorem we now have a symplectic embedding of a small tubular neighborhood of the zero-section of  $T^*(S^{2k+1})$  into  $\mathbb{C}P^k \times \mathbb{C}^{k+1}$ . Using homotheties along the cotangent fibers of  $T^*(S^{2k+1})$  we may assume that the Lagrangian  $L$  lies in the above small tubular neighborhood of the zero-section. Thus we obtain a Lagrangian embedding of  $L$  into  $\mathbb{C}P^k \times \mathbb{C}^{k+1}$ . From now on we shall view  $L$  as a Lagrangian submanifold of  $\mathbb{C}P^k \times \mathbb{C}^{k+1}$ .

Note that  $L \subset \mathbb{C}P^k \times \mathbb{C}^{k+1}$  is monotone since  $\mathbb{C}P^k$  is a monotone symplectic manifold and  $H_1(L, \mathbb{Z}) = 0$ . A simple computation shows that the minimal Maslov number of  $L$  is  $N_L = 2k + 2$ . Thus  $N_L \geq 2$  and the Floer homology of  $L$  is well defined.

We now claim that  $HF(L, L) = 0$ . Indeed,  $L$  can be disjoint from itself by a large enough linear translation along the  $\mathbb{C}^{k+1}$ -factor, and linear translations are Hamiltonian.

We now compare this to a computation using Oh's spectral sequence. We use here the notations from Section 2. Since  $N_L = 2k + 2$  and  $\dim L = 2k + 1$  we must have  $\partial_j = 0 \quad \forall j \geq 2$ , hence  $d_F = \partial_0 + \partial_1$ . Therefore  $H^*([\partial_1]) = HF(L, L)$ . Since  $HF(L, L) = 0$  this implies that  $H^{i+2k+1}(L, \mathbb{Z}_2) \xrightarrow{[\partial_1]} H^i(L; \mathbb{Z}_2) \xrightarrow{[\partial_1]} H^{i-2k-1}(L; \mathbb{Z}_2)$  is an exact sequence  $\forall 0 \leq i \leq 2k + 1$ . As  $\dim L = 2k + 1$  we immediately get that  $H^i(L, \mathbb{Z}_2) = 0 \quad \forall 1 \leq i \leq 2k$  and  $H^0(L, \mathbb{Z}_2) = H^{2k+1}(L, \mathbb{Z}_2) = \mathbb{Z}_2$ , or in other words that  $H^*(L, \mathbb{Z}_2) \cong H^*(S^{2k+1}, \mathbb{Z}_2)$ .  $\square$

*Proof of Theorem 2.* Denote by  $i : L \hookrightarrow T^*(Lens_m^{2k+1})$  the inclusion. We claim that  $i_* : \pi_1(L) \rightarrow \pi_1(T^*(Lens_m^{2k+1})) \cong \mathbb{Z}_m$  is an isomorphism. Indeed, suppose on the contrary that  $i_* : \pi_1(L) \rightarrow \pi_1(T^*(Lens_m^{2k+1}))$  is not surjective. Let  $Y \rightarrow Lens_m^{2k+1}$  be the covering associated to the

subgroup  $i_*(\pi_1(L)) \subset \pi_1(T^*(Lens_m^{2k+1})) \cong \pi_1(Lens_m^{2k+1})$ . Consider the corresponding covering  $T^*(Y) \rightarrow T^*(Lens_m^{2k+1})$ , and let  $j : L \hookrightarrow T^*(Y)$  be a lifting of  $i : L \hookrightarrow T^*(Lens_m^{2k+1})$  (it exists because  $Y$  is chosen to be a covering associated to the subgroup  $i_*(\pi_1(L)) \subset \pi_1(T^*(Lens_m^{2k+1})) \cong \pi_1(Lens_m^{2k+1})$ , and it is obviously a Lagrangian embedding). Since  $i_*$  is not surjective there exists a nontrivial deck transformation  $\sigma : Y \rightarrow Y$ . We claim that  $\sigma$  must be isotopic to the identity. To see this, note that all connected coverings of  $Lens_m^{2k+1}$  are determined, up to an isomorphism, by a subgroup  $\mathbb{Z}_d \hookrightarrow \pi_1(Lens_m^{2k+1}) \cong \mathbb{Z}_m$ . Hence every such covering is isomorphic to  $Lens_m^{2k+1} \rightarrow Lens_m^{2k+1}$ , where  $d|m$ . But for such coverings it is easy to check that all deck transformations are isotopic to the identity. Thus  $\sigma$  is isotopic to the identity, and therefore the canonical lift of  $\sigma$ ,  $\Phi_\sigma : T^*(Y) \rightarrow T^*(Y)$  is Hamiltonian. Now  $\Phi_\sigma$  itself is a nontrivial deck transformation for the covering  $T^*(Y) \rightarrow T^*(Lens_m^{2k+1})$ . Therefore since  $j : L \hookrightarrow T^*(Y)$  is a lifting of  $i : L \hookrightarrow T^*(Lens_m^{2k+1})$  we must have  $\Phi_\sigma(j(L)) \cap j(L) = \emptyset$ . But this is impossible since  $j(L) \subset T^*(Y)$  is an exact Lagrangian. We therefore conclude that  $i_* : \pi_1(L) \rightarrow \pi_1(T^*(Lens_m^{2k+1}))$  is surjective and therefore an isomorphism.

Consider now the universal coverings  $\tilde{L} \rightarrow L$  and  $T^*(S^{2k+1}) \rightarrow T^*(Lens_m^{2k+1})$ . As  $i_*$  is an isomorphism we can lift  $\tilde{L}$  into  $T^*(S^{2k+1})$  namely we have a commutative diagram

$$\begin{array}{ccc}
 \tilde{L} & \hookrightarrow & T^*(S^{2k+1}) \\
 \downarrow & & \downarrow \\
 L & \hookrightarrow & T^*(Lens_m^{2k+1})
 \end{array}$$

where the upper horizontal map is a Lagrangian embedding. By Theorem 1 we have  $H^*(\tilde{L}; \mathbb{Z}_2) \cong H^*(S^{2k+1}; \mathbb{Z}_2)$ .

Now we use the Cartan-Leray spectral sequence for coverings (see e.g. [M]), in order to derive  $H^*(L, \mathbb{Z}_2)$  from  $H^*(\tilde{L}, \mathbb{Z}_2)$ . Let us outline the main steps in this computation.

Recall that given a covering  $X \rightarrow Y$  with group  $\pi$  and a commutative group  $A$ , there exists a spectral sequence with  $E_{p,q}^2 = H_p(\pi, H_q(X, A))$  that converges to  $H_*(Y, A)$ . Here  $H_p(\pi, H_q(X, A))$  stands for the homology of the group  $\pi$  with coefficients in the  $\pi$ -module  $H_q(X, A)$ . (Namely,  $H_p(\pi, H_q(X, A)) = Tor_p^{\mathbb{Z}[\pi]}(\mathbb{Z}, H_q(X, A))$ .)

In our situation we have  $A = \mathbb{Z}_2$ , and the covering  $\tilde{L} \rightarrow L$  with group  $\pi = \mathbb{Z}_m$ . Since  $H_*(\tilde{L}, \mathbb{Z}_2) \cong H_*(S^{2k+1}, \mathbb{Z}_2)$  and since  $\pi$  acts trivially on  $H_0(\tilde{L}, \mathbb{Z}_2)$  and on  $H_{2k+1}(\tilde{L}, \mathbb{Z}_2)$ , the beginning of the spectral sequence looks as follows:

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2k+2$	0	0	$\dots$	0	0	0	$\dots$
$2k+1$	$H_0(\mathbb{Z}_m, \mathbb{Z}_2)$	$H_1(\mathbb{Z}_m, \mathbb{Z}_2)$	$\dots$	$H_{2k}(\mathbb{Z}_m, \mathbb{Z}_2)$	$H_{2k+1}(\mathbb{Z}_m, \mathbb{Z}_2)$	0	$\dots$
$2k$	0	0	$\dots$	0	0	0	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$
1	0	0	$\dots$	0	0	0	$\dots$
0	$H_0(\mathbb{Z}_m, \mathbb{Z}_2)$	$H_1(\mathbb{Z}_m, \mathbb{Z}_2)$	$\dots$	$H_{2k}(\mathbb{Z}_m, \mathbb{Z}_2)$	$H_{2k+1}(\mathbb{Z}_m, \mathbb{Z}_2)$	0	$\dots$
	0	1	$\dots$	$2k$	$2k+1$		

Since the differential at the  $r$ 'th step is  $d_r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$  we get  $H_i(L, \mathbb{Z}_2) = \bigoplus_{p+q=i} E_{p,q}^\infty \cong H_i(\mathbb{Z}_m, \mathbb{Z}_2)$  for every  $0 \leq i \leq 2k$ , and of course  $H_{2k+1}(L, \mathbb{Z}_2) = \mathbb{Z}_2$ . A straightforward computation of  $H_i(\mathbb{Z}_m, \mathbb{Z}_2)$  now gives:

$$H_i(L, \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & m \text{ even} \\ 0 & m \text{ odd} \end{cases}$$

for every  $0 \leq i \leq 2k$ . On the other hand, this is precisely the homology  $H_i(Lens_m^{2k+1}, \mathbb{Z}_2)$ .  $\square$

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