

NON-ISOMETRIC DOMAINS WITH THE SAME MARVIZI-MELROSE INVARIANTS

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ABSTRACT. For any strictly convex planar domain $\Omega \subset \mathbb{R}^2$ with a C^∞ boundary one can associate an infinite sequence of spectral invariants introduced by Marvizi-Merlose [5]. These invariants can generically be determined using the spectrum of the Dirichlet problem of the Laplace operator. A natural question asks if this collection is sufficient to determine Ω up to isometry. In this paper we give a counterexample, namely, we present two non-isometric domains Ω and $\bar{\Omega}$ with the same collection of Marvizi-Melrose invariants. Moreover, each domain has countably many periodic orbits $\{S^n\}_{n \geq 1}$ (resp. $\{\bar{S}^n\}_{n \geq 1}$) of period going to infinity such that S^n and \bar{S}^n have the same period and perimeter for each n .

Consider a C^∞ smooth strictly convex planar domain $\Omega \subset \mathbb{R}^2$. Let us start by introducing the *Length Spectrum* of a domain Ω . The length spectrum of Ω is given by the set of lengths of its periodic orbits, counted with multiplicity:

$$\mathcal{L}(\Omega) := \mathbb{N}\{\text{lengths of closed geodesics in } \Omega\} \cup \mathbb{N}|\partial\Omega|,$$

where $|\partial\Omega|$ denotes the length of the boundary of Ω . Generically this collection can be determined from the spectrum of the Laplace operator in Ω with Dirichlet boundary condition (similarly for Neumann boundary one):

$$(1) \quad \begin{cases} \Delta f = \lambda f & \text{in } \Omega \\ f|_{\partial\Omega} = 0. \end{cases}$$

From the physical point of view, the eigenvalues λ 's are the eigenfrequencies of the membrane Ω with a fixed boundary. There is the following relation between the Laplace spectrum and the length spectrum (see e.g. [1, 6]). Call the function

$$w(t) := \sum_{\lambda_i \in \text{spec}\Delta} \cos(t\sqrt{-\lambda_i}),$$

the wave trace. Then, the wave trace $w(t)$ is a well-defined generalized function (distribution) of t , smooth away from the length spectrum, namely,

$$(2) \quad \text{sing. supp.}(w(t)) \subseteq \pm\mathcal{L}(\Omega) \cup \{0\}.$$

So if $l > 0$ belongs to the singular support of this distribution, then there exists either a closed billiard trajectory of length l , or a closed geodesic of length l in the boundary of the billiard table.

Generically, equality holds in (2). More precisely, if no two distinct orbits have the same length and the Poincaré map of any periodic orbit is non-degenerate, then the singular support of the wave trace coincides with $\pm\mathcal{L}(\Omega) \cup \{0\}$ (see e.g. [6]). This theorem implies that, at least for generic domains, one can recover the length spectrum from the Laplace one.

This relation between periodic orbits and spectral properties of the domain, immediately leads to a famous inverse spectral problem:

Can one hear the shape of a drum?,

as formulated in a very suggestive way by M. Kac [3] (although the problem had been already stated by H. Weyl). More precisely, does the spectrum $\text{Spec } \Delta$ determine Ω up to isometry? This question has not been completely solved yet: there are negative and positive answers (see [9, 10]).

S. Marvizi and R. Melrose [5] studied the asymptotics of the lengths of n -periodic billiard trajectories in a smooth strictly convex plane domain as $n \rightarrow \infty$. Let L_n be the supremum and l_n the infimum of the perimeters of simple billiard n -gons. The following theorem was proved in [5]:

Theorem. *For any positive integer k we have*

$$\lim n^k(L_n - l_n) = 0 \quad n \rightarrow \infty$$

for any positive k . Moreover, L_n has an asymptotic expansion as $n \rightarrow \infty$:

$$L_n \sim l_0 + \sum_{k=1}^{\infty} \frac{\ell_k}{n^{2k}},$$

where l_0 is the length of the billiard table and ℓ_k 's are constants, depending on the curvature of the table.

This collection $\{\ell_k\}_{k \geq 0}$ is sometimes called Marvizi-Melrose spectral invariants. These coefficients are closely related to expansion at the origin of so-called Mather's β -function (see [7, 8]). A natural question is

Do Marvizi-Melrose spectral invariants $\{\ell_k\}_{k \geq 0}$ determine a strictly convex domain (up to isometry)?

In this paper we provide a negative answer, namely,

Theorem 1. *There exist two C^∞ strictly convex planar domain Ω , $\Omega' \subset \mathbb{R}^2$ which are non-isometric, but have the same Marvizi-Melrose invariants. Moreover, there is a sequence $q_n \rightarrow \infty$ such that for each $n \geq 1$ there are periodic orbits of period q_n for both domains of the same perimeter.*

We note that originally Marvizi-Melrose [5] derived their invariants $\{\ell_k\}_{k \geq 0}$ as integrated quantities. If s is the length parametrization of the boundary and $\rho(s)$ is its radius of curvature, then

$$\ell_1 = -2 \int_0^{\ell(\partial\Omega)} \rho^{2/3}(s) ds$$

$$\ell_2 = \frac{1}{1080} \int_0^{\ell(\partial\Omega)} (9\rho^{4/3}(s) + 8\rho^{-8/3}\dot{\rho}^2(s)) ds$$

and so on.

We construct domains Ω and Ω' using the same “building blocks”, namely, there is a partition of the boundary of both domains such that parts are isometric (see Figure 1 below). Since these invariants are integrated quantities of products of fractional powers of ρ and its derivatives, the first part of our result can be derived from the derivations of Marvizi-Melrose [5]. We, however, feel that our “moreover” construction in Theorem 1 is interesting by itself.

1. CONSTRUCTION OF NON-ISOMETRIC DOMAINS WITH THE SAME MARVIZI-MELROSE SPECTRAL INVARIANTS

Our convex billiard tables Ω and Ω' will consist of the same “building blocks”, which are “glued” together in different order. To be more precise, these building blocks will be smooth curves $\gamma : [0, a] \rightarrow \mathbb{R}^2$, and we will require that such γ satisfies the following:

- (1) γ is regular, simple and non-closed.
- (2) γ is symmetric with respect to the line passing through $\gamma(a/2)$ and normal to γ , that is, denoting by $R_\gamma : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ the reflection of \mathbb{R}^2 with respect to the line passing through $\gamma(a/2)$ and orthogonal to γ at $\gamma(a/2)$, we have $R_\gamma(\gamma(t)) = \gamma(a - t)$ for all $t \in [0, a]$.
- (3) $\arg \gamma'(t)$ is monotone increasing (mod 2π) with a positive speed, when $t \in [0, a]$ (for given $z \in \mathbb{R}^2 \setminus \{(0, 0)\}$, the notation $\theta = \arg z$ means that $z = (r \cos \theta, r \sin \theta)$ for some $r > 0$).

Given building blocks $\gamma_1 : [0, a_1] \rightarrow \mathbb{R}^2$ and $\gamma_2 : [0, a_2] \rightarrow \mathbb{R}^2$, we define their gluing to be a curve $\gamma : [0, a_1 + a_2] \rightarrow \mathbb{R}^2$ defined as follows. First, let T be the orientation preserving isometry of \mathbb{R}^2 such that for $\tilde{\gamma}_2 := T \circ \gamma_2$ we have $\tilde{\gamma}_2(0) = \gamma_1(a_1)$ and $\tilde{\gamma}_2'(0) = \gamma_1'(a_1)$. Then we define γ by $\gamma(t) = \gamma_1(t)$ for $t \in [0, a_1]$, and $\gamma(t) = \tilde{\gamma}_2(t - a_1)$ for $t \in [a_1, a_1 + a_2]$. We will use the notation $\gamma = \gamma_1 \sharp \gamma_2$. Notice that γ does not have to be C^∞ -smooth in general. However, in all examples that we will be considering below, this will always be the case.

We will call a smooth simple closed curve bounding a smooth strictly convex domain a *billiard table boundary*. Given a building block γ we can think of it as a “wall”, with respect to which we can play billiard. We say that an angle $\theta \in (0, \frac{\pi}{2})$ *matches* γ if the billiard trajectory starting at $\gamma(0)$ and making the angle θ with γ at $\gamma(0)$, passes through $\gamma(a/2)$ (and in particular, arrives to $\gamma(a)$). More precisely, there exist

$$t_0 = 0 < t_1 < \dots < t_{2p} = a,$$

such that

$$\angle(\gamma(t_i) - \gamma(t_{i-1}), \gamma'(t_i)) = \angle(\gamma'(t_i), \gamma(t_{i+1}) - \gamma(t_i))$$

for $i = 1, 2, \dots, 2p - 1$, we have $t_p = a/2$ and $\angle(\gamma'(t_0), \gamma(t_1) - \gamma(t_0)) = \theta$. Clearly, in this case we also have $\angle(\gamma(t_{2p}) - \gamma(t_{2p-1}), \gamma'(t_{2p})) = \theta$.

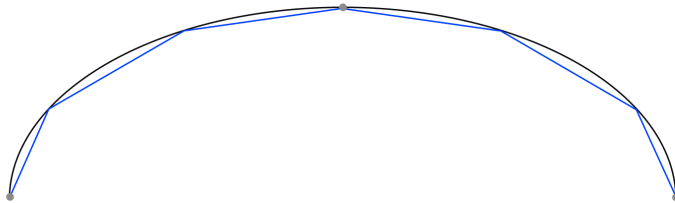


FIGURE 1.

A billiard ball trajectory matching a building block.

Assume that we have building blocks $\gamma_1, \dots, \gamma_n$, such that $\gamma = \gamma_1 \# \dots \# \gamma_n$ is a billiard table boundary. Then, for any angle θ which matches each γ_k , we obtain a closed billiard trajectory for γ , which starts at $\gamma(0)$ with angle θ . Clearly, if we have a permutation $\gamma_{k_1}, \gamma_{k_2}, \dots, \gamma_{k_n}$ of our building blocks, such that $\tilde{\gamma} := \gamma_{k_1} \# \dots \# \gamma_{k_n}$ is a billiard table boundary, then of course, we also obtain a closed billiard trajectory for γ , which starts at $\gamma(0)$ with angle θ .

The idea of our example is to find building blocks $\gamma_1, \dots, \gamma_n$ and a permutation k_1, \dots, k_n of the indices $1, \dots, n$, such that $\gamma = \gamma_1 \# \dots \# \gamma_n$ and $\tilde{\gamma} = \gamma_{k_1} \# \dots \# \gamma_{k_n}$ are *different* (i.e. non-congruent) billiard table boundaries, and such that for an infinite decreasing sequence $\theta_1, \theta_2, \dots \in (0, \frac{\pi}{2})$ of angles, converging to 0, each θ_k matches each γ_j . Then from the mentioned above, it follows that γ and $\tilde{\gamma}$ admit an infinite sequence of pairs of simple closed billiard trajectories τ_1, τ_2, \dots and $\tilde{\tau}_1, \tilde{\tau}_2, \dots$, such that τ_k has the same length and the same number of bouncing points as $\tilde{\tau}_k$, for each k . In that case, the Marvizi-Melrose invariants of γ and $\tilde{\gamma}$ coincide, since they are defined by the asymptotic behaviour of lengths of simple billiard n -gons.

To construct such building blocks, we start with some $n \geq 4$ and initial collection $\gamma_1, \dots, \gamma_n$ of building blocks and a permutation k_1, \dots, k_n such that $\gamma_1 \# \dots \# \gamma_n$ and $\gamma_{k_1} \# \dots \# \gamma_{k_n}$ are *non-congruent* billiard table boundaries. To obtain an example of such collection of building blocks and a permutation, one can simply look first at $\gamma_k : [0, \frac{2\pi}{n}] \rightarrow \mathbb{R}^2 \cong \mathbb{C}$, $\gamma_k(t) = e^{it+2\pi ki/n}$, and take any nontrivial permutation k_1, \dots, k_n which is not of the form $k_\ell = \ell + a \pmod{n}$ or $k_\ell = a - \ell \pmod{n}$. Of course, in this case $\gamma_1 \# \dots \# \gamma_n$ and $\gamma_{k_1} \# \dots \# \gamma_{k_n}$ are both congruent to the unit circle, but if we slightly perturb each γ_k on a compact subset of $(0, \frac{2\pi}{n})$ (keeping it to be a building block), then one can achieve the non-congruence of $\gamma_1 \# \dots \# \gamma_n$ and $\gamma_{k_1} \# \dots \# \gamma_{k_n}$.

Remark 2. If in general, $\gamma_1, \dots, \gamma_n$ are building blocks, $\gamma_k : [0, a_k] \rightarrow \mathbb{R}^2$, such that $\gamma_1 \# \dots \# \gamma_n$ is a billiard table boundary, and if we perturb each γ_k on a compact subset of $(0, a_k)$ while keeping it being a building block, then $\gamma_1 \# \dots \# \gamma_n$ remains to be a billiard table boundary.

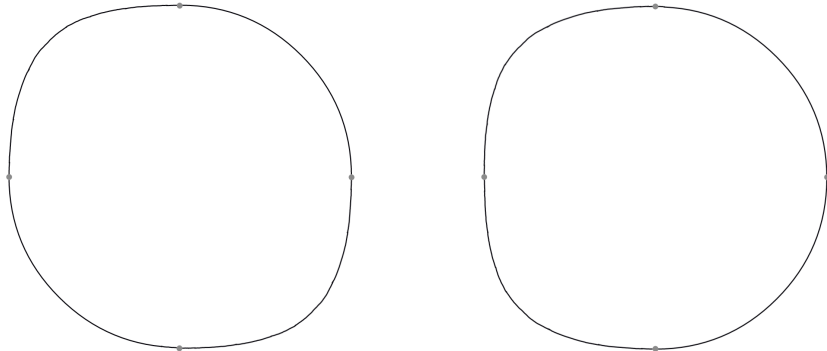


FIGURE 2.

On the left: $\gamma_1 \# \gamma_2 \# \gamma_3 \# \gamma_4$. On the right: $\gamma_1 \# \gamma_3 \# \gamma_2 \# \gamma_4$.

On the next step we make infinitely many small steps on each of which we further perturb each γ_k , such that the sizes of the perturbations decay very fast, so that in the limit we obtain C^∞ -smooth building blocks as well. We will use the following lemma:

Lemma 3. *Let $\gamma : [0, a] \rightarrow \mathbb{R}^2$ be a building block, and let $[b, c] \subset (0, a/2)$ be a closed interval. Then for any small enough angle $\theta \in (0, \pi/2)$ one can find an arbitrarily C^∞ -small perturbation (meaning that the size of the perturbation converges to 0 as $\theta \rightarrow 0$) $\tilde{\gamma}$ of γ on $[b, c] \cup [a - c, a - b]$, such that $\tilde{\gamma}$ is a building block, and such that θ matches $\tilde{\gamma}$.*

Proof. We defer a proof to Section 1.1. □

Let us describe more precisely this perturbation scheme. Fix some small enough $\epsilon > 0$. At the first step, by the lemma, one can find a small $\theta_1 \in (0, \pi/2)$, such that for a small perturbation $\tilde{\gamma}_k$ of each γ_k on a compact subset of $(0, \frac{2\pi}{n})$, θ_1 matches each $\tilde{\gamma}_k$. We can assume that $\|\gamma_k - \tilde{\gamma}_k\|_{C^0} < \epsilon$. Now re-define each γ_k to be $\tilde{\gamma}_k$, and pass to the second step.

Now assume that we have made m steps, and have already obtained some angles $\theta_1, \theta_2, \dots, \theta_m \in (0, \pi/2)$ such that each θ_j matches each γ_k . Let us describe the perturbation that we make on the step $(m + 1)$. For each $1 \leq k \leq n$, and for each $1 \leq j \leq m$, let τ_{jk} be the billiard trajectory with respect to the “wall” γ_k , which starts at $\gamma_k(0)$ at the angle θ_j with $\gamma'_k(0)$. Now for every k , look at all the bouncing points of all τ_{jk} , $1 \leq j \leq m$, and choose a closed interval $[b_k, c_k] \subset (0, \pi/n)$, such that $\gamma_k([b_k, c_k] \cup [2\pi/n - c_k, 2\pi/n - b_k])$ does not contain any of these bouncing points. Then, by the lemma, for each k there exists a small perturbation $\tilde{\gamma}_k$ of γ_k on $[b_k, c_k] \cup [2\pi/n - c_k, 2\pi/n - b_k]$, with $\|\gamma_k - \tilde{\gamma}_k\|_{C^m} < \epsilon/2^m$, such that for a small angle $\theta_{m+1} \in (0, \theta_m)$, θ_{m+1} matches $\tilde{\gamma}_k$ for every k . Now re-define each γ_k to be $\tilde{\gamma}_k$, and pass to the next step.

Note that the sizes of the perturbations decay very fast, so that our changing collection $\gamma_1, \dots, \gamma_n$ of building blocks, converges to some limiting collection of building blocks, which we again denote by $\gamma_1, \dots, \gamma_n$. Of course, by our construction procedure, now each θ_j matches each γ_k . Moreover, since on each step m , the C^0 distance between the initial γ_k and the perturbed γ_k is smaller than $\epsilon/2^m$, we conclude that the C^0 distance between each starting γ_k (which we had before performing the perturbation scheme) and the limiting γ_k is less than 2ϵ . Therefore, if ϵ is small enough, then for the limiting building blocks $\gamma_1, \dots, \gamma_n$, we still get that $\gamma_1 \sharp \dots \sharp \gamma_n$ and $\gamma_{k_1} \sharp \dots \sharp \gamma_{k_n}$ are not congruent.

1.1. A proof of Lemma 3. Let Ω be a strictly convex domain; recall that s denotes the arc-length parametrization of $\partial\Omega$ and denote with $\rho(s)$ its radius of curvature at s . Observe that if Ω is C^r , then ρ is C^{r-2} . Define the *Lazutkin parametrization* of the boundary:

$$(3) \quad x(s) = \int_0^s \rho(\sigma)^{-2/3} d\sigma.$$

We call *the Lazutkin map* the following change of variables:

$$(4) \quad \Psi_L : (s, \varphi) \mapsto (x = x(s), y(s, \varphi) = 4\rho(s)^{1/3} \sin(\varphi/2)).$$

Consider now the billiard map in Lazutkin coordinates $f_L = \Psi_L \circ f \circ \Psi_L^{-1}$; then f_L has the following form (see e.g. [4, (1.4)]):

$$(5) \quad f_L : (x, y) \rightarrow (x + y + y^3 g(x, y), y + y^4 h(x, y)),$$

where g and h can be expressed analytically in terms of derivatives of the curvature radius ρ up to order 3: hence, if Ω is C^r , g, h are C^{r-5} . In the case of C^∞ domains all functions stay C^∞ . We need the following

Lemma 4. *Let Ω be a C^5 strictly convex domain; $\gamma \subset \partial\Omega$ be a connected closed segment of the boundary. For $k, N \in \mathbb{Z}$, $N > 2$, let $(x_k, y_k) = f_L^k(x_0, y_0)$, and $\{x_k\}_{k=0}^N \subset \gamma$. Then there exists $C > 0$ depending on $\|\rho\|_{C^3}$ and independent of N , such that for small enough y_0 we have*

$$(6) \quad |y_k - y_0| < \frac{C}{N^3}, \quad |\tilde{x}_k - \tilde{x}_0 - ky_0| < \frac{C}{N^2}.$$

for any $0 < k \leq N$.

This lemma is proven for periodic orbits in [2], but the same proof applies to orbits glancing only at a part of the boundary of a C^5 strictly convex domain.

Since a building block is symmetric, it is sufficient to construct a symmetric perturbation such that an orbit emanating from $x_0 = \gamma(0)$ with small angle $\theta > 0$ will hit the symmetry point $\gamma(a/2)$.

Consider a smooth variation $\gamma_\delta : [0, a] \rightarrow \mathbb{R}^2$ of γ on $[b, c] \cup [a - c, a - b]$, when $\delta \in (-\delta_0, \delta_0)$, such that each γ_δ is a building block, $\gamma_0 = \gamma$, and moreover the Lazutkin perimeter of γ_δ is non-constant as a function of δ on any neighbourhood of $\delta = 0$ (in a sense, the family γ_δ varies the Lazutkin

perimeter). By Lemma 4 and by the intermediate value theorem, for small enough y_0 , one can always find γ_δ on which we have $x_0 = \gamma_\delta(0)$ and $x_N = \gamma_\delta(a/2)$ for some N (for convenience, one may consider a smooth family of strictly convex domains Ω_δ such that γ_δ is a part of Ω_δ). Moreover, we may choose $\delta = o(1)$ when $y_0 \rightarrow 0$.

Indeed, for any given $\lambda \in (0, \delta_0)$, choose some $-\lambda < \delta_1 < \delta_2 < \lambda$ such that the Lazutkin perimeters of γ_{δ_1} and γ_{δ_2} are different. Then, for small enough y_0 and any $\delta \in [\delta_1, \delta_2]$ consider the billiard ball trajectory $\tau(\delta)$ with bouncing points $x_0(\delta) = \gamma_\delta(0), x_1(\delta), \dots, x_k(\delta)$ on γ_δ , with angle y_0 at the first bouncing point $x_0(\delta)$. Here $k = k(\delta, y_0)$ is the last bouncing point before the trajectory escapes γ_δ . From Lemma 4 it follows that we have $\lim_{y_0 \rightarrow 0} y_0 k(\delta, y_0) = L_\delta$, where L_δ is the Lazutkin perimeter of γ_δ . Since $L_{\delta_1} \neq L_{\delta_2}$, for small enough y_0 we get $k(\delta_1, y_0) \neq k(\delta_2, y_0)$. Hence the function $\delta \mapsto k(\delta, y_0)$ has a discontinuity point $\delta_3 \in [\delta_1, \delta_2]$, at which we must have $x_k(\delta_3) = \gamma_{\delta_3}(a/2)$, where $k = k(\delta_3, y_0)$. This completes the proof of the lemma.

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