Unboundedness of the first eigenvalue of the Laplacian in the symplectic category

Lev Buhovsky[1]

August 23, 2013

Abstract

Given a closed symplectic manifold (M^{2n}, ω) of dimension $2n \geqslant 4$, we consider all Riemannian metrics on M, which are compatible with the symplectic structure ω. For each such metric g, we look at the first eigenvalue λ_1 of the Laplacian associated with it. We show that λ_1 can be made arbitrarily large, when we vary g. This generalizes previous results of Polterovich, and of Mangoubi.

1 Introduction and main results

The current paper addresses the discussion on rigidity versus flexibility of the first eigenvalue of the Laplacian. The first result in this direction was proved by Hersch [He]:

Theorem 1.1. *Let (S^2, g) be the 2-sphere equipped with a Riemannian metric g. Then,*

$$\lambda_1(S^2, g)Area(S^2, g) \leqslant 8\pi,$$

where $\lambda_1(S^2, g)$ is the first positive eigenvalue of the Laplacian on (S^2, g).

In Theorem 1.1 the equality is known to occur if and only if (S^2, g) is the standard round sphere.

Theorem 1.1 was extended to the case of a general closed surface, by Yang and Yau [Y-Y]:

[1]The author also uses the spelling "Buhovski" for his family name.

[2]Electronic version of an article published as Journal of Topology and Analysis, Vol. 5, No. 1 (2013), pp 13-56, DOI: 10.1142/S1793525313500040, © copyright World Scientific Publishing Company, [http://www.worldscientific.com/worldscinet/jta].

Theorem 1.2. *Let (Σ, g) be a closed Riemannian surface. Then*

$$\lambda_1(\Sigma, g)Area(\Sigma, g) \leqslant 8\pi(genus(\Sigma) + 1).$$

In Theorem 1.2, however, the upper bound is not optimal.

These results reflect a rigidity phenomenon in dimension 2, stating that in this case λ_1 is bounded, when we run over all Riemannian metrics that have a given volume. In contrast to the dimension 2 case, in higher dimensions, for the case of a "fixed volume form category", we have the following flexibility result of Colbois and Dodziuk [C-D]:

Theorem 1.3. *Let M be a closed manifold of dimension > 2, equipped with a volume form Ω. Consider the class of all Riemannian metrics on M having Ω as their volume form. Then this class admits metrics with arbitrarily large λ_1.*

However, if one restricts to a fixed conformal class of metrics, then we get a rigidity for λ_1, as the following result of Friedlander and Nadirashvili [F-N] shows (see also the work of El Soufi and Ilias [E-I]):

Theorem 1.4. *Let (M, g) be a closed Riemannian manifold of dimension d. Then*

$$\lambda_1(fg)\mathrm{Vol}(M, fg)^{\frac{2}{d}} \leqslant C(g),$$

where f is any positive function on M and $C(g)$ is a constant independent of f.

The latter Theorem 1.4 can be seen as a generalization of Theorem 1.1, due to the Uniformization Theorem for the Riemann sphere.

As it turns out, Theorems 1.3 and 1.4 do not give us a full variety of ways in which one can generalize the 2-dimensional setting of Theorems 1.1 and 1.2. In [P], Polterovich proposes to look at a symplectic side of this story. For a given closed symplectic manifold (M, ω), he considers the Kähler, and the quasi-Kähler categories. In the Kähler case, Polterovich looks at the collection of all Riemannian metrics g on M, such that $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$, where J is a complex structure (i.e. an integrable almost complex structure) on M. In the quasi-Kähler case, Polterovich considers the collection of all Riemannian metrics g on M, such that $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$, where J is an almost (i.e. not necessarily integrable) complex structure on M. As it turns out, these two settings exhibit an opposite type of behaviour in terms of λ_1. Namely, in the Kähler case we meet with a rigidity phenomenon, whereas in the quasi-Kähler case we have examples of flexibility, as the following two theorems show [P]:

Theorem 1.5. *Let (M, ω) be a closed symplectic manifold, such that ω is a rational form. Let g be a Kähler metric whose Kähler form is ω. Then*

$$\lambda_1 \leqslant C(\omega),$$

where $C(\omega)$ is independent of g.

Theorem 1.6. *Let $(\mathbb{T}^4, \sigma)$ be the standard symplectic 4-torus. Let (M, ω) be a closed symplectic manifold. Then, on $(\mathbb{T}^4 \times M, \sigma \oplus \omega)$ there exists a quasi-Kähler structure with arbitrarily large λ_1.*

In the case of the Kähler category, it is still an open question whether Theorem 1.5 is true for any closed symplectic manifold (M, ω). In the quasi-Kähler case, Theorem 1.6 was generalized by Mangoubi [M]:

Theorem 1.7. *Let $(\mathbb{T}^2, \sigma)$ be the standard symplectic 2-torus. Let (M, ω) be a closed symplectic manifold. Then, on $(\mathbb{T}^2 \times M, \sigma \oplus \omega)$ there exists a quasi-Kähler structure with arbitrarily large λ_1.*

The following conjecture was raised in [M]:

Conjecture 1.8. *Let (M, ω) be a closed symplectic manifold of dimension $\geqslant 4$. Then, there exists a quasi-Kähler structure on it with arbitrarily large λ_1.*

The proof of Theorem 1.6 in [P] is based on a construction of an isotropic plane distribution on $(\mathbb{T}^4 \times M, \sigma \oplus \omega)$, which satisfies Hörmander condition. After providing the construction, Polterovich fixes some Riemannian metric on $\mathbb{T}^4 \times M$ which is compatible with $\sigma \oplus \omega$, and applies to it a "stretching the neck"-type procedure associated with the constructed distribution, and thus provides us with a new Riemannian metric on $\mathbb{T}^4 \times M$. Then finally the Hörmander theory [Ho] is applied in order to show that by such a procedure one might get a desired Riemannian metric on $\mathbb{T}^4 \times M$, and this finishes the proof of Theorem 1.6.

Mangoubi, in order to prove Theorem 1.7, generalizes the approach of Polterovich by expanding it to non-regular distributions. Mangoubi proves, that on $(\mathbb{T}^2 \times M, \sigma \oplus \omega)$ there exists an isotropic singular plane distribution that satisfies the Hörmander condition, by providing the needed construction. After establishing this, Mangoubi constructs a Riemannian metric on $\mathbb{T}^2 \times M$ by the way which is similar to the "stretching the neck"-type procedure in the approach of Polterovich, and concludes Theorem 1.7 by showing that this is a desired metric. However, the last step of the proof is technically more difficult than the one in the case of Theorem 1.6. In order to overcome these difficulties, Mangoubi applies the theory of anisotropic Sobolev spaces as developed in [R-S], and the machinery of fractional Sobolev Spaces also known as Bessel Potential Spaces.

The following conjecture was raised in [M]:

Conjecture 1.9. *Let (M, ω) be a closed symplectic manifold of dimension $\geqslant 4$. Then one can find on (M, ω) an isotropic singular distribution that satisfies the Hörmander condition.*

A positive answer to Conjecture 1.9 will imply Conjecture 1.8 (see [M]). Interestingly, a negative answer to Conjecture 1.9 would yield a new type of symplectic rigidity.

In this paper we concentrate on the quasi-Kähler situation. We affirmatively answer Conjecture 1.8, and prove the following

Theorem 1.10. *Let (M^{2n}, ω) be a closed symplectic manifold of dimension $2n \geqslant 4$. Then there exist Riemannian metrics g on M, compatible with the symplectic structure ω, having arbitrarily large λ_1.*

The proof of Theorem 1.10 relies on the following local result (see below the section describing the notations that we use here):

Proposition 1.11. *For any $R > 0$ and for any $\epsilon > 0$ there exists $\frac{R}{2} < r < R$, and a Riemannian metric g on the domain*

$$D^{2n}_{r,R} = \{x \in \mathbb{R}^{2n} \,|\, r < |x| < R\},$$

which is compatible with the standard symplectic structure ω_{std} on $D^{2n}_{r,R}$, such that g coincides with the euclidean metric on a neighborhood of the boundary of $D^{2n}_{r,R}$, and such that for any smooth function $f : D^{2n}_{r,R} \to \mathbb{R}$ satisfying

$$\int_{D^{2n}_{r,R}} \|\nabla_g f\|^2_g \, dg^{2n}_{std} \leqslant 1,$$

there exists some $E \in \mathbb{R}$, such that for any $r < u < R$ we have

$$\int_{S^{2n-1}_u} |f - E|^2 \, dg^{2n-1}_{std} \leqslant \epsilon,$$

where

$$S^{2n-1}_u = \{x \in \mathbb{R}^{2n} \,|\, |x| = u\},$$

and g_{std} is the euclidean metric on $D^{2n}_{r,R}$.

In our approach, similarly to previous approaches [P, M], we construct the desired Riemannian metric with the help of a "stretching the neck"-type procedure. However, in our approach we use ideas that are different from the construction of an isotropic distribution that satisfies the Hörmander condition, as it was done by Polterovich [P], and later generalized to the case of singular distributions by Mangoubi [M].

The advantage of our approach over previous approaches [P, M], is that it makes it possible to prove an analogue of the symplectic flexibility of the first eigenvalue of the Laplacian, in the case of the *open* ball $B = \{x \in \mathbb{R}^{2n} \,|\, |x| < 1\} \subset \mathbb{R}^{2n}$ endowed with a symplectic structure ω, under certain weak enough assumptions on the behaviour of ω near the boundary of B. The latter fact implies the statement of Theorem 1.10 for an arbitrary closed symplectic manifold (M, ω), helping to avoid possible complications with the topology of M. See section 6.1 for more details.

Structure of the paper

In section 2 we sketch an outline of the proof of Theorem 1.10. In section 3 we prove a number of preliminary lemmas, which are used later in the proofs of Proposition 1.11 and Theorem 1.10. The proofs of lemmas from section 3 are quite standard and straightforward, and can be omitted by the reader. In section 4 we prove a local result - Proposition 1.11. Proposition 1.11 is the central ingredient in the proof of Theorem 1.10 in section 5. Finally, in section 6.1 we compare between our approach and previous approaches [P, M], and in section 6.2 we discuss the symplectic flexibility of the first Dirichlet and the first nonzero Neumann eigenvalues.

Notations

Looking at the euclidean space $\mathbb{R}^d$, by $|\cdot|$ we denote the euclidean norm, and by $\langle\cdot,\cdot\rangle$ we denote the scalar product on $\mathbb{R}^d$. We use the notation g_{std} for the standard euclidean metric on $\mathbb{R}^d$: $g_{std}(u,v) = \langle u,v\rangle$, at each point of $\mathbb{R}^d$. For $r > 0$, we denote by

$$S_r^{d-1} = \{x \in \mathbb{R}^d \,|\, |x| = r\}$$

the $(d-1)$-dimensional sphere of radius r centered at the origin, in $\mathbb{R}^d$. For $r > 0$ and $x \in \mathbb{R}^d$, we denote by

$$B_r^d(x) = \{y \in \mathbb{R}^d \,|\, |y - x| < r\}$$

the open ball of radius r centered at x, and by

$$\overline{B}_r^d(x) = \{y \in \mathbb{R}^d \,|\, |y - x| \leqslant r\}$$

the closed ball of radius r centered at x.

On the unit sphere $S^{2n-1} \subset \mathbb{R}^{2n}$ centered at the origin, consider the spherical Riemannian metric that is induced from the euclidean metric on $\mathbb{R}^{2n}$. For any $x \in S^{2n-1}$ and $\rho > 0$ we denote by $B_\rho^S(x) \subseteq S^{2n-1}$ the ball of radius ρ centered at x, with respect to the spherical Riemannian metric on S^{2n-1}. We call $B_\rho^S(x)$ a "spherical cap", or a "spherical ball". We denote by $B_\rho^S \subseteq S^{2n-1}$ the spherical ball of radius ρ centered at the point $(1,0,0,...,0) \in S^{2n-1}$. In the sequel we also consider hypersurfaces of the form

$$rB_\rho^S(x) = \{ry \,|\, y \in B_\rho^S(x)\} \subseteq S_r^{2n-1},$$

for $r, \rho > 0$, and $x \in S^{2n-1}$. We call them spherical caps (spherical balls) as well.

In the paper, we sometimes use the polar coordinates notation (r,θ) for the point $r\theta \in \mathbb{R}^{2n}$ (here $r \in [0,\infty)$, $\theta \in S^{2n-1}$).

We denote by ω_{std} the standard symplectic form on $\mathbb{R}^{2n}$. For given $0 < r < R < \infty$, we denote the annulus

$$D^{2n}_{r,R} = \{x \in \mathbb{R}^{2n} \,|\, r < |x| < R\}.$$

Given a smooth manifold X^d, and a Riemannian metric g on X, we denote by $\|\cdot\|_g$ the norm on TX induced by g. For a differentiable function $f : X \to \mathbb{R}$, by $\nabla_g f$ we denote the gradient of f with respect to the metric g, so $\nabla_g f(x) \in T_x X$ for every $x \in X$. For $0 \leqslant k \leqslant d$, a k-dimensional submanifold $\Sigma \subseteq X$, and a continuous function $f : \Sigma \to \mathbb{R}$, we denote by $\int_\Sigma f\, dg^k$ the integral of f over Σ with respect to the volume density on Σ induced by g. We will use the notation $\mathrm{Vol}_g(\Sigma)$ for $\int_\Sigma 1\, dg^k$. For a continuous function $h : \Sigma \to \mathbb{R}$, we will say that h is almost equal to some $E \in \mathbb{R}$ in the $L^2(g)$ sense, if $\int_\Sigma |h - E|^2\, dg^k$ is small; we will say that h is almost equal to some $E \in \mathbb{R}$ in the average $L^2(g)$ sense, if $\frac{1}{\mathrm{Vol}_g(\Sigma)} \int_\Sigma |h - E|^2\, dg^k$ is small. Given two k-dimensional submanifolds $\Sigma_1, \Sigma_2 \subseteq X$ together with a diffeomorphism $\psi : \Sigma_1 \to \Sigma_2$, and continuous functions $h_1 : \Sigma_1 \to \mathbb{R}$, $h_2 : \Sigma_2 \to \mathbb{R}$, we will say that h_2 is close to h_1 in the $L^2(g)$ sense when we identify Σ_2 with Σ_1 via the map ψ, if $\int_{\Sigma_1} |\psi^* h_2 - h_1|^2 dg^k$ is small.

If $\Sigma \subseteq \mathbb{R}^d$ is a k-dimensional submanifold, then by $\mathrm{Vol}(\Sigma)$ we mean $\mathrm{Vol}_{g_{std}}(\Sigma)$.

For a nice (e.g. open) subset $\Sigma \subseteq S^{2n-1}$, and a continuous function $f : \Sigma \to \mathbb{R}$, we will also use the notation $\int_\Sigma f(\theta)\, d\theta$ for $\int_\Sigma f\, dg^{2n-1}_{std}$.

Acknowledgements

I thank Michael Entov, Dan Mangoubi, Leonid Polterovich and Felix Schlenk for their encouragement, interest in this work, and helpful comments and suggestions. I thank Yaron Ostrover for pointing my attention to Conjecture 1.8. I thank Alexander Bykhovsky and Sobhan Seyfaddini for improving the style of the paper. I thank the referee for useful suggestions which helped to reduce the size of the paper. The author was partially supported by the NSF Grant DMS-1105813.

2 Outline of the proof

In this section we provide an explanation of the proof of Theorem 1.10. Recall that one can express $\lambda_1(g)$ as the minimum of

$$\frac{\int_M \|\nabla_g f\|_g^2\, dg^{2n}}{\int_M f^2\, dg^{2n}},$$

where we run over all nonzero functions $f : M \to \mathbb{R}$ having zero mean: $\int_M f dg^{2n} = 0$. Hence the first eigenvalue is large if and only if, for any smooth function $f : M \to \mathbb{R}$ satisfying

$$\int_M f dg^{2n} = 0, \int_M \|\nabla_g f\|_g^2 \, dg^{2n} \leqslant 1,$$

we have that f is "almost zero" on M in the $L^2(g)$ sense, or in other words, that $\int_M |f|^2 \, dg^{2n}$ is small.

In the proof of Theorem 1.10, we avoid possible complications with the topology of M, by first proving a local result (Proposition 1.11), and then by passing to any closed symplectic manifold via a smooth triangulation. Below in section 2.3, we briefly explain the proof of Proposition 1.11, and in section 2.2 we briefly explain how we reduce Theorem 1.10 to Proposition 1.11. Section 2.1 explains the "compressing the neck" procedure, that is used in the proofs of Proposition 1.11 and of Theorem 1.10. We direct the reader to section 2.1 first.

2.1 "Compressing the neck" - explanation

We use the following idea (similar constructions were used in [P], [M]). Let (M, ω) be a symplectic manifold (open or closed). Assume that we have fixed a Riemannian metric g_0 on M, which is compatible with the symplectic structure ω, and denote by J_0 the almost complex structure on M, associated with ω and g_0. Let $U \subseteq M$ be an open subset of M, let Y be a smooth nonzero vector field defined on $\overline{U} \subseteq M$, and let $\Sigma \subset \overline{U}$ be a $(2n-1)$-dimensional smooth hypersurface, such that Σ is a proper subset of M. Denote by ψ^t the flow of Y, and assume that for some $T > 0$, we have $\psi^t(\Sigma) \subset U$ for any $t \in (0, T)$. Denote $\Sigma' = \psi^T(\Sigma) \subset \overline{U}$. Given all this setting, we can obtain a new Riemannian metric g on M by deforming the metric g_0 as follows: choose a smooth function $b : M \to \mathbb{R}$, such that $b(x) \geqslant 1$ on M, such that $b(x) = 1$ on some open set containing $M \setminus U$, and such that the function $b(\cdot)$ is very large on almost all of U. Then considering the g_0-orthogonal decomposition

$$TU = Span(Y) \oplus Span(J_0 Y) \oplus L,$$

for any $x \in U$ we define

$$g|_x = b(x)^{-1} g_0|_x \oplus b(x) g_0|_x \oplus g_0|_x,$$

and for any $x \notin U$ we set $g|_x = g_0|_x$. Clearly g is compatible with ω as well. By choosing an appropriate function $b(\cdot)$, we can achieve that the hypersurface Σ will become very close to Σ', in metric g, since for any $x \in \Sigma$, the flow trajectory $\{\psi^t(x) \,|\, t \in [0, T]\}$ becomes very short in the metric g. Then, one can easily check that as a consequence, we get the following: for any continuous function $f : \overline{U} \to \mathbb{R}$ which is smooth inside U, and which satisfies

$$\int_U \|\nabla_g f\|_g^2 \, dg^{2n} \leqslant 1,$$

we have that the restriction $f|_{\Sigma'}$ is very close to the restriction $f|_{\Sigma}$, in the $L^2(g_0)$ sense, when we identify Σ' with Σ with help of the map ψ^T. This way of passing from the metric g_0 to the metric g reminds the so-called "stretching the neck" procedure, but as we can see, its purpose is rather to "compress" than to "stretch". Along section 2, we will call it "compressing the neck on U along the vector field Y".

2.2 From local result to global

Here we briefly describe how we reduce Theorem 1.10 to a local result (Proposition 1.11).

2.2.1 Sketch of the construction

Choose a smooth triangulation of M. Let $\{\Delta_\alpha \,|\, \alpha \in I\}$ be all the open simplices of this triangulation. Choose a Riemannian metric g_0 on M, such that for each $\alpha \in I$, there exists a Darboux neighborhood inside Δ_α, on which g_0 coincides with the euclidean metric.

The desired metric g on M will be constructed by deforming g_0 on a proper subset of Δ_α, for each $\alpha \in I$. For a given $\alpha \in I$, let us describe the way in which we deform g_0 inside Δ_α. For the sake of convenience, we will actually work not on Δ_α, but on the open unit ball $B_1^{2n}(0) \subset \mathbb{R}^{2n}$. In order to make this switch, we use Lemma 3.1 (section 3), which implies that there exists a bi-Lipschitz homeomorphism $\Psi_\alpha : \overline{\Delta_\alpha} \to \overline{B}_1^{2n}(0)$, such that its restriction to Δ_α is a diffeomorphism onto the open unit ball $B_1^d(0)$, and such that its restriction to Δ'_α is a diffeomorphism onto the image, where Δ'_α is the union of Δ_α with all of its open faces. Because of our choice of the metric g_0, WLOG we may assume that the pushforward $\omega_\alpha = (\Psi_\alpha)_*\omega$ of ω from Δ_α to $B_1^{2n}(0)$, and the pushforward $g_{0,\alpha} = (\Psi_\alpha)_* g_0$ of g_0 from Δ'_α to $\Psi_\alpha(\Delta'_\alpha)$, coincide with ω_{std} and g_{std} near the origin $0 \in \overline{B}_1^{2n}(0)$, respectively. Hence we can find some $R_0 > 0$ such that $\omega_\alpha = \omega_{std}$ and $g_{0,\alpha} = g_{std}$ on $B_{R_0}^{2n}(0)$, for each $\alpha \in I$.

Take $0 < R \leqslant R_0$ small enough. By Proposition 1.11, there exists $\frac{R}{2} < r < R$, and a metric g_{loc} on the domain

$$D_{r,R}^{2n} = \{x \in \mathbb{R}^{2n} \,|\, r < |x| < R\},$$

which is compatible with ω_{std}, and is standard near the boundary, such that for any smooth function $f : D^{2n}_{r,R} \to \mathbb{R}$ with

$$\int_{D^{2n}_{r,R}} \|\nabla_g f\|_g^2 \, dg^{2n}_{std} \leqslant 1,$$

there exists some $E \in \mathbb{R}$, such that for any $r < u < R$, the restriction of the function f to S^{2n-1}_u, is very close to the constant function E, in the $L^2(g_{std})$ sense. Denote by $X(x) = -\frac{x}{|x|}$ the "minus-radial vector field" on $\mathbb{R}^{2n} \setminus \{0\}$. Now let us explain how we deform the metric $g_{0,\alpha}$ to a metric g_α, inside Δ_α. At a first step, we define a preliminary metric on $B^{2n}_1(0)$ by starting with the metric $g_{0,\alpha}$ on $B^{2n}_1(0)$, and changing it on $D^{2n}_{r,R}$ to be equal to g_{loc}. Then we define g_α on $B^{2n}_1(0)$ by starting with this preliminary metric on $B^{2n}_1(0)$, and applying the "compressing the neck on $D^{2n}_{R,1}$ along the vector field X".

Finally, we define the metric g on M to be equal to $(\Psi_\alpha)^* g_\alpha$ on each Δ_α, and set $g = g_0$ on $M \setminus (\cup_{\alpha \in I} \Delta_\alpha)$.

2.2.2 Sketch of the proof

Let us show that the metric g will have arbitrarily large λ_1, provided that R is small enough and that the neck compression is applied.

Let $f : M \to \mathbb{R}$ be a smooth function with

$$\int_M f \, dg^{2n} = 0,$$

and

$$\int_M \|\nabla_g f\|_g^2 \, dg^{2n} \leqslant 1.$$

Then for any $\alpha \in I$, consider $f_\alpha : \overline{B}^{2n}_1(0) \to \mathbb{R}$ defined by $f_\alpha = (\Psi_\alpha)_* f$. Then f_α is smooth inside $B^{2n}_1(0)$, and is continuous on $\overline{B}^{2n}_1(0)$. We have

$$\int_{\Delta_\alpha} \|\nabla_g f\|_g^2 \, dg^{2n}_0 = \int_{\Delta_\alpha} \|\nabla_g f\|_g^2 \, dg^{2n} \leqslant 1,$$

and hence

$$\int_{B^{2n}_1(0)} \|\nabla_{g_\alpha} f_\alpha\|_{g_\alpha}^2 \, dg^{2n}_{0,\alpha} = \int_{B^{2n}_1(0)} \|\nabla_{g_\alpha} f_\alpha\|_{g_\alpha}^2 \, dg^{2n}_\alpha \leqslant 1. \tag{2.2.1}$$

Now, since $g_\alpha = g_{loc}$ on $D^{2n}_{r,R}$, by Proposition 1.11 we conclude that there exists a constant $E_\alpha \in \mathbb{R}$, such that for any $r < u < R$ (and hence, by continuity, also for $u = r, R$), the restriction of the function f_α to S^{2n-1}_u, is very close to the constant function E_α, in the $L^2(g_{std})$ sense. Then, since we have compressed the neck on

$D^{2n}_{R,1}$, we get that for any $u \in (R, 1)$, the restriction of f_α to S^{2n-1}_u is very close to the restriction of f_α to S^{2n-1}_R, in the $L^2(g_{std})$ sense, when we identify S^{2n-1}_u with S^{2n-1}_R via a homothety (which is part of the flow of the vector field X). Hence we conclude that also for $u \in (R, 1)$ (and hence, by continuity, for $u = 1$ as well), the restriction of f_α to S^{2n-1}_u, is very close to the constant function E_α, in the $L^2(g_{std})$ sense. Therefore, integrating over the radius, we conclude that the restriction of f_α to $D^{2n}_{r,1}$, is very close to the constant function E_α, in the $L^2(g_{std})$ sense. Finally, from the fact that the restriction of f_α to S^{2n-1}_r, is very close to the constant function E_α, in the $L^2(g_{std})$ sense, and from (2.2.1), since r is small we conclude that the restriction of f_α to $B^{2n}_r(0)$, is very close to the constant function E_α, in the $L^2(g_{std})$ sense (at this point we use Lemma 3.5 from section 3).

Hence we get the following:
1) The restriction of f_α to $S^{2n-1} = S^{2n-1}_1 = \partial\overline{B}^{2n}_1(0)$, is very close to the constant function E_α, in the $L^2(g_{std})$ sense, and hence in the $L^2(g_{0,\alpha})$ sense.
2) The function f_α is very close to the constant function E_α, in the $L^2(g_{std})$ sense, and hence in the $L^2(g_{0,\alpha})$ sense, on $B^{2n}_1(0)$.

Going back to the manifold M with help of maps Ψ_α, $\alpha \in I$, we get:
1') The restriction of f to $\partial\Delta_\alpha$, is very close to the constant function E_α, in the $L^2(g_0)$ sense.
2') The restriction of f to Δ_α, is very close to the constant function E_α, in the $L^2(g_0)$ sense.

Now we use 1') to conclude that in fact all E_α are close real numbers. Indeed, considering any two adjacent simplices Δ_α and Δ_β having a common face Σ, from 1') we get that the restriction of f to Σ, is very close to both E_α and E_β, in the $L^2(g_0)$ sense. Hence for any two adjacent simplices Δ_α and Δ_β, we have that E_α is close to E_β. Now, since our triangulation is fixed, and since we have a finite number of simplices in our triangulation, we conclude that all E_α, $\alpha \in I$, are close real numbers.

Now fix any $E \in \mathbb{R}$ which is close to all E_α, $\alpha \in I$ (we can take E to be equal to any E_γ). Then from 2') we get that for any $\alpha \in I$, the restriction of f to Δ_α, is very close to E, in the $L^2(g_0)$ sense. But this implies that in fact, f is very close to E on M, in the $L^2(g_0)$ sense, and hence in the $L^2(g)$ sense. Finally, since f is normalized: $\int_M f\, dg^{2n} = 0$, we get that E is very small, and hence we conclude the statement of the theorem.

2.3 Local result

Proposition 1.11 tells us, that we can deform the standard euclidean metric on an annulus $D^{2n}_{r,R}$ (for some $\frac{R}{2} < r < R$), such that we will get again a Riemannian metric g on $D^{2n}_{r,R}$ that is compatible with the standard symplectic structure ω_{std} on $D^{2n}_{r,R}$, and such that any smooth function on $D^{2n}_{r,R}$ having the L^2-norm of its g-gradient bounded by 1, is almost constant on the cocentric spheres

$$S^{2n-1}_u = \{x \in \mathbb{R}^{2n} \,|\, |x| = u\},$$

in the $L^2(g_{std})$ sense, where $u \in (r, R)$, and the constant is the same for all $u \in (r, R)$. In our construction, the volume of the annulus $D^{2n}_{r,R}$ is divided into two sub-annuli $D^{2n}_{r,r'}$ and $D^{2n}_{r',R}$ (where $r < r' < R$), where these sub-annuli play different roles in the construction and in the proof. The sub-annulus $D^{2n}_{r',R}$ is chosen to be of width ϵ (i.e. $R - r' = \epsilon$), and the width $r' - r$ of the sub-annulus $D^{2n}_{r,r'}$ is much smaller relative to ϵ. On $D^{2n}_{r',R}$ we choose the metric g to be equal to the standard euclidean metric g_{std}, while on $D^{2n}_{r,r'}$ we construct g by deforming the euclidean metric g_{std} so that the metric g occurs to be "mixed enough" (the precise meaning of this will be clear in the sequel). In the proof that g is the desired metric, the roles of the sub-annuli $D^{2n}_{r',R}$ and $D^{2n}_{r,r'}$ are different. Let us give a rough explanation of this point. Assume that $f : D^{2n}_{r,R} \to \mathbb{R}$ is a smooth function with

$$\int_{D^{2n}_{r,R}} \|\nabla_g f\|^2_g \, dg^{2n}_{std} \leqslant 1. \tag{2.3.2}$$

Then we use the sub-annulus $D^{2n}_{r',R}$ and (2.3.2) to show, that on the sphere $S^{2n-1}_{r'}$ (which is a part of its boundary), there exists a small piece of volume (which is in fact a spherical cap) of size having rate ϵ, such that the restriction of f to it is almost constant in the average $L^2(g_{std})$ sense (Lemma 4.3 in section 4). Then, we use the fact that g is "mixed enough" on $D^{2n}_{r,r'}$, to show that condition (2.3.2) implies that in fact, f is almost constant on concentric spheres S^{2n-1}_u for $u \in (r, r']$, in the $L^2(g_{std})$ sense. Then, using the fact that g is standard on $D^{2n}_{r',R}$ and that the width $R - r'$ of $D^{2n}_{r',R}$ is small, we easily show that (2.3.2) implies that f is almost constant on concentric spheres S^{2n-1}_u in the $L^2(g_{std})$ sense, also for $u \in (r', R)$. To be more precise, we conclude that there exists some $E \in \mathbb{R}$ such that for each $u \in (r, R)$, the restriction of f to S^{2n-1}_u is close to E in the $L^2(g_{std})$ sense, up to $C\epsilon$, where $C = C(n, R)$. Since ϵ can be arbitrary, by replacing ϵ by $\frac{\epsilon}{C}$, we conclude the proposition.

Let us go over the construction of g on $D^{2n}_{r,R}$, and explain the role of $D^{2n}_{r',R}$ and $D^{2n}_{r,r'}$ in the proof in more details.

2.3.1 Sketch of the construction

Consider the sphere $S^{2n-1} \subset \mathbb{R}^{2n}$, and denote by $\tilde{H}$ the Hopf vector field on S^{2n-1}: $\tilde{H}(x) = Jx$ for any $x \in S^{2n-1}$, where J is the standard complex structure on $\mathbb{R}^{2n}$. Choose an isometry $\tilde{\alpha} : S^{2n-1} \to S^{2n-1}$ of the sphere, such that the pushforward $\tilde{\alpha}_* \tilde{H}$ of the Hopf vector field $\tilde{H}$, is transversal to the Hopf vector field $\tilde{H}$ at some point $x_1 \in S^{2n-1}$, and hence for some spherical cap $S := B^S_\rho(x_1) \subset S^{2n-1}$ around x_1, the vector field $\tilde{\alpha}_* \tilde{H}$ is transversal to the Hopf vector field $\tilde{H}$ on the closure $\overline{S}$. The radius ρ of the cap S can be chosen to depend only on the dimension $2n-1$. Then we can choose and fix a certain finite collection $\{B^S_{2\epsilon}(x) \,|\, x \in \mathcal{P}\}$ of non-intersecting spherical caps of radius 2ϵ inside S, where $\mathcal{P} \subset S \subset S^{2n-1}$ is a certain finite set of points, such that distance from each such $B^S_{2\epsilon}(x)$ (for $x \in \mathcal{P}$) to the boundary ∂S is bounded away from 0, and such that the cardinality $|\mathcal{P}|$ of this collection has rate $\frac{1}{\epsilon^{2n-1}}$. We show (Lemma 4.1 in section 4) that on S^{2n-1} there exists a smooth time dependent vector field $\tilde{Y}^t$, $t \in [0, T]$, such that $\tilde{Y}^t$ is sufficiently C^0-close to the vector field $\tilde{\alpha}_* \tilde{H}$, such that the flow $\tilde{\psi}^t$, $t \in [0, T]$ of $\tilde{Y}^t$ is volume preserving, and such that for any cap in $\{B^S_{2\epsilon}(x) \,|\, x \in \mathcal{P}\}$, there exists a collection of time moments $t_i \in (0, T)$, $i = 1, 2, ..., N$, so that the preimages of this cap under $\tilde{\psi}^{t_i}$, $i = 1, 2, ..., N$, cover the sphere nearly uniformly. Observe that if $\tilde{Y}^t$, $t \in [0, T]$ is sufficiently C^0-close to $\tilde{\alpha}_* \tilde{H}$, then $\tilde{Y}^t$, $t \in [0, T]$ must be transversal to the Hopf vector field $\tilde{H}$ on $\overline{S}$, as well. This observation will be used in the sequel. Now, given this time dependent vector field $\tilde{Y}^t$, $t \in [0, T]$ on S^{2n-1}, for $\delta > 0$ small enough, we set $r' = R - \epsilon$, and $r = r' - T\delta$, and we define a (time independent) vector field Y_δ on $\overline{D^{2n}_{r,r'}}$ by

$$Y_\delta(r' - \delta t, \theta) = -\delta\theta + (r' - \delta t)\tilde{Y}^t(\theta),$$

for $t \in [0, T]$ and $\theta \in S^{2n-1}$. In other words, we obtain the time independent vector field Y_δ on $\overline{D^{2n}_{r,r'}}$ by "spreading" the time dependent vector field $\tilde{Y}^t$, $t \in [0, T]$ through the family of spheres S^{2n-1}_u, $u \in [r, r']$ (and so the radius $u = r' - \delta t$ plays the role of time), and then by adding a small component (of amount δ) in the minus-radial direction. Note that as a consequence, if we look at the flow ψ^t_δ of the (time independent) vector field Y_δ, applied to the sphere $S^{2n}_{r'}$, we get just a composition of homotheties of $\mathbb{R}^{2n}$ with the flow $\tilde{\psi}^t$ of the (time dependent) vector field $\tilde{Y}^t$:

$$\psi^t_\delta(r', \theta) = (r' - \delta t, \tilde{\psi}^t(\theta)).$$

Now we construct the metric g on $D^{2n}_{r,r'}$ by starting with the standard euclidean metric g_{std} on $D^{2n}_{r,r'}$, and then "compressing the neck on $D^{2n}_{r,r'}$ along the vector field Y_δ". On $D^{2n}_{r,R} \setminus D^{2n}_{r,r'}$ we set $g = g_{std}$.

2.3.2 Sketch of the proof

Since the vector field $\tilde{Y}^t$, $t \in [0, T]$, is transversal to the Hopf vector field $\tilde{H}$ on $\overline{S}$, then for small enough $\delta > 0$ we are able to find a certain smooth vector field X_δ (which has a bounded euclidean norm, uniformly on δ) on $[r, r'] \cdot \overline{S} \subseteq \overline{D^{2n}_{r,r'}}$, which is orthogonal to $Span(Y_\delta, JY_\delta)$ at every point of $[r, r'] \cdot \overline{S}$, and which radial component equals -1, or in other words

$$X_\delta(r' - \delta t, \theta) = -\theta + (r' - \delta t)\tilde{X}^t_\delta(\theta),$$

for any $t \in [0,T]$ and $\theta \in \overline{S}$, where $\tilde{X}^t_\delta$, $t \in [0,T]$, is a certain time dependent vector field on $\overline{S} \subset S^{2n-1}$ which is tangent to the sphere S^{2n-1}. Note that since X_δ is orthogonal to $Span(Y_\delta, JY_\delta)$, it follows that its g-norm coincides with its euclidean norm at each point of $[r,r'] \cdot \overline{S}$, and hence is bounded, uniformly on δ. The flow σ^s_δ of X_δ (which of course, might be defined only partially), satisfies

$$\sigma^s_\delta(r',\theta) = (r'-s, \tilde{\sigma}^s_\delta(\theta)),\ s \in [0,\delta T),$$

where $\tilde{\sigma}^s_\delta$, $s \in [0,\delta T)$ is the flow of the time dependent vector field $\tilde{X}^{\frac{s}{\delta}}_\delta$, $s \in [0,\delta T)$. Note first, that since the time range for the parameter s is small (of length δT), and since the euclidean norm of our vector field X_δ (and hence also of the vector field $\tilde{X}^t_\delta$) is bounded uniformly on δ, it follows that for δ small enough, the flow $\tilde{\sigma}^s_\delta(\theta)$, $s \in [0,\delta T)$ is well defined when the distance from $\theta \in S$ to the boundary ∂S is bounded away from 0, and moreover the flow $\tilde{\sigma}^s_\delta(\theta)$, $s \in [0,\delta T)$ is arbitrarily C^0-close to the identity when δ is small enough. Secondly, we show that in fact, one can choose X_δ for small $\delta > 0$ as above, such that in addition, the flow $\tilde{\sigma}^s_\delta$, $s \in [0,\delta T)$ is "almost volume preserving" when δ is small enough.

Now let $f : D^{2n}_{r,R} \to \mathbb{R}$ be a smooth function such that (2.3.2) holds. First, by looking at the restriction of f to $D^{2n}_{r',R}$, and using (2.3.2), we show (Lemma 4.3 in section 4) that there exists a certain spherical cap $B^S_{2\epsilon}(x_2)$ from our collection of caps (i.e. $x_2 \in \mathcal{P}$), such that on $r'B^S_{2\epsilon}(x_2) \subset S^{2n}_{r'}$, the function f is almost constant (denote this constant by E) in the average $L^2(g_{std})$ sense.

Now for some $s \in (0,\delta T)$, apply the map σ^s_δ (which belongs to the flow of X_δ) to $r'B^S_{2\epsilon}(x_2) \subset S^{2n}_{r'}$. Since δ (and hence s) is small enough, and since the vector field X_δ is bounded uniformly on δ, we get that the hypersurface $r'B^S_{2\epsilon}(x_2)$ is very close to the hypersurface $\sigma^s_\delta(r'B^S_{2\epsilon}(x_2)) \subset S^{2n}_{r'-s}$, in the metric g. Therefore we can conclude that the restriction of f to $\sigma^s_\delta(r'B^S_{2\epsilon}(x_2))$ is very close to the restriction of f to $r'B^S_{2\epsilon}(x_2)$, in the $L^2(g_{std})$ sense, when we identify $\sigma^s_\delta(r'B^S_{2\epsilon}(x_2))$ with $r'B^S_{2\epsilon}(x_2)$ via the map σ^s_δ (note that here we did not use a "compressing the neck" procedure - it is not necessary since the hypersurfaces $r'B^S_{2\epsilon}(x_2)$ and $\sigma^s_\delta(r'B^S_{2\epsilon}(x_2))$ are already close in the metric g). Hence we conclude that the restriction of f to $\sigma^s_\delta(r'B^S_{2\epsilon}(x_2))$ is almost equal to E in the average $L^2((\sigma^s_\delta)_* g_{std})$ sense (i.e. when we consider the L^2 norm with respect to the pushforward by the map σ^s_δ, of the standard spherical volume density $dg^{2n-1}_{std}|_{r'B^S_{2\epsilon}(x_2)}$ from $r'B^S_{2\epsilon}(x_2)$ to $\sigma^s_\delta(r'B^S_{2\epsilon}(x_2))$). Now, since $\tilde{\sigma}^s_\delta$ is "almost volume preserving", we conclude that in fact, the restriction of f to $\sigma^s_\delta(r'B^S_{2\epsilon}(x_2))$ is almost equal to E in the average $L^2(g_{std})$ sense. Since for small δ, the map $\tilde{\sigma}^s_\delta(\theta)$ is arbitrarily C^0-close to the identity, we conclude that $\sigma^s_\delta(r'B^S_{2\epsilon}(x_2)) \supseteq (r'-s)B^S_\epsilon(x_2)$, and therefore in particular, on $(r'-s)B^S_\epsilon(x_2)$ the function f is almost equal to the constant E, in the average $L^2(g_{std})$ sense (see Lemma 4.4 in section 4).

We have used the flow of X_δ to show that for each $s \in (0,\delta T)$, the restriction of f to $(r'-s)B^S_\epsilon(x_2)$, is almost equal to the constant E, in the average $L^2(g_{std})$

sense. We can rephrase it by saying that for each $t \in (0, T)$, the restriction of f to $(r' - \delta t)B^S_\epsilon(x_2)$, is almost equal to the constant E, in the average $L^2(g_{std})$ sense. Now let us use the vector field Y_δ, for a similar purpose. Note that we have

$$\psi^t_\delta(S^{2n-1}_{r'}) = S^{2n-1}_{r'-\delta t}.$$

Since we have compressed the neck on $D^{2n}_{r,r'}$ along Y_δ, we can conclude that the restriction of f to $S^{2n-1}_{r'-\delta t}$ is very close to the restriction of f to $S^{2n-1}_{r'}$ in the $L^2(g_{std})$ sense, when we identify $S^{2n-1}_{r'-\delta t}$ with $S^{2n-1}_{r'}$ via the map ψ^t_δ. In particular, the restriction of f to $(r' - \delta t)B^S_\epsilon(x_2) \subset S^{2n-1}_{r'-\delta t}$ is very close to the restriction of f to $(\psi^t_\delta)^{-1}((r' - \delta t)B^S_\epsilon(x_2)) \subset S^{2n-1}_{r'}$, in the $L^2(g_{std})$ sense, when we identify $(r' - \delta t)B^S_\epsilon(x_2)$ with $(\psi^t_\delta)^{-1}((r' - \delta t)B^S_\epsilon(x_2))$ via the map ψ^t_δ. The map $\tilde{\psi}^t$ is volume preserving, and hence the restriction of ψ^t_δ to $S^{2n-1}_{r'}$ is conformally volume preserving, as a map from $S^{2n-1}_{r'}$ to $S^{2n-1}_{r'-\delta t}$. Also recall that the restriction of f to $(r' - \delta t)B^S_\epsilon(x_2)$, is almost equal to the constant E, in the average $L^2(g_{std})$ sense. Hence we can conclude from the described above, that the restriction of f to $(\psi^t_\delta)^{-1}((r' - \delta t)B^S_\epsilon(x_2)) \subset S^{2n-1}_{r'}$, is almost equal to the constant E, in the average $L^2(g_{std})$ sense (see Lemmas 4.5, 4.6 in section 1.11).

So we finally conclude that for any $t \in (0, T)$, the restriction of f to $(\psi^t_\delta)^{-1}((r' - \delta t)B^S_\epsilon(x_2)) \subset S^{2n-1}_{r'}$, is almost equal to the constant E, in the average $L^2(g_{std})$ sense. Now, by one of the properties of the flow $\tilde{\psi}^t$ described above, for our point $x_2 \in \mathcal{P}$, there exists a collection of time moments $t_1, t_2, ..., t_N \in (0, T)$, such that the preimages $(\tilde{\psi}^{t_i})^{-1}(B^S_\epsilon(x_2))$, $i = 1, 2, ..., N$, cover the sphere S^{2n-1} nearly uniformly. Clearly this can be rephrased by saying that the preimages $(\psi^{t_i}_\delta)^{-1}((r' - \delta t_i)B^S_\epsilon(x_2)) \subset S^{2n-1}_{r'}$, $i = 1, 2, ..., N$, cover the sphere $S^{2n-1}_{r'}$ nearly uniformly. Therefore, since the restriction of f to each such preimage $(\psi^{t_i}_\delta)^{-1}((r' - \delta t_i)B^S_\epsilon(x_2))$, is almost equal to the constant E, in the average $L^2(g_{std})$ sense, we conclude that in fact, the restriction of f to the whole sphere $S^{2n-1}_{r'}$, is almost equal to E, in the $L^2(g_{std})$ sense. Having this in mind, it is already easy to derive the statement of Proposition 1.11. Indeed, if $u \in (r, r')$, then writing $u = r' - \delta t$ for $t \in (0, T)$, we can use the vector field Y_δ once again, identifying $S^{2n-1}_{r'}$ with S^{2n-1}_u with help of the map ψ^t_δ, to conclude that the restriction of f to the whole sphere S^{2n-1}_u, is almost equal to E, in the $L^2(g_{std})$ sense. If we have $u \in (r, R)$, then we can use the minus-radial vector field $X(x) = -\frac{x}{|x|}$ on $D_{r',R}$, and its flow, identifying the sphere S^{2n-1}_u with the sphere $S^{2n-1}_{r'}$, to conclude that the restriction of f to the whole sphere S^{2n-1}_u, is almost equal to E, in the $L^2(g_{std})$ sense. So finally, for any $u \in (r, R)$, the restriction of f to the whole sphere S^{2n-1}_u, is almost equal to E, in the $L^2(g_{std})$ sense. More precisely, we have shown, that by taking sufficiently small δ, and by appropriately applying the "compressing the neck", we get that for any smooth function $f : D^{2n}_{r,R} \to \mathbb{R}$ satisfying (2.3.2), there exists some $E \in \mathbb{R}$, such that for any $u \in (r, R)$, the restriction of f to S^{2n-1}_u, is close to E in the $L^2(g_{std})$ sense, up to $C\epsilon$, where $C = C(n, R)$. Since we have freedom in the choice of ϵ, we conclude Proposition 1.11.

3 Some preliminary lemmata

Lemma 3.1. *Consider an open bounded convex polytope $K \subset \mathbb{R}^d$. Denote by K' the union of K with all of its open faces. Then there exists a bi-Lipschitz homeomorphism $\overline{K} \to \overline{B}_1^d(0)$, such that its restriction to K is a diffeomorphism onto the open unit ball $B_1^d(0)$, and such that its restriction to K' is a diffeomorphism onto the image of K'.*

The proof of Lemma 3.1 is completely standard, and therefore we omit it.

Lemma 3.2. *Let $l > 0$, and consider a d-dimensional open cube $(-l,l)^d \subset \mathbb{R}^d$, endowed with coordinates $(x_1, ..., x_d)$ and with the Riemannian metric g_{std} that comes from the Euclidean metric on $\mathbb{R}^d$. Let $\varepsilon > 0$, and let $f : (-l,l)^d \to \mathbb{R}$ be a smooth function, such that*

$$\int_{(-l,l)^d} |\nabla f|^2 \, dg_{std}^d \leqslant \varepsilon.$$

Then there exists some $E \in \mathbb{R}$ such that for any $-1 < x_1 < 1$ we have

$$\int_{\{x_1\} \times (-l,l)^{d-1}} |f - E|^2 dg_{std}^{d-1} \leqslant C\varepsilon l,$$

and moreover we have

$$\int_{(-l,l)^d} |f - E|^2 dg_{std}^d \leqslant C\varepsilon l^2,$$

where $C = C(d)$.

Proof of Lemma 3.2. We prove the lemma by induction on the dimension d. First consider $d = 1$. In this case take $E = f(0)$. Then for any $x \in (0,l)$ we have

$$|f(x) - E|^2 = |f(x) - f(0)|^2 = \left| \int_0^x f'(t) \, dt \right|^2 \leqslant x \int_0^x |f'(t)|^2 \, dt \leqslant l \int_{-l}^l |f'(t)|^2 \, dt \leqslant l\varepsilon.$$

Similarly we get $|f(x) - E|^2 \leqslant l\varepsilon$ for $x \in (-1, 0)$. As a consequence, we have

$$\int_{-l}^l |f(x) - E|^2 \, dx \leqslant 2l^2\varepsilon.$$

This settles the case of $d = 1$.

Now assume that $d \geqslant 2$, and that the lemma is true when the dimension is $d - 1$, and let us prove it for the dimension d. Let $f : (-l,l)^d \to \mathbb{R}$ be a smooth function, such that

$$\int_{(-l,l)^d} |\nabla f|^2 \, dg_{std}^d \leqslant \varepsilon.$$

For any $t \in (-l,l)$ define the function $g_t : (-l,l)^{d-1} \to \mathbb{R}$ as

$$g_t(x_2, ..., x_n) = f(t, x_2, x_3, ..., x_n).$$

Then

$$\int_{-l}^{l}\left(\int_{(-l,l)^{d-1}} |\nabla g_t(x_2,...,x_n)|^2\, dx_2 \dots dx_d\right) dt$$
$$\leqslant \int_{(-l,l)^d} |\nabla f|^2\, dg_{std}^d \leqslant \varepsilon.$$

Hence there exists some $t_1 \in (-l,l)$ such that

$$\int_{(-l,l)^{d-1}} |\nabla g_{t_1}(x_2,...,x_n)|^2\, dx_2 \dots dx_d \leqslant \frac{\varepsilon}{2l}.$$

Now, by the induction hypothesis applied to g_{t_1}, there exists $E \in \mathbb{R}$ such that

$$\int_{(-l,l)^{d-1}} |g_{t_1}(x_2,...,x_n) - E|^2\, dx_2 \dots dx_d$$

$$= \int_{(-1,1)^{d-1}} |f(t_1,x_2,...,x_n) - E|^2\, dx_2 \dots dx_d \leqslant C\frac{\varepsilon}{2l}l^2 = \frac{C}{2}\varepsilon l.$$

Now let $t \in (-l,l)$ be different from t_1. WLOG assume that $t \in (t_1,l)$. Then we have

$$\int_{(-l,l)^{d-1}} |f(t,x_2,...,x_n) - f(t_1,x_2,...,x_n)|^2\, dx_2 \dots dx_d$$

$$= \int_{(-l,l)^{d-1}} \left|\int_{t_1}^{t} \frac{\partial}{\partial x_1} f(x_1,x_2,...,x_n)\, dx_1\right|^2 dx_2 \dots dx_d$$

$$\leqslant \int_{(-l,l)^{d-1}} (t-t_1)\left(\int_{t_1}^{t} \left|\frac{\partial}{\partial x_1} f(x_1,x_2,...,x_n)\right|^2 dx_1\right) dx_2 \dots dx_d$$

$$\leqslant 2l \int_{(-l,l)^{d-1}} \left(\int_{t_1}^{t} \left|\frac{\partial}{\partial x_1} f(x_1,x_2,...,x_n)\right|^2 dx_1\right) dx_2 \dots dx_d$$

$$\leqslant 2l \int_{(-l,l)^{d-1}} \left(\int_{t_1}^{t} |\nabla f(x_1,x_2,...,x_n)|^2\, dx_1\right) dx_2 \dots dx_d$$

$$\leqslant 2l \int_{(-l,l)^{d}} |\nabla f(x_1,x_2,...,x_n)|^2\, dx_1\, dx_2 \dots dx_d \leqslant 2\varepsilon l.$$

Hence we conclude that

$$\int_{(-l,l)^{d-1}} |f(t,x_2,...,x_n) - E|^2\, dx_2 \dots dx_d$$

$$\leqslant 2\Big(\int_{(-l,l)^{d-1}} |f(t,x_2,...,x_n) - f(t_1,x_2,...,x_n)|^2\, dx_2 \dots dx_d$$
$$+ \int_{(-l,l)^{d-1}} |f(t_1,x_2,...,x_n) - E|^2\, dx_2 \dots dx_d\Big)$$

$$\leqslant 2\left(2\varepsilon l + \frac{C}{2}\varepsilon l\right) = (C+4)\varepsilon l.$$

Similarly, one checks that

$$\int_{(-l,l)^{d-1}} |f(t, x_2, ..., x_n) - E|^2 \, dx_2 \ldots dx_d \leqslant (C+4)\varepsilon l$$

also for $t \in (-l, t_1)$. Finally, integrating over t, we get

$$\begin{aligned}\int_{-l}^{l} \int_{(-l,l)^{d-1}} & |f(t, x_2, ..., x_n) - E|^2 \, dx_2 \ldots dx_d \, dt \\ &= \int_{(-l,l)^d} |f - E|^2 dg_{std}^d \\ &\leqslant 2(C+4)\varepsilon l^2.\end{aligned}$$

□

From Lemmas 3.1 and 3.2 we conclude the following

Corollary 3.3. *Let $r > 0$. Consider the domain $U = (-r, r) \times B_r^{d-1}(0) \subset \mathbb{R}^d$, where*

$$B_r^{d-1}(0) = \{x \in \mathbb{R}^{d-1} \,|\, |x| < r\} \subset \mathbb{R}^{d-1},$$

and the metric g_{std} on U that comes from the euclidean metric on $\mathbb{R}^d$. Let $\varepsilon > 0$, and let $f : U \to \mathbb{R}$ be a smooth function, such that

$$\int_U |\nabla f|^2 \, dg_{std}^d \leqslant \varepsilon.$$

Then there exists some $E \in \mathbb{R}$, such that for any $-r < x_1 < r$ we have

$$\int_{\{x_1\}\times B_r^{d-1}(0)} |f - E|^2 dg_{std}^{d-1} \leqslant C\varepsilon r,$$

where $C = C(d)$.

Remark 3.4. *Lemma 3.2 and Corollary 3.3 are slight refinements of the Poincaré lemma for the cube $(-l, l)^d \subset \mathbb{R}^d$, and for the domain $U = (-r, r) \times B_r^{d-1}(0) \subset \mathbb{R}^d$, respectively. In fact, we could give a shorter proof of Corollary 3.3, by referencing to the Poincaré lemma. However, we decided not to do so, for the sake of keeping the exposition self-contained.*

Lemma 3.5. *Let $r, \epsilon > 0$, and let $f : \overline{B}_r^d(0) \to \mathbb{R}$ be a continuous function which is smooth on $B_r^d(0)$, and such that for some $E \in \mathbb{R}$ we have*

$$\int_{S_r^{d-1}} |f - E|^2 dg_{std}^{d-1} \leqslant \epsilon,$$

$$\int_{B_r^d(0)} |\nabla f|^2 dg_{std}^d \leqslant 1.$$

Then

$$\int_{B_r^d(0)} |f-E|^2 dg_{std}^d \leqslant C(r^2+\epsilon r),$$

for some constant $C = C(d)$.

Proof of Lemma 3.5. Define the function $F : \overline{B}_1^d(0) \to \mathbb{R}$ as $F(x) = r^{\frac{d}{2}} f(rx)$, and denote $E' = r^{\frac{d}{2}} E$. Then we have

$$\int_{S_1^{d-1}} |F-E'|^2 dg_{std}^{d-1} = r \int_{S_r^{d-1}} |f-E|^2 dg_{std}^{d-1} \leqslant r\epsilon,$$

and

$$\int_{B_1^d(0)} |\nabla F|^2 dg_{std}^d = r^2 \int_{B_r^d(0)} |\nabla f|^2 dg_{std}^d \leqslant r^2.$$

Now, by Lemma 3.1, there exists a bi-Lipschitz homeomorphism $[-1,1]^d \to \overline{B}_1^d(0)$, such that its restriction to $(-1,1)^d$ is a diffeomorphism onto the open unit ball $B_1^d(0)$, and such that its restriction to the union of $(-1,1)^d$ with all of open faces of $(-1,1)^d$, is a diffeomorphism onto the image, and let $C = C(d) \geqslant 1$ be a bi-Lipschitz constant of this homeomorphism. Denote by $H : [-1,1]^d \to \mathbb{R}$ the pullback of F under this homeomorphism. Then we obtain

$$\int_{\partial[-1,1]^d} |H-E'|^2 dg_{std}^{d-1} \leqslant C^{d-1} r\epsilon, \tag{3.1}$$

and

$$\int_{(-1,1)^d} |\nabla H|^2 dg_{std}^d \leqslant C^{d+2} r^2. \tag{3.2}$$

Now, due to (3.2) and Lemma 3.2, we conclude that there exists some $E'' \in \mathbb{R}$, such that for any $-1 < x_1 < 1$ we have

$$\int_{\{x_1\}\times(-1,1)^{d-1}} |H-E''|^2 dg_{std}^{d-1} \leqslant C' r^2, \tag{3.3}$$

and moreover we have

$$\int_{(-1,1)^d} |H-E''|^2 dg_{std}^d \leqslant C' r^2, \tag{3.4}$$

where $C' = C'(d)$.

Then on one hand, from (3.3) and the uniform continuity of H we get

$$\int_{\{1\}\times(-1,1)^{d-1}} |H-E''|^2 dg_{std}^{d-1} \leqslant C' r^2. \tag{3.5}$$

On the other hand, from (3.1) we get

$$\int_{\{1\}\times(-1,1)^{d-1}} |H-E'|^2 dg_{std}^{d-1} \leqslant C^{d-1} r\epsilon. \tag{3.6}$$

Hence from (3.5), (3.6) we conclude

$$2^{d-1}|E'-E''|^2 \leqslant 2\Big(\int_{\{1\}\times(-1,1)^{d-1}} |H-E''|^2 dg_{std}^{d-1} + \int_{\{1\}\times(-1,1)^{d-1}} |E'-H|^2 dg_{std}^{d-1}\Big)$$
$$\leqslant 2C'r^2 + 2C^{d-1}r\epsilon.$$

Therefore, by (3.4) we get

$$\int_{(-1,1)^d} |H-E'|^2 dg_{std}^d \leqslant 2\Big(\int_{(-1,1)^d} |H-E''|^2 dg_{std}^d + 2^d|E''-E'|^2\Big)$$
$$\leqslant 2(C'r^2 + 4C'r^2 + 4C^{d-1}r\epsilon) = 10C'r^2 + 8C^{d-1}r\epsilon.$$

Now, going back to the function F, we conclude

$$\int_{B_1^d(0)} |F-E'|^2 dg_{std}^d \leqslant C^d(10C'r^2 + 8C^{d-1}r\epsilon) \leqslant C''(r^2 + r\epsilon),$$

for $C'' = \max(10C^dC', 8C^{2d-1})$. Therefore we finally get

$$\int_{B_r^d(0)} |f-E|^2 dg_{std}^d \leqslant C''(r^2 + \epsilon r).$$

□

4 Proof of Proposition 1.11

First of all, WLOG, in the proof of this proposition, we may assume that ϵ is small enough.

Restricting to $S^{2n-1} \subset \mathbb{R}^{2n}$, we denote the Hopf vector field by $\tilde{H}(x) = Jx$. We can find an isometry $\tilde{\alpha} : S^{2n-1} \to S^{2n-1}$ of the sphere, such that the pushforward $\tilde{\alpha}_*\tilde{H}$ of the Hopf vector field $\tilde{H}$, is transversal to the Hopf vector field $\tilde{H}$ at some point $x_1 \in S^{2n-1}$, and hence for some spherical cap $S = B_\rho^S(x_1) \subset S^{2n-1}$ around x_1, the vector field $\tilde{\alpha}_*\tilde{H}$ is transversal to the Hopf vector field $\tilde{H}$ on the closure $\overline{S}$. Note that the radius ρ of the cap can be chosen to depend only on the dimension $2n-1$. Consider the spherical cap $B_{\frac{\rho}{3}}^S(x_1) \subset S^{2n-1}$, and choose a maximal set of points $\mathcal{P} \subset B_{\frac{\rho}{3}}^S(x_1)$ with the property that the spherical distance between any 2 distinct points of $\mathcal{P}$ is greater or equal to 4ϵ. Since the spherical balls of radius 4ϵ centered at the points of $\mathcal{P}$, cover $B^S_{\frac{\rho}{3}}(x_1)$, we conclude that the cardinality of $\mathcal{P}$ satisfies

$$|\mathcal{P}| \geqslant \frac{\mathrm{Vol}(B^S_{\frac{\rho}{3}}(x_1))}{\mathrm{Vol}(B^S_{4\epsilon})} \geqslant \frac{c(n)}{\epsilon^{2n-1}},$$

where $\mathrm{Vol}(\cdot)$ is evaluated with respect to the volume density g^{2n-1}_{std}.

Lemma 4.1. *There exists some $T > 0$ and a smooth volume preserving flow $\tilde{\psi}^t$, $t \in [0, T]$ on S^{2n-1}, generated by a time dependent vector field $\tilde{Y}^t$, $t \in [0, T]$ on S^{2n-1}, such that $\tilde{Y}^t$ is sufficiently C^0-close to the pushforward $\tilde{\alpha}_* \tilde{H}$ of the Hopf vector field on S^{2n-1}, and such that the flow $\tilde{\psi}^t$ satisfies the following:*
Take any $x \in \mathcal{P}$, and denote by $\chi : S^{2n-1} \to \mathbb{R}$ the characteristic function of $B^S_\epsilon(x)$. Then there exist some $t_1, t_2, ..., t_N \in (0, T)$ such that

$$\frac{1}{N} \sum_{k=1}^{N} (\tilde{\psi}^{t_k})^* \chi \geqslant c'(n) \frac{\mathrm{Vol}(B^S_\epsilon(x))}{\mathrm{Vol}(S^{2n-1})}$$

on S^{2n-1}, where $\mathrm{Vol}(\cdot)$ is evaluated with respect to the volume density g^{2n-1}_{std}, and $c'(n) > 0$ is some positive constant that depends only on n.

Proof of Lemma 4.1. Along the proof we will use the notation χ_z for the characteristic function $\chi_z : S^{2n-1} \to \mathbb{R}$ of $B^S_\epsilon(z) \subseteq S^{2n-1}$, where $z \in S^{2n-1}$.

Let $\mathcal{Q}$ be a maximal set of points of S^{2n-1} with the property that the spherical distance between any two points of $\mathcal{Q}$ is at least ϵ. Then first of all, spherical balls of radius ϵ centered at points of $\mathcal{Q}$ cover S^{2n-1}. Secondly, spherical balls of radius $\frac{\epsilon}{2}$ centered at the points of $\mathcal{Q}$, do not intersect pairwise, which means that

$$|\mathcal{Q}| \leqslant \frac{\mathrm{Vol}(S^{2n-1})}{\mathrm{Vol}(B^S_{\frac{\epsilon}{2}})}.$$

Therefore we have

$$\frac{1}{|\mathcal{Q}|} \sum_{y \in \mathcal{Q}} \chi_y \geqslant \frac{1}{|\mathcal{Q}|} \geqslant \frac{\mathrm{Vol}(B^S_{\frac{\epsilon}{2}})}{\mathrm{Vol}(S^{2n-1})} \geqslant c'(n) \frac{\mathrm{Vol}(B^S_\epsilon)}{\mathrm{Vol}(S^{2n-1})}$$

on S^{2n-1}, where $c'(n)$ depends only on n.

Now let $x \in \mathcal{P}$ and $y \in \mathcal{Q}$ be any two points. Then there clearly exists a smooth flow $\tilde{\phi}^t_{x,y} : S^{2n-1} \to S^{2n-1}$, $t \in [0, 1]$, consisting of isometries of S^{2n-1}, such that $\tilde{\phi}^t_{x,y}$ is the identity map when t is sufficiently close to 0 or 1, and such that

$$(\tilde{\phi}^{\frac{1}{2}}_{x,y})^* \chi_x = \chi_y.$$

Denote by $\tilde{\xi}^t : S^{2n-1} \to S^{2n-1}$ the flow of $\tilde{\alpha}_* \tilde{H}$ - this is also a flow of isometries of S^{2n-1}, and we have that $\tilde{\xi}^{2\pi}$ is the identity diffeomorphism of S^{2n-1}. Now take $M_{x,y} \in \mathbb{N}$ to be sufficiently large, and define the flow $\tilde{\psi}^t_{x,y} : S^{2n-1} \to S^{2n-1}$, $t \in [0, 4M_{x,y}\pi]$, as

$$\tilde{\psi}^t_{x,y} = \tilde{\xi}^t \circ \tilde{\phi}_{x,y}^{\frac{t}{4M_{x,y}\pi}}.$$

If we take $M_{x,y}$ to be sufficiently large, then the vector field that generates the flow $\tilde{\psi}^t_{x,y}$ will be sufficiently C^0-close to $\tilde{\alpha}_* \tilde{H}$. In addition, we have that $\tilde{\psi}^t_{x,y}$ equals to $\tilde{\xi}^t$ when t is close to the endpoints 0 and $4M_{x,y}\pi$, so that in particular $\tilde{\psi}^{4M_{x,y}\pi}_{x,y}$ is the identity diffeomorphism of S^{2n-1}, and also we have that

$$(\tilde{\psi}^{2M_{x,y}\pi}_{x,y})^* \chi_x = \chi_y.$$

Now define the flow $\tilde{\psi}^t$, $t \in [0, T]$ to be the concatenation of flows $\tilde{\psi}^t_{x,y}$, when we run over all $x \in \mathcal{P}$ and $y \in \mathcal{Q}$. We claim that $\tilde{\psi}^t$ is a desired flow. Indeed, first of all it is smooth since $\tilde{\psi}^t_{x,y}$ equals to $\tilde{\xi}^t$ when t is close to the endpoints 0 and $4M_{x,y}\pi$, for every $x \in \mathcal{P}$ and $y \in \mathcal{Q}$. Secondly, the vector field that generates $\tilde{\psi}^t$, is sufficiently C^0-close to $\tilde{\alpha}_* \tilde{H}$. Fixing any $x \in \mathcal{P}$, we have that for any $y \in \mathcal{Q}$ there exists $t_y \in (0, T)$ such that $(\tilde{\psi}^{t_y})^* \chi_x = \chi_y$. Therefore we have

$$\frac{1}{|\mathcal{Q}|} \sum_{y \in \mathcal{Q}} (\tilde{\psi}^{t_y})^* \chi_x = \frac{1}{|\mathcal{Q}|} \sum_{y \in \mathcal{Q}} \chi_y \geqslant c'(n) \frac{\mathrm{Vol}(B^S_\epsilon)}{\mathrm{Vol}(S^{2n-1})} = c'(n) \frac{\mathrm{Vol}(B^S_\epsilon(x))}{\mathrm{Vol}(S^{2n-1})}$$

on S^{2n-1}. Finally, the flow $\tilde{\psi}^t$ consists of isometries of S^{2n-1}, and hence is volume preserving.

□

Consider the time dependent vector field $\tilde{Y}^t$ and its flow $\tilde{\psi}^t$ on S^{2n-1}, guaranteed by Lemma 4.1. Since the vector field $\tilde{Y}^t$ is sufficiently C^0-close to $\tilde{\alpha}_* \tilde{H}$, then $\tilde{Y}^t$ is also transversal to the Hopf vector field $\tilde{H}$ on the closure $\overline{S}$. Choose a sufficiently small $\delta > 0$, and denote

$$r' = R - \epsilon,$$

$$r = r' - T\delta = R - \epsilon - T\delta.$$

Clearly, if ϵ and δ are small enough, then $r > \frac{R}{2}$. Define the (time independent) vector field Y_δ on $\overline{D^{2n}_{r,r'}}$, which has the form

$$Y_\delta(r' - \delta t, \theta) = -\delta\theta + (r' - \delta t)\tilde{Y}^t(\theta),$$

where $t \in [0, T]$ and $\theta \in S^{2n-1}$. Consider a smooth function $a : \mathbb{R} \to \mathbb{R}$, such that $a(t) \geqslant 1$ for any $t \in \mathbb{R}$, such that $a(t) = 1$ for any $t \notin (\frac{\delta^2}{2}, T - \frac{\delta^2}{2})$, and such that $a(t)$ is large enough on $[\delta^2, T - \delta^2]$. Now define $b : D^{2n}_{r,r'} \to \mathbb{R}$ as

$$b(x) = a\left(\frac{r' - |x|}{\delta}\right).$$

Let us give the definition of the metric g on $D^{2n}_{r,R}$. On $D^{2n}_{r,R} \setminus D^{2n}_{r,r'}$ we set $g = g_{std}$. Now consider $D^{2n}_{r,r'}$. Looking at the g_{std}-orthogonal decomposition

$$TD^{2n}_{r,r'} = Span(Y_\delta) \oplus Span(JY_\delta) \oplus L,$$

we define

$$g|_x = b(x)^{-1} g_{std}|_x \oplus b(x) g_{std}|_x \oplus g_{std}|_x$$

for any $x \in D^{2n}_{r,r'}$.

Our main statement is the following:

Claim 4.2. *If we pick sufficiently small δ, and then choose the function $a(\cdot)$ to be large enough on $[\delta^2, T-\delta^2]$, then the constructed metric g will satisfy the following: For any smooth function $f : D^{2n}_{r,R} \to \mathbb{R}$ with*

$$\int_{D^{2n}_{r,R}} \|\nabla_g f\|^2_g \, dg^{2n}_{std} \leqslant 1,$$

there exists some $E \in \mathbb{R}$, such that for any $r < u < R$ we have

$$\int_{S^{2n-1}_u} |f - E|^2 \, dg^{2n-1}_{std} \leqslant C\epsilon,$$

where $C = C(n, R)$ depends only on n and R.

The rest of the proof of Proposition 1.11 will be devoted for proving Claim 4.2. In the sequel we will assume that we have chosen δ to be small enough, and the function $a(\cdot)$ to be sufficiently large on $[\delta^2, T-\delta^2]$.

We denote by ψ^t_δ the flow of the vector field Y_δ. In polar coordinates we have

$$\psi^t_\delta(r', \theta) = (r' - \delta t, \tilde{\psi}^t(\theta)),$$

for $t \in [0, T]$.

Based on Corollary 3.3 (section 3), we are able to prove the following

Lemma 4.3. *Let $f : D^{2n}_{r,R} \to \mathbb{R}$ be a smooth function, satisfying*

$$\int_{D^{2n}_{r,R}} \|\nabla_g f\|^2_g \, dg^{2n}_{std} \leqslant 1.$$

Then there exists a point $x_2 \in \mathcal{P}$ such that for some $E \in \mathbb{R}$ we have

$$\frac{1}{\mathrm{Vol}(r' B^S_{2\epsilon}(x_2))} \int_{r' B^S_{2\epsilon}(x_2)} |f - E|^2 \, dg^{2n-1}_{std} \leqslant C\epsilon,$$

where $C = C(n, R)$.

Proof of Lemma 4.3. For $x \in S^{2n-1}$ denote $U_{x,\epsilon} = (r', R) \cdot B^S_{2\epsilon}(x) \subseteq D^{2n}_{r',R}$. Note that in polar coordinates $U_{x,\epsilon}$ is

$$U_{x,\epsilon} = (R - \epsilon, R) \times B^S_{2\epsilon}(x) \subset (0, \infty) \times S^{2n-1}.$$

Define the following domain in $\mathbb{R}^{2n}$:

$$U_\epsilon = (-\epsilon, \epsilon) \times B^{2n-1}_\epsilon(0) = (-\epsilon, \epsilon) \times \{y \in \mathbb{R}^{2n-1} \mid |y| < \epsilon\} \subset \mathbb{R}^{2n}.$$

It is easy to see that for small enough ϵ, for any $x \in S^{2n-1}$ there exists a diffeomorphism $\Psi_x : U_{x,\epsilon} \to U_\epsilon$, such that we have

$$\frac{1}{C^2} g_{std} \leqslant \Psi_x^* g_{std} \leqslant C^2 g_{std},$$

where $C = C(n, R)$, and such that in polar coordinates the map Ψ_x has the form

$$\Psi_x(u, \theta) = \left(2\left(R - \frac{\epsilon}{2} - u\right), \tilde{\Psi}_x(\theta)\right),$$

for some diffeomorphism $\tilde{\Psi}_x : B^S_{2\epsilon}(x) \to B^{2n-1}_\epsilon(0)$.

Now, since the distance between any two distinct points of $\mathcal{P}$ is greater or equal to 4ϵ, it follows that all $U_{x,\epsilon}$, for $x \in \mathcal{P}$, do not intersect pairwise. Therefore we conclude that

$$\sum_{x \in \mathcal{P}} \int_{U_{x,\epsilon}} |\nabla f|^2 \, dg^{2n}_{std} \leqslant \int_{D^{2n}_{r',R}} |\nabla f|^2 \, dg^{2n}_{std} = \int_{D^{2n}_{r',R}} \|\nabla_g f\|^2_g \, dg^{2n}_{std} \leqslant \int_{D^{2n}_{r,R}} \|\nabla_g f\|^2_g \, dg^{2n}_{std} \leqslant 1.$$

Keeping in mind that $|\mathcal{P}| \geqslant \frac{c(n)}{\epsilon^{2n-1}}$, we conclude that there exists some $x_2 \in \mathcal{P}$ such that

$$\int_{U_{x_2,\epsilon}} |\nabla f|^2 \, dg^{2n}_{std} \leqslant \frac{1}{|\mathcal{P}|} \leqslant C' \epsilon^{2n-1},$$

where $C' = C'(n)$. Now look at $\Psi := \Psi_{x_2} : U_{x_2,\epsilon} \to U_\epsilon$. Define the function $h : U_\epsilon \to \mathbb{R}$ as $h = f \circ \Psi^{-1}$. Then we have

$$\int_{U_\epsilon} |\nabla h|^2 \, dg^{2n}_{std} \leqslant C^{2n+2} C' \epsilon^{2n-1}.$$

Now, applying Corollary 3.3, we conclude that there exists $E \in \mathbb{R}$ such that

$$\int_{\{x_1\} \times B^{2n-1}_\epsilon(0)} |h - E|^2 dg^{2n-1}_{std} \leqslant C'' \epsilon^{2n}$$

for any $x_1 \in (-\epsilon, \epsilon)$, where $C'' = C''(n, R)$. Going back to $f = h \circ \Psi$, we get that for any $u \in (r', R)$ we have

$$\int_{uB^S_{2\epsilon}(x_2)} |f - E|^2 \, dg^{2n-1}_{std} \leqslant C^{2n-1} C'' \epsilon^{2n}.$$

Hence by continuity we have

$$\int_{r' B^S_{2\epsilon}(x_2)} |f - E|^2 \, dg^{2n-1}_{std} \leqslant C^{2n-1} C'' \epsilon^{2n}.$$

Finally, keeping in mind that $\mathrm{Vol}(r' B^S_{2\epsilon}(x_2)) \geqslant c' \epsilon^{2n-1}$ for $c' = c'(n, R)$, we conclude the statement of the lemma. $\square$

Lemma 4.4. *Let f and $x_2 \in \mathcal{P}$ be as in Lemma 4.3. Denote $B^t_\epsilon = (r' - \delta t) B^S_\epsilon(x_2)$, for $t \in (0, T)$. Then provided that δ is small enough, we will have*

$$\frac{1}{\mathrm{Vol}(B^t_\epsilon)} \int_{B^t_\epsilon} |f - E|^2 \, dg^{2n-1}_{std} \leqslant C\epsilon,$$

for all $t \in (0, T)$. (In this lemma the constant $C = C(n, R)$ might be different from the one in Lemma 4.3).

Proof of Lemma 4.4. For some $x \in \mathbb{R}^{2n} \setminus \{0\}$ and a nonzero tangent vector $Y \in T_x(\mathbb{R}^{2n}) \setminus \{0\}$, denote by $Z = \iota(x; Y) \in T_x(\mathbb{R}^{2n})$ the vector that satisfies

$$\langle Z, Y \rangle = \langle Z, JY \rangle = 0,$$

$$\langle Z, X \rangle = 1,$$

and that minimizes the Euclidean distance $|Z - X|$, where $X = -\frac{x}{|x|} \in \mathbb{R}^{2n} \cong T_x(\mathbb{R}^{2n})$. It is easy to see that $\iota(x; Y)$ is well defined when $x \notin Span(Y, JY)$, which is equivalent to $Y \notin Span(x, Jx)$, and we will apply ι only on that case. Clearly, $\iota(x; Y)$ depends on x and Y in a smooth way, on its domain of definition.

Define the vector field

$$X_\delta(x) = \iota(x; Y_\delta(x)),$$

on $x \in [r, r'] \cdot \overline{S} = [r, r'] \cdot \overline{B^S_\rho(x_1)}$ (for small δ, $\iota(x; Y_\delta(x))$ is well defined on $x \in [r, r'] \cdot \overline{S} = [r, r'] \cdot \overline{B^S_\rho(x_1)}$). Note that since X_δ is orthogonal to $Span(Y_\delta, JY_\delta)$ on $(r, r'] \cdot \overline{S} = (r, r'] \cdot \overline{B^S_\rho(x_1)}$, it follows that

$$\|X_\delta\|_g = |X_\delta|$$

on $(r, r'] \cdot \overline{S} = (r, r'] \cdot \overline{B^S_\rho(x_1)}$.

Denote by σ^s_δ the flow of the vector field X_δ. We have

$$X_\delta(r' - \delta t, \theta) = -\theta + (r' - \delta t) \tilde{X}^t_\delta(\theta),$$

for $t \in [0, T]$, where $\tilde{X}^t_\delta(\theta)$ is a time-dependent vector field which is tangent to the unit sphere S^{2n-1}. Note that $\tilde{X}^t_\delta(\theta)$ is well defined for $\theta \in \overline{S} = \overline{B^S_\rho(x_1)}$ and $t \in [0, T]$, when δ is small enough. For the flow σ^s_δ of X_δ we have

$$\sigma^s_\delta(r', \theta) = (r' - s, \tilde{\sigma}^s_\delta(\theta)),$$

for $s \in [0, \delta T]$, where $\tilde{\sigma}_\delta^s(\theta)$ is a flow on $S = B_\rho^S(x_1) \subseteq S^{2n-1}$ (probably, only partially defined), which is generated by the vector field $\tilde{X}_\delta^{\frac{s}{\delta}}(\theta)$, $s \in [0, \delta T]$.

Since ϵ is small and $\mathcal{P} \subset B_{\frac{\rho}{3}}^S(x_1)$, then for any $x \in \mathcal{P}$ we have $B_{2\epsilon}^S(x) \subseteq B_{\frac{\rho}{2}}^S(x_1)$, and in particular, $B_{2\epsilon}^S(x_2) \subseteq B_{\frac{\rho}{2}}^S(x_1)$. Look at the vector field

$$X_\delta(r' - \delta t, \theta) = -\theta + (r' - \delta t)\tilde{X}_\delta^t(\theta).$$

The flow of X_δ satisfies

$$\sigma_\delta^s(r', \theta) = (r' - s, \tilde{\sigma}_\delta^s(\theta)),$$

for $s \in [0, \delta T]$. The flow $\tilde{\sigma}_\delta^s$ is generated by the vector field $\tilde{X}_\delta^{\frac{s}{\delta}}$, when $s \in [0, \delta T]$. Let us make a time rescaling, concentrating on the time parameter $t = \frac{s}{\delta}$. Define the flow $\tilde{\varsigma}_\delta^t$ on $S = B_\rho^S(x_1) \subseteq S^{2n-1}$ (which could be well defined only on a part of S), as $\tilde{\varsigma}_\delta^t(\theta) = \tilde{\sigma}_\delta^{\delta t}(\theta)$, when $t \in [0, T]$. Then the flow $\tilde{\varsigma}_\delta^t$ is generated by the vector field $\delta \tilde{X}_\delta^t$. Up till now we had a family of vector fields $\delta \to \tilde{X}_\delta^t$ on $\overline{S} \subseteq S^{2n-1}$, defined for small $\delta > 0$. However, it is quite easy to see that we can extend this family also to $\delta = 0$ in a natural way. Indeed, we have that

$$\tilde{X}_\delta^t(\theta) = \theta + \iota((r' - \delta t, \theta); Y_\delta(r' - \delta t, \theta)) = \theta + \iota((r' - \delta t)\theta; -\delta\theta + (r' - \delta t)\tilde{Y}^t(\theta)).$$

Hence if we define

$$\tilde{X}_0^t(\theta) = \theta + \iota(r'\theta; r'\tilde{Y}^t(\theta)),$$

then $\tilde{X}_\delta^t(\theta)$ is well defined for small $\delta \geqslant 0$, for $t \in [0, T]$ and for $\theta \in \overline{S}$, and depends on δ, t, and θ, in a smooth way.

So we get that the flow $\tilde{\varsigma}_\delta^t$ is generated by the vector field $\delta \tilde{X}_\delta^t$, and that the family of vector fields $\tilde{X}_\delta^t(\theta)$ on $\overline{S}$, depends on small $\delta \geqslant 0$, on $\theta \in \overline{S}$, and on $t \in [0, T]$, in a smooth way. From here we can conclude the following:
1) For δ small enough, the flow $\tilde{\varsigma}_\delta^t(\theta)$ is well defined for $\theta \in B_{\frac{\rho}{2}}^S(x_1)$ and $t \in [0, T]$, and we have $\tilde{\varsigma}_\delta^t(\theta) \in S = B_\rho^S(x_1)$ for any $\theta \in B_{\frac{\rho}{2}}^S(x_1)$ and $t \in [0, T]$.
2) Moreover, if δ is small enough, then for any $x \in B_{\frac{\rho}{3}}^S(x_1)$ we will have $\tilde{\varsigma}_\delta^t(B_{2\epsilon}^S(x)) \supseteq B_\epsilon^S(x)$ for all $t \in [0, T]$.
3) Finally, if we choose sufficiently small δ, then for any $t \in [0, T]$, the Jacobian of the map $B_{\frac{\rho}{2}}^S(x_1) \to S^{2n-1}$ given by $\theta \mapsto \tilde{\varsigma}_\delta^t(\theta)$, will be arbitrarily close to 1, uniformly on $\theta \in B_{\frac{\rho}{2}}^S(x_1)$ and $t \in [0, T]$. In particular, if δ is small enough, then the Jacobian of $\tilde{\varsigma}_\delta^t$ lies between $\frac{1}{2}$ and 2, at any point $\theta \in B_{\frac{\rho}{2}}^S(x_1)$, for any $t \in [0, T]$.

Now assume that δ is small enough so that 1), 2), 3) above are satisfied. Then, translating these properties to the flow of $\tilde{\sigma}_\delta^s(\theta)$, we get:
1') The flow $\tilde{\sigma}_\delta^s(\theta)$ is well defined for $\theta \in B_{\frac{\rho}{2}}^S(x_1)$ and $s \in [0, \delta T]$, and we have $\tilde{\sigma}_\delta^s(\theta) \in S = B_\rho^S(x_1)$ for any $\theta \in B_{\frac{\rho}{2}}^S(x_1)$ and $s \in [0, \delta T]$.
2') For any $x \in B_{\frac{\rho}{3}}^S(x_1)$ we have $\tilde{\sigma}_\delta^s(B_{2\epsilon}^S(x)) \supseteq B_\epsilon^S(x)$ for all $s \in [0, \delta T]$.

3') For any $s \in [0, \delta T]$, the Jacobian of the map $B^S_{\frac{\rho}{2}}(x_1) \to S^{2n-1}$ given by $\theta \mapsto \tilde{\sigma}^s_\delta(\theta)$, lies between $\frac{1}{2}$ and 2, at any $\theta \in B^S_{\frac{\rho}{2}}(x_1)$.

We are now ready to prove our lemma. Define the function $F : [0, \delta T) \times B^S_{2\epsilon}(x_2) \to \mathbb{R}$ as

$$F(s, \theta) = f(\sigma^s_\delta(r', \theta)) = f(r' - s, \tilde{\sigma}^s_\delta(\theta)).$$

For any $0 < s_1 < \delta T$ we have

$$\begin{aligned}
\int_{B^S_{2\epsilon}(x_2)} |f(r' - s_1, \tilde{\sigma}^{s_1}_\delta(\theta)) - f(r', \theta)|^2 \, d\theta &= \int_{B^S_{2\epsilon}(x_2)} |F(s_1, \theta) - F(0, \theta)|^2 \, d\theta \\
= \int_{B^S_{2\epsilon}(x_2)} \left| \int_0^{s_1} \frac{\partial}{\partial s} F(s, \theta) \, ds \right|^2 d\theta &\leqslant \int_{B^S_{2\epsilon}(x_2)} s_1 \int_0^{s_1} \left| \frac{\partial}{\partial s} F(s, \theta) \right|^2 ds \, d\theta \\
&= s_1 \int_{B^S_{2\epsilon}(x_2)} \int_0^{s_1} |L_{X_\delta} f(\sigma^s_\delta(r', \theta))|^2 \, ds \, d\theta \\
&\leqslant \delta T \int_{B^S_{2\epsilon}(x_2)} \int_0^{s_1} \|\nabla_g f(\sigma^s_\delta(r', \theta))\|_g^2 \cdot \|X_\delta(\sigma^s_\delta(r', \theta))\|_g^2 \, ds \, d\theta \\
&= \delta T \int_{B^S_{2\epsilon}(x_2)} \int_0^{s_1} \|\nabla_g f(\sigma^s_\delta(r', \theta))\|_g^2 \cdot |X_\delta(\sigma^s_\delta(r', \theta))|^2 \, ds \, d\theta.
\end{aligned}$$

We have

$$X_\delta(u, \theta) = \iota((u, \theta); Y_\delta(u, \theta)) = \iota(u\theta; -\delta\theta + u\tilde{Y}^{\frac{r'-u}{\delta}}(\theta)),$$

for $u \in [r, r']$ and $\theta \in \overline{S} = \overline{B^S_\rho(x_1)}$. Therefore, if $\tilde{Y}^t$ is sufficiently C^0-close to $\tilde{\alpha}_* \tilde{H}$, and if ϵ, δ are small enough, then because of continuous dependence of $\iota(\cdot\,;\cdot)$ on its arguments, we can conclude that $X_\delta(u, \theta)$ is C^0-close to $\iota(R\theta; R\tilde{\alpha}_* \tilde{H}(\theta)) = \iota(\theta; \tilde{\alpha}_* \tilde{H}(\theta))$, and hence in this case we have $|X_\delta(u, \theta)|^2 \leqslant C$ for any $u \in [r, r']$ and $\theta \in \overline{S} = \overline{B^S_\rho(x_1)}$, where $C = C(n)$. Hence returning to our chain of estimates, we get

$$\int_{B^S_{2\epsilon}(x_2)} |f(r' - s_1, \tilde{\sigma}^{s_1}_\delta(\theta)) - f(r', \theta)|^2 \, d\theta \leqslant \delta T C \int_{B^S_{2\epsilon}(x_2)} \int_0^{s_1} \|\nabla_g f(\sigma^s_\delta(r', \theta))\|_g^2 \, ds \, d\theta.$$

Now, because of 3'), the Jacobian of the map $\Phi : (0, \delta T) \times B^S_{2\epsilon}(x_2) \to D_{r,r'}$ given by $(s, \theta) \mapsto \sigma^s_\delta(r', \theta) = (r' - s, \tilde{\sigma}^s_\delta(\theta))$, is greater or equal to $\frac{(r'-s)^{2n-1}}{2}$, which is greater than $\frac{r^{2n-1}}{2}$, which in turn, is greater than $\frac{\left(\frac{R}{2}\right)^{2n-1}}{2} = \frac{R^{2n-1}}{2^{2n}}$, for small ϵ and δ. Hence returning to our chain of estimates, we get

$$\begin{aligned}
&\int_{B^S_{2\epsilon}(x_2)} |f(r' - s_1, \tilde{\sigma}^{s_1}_\delta(\theta)) - f(r', \theta)|^2 \, d\theta \\
&\leqslant \delta \frac{2^{2n} T C}{R^{2n-1}} \int_{\Phi((0, \delta T) \times B^S_{2\epsilon}(x_2))} \|\nabla_g f\|_g^2 \, dg^{2n}_{std} \qquad (4.1) \\
&\leqslant \delta \frac{2^{2n} T C}{R^{2n-1}} \int_{D^{2n}_{r,R}} \|\nabla_g f\|_g^2 \, dg^{2n}_{std} \leqslant \delta \frac{2^{2n} T C}{R^{2n-1}}.
\end{aligned}$$

Now, by Lemma 4.3 we have

$$\frac{1}{\mathrm{Vol}(r'B^S_{2\epsilon}(x_2))}\int_{r'B^S_{2\epsilon}(x_2)}|f-E|^2\,dg^{2n-1}_{std}\leqslant C'\epsilon,$$

where $C'=C'(n,R)$, and hence

$$\frac{1}{\mathrm{Vol}(B^S_{2\epsilon}(x_2))}\int_{B^S_{2\epsilon}(x_2)}|f(r',\theta)-E|^2\,d\theta\leqslant C'\epsilon. \tag{4.2}$$

Therefore from (4.1) and (4.2) we conclude that

$$\frac{1}{\mathrm{Vol}(B^S_{2\epsilon}(x_2))}\int_{B^S_{2\epsilon}(x_2)}|f(r'-s_1,\tilde{\sigma}^{s_1}_\delta(\theta))-E|^2\,d\theta$$

$$\leqslant 2\bigg(\frac{1}{\mathrm{Vol}(B^S_{2\epsilon}(x_2))}\int_{B^S_{2\epsilon}(x_2)}|f(r'-s_1,\tilde{\sigma}^{s_1}_\delta(\theta))-f(r',\theta)|^2\,d\theta$$

$$+\frac{1}{\mathrm{Vol}(B^S_{2\epsilon}(x_2))}\int_{B^S_{2\epsilon}(x_2)}|f(r',\theta)-E|^2\,d\theta\bigg)\leqslant 2\left(\delta\frac{2^{2n}TC}{R^{2n-1}\mathrm{Vol}(B^S_{2\epsilon}(x_2))}+C'\epsilon\right)\leqslant 3C'\epsilon,$$

where the latter inequality is true if δ is small enough. Now, from 3') we know that the Jacobian of the map $\tilde{\sigma}^{s_1}_\delta(\theta):B^S_{\frac{\rho}{2}}(x_1)\to S^{2n-1}$ is not greater than 2, hence we conclude

$$\frac{1}{\mathrm{Vol}(B^S_{2\epsilon}(x_2))}\int_{\tilde{\sigma}^{s_1}_\delta(B^S_{2\epsilon}(x_2))}|f(r'-s_1,\theta)-E|^2\,d\theta$$

$$\leqslant\frac{2}{\mathrm{Vol}(B^S_{2\epsilon}(x_2))}\int_{B^S_{2\epsilon}(x_2)}|f(r'-s_1,\tilde{\sigma}^{s_1}_\delta(\theta))-E|^2\,d\theta\leqslant 6C'\epsilon.$$

Because of 2') we have $\tilde{\sigma}^{s_1}_\delta(B^S_{2\epsilon}(x_2))\supseteq B^S_\epsilon(x_2)$, so we get

$$\frac{1}{\mathrm{Vol}(B^S_{2\epsilon}(x_2))}\int_{B^S_{\epsilon}(x_2)}|f(r'-s_1,\theta)-E|^2\,d\theta$$

$$\leqslant\frac{1}{\mathrm{Vol}(B^S_{2\epsilon}(x_2))}\int_{\tilde{\sigma}^{s_1}_\delta(B^S_{2\epsilon}(x_2))}|f(r'-s_1,\theta)-E|^2\,d\theta\leqslant 6C'\epsilon.$$

Therefore we finally obtain

$$\frac{1}{\mathrm{Vol}(B^S_{\epsilon}(x_2))}\int_{B^S_{\epsilon}(x_2)}|f(r'-s_1,\theta)-E|^2\,d\theta$$

$$=\frac{\mathrm{Vol}(B^S_{2\epsilon}(x_2))}{\mathrm{Vol}(B^S_{\epsilon}(x_2))}\cdot\frac{1}{\mathrm{Vol}(B^S_{2\epsilon}(x_2))}\cdot\int_{B^S_{\epsilon}(x_2)}|f(r'-s_1,\theta)-E|^2\,d\theta$$

$$\leqslant 6C'\frac{\mathrm{Vol}(B^S_{2\epsilon}(x_2))}{\mathrm{Vol}(B^S_{\epsilon}(x_2))}\epsilon\leqslant C''\epsilon,$$

for $C'' = C''(n,R)$. The latter means that for any $t \in (0,T)$, for $B^t_\epsilon = (r' - \delta t) B^S_\epsilon(x_2)$ we have

$$\frac{1}{\mathrm{Vol}(B^t_\epsilon)} \int_{B^t_\epsilon} |f - E|^2 \, dg^{2n-1}_{std} \leqslant C'' \epsilon,$$

$\square$

Lemma 4.5. *Let $f : D^{2n}_{r,R} \to \mathbb{R}$ be a smooth function satisfying*

$$\int_{D^{2n}_{r,R}} \|\nabla_g f\|^2_g \, dg^{2n}_{std} \leqslant 1.$$

Then for any $0 \leqslant t_1 < t_2 < T$ we have

$$\int_{S^{2n-1}} |f(r' - \delta t_2, \tilde{\psi}^{t_2}(\theta)) - f(r' - \delta t_1, \tilde{\psi}^{t_1}(\theta))|^2 \, d\theta \leqslant C\delta.$$

(In this lemma the constant $C = C(n,R)$ might be different from those in lemmas 4.3, 4.4).

Proof of Lemma 4.5. This Lemma is true since we compressed the neck along Y_δ.

Define the function $F : [0,T) \times S^{2n-1} \to \mathbb{R}$ as

$$F(t,\theta) = f(\psi^t_\delta(r',\theta)) = f(r' - \delta t, \tilde{\psi}^t(\theta)).$$

Then for any $0 \leqslant t_1 < t_2 < T$ we have

$$\begin{aligned}
&\int_{S^{2n-1}} |f(r' - \delta t_2, \tilde{\psi}^{t_2}(\theta)) - f(r' - \delta t_1, \tilde{\psi}^{t_1}(\theta))|^2 \, d\theta \\
&\leqslant \int_{S^{2n-1}} (t_2 - t_1) \int_{t_1}^{t_2} \left| \frac{\partial}{\partial t} F(t,\theta) \right|^2 \, dt \, d\theta \\
&= (t_2 - t_1) \int_{S^{2n-1}} \int_{t_1}^{t_2} \left| L_{Y_\delta} f(\psi^t_\delta(r',\theta)) \right|^2 \, dt \, d\theta \\
&\leqslant (t_2 - t_1) \int_{S^{2n-1}} \int_{t_1}^{t_2} \|\nabla_g f(\psi^t_\delta(r',\theta))\|^2_g \cdot \|Y_\delta(\psi^t_\delta(r',\theta))\|^2_g \, dt \, d\theta \\
&= (t_2 - t_1) \int_{S^{2n-1}} \int_{t_1}^{t_2} \|\nabla_g f(\psi^t_\delta(r',\theta))\|^2_g \cdot |Y_\delta(\psi^t_\delta(r',\theta))|^2 \cdot b(\psi^t_\delta(r',\theta))^{-2} \, dt \, d\theta \\
&= (t_2 - t_1) \int_{S^{2n-1}} \int_{t_1}^{t_2} \|\nabla_g f(\psi^t_\delta(r',\theta))\|^2_g \cdot |Y_\delta(\psi^t_\delta(r',\theta))|^2 \cdot a(t)^{-2} \, dt \, d\theta \\
&\leqslant \frac{t_2 - t_1}{\left(\min_{t \in [t_1,t_2]} a(t)\right)^2} \int_{S^{2n-1}} \int_{t_1}^{t_2} \|\nabla_g f(\psi^t_\delta(r',\theta))\|^2_g \cdot |Y_\delta(\psi^t_\delta(r',\theta))|^2 \, dt \, d\theta\,.
\end{aligned}$$

We have

$$|Y_\delta(u,\theta)|^2 = |-\delta\theta + u\tilde{Y}^{\frac{r'-u}{\delta}}(\theta)|^2 = \delta^2 + u^2|\tilde{Y}^{\frac{r'-u}{\delta}}(\theta)|^2,$$

for $u \in [r, r']$ and $\theta \in S^{2n-1}$. According to the property that $\tilde{Y}^t$ is sufficiently C^0-close to $\tilde{\alpha}_*\tilde{H}$ (Lemma 4.1), the norm $|\tilde{Y}^t(\theta)|$ is sufficiently close to 1, so we may assume that $|\tilde{Y}^t(\theta)| \leqslant 2$ for all $t \in [0,T]$ and $\theta \in S^{2n-1}$. Hence we have

$$|Y_\delta(u,\theta)|^2 = \delta^2 + u^2|\tilde{Y}^{\frac{r'-u}{\delta}}(\theta)|^2 \leqslant \delta^2 + 4u^2 \leqslant \delta^2 + 4r'^2,$$

for $u \in [r, r']$ and $\theta \in S^{2n-1}$. Therefore, returning to our chain of estimates, we get

$$\int_{S^{2n-1}} |f(r'-\delta t_2, \tilde{\psi}^{t_2}(\theta)) - f(r'-\delta t_1, \tilde{\psi}^{t_1}(\theta))|^2 \, d\theta$$

$$\leqslant \frac{(t_2-t_1)(\delta^2+4r'^2)}{\left(\min_{t\in[t_1,t_2]} a(t)\right)^2} \int_{S^{2n-1}} \int_{t_1}^{t_2} \|\nabla_g f(\psi_\delta^t(r',\theta))\|_g^2 \, dt \, d\theta \,.$$

The Jacobian of the map $(0,T) \times S^{2n-1} \to D^{2n}_{r,r'}$ given by

$$(t,\theta) \mapsto \psi_\delta^t(r',\theta) = (r'-\delta t, \tilde{\psi}^t(\theta)),$$

equals to $\delta(r'-\delta t)^{2n-1}$, since the flow $\tilde{\psi}^t$ on S^{2n-1} is volume preserving. Hence this Jacobian is greater than $\delta(r'-\delta T)^{2n-1} = \delta r^{2n-1}$ at every point of $(0,T)\times S^{2n-1}$. So returning again to our chain of estimates, we conclude that

$$\int_{S^{2n-1}} |f(r'-\delta t_2, \tilde{\psi}^{t_2}(\theta)) - f(r'-\delta t_1, \tilde{\psi}^{t_1}(\theta))|^2 \, d\theta$$

$$\leqslant \frac{t_2-t_1}{\delta\left(\min_{t\in[t_1,t_2]} a(t)\right)^2} \cdot \frac{\delta^2+4r'^2}{r^{2n-1}} \cdot \int_{D^{2n}_{r,r'}} \|\nabla_g f\|_g^2 \, dg^{2n}_{std}$$

$$\leqslant \frac{t_2-t_1}{\delta\left(\min_{t\in[t_1,t_2]} a(t)\right)^2} \cdot \frac{\delta^2+4r'^2}{r^{2n-1}} \cdot \int_{D^{2n}_{r,R}} \|\nabla_g f\|_g^2 \, dg^{2n}_{std} \leqslant \frac{t_2-t_1}{\delta\left(\min_{t\in[t_1,t_2]} a(t)\right)^2} \cdot \frac{\delta^2+4r'^2}{r^{2n-1}}.$$

Now, provided that ϵ, δ are small enough, we have

$$\frac{\delta^2+4r'^2}{r^{2n-1}} < \frac{5R^2}{(\frac{1}{2}R)^{2n-1}} =: C.$$

So we conclude that

$$\int_{S^{2n-1}} |f(r'-\delta t_2, \tilde{\psi}^{t_2}(\theta)) - f(r'-\delta t_1, \tilde{\psi}^{t_1}(\theta))|^2 \, d\theta \leqslant C \frac{t_2-t_1}{\delta\left(\min_{t\in[t_1,t_2]} a(t)\right)^2},$$

for any $0 \leqslant t_1 < t_2 < T$. In particular, for $0 \leqslant t_1 < t_2 \leqslant \delta^2$ we get

$$\int_{S^{2n-1}} |f(r'-\delta t_2, \tilde{\psi}^{t_2}(\theta)) - f(r'-\delta t_1, \tilde{\psi}^{t_1}(\theta))|^2 \, d\theta \leqslant C \frac{t_2-t_1}{\delta\left(\min_{t\in[t_1,t_2]} a(t)\right)^2}$$

$$\leqslant C\frac{\delta^2}{\delta} = C\delta.$$

Analogously, for any $T-\delta^2 \leqslant t_1 < t_2 < T$ we have

$$\int_{S^{2n-1}} |f(r'-\delta t_2, \tilde{\psi}^{t_2}(\theta)) - f(r'-\delta t_1, \tilde{\psi}^{t_1}(\theta))|^2 \, d\theta \leqslant C\frac{t_2 - t_1}{\delta\left(\min_{t\in[t_1,t_2]} a(t)\right)^2}$$

$$\leqslant C\frac{\delta^2}{\delta} = C\delta.$$

Finally, for $\delta^2 \leqslant t_1 < t_2 \leqslant T-\delta^2$ we have

$$\int_{S^{2n-1}} |f(r'-\delta t_2, \tilde{\psi}^{t_2}(\theta)) - f(r'-\delta t_1, \tilde{\psi}^{t_1}(\theta))|^2 \, d\theta \leqslant C\frac{t_2 - t_1}{\delta\left(\min_{t\in[t_1,t_2]} a(t)\right)^2}$$

$$\leqslant C\frac{T}{\delta\left(\min_{t\in[t_1,t_2]} a(t)\right)^2}.$$

If we choose the function $a(\cdot)$ to be sufficiently large on $[\delta^2, T-\delta^2]$ (it is enough to require $a(t) \geqslant \frac{\sqrt{T}}{\delta}$ for $t \in [\delta^2, T-\delta^2]$), then we will get

$$\int_{S^{2n-1}} |f(r'-\delta t_2, \tilde{\psi}^{t_2}(\theta)) - f(r'-\delta t_1, \tilde{\psi}^{t_1}(\theta))|^2 \, d\theta$$

$$\leqslant C\frac{T}{\delta\left(\min_{t\in[t_1,t_2]} a(t)\right)^2} \leqslant C\delta.$$

These three cases, combined together, imply that for any $0 \leqslant t_1 < t_2 < T$ we have

$$\int_{S^{2n-1}} |f(r'-\delta t_2, \tilde{\psi}^{t_2}(\theta)) - f(r'-\delta t_1, \tilde{\psi}^{t_1}(\theta))|^2 \, d\theta \leqslant 9C\delta.$$

$\square$

Lemma 4.6. *Let f, x_2 and B^t_ϵ be as in Lemma 4.4. Then for any $t \in (0,T)$, looking at the preimage $(\psi^t_\delta)^{-1}(B^t_\epsilon) \subset S^{2n-1}_{r'}$, we have*

$$\frac{1}{\mathrm{Vol}((\psi^t_\delta)^{-1}(B^t_\epsilon))} \int_{(\psi^t_\delta)^{-1}(B^t_\epsilon)} |f-E|^2 \, dg^{2n-1}_{std} \leqslant C\epsilon.$$

(In this lemma the constant $C = C(n,R)$ might be different from those in lemmas 4.3, 4.4, 4.5).

Proof of Lemma 4.6. By Lemma 4.4, we have

$$\frac{1}{\mathrm{Vol}(B^t_\epsilon)} \int_{B^t_\epsilon} |f-E|^2 \, dg^{2n-1}_{std} \leqslant C\epsilon,$$

which means

$$\frac{1}{\mathrm{Vol}(B^S_\epsilon(x_2))}\int_{B^S_\epsilon(x_2)}|f(r'-\delta t,\theta)-E|^2\,d\theta\leqslant C\epsilon,\tag{4.3}$$

for all $t\in(0,T)$. By Lemma 4.5, for any $0\leqslant t_1<t_2<T$ we have

$$\int_{S^{2n-1}}|f(r'-\delta t_2,\tilde{\psi}^{t_2}(\theta))-f(r'-\delta t_1,\tilde{\psi}^{t_1}(\theta))|^2\,d\theta\leqslant C'\delta,$$

so in particular taking some $t\in(0,T)$ and considering $t_1=0$, $t_2=t$, we get

$$\int_{S^{2n-1}}|f(r'-\delta t,\tilde{\psi}^t(\theta))-f(r',\theta)|^2\,d\theta\leqslant C'\delta,$$

which implies that

$$\frac{1}{\mathrm{Vol}(B^S_\epsilon(x_2))}\int_{(\tilde{\psi}^t)^{-1}(B^S_\epsilon(x_2))}|f(r'-\delta t,\tilde{\psi}^t(\theta))-f(r',\theta)|^2\,d\theta\leqslant\frac{C'\delta}{\mathrm{Vol}(B^S_\epsilon(x_2))}.$$

Also, since the flow $\tilde{\psi}^t:S^{2n-1}\to S^{2n-1}$ is volume preserving, it follows from (4.3) that

$$\begin{aligned}&\frac{1}{\mathrm{Vol}(B^S_\epsilon(x_2))}\int_{(\tilde{\psi}^t)^{-1}(B^S_\epsilon(x_2))}|f(r'-\delta t,\tilde{\psi}^t(\theta))-E|^2\,d\theta\\&=\frac{1}{\mathrm{Vol}(B^S_\epsilon(x_2))}\int_{B^S_\epsilon(x_2)}|f(r'-\delta t,\theta)-E|^2\,d\theta\leqslant C\epsilon.\end{aligned}$$

Hence we conclude

$$\begin{aligned}&\frac{1}{\mathrm{Vol}(B^S_\epsilon(x_2))}\int_{(\tilde{\psi}^t)^{-1}(B^S_\epsilon(x_2))}|f(r',\theta)-E|^2\,d\theta\\
&\leqslant 2\Bigg(\frac{1}{\mathrm{Vol}(B^S_\epsilon(x_2))}\int_{(\tilde{\psi}^t)^{-1}(B^S_\epsilon(x_2))}|f(r'-\delta t,\tilde{\psi}^t(\theta))-E|^2\,d\theta\\
&+\frac{1}{\mathrm{Vol}(B^S_\epsilon(x_2))}\int_{(\tilde{\psi}^t)^{-1}(B^S_\epsilon(x_2))}|f(r'-\delta t,\tilde{\psi}^t(\theta))-f(r',\theta)|^2\,d\theta\Bigg)\\
&\leqslant 2\left(C\epsilon+\frac{C'\delta}{\mathrm{Vol}(B^S_\epsilon(x_2))}\right).\end{aligned}$$

If δ is small enough, then we will have

$$2\left(C\epsilon+\frac{C'\delta}{\mathrm{Vol}(B^S_\epsilon(x_2))}\right)\leqslant 3C\epsilon.$$

Therefore we conclude that

$$\frac{1}{\mathrm{Vol}(B^S_\epsilon(x_2))}\int_{(\tilde{\psi}^t)^{-1}(B^S_\epsilon(x_2))}|f(r',\theta)-E|^2\,d\theta\leqslant 3C\epsilon,$$

or in other words,

$$\frac{1}{\mathrm{Vol}((\psi^t_\delta)^{-1}(B^t_\epsilon))}\int_{(\psi^t_\delta)^{-1}(B^t_\epsilon)}|f-E|^2\,dg^{2n-1}_{std}\leqslant 3C\epsilon.$$

□

Proof of proposition

Let us finally conclude Claim 4.2 stated above. Let g be the metric on $D^{2n}_{r,R}$ defined as above, and assume that δ is small enough and that the function $a(\cdot)$ is large enough on $[\delta^2, T-\delta^2]$. Let $f : D^{2n}_{r,R} \to \mathbb{R}$ be a smooth function satisfying

$$\int_{D^{2n}_{r,R}} \|\nabla_g f\|_g^2 \, dg^{2n}_{std} \leqslant 1.$$

By Lemma 4.3, there exists a point $x_2 \in \mathcal{P}$ and some $E \in \mathbb{R}$ such that

$$\frac{1}{\mathrm{Vol}(r' B^S_{2\epsilon}(x_2))} \int_{r' B^S_{2\epsilon}(x_2)} |f - E|^2 \, dg^{2n-1}_{std} \leqslant C\epsilon.$$

Then, by Lemma 4.6, for any $t \in (0, T)$, looking at the preimage $(\psi^t_\delta)^{-1}(B^t_\epsilon) \subseteq S^{2n-1}_{r'}$ of $B^t_\epsilon = (r' - \delta t) B^S_\epsilon(x_2)$, we have

$$\frac{1}{\mathrm{Vol}((\psi^t_\delta)^{-1}(B^t_\epsilon))} \int_{(\psi^t_\delta)^{-1}(B^t_\epsilon)} |f - E|^2 \, dg^{2n-1}_{std} \leqslant C'\epsilon,$$

where $C' = C'(n, R)$. But since $\tilde{\psi}^t$ is a volume preserving flow, we have

$$\begin{aligned}
&\frac{1}{\mathrm{Vol}((\psi^t_\delta)^{-1}(B^t_\epsilon))} \int_{(\psi^t_\delta)^{-1}(B^t_\epsilon)} |f - E|^2 \, dg^{2n-1}_{std} \\
&= \frac{1}{\mathrm{Vol}((\tilde{\psi}^t)^{-1}(B^S_\epsilon(x_2)))} \int_{(\tilde{\psi}^t)^{-1}(B^S_\epsilon(x_2))} |f(r', \theta) - E|^2 \, d\theta \\
&= \frac{1}{\mathrm{Vol}(B^S_\epsilon)} \int_{(\tilde{\psi}^t)^{-1}(B^S_\epsilon(x_2))} |f(r', \theta) - E|^2 \, d\theta \\
&= \int_{S^{2n-1}} \left(\frac{1}{\mathrm{Vol}(B^S_\epsilon)} (\tilde{\psi}^t)^* \chi(\theta) \right) |f(r', \theta) - E|^2 \, d\theta,
\end{aligned}$$

and so

$$\int_{S^{2n-1}} \left(\frac{1}{\mathrm{Vol}(B^S_\epsilon)} (\tilde{\psi}^t)^* \chi(\theta) \right) |f(r', \theta) - E|^2 \, d\theta \leqslant C'\epsilon, \tag{4.4}$$

where $\chi : S^{2n-1} \to \mathbb{R}$ is the characteristic function of $B^S_\epsilon(x_2)$. Now, by Lemma 4.1, there exist some $t_1, t_2, ..., t_N \in (0, T)$ such that

$$\frac{1}{N} \sum_{k=1}^{N} (\tilde{\psi}^{t_k})^* \chi \geqslant c \frac{\mathrm{Vol}(B^S_\epsilon(x_2))}{\mathrm{Vol}(S^{2n-1})} \tag{4.5}$$

on S^{2n-1}. Averaging (4.4) over $t = t_1, ..., t_N$, and using (4.5), we get

$$\frac{c}{\mathrm{Vol}(S^{2n-1})} \int_{S^{2n-1}} |f(r', \theta) - E|^2 \, d\theta$$

$$= \int_{S^{2n-1}} \frac{1}{\mathrm{Vol}(B^S_\epsilon)} \frac{c \cdot \mathrm{Vol}(B^S_\epsilon(x_2))}{\mathrm{Vol}(S^{2n-1})} |f(r',\theta) - E|^2 \, d\theta$$

$$\leqslant \int_{S^{2n-1}} \left(\frac{1}{\mathrm{Vol}(B^S_\epsilon)} \frac{1}{N} \sum_{k=1}^{N} (\tilde{\psi}^{t_k})^* \chi(\theta) \right) |f(r',\theta) - E|^2 \, d\theta \leqslant C'\epsilon,$$

and hence

$$\int_{S^{2n-1}} |f(r',\theta) - E|^2 \, d\theta \leqslant C''\epsilon, \tag{4.6}$$

where

$$C'' = \frac{C' \mathrm{Vol}(S^{2n-1})}{c}.$$

Now, fixing any $t \in (0, T)$, and applying Lemma 4.5 for $t_1 = 0$, $t_2 = t$, we get

$$\int_{S^{2n-1}} |f(r' - \delta t, \tilde{\psi}^t(\theta)) - f(r',\theta)|^2 \, d\theta \leqslant C'''\delta,$$

which together with (4.6) gives us

$$\int_{S^{2n-1}} |f(r' - \delta t, \theta) - E|^2 \, d\theta = \int_{S^{2n-1}} |f(r' - \delta t, \tilde{\psi}^t(\theta)) - E|^2 \, d\theta$$

$$\leqslant 2 \left(\int_{S^{2n-1}} |f(r' - \delta t, \tilde{\psi}^t(\theta)) - f(r',\theta)|^2 \, d\theta + \int_{S^{2n-1}} |f(r',\theta) - E|^2 \, d\theta \right)$$

$$\leqslant 2(C'''\delta + C''\epsilon) \leqslant 3C''\epsilon,$$

since $\tilde{\psi}^t$ is volume preserving, and δ is small enough. Thus we have proved that

$$\int_{S^{2n-1}} |f(u,\theta) - E|^2 \, d\theta \leqslant 3C''\epsilon, \tag{4.7}$$

for any $u \in (r, r')$. Now consider the case when $u \in (r', R)$. Define the vector field X on $\mathbb{R}^{2n} \setminus \{0\}$, as $X(x) = -\frac{x}{|x|}$ for $x \in \mathbb{R}^{2n} \setminus \{0\}$. Then keeping in mind that $g = g_{std}$ on $D^{2n}_{r',R}$, we obtain

$$\int_{S^{2n-1}} |f(u,\theta) - f(r',\theta)|^2 \, d\theta$$

$$= \int_{S^{2n-1}} \left| \int_{r'}^{u} \frac{\partial}{\partial s} f(s,\theta) \, ds \right|^2 d\theta \leqslant \int_{S^{2n-1}} (u - r') \int_{r'}^{u} \left| \frac{\partial}{\partial s} f(s,\theta) \right|^2 ds \, d\theta$$

$$= (u - r') \int_{S^{2n-1}} \int_{r'}^{u} |L_X f(s,\theta)|^2 \, ds \, d\theta$$

$$\leqslant (u - r') \int_{S^{2n-1}} \int_{r'}^{u} \|\nabla_g f(s,\theta)\|_g^2 \cdot \|X(s,\theta)\|_g^2 \, ds \, d\theta$$

$$= (u - r') \int_{S^{2n-1}} \int_{r'}^{u} \|\nabla_g f(s,\theta)\|_g^2 \, ds \, d\theta.$$

The Jacobian of the map $(r', R) \times S^{2n-1} \to D^{2n}_{r',R}$ given by

$$(s, \theta) \mapsto s\theta,$$

equals to s^{2n-1}, and hence is greater than r'^{2n-1} at every point of $(r', R) \times S^{2n-1}$. So returning again to our chain of estimates, we conclude that

$$\int_{S^{2n-1}} |f(u,\theta) - f(r',\theta)|^2 \, d\theta \leqslant (u - r') \int_{S^{2n-1}} \int_{r'}^{u} \|\nabla_g f(s,\theta)\|_g^2 \, ds \, d\theta$$

$$\leqslant \frac{u - r'}{r'^{2n-1}} \int_{D^{2n}_{r',u}} \|\nabla_g f\|_g^2 \, dg^{2n}_{std} \leqslant \frac{u - r'}{r'^{2n-1}} \int_{D^{2n}_{r,R}} \|\nabla_g f\|_g^2 \, dg^{2n}_{std} \leqslant \frac{u - r'}{r'^{2n-1}}.$$

But we have $u - r' < R - r' = \epsilon$, and for small ϵ, δ we have $r'^{2n-1} \geqslant \left(\frac{R}{2}\right)^{2n-1}$, hence we get

$$\int_{S^{2n-1}} |f(u,\theta) - f(r',\theta)|^2 \, d\theta \leqslant \left(\frac{2}{R}\right)^{2n-1} \epsilon. \tag{4.8}$$

Therefore, from (4.6) and (4.8) we get

$$\begin{aligned} &\int_{S^{2n-1}} |f(u,\theta) - E|^2 \, d\theta \\ &\leqslant 2\left(\int_{S^{2n-1}} |f(r',\theta) - E|^2 \, d\theta + \int_{S^{2n-1}} |f(u,\theta) - f(r',\theta)|^2 \, d\theta\right) \\ &\leqslant 2\left(C''\epsilon + \left(\frac{2}{R}\right)^{2n-1} \epsilon\right) = \left(2C'' + \frac{2^{2n}}{R^{2n-1}}\right)\epsilon, \end{aligned} \tag{4.9}$$

for any $u \in (r', R)$. Combining (4.6), (4.7), and (4.9), we conclude that for any $u \in (r, R)$ we have

$$\int_{S^{2n-1}} |f(u,\theta) - E|^2 \, d\theta \leqslant \left(3C'' + \frac{2^{2n}}{R^{2n-1}}\right)\epsilon.$$

Hence for any $u \in (r, R)$ we have

$$\int_{S^{2n-1}_u} |f - E|^2 \, dg^{2n-1}_{std} = u^{2n-1} \int_{S^{2n-1}} |f(u,\theta) - E|^2 \, d\theta$$

$$\leqslant R^{2n-1}\left(3C'' + \frac{2^{2n}}{R^{2n-1}}\right)\epsilon = (3C''R^{2n-1} + 2^{2n})\epsilon.$$

This finishes the proof of Claim 4.2, and hence of the proposition.

5 Proof of Theorem 1.10

Choose a smooth triangulation of M, and let $\Delta_\alpha \subseteq M$, $\alpha \in I$, be the open simplices of this triangulation. Choose a Riemannian metric g_0 on M, such that for each $\alpha \in I$ there exists a Darboux neighborhood inside Δ_α, on which g_0 coincides with the euclidean metric.

For $\alpha \in I$, denote by Δ'_α the union of Δ_α with all of its open faces. Then for each $\alpha \in I$, by Lemma 3.1 (section 3), there exists a bi-Lipschitz homeomorphism $\Psi_\alpha : \overline{\Delta_\alpha} \to \overline{B}_1^{2n}(0)$, such that Ψ_α is a diffeomorphism from Δ_α onto $B_1^{2n}(0)$, and also is a diffeomorphism from Δ'_α onto the image. Due to our choice of g_0, WLOG we may assume that the pushforward ω_α of the symplectic structure ω from Δ_α to $B_1^{2n}(0)$ by the map Ψ_α, equals ω_{std} (i.e. is standard) near the origin, and that the pushforward $g_{0,\alpha}$ of the metric g_0 from Δ'_α to its image $\Psi_\alpha(\Delta'_\alpha) \subset \overline{B}_1^{2n}(0)$ by the map Ψ_α, coincides with the standard euclidean metric g_{std} near the origin. Hence we can find some $0 < R_0 < 1$, such that $\omega_\alpha = \omega_{std}$ and $g_{0,\alpha} = g_{std}$ on $B_{R_0}^{2n}(0)$, for all $\alpha \in I$. Let $C \geqslant 1$ be a bi-Lipschitz constant for all Ψ_α, $\alpha \in I$, when we consider the metric g_0 on $\overline{\Delta_\alpha}$, and the metric g_{std} on $\overline{B}_1^{2n}(0)$. Then we get

$$\frac{1}{C^2} g_{std} \leqslant g_{0,\alpha} \leqslant C^2 g_{std}$$

on $\Psi_\alpha(\Delta'_\alpha)$, for each $\alpha \in I$.

Now pick any $0 < R \leqslant R_0$. After choosing R, pick a small enough $\epsilon > 0$. Then by Proposition 1.11, there exists $\frac{R}{2} < r < R$, and a metric g_{loc} on the domain

$$D_{r,R}^{2n} = \{x \in \mathbb{R}^{2n} \,|\, r < |x| < R\},$$

having all the desired properties. Consider the “minus-radial vector field” $X(x) = -\frac{x}{|x|}$ on $D_{R,1}^{2n}$. Pick a sufficiently small $\delta' > 0$, and choose a smooth function $\hat{a} : \mathbb{R} \to \mathbb{R}$, such that $\hat{a}(u) = 1$ for $u \notin (R + \frac{\delta'}{2}, 1 - \frac{\delta'}{2})$, such that $\hat{a}(u) \geqslant 1$ for all $u \in \mathbb{R}$, and such that $\hat{a}(u)$ is sufficiently large on $[R+\delta', 1-\delta']$. Define $\hat{b} : B_1^{2n}(0) \to \mathbb{R}$ as $\hat{b}(x) = \hat{a}(|x|)$. Denote by $J_{0,\alpha}$ the almost complex structure that relates ω_α and $g_{0,\alpha}$. Now we define the metric g_α on $B_1^{2n}(0)$ as follows: on $D_{r,R}^{2n}$ we set $g_\alpha = g_{loc}$; on $D_{R,1}^{2n}$, looking at the $g_{0,\alpha}$-orthogonal decomposition

$$TD_{R,1}^{2n} = Span(X) \oplus Span(J_{0,\alpha}X) \oplus L,$$

we define

$$g_\alpha|_x = \hat{b}(x)^{-1} g_{0,\alpha}|_x \oplus \hat{b}(x) g_{0,\alpha}|_x \oplus g_{0,\alpha}|_x$$

at each $x \in D_{R,1}^{2n}$; finally, on $B_1^{2n}(0) \setminus (D_{r,R}^{2n} \cup D_{R,1}^{2n})$ we set $g_\alpha = g_{0,\alpha} = g_{std}$. Clearly, g_α is a smooth Riemannian metric on $B_1^{2n}(0)$, is compatible with ω_α, and coincides with $g_{0,\alpha}$ near the boundary of $B_1^{2n}(0)$. Also, exactly as in the case of Lemma 4.5 (section 4), but now using the flow of the vector field X, one can prove the following

Claim 5.1. *If the function $\hat{a}(\cdot)$ is sufficiently large on $[R+\delta', 1-\delta']$, then for any smooth function $h : D^{2n}_{R,1} \to \mathbb{R}$ satisfying*

$$\int_{D^{2n}_{R,1}} \|\nabla_{g_\alpha} h\|^2_{g_\alpha} \, dg^{2n}_{std} \leqslant 1,$$

we have the following: for any $R < u_1 < u_2 < 1$,

$$\int_{S^{2n-1}} |h(u_2, \theta) - h(u_1, \theta)|^2 \, d\theta \leqslant C^2 C' \delta',$$

where $C' = C'(n, R)$.

Finally, we define the metric g on M as follows: for any $\alpha \in I$, on Δ_α we set $g = \Psi^*_\alpha g_\alpha$; on $M \setminus (\cup_{\alpha \in I} \Delta_\alpha)$ we set $g = g_0$. Clearly g is a smooth Riemannian metric on M, compatible with ω. We claim that the metric g will have arbitrarily large λ_1, once we take R to be small enough, and then pick ϵ, δ' to be sufficiently small, and $\hat{a}(\cdot)$ to be sufficiently large on $[R+\delta', 1-\delta']$. Let us show this.

Let $f : M \to \mathbb{R}$ be a smooth function with

$$\int_M f \, dg^{2n} = 0,$$

and

$$\int_M \|\nabla_g f\|^2_g \, dg^{2n} \leqslant 1.$$

Then, for any $\alpha \in I$, define $f_\alpha : \overline{B}^{2n}_1(0) \to \mathbb{R}$ as $f_\alpha = (\Psi_\alpha)_* f$ - the pushforward of f by Ψ_α. Then keeping in mind that $\omega_\alpha = \omega_{std}$ on $B^{2n}_R(0)$, and that $g_\alpha = g_{std}$ on $B^{2n}_r(0)$, we get that

$$\begin{aligned} \int_{B^{2n}_r(0)} |\nabla f_\alpha|^2 \, dg^{2n}_{std} &= \int_{B^{2n}_r(0)} \|\nabla_{g_\alpha} f_\alpha\|^2_{g_\alpha} \, dg^{2n}_\alpha \\ \leqslant \int_{B^{2n}_1(0)} \|\nabla_{g_\alpha} f_\alpha\|^2_{g_\alpha} \, dg^{2n}_\alpha = \int_{\Delta_\alpha} \|\nabla_g f\|^2_g \, dg^{2n} &\leqslant \int_M \|\nabla_g f\|^2_g \, dg^{2n} \leqslant 1, \end{aligned} \tag{5.1}$$

that

$$\begin{aligned} \int_{D^{2n}_{r,R}} \|\nabla_{g_{loc}} f_\alpha\|^2_{g_{loc}} \, dg^{2n}_{std} &= \int_{D^{2n}_{r,R}} \|\nabla_{g_\alpha} f_\alpha\|^2_{g_\alpha} \, dg^{2n}_\alpha \\ \leqslant \int_{B^{2n}_1(0)} \|\nabla_{g_\alpha} f_\alpha\|^2_{g_\alpha} \, dg^{2n}_\alpha = \int_{\Delta_\alpha} \|\nabla_g f\|^2_g \, dg^{2n} &\leqslant \int_M \|\nabla_g f\|^2_g \, dg^{2n} \leqslant 1, \end{aligned} \tag{5.2}$$

and that

$$
\begin{aligned}
\int_{D^{2n}_{R,1}} \|\nabla_{g_\alpha} f_\alpha\|^2_{g_\alpha}\, dg^{2n}_{std} \leqslant C^{2n} \int_{D^{2n}_{R,1}} \|\nabla_{g_\alpha} f_\alpha\|^2_{g_\alpha}\, dg^{2n}_{0,\alpha} &= C^{2n} \int_{D^{2n}_{R,1}} \|\nabla_{g_\alpha} f_\alpha\|^2_{g_\alpha}\, dg^{2n}_{\alpha} \\
\leqslant C^{2n} \int_{B^{2n}_1(0)} \|\nabla_{g_\alpha} f_\alpha\|^2_{g_\alpha}\, dg^{2n}_{\alpha} &= C^{2n} \int_{\Delta_\alpha} \|\nabla_g f\|^2_g\, dg^{2n} \\
&\leqslant C^{2n} \int_M \|\nabla_g f\|^2_g\, dg^{2n} \leqslant C^{2n}.
\end{aligned} \tag{5.3}
$$

Applying Proposition 1.11 to (5.2), we conclude that there exists some $E_\alpha \in \mathbb{R}$, such that for any $u \in (r, R)$ we have

$$
\int_{S^{2n-1}_u} |f_\alpha - E_\alpha|^2\, dg^{2n-1}_{std} \leqslant \epsilon, \tag{5.4}
$$

which implies that

$$
\begin{aligned}
\int_{S^{2n-1}} |f_\alpha(u,\theta) - E_\alpha|^2\, d\theta &= \frac{1}{u^{2n-1}} \int_{S^{2n-1}_u} |f_\alpha - E_\alpha|^2\, dg^{2n-1}_{std} \\
&\leqslant \frac{\epsilon}{u^{2n-1}} \leqslant \frac{\epsilon}{r^{2n-1}} \leqslant \frac{2^{2n-1}\epsilon}{R^{2n-1}},
\end{aligned} \tag{5.5}
$$

for any $u \in (r, R)$. Note that by a continuity reason, (5.4) and (5.5) hold also for $u = r, R$.

Applying our Claim 5.1 above to (5.3), we conclude that for any $R < u_1 < u_2 < 1$ we have

$$
\int_{S^{2n-1}} |f_\alpha(u_2,\theta) - f_\alpha(u_1,\theta)|^2\, d\theta \leqslant C^{2n+2} C' \delta'. \tag{5.6}
$$

By a continuity reason, (5.6) holds for any $R \leqslant u_1 \leqslant u_2 \leqslant 1$.

We have that (5.5) is valid for $u = R$, and (5.6) holds when $u_1 = R$ and $R \leqslant u_2 \leqslant 1$. Hence for any $u \in [R, 1]$ we have

$$
\begin{aligned}
&\int_{S^{2n-1}} |f_\alpha(u,\theta) - E_\alpha|^2\, d\theta \\
\leqslant 2\Big(\int_{S^{2n-1}} |f_\alpha(u,\theta) - f_\alpha(R,\theta)|^2\, d\theta &+ \int_{S^{2n-1}} |f_\alpha(R,\theta) - E_\alpha|^2\, d\theta \Big) \\
&\leqslant 2C^{2n+2} C' \delta' + \frac{2^{2n}}{R^{2n-1}} \epsilon.
\end{aligned} \tag{5.7}
$$

Therefore from (5.5) and (5.7) we conclude that

$$\int_{S^{2n-1}} |f_\alpha(u,\theta) - E_\alpha|^2 \, d\theta \leqslant 2C^{2n+2}C'\delta' + \frac{2^{2n}}{R^{2n-1}}\epsilon, \tag{5.8}$$

for any $u \in [r,1]$. This, in turn, implies that

$$\begin{aligned} \int_{S_u^{2n-1}} |f_\alpha - E_\alpha|^2 \, dg_{std}^{2n-1} &= u^{2n-1} \int_{S^{2n-1}} |f_\alpha(u,\theta) - E_\alpha|^2 \, d\theta \\ \leqslant u^{2n-1} \left(2C^{2n+2}C'\delta' + \frac{2^{2n}}{R^{2n-1}}\epsilon \right) &\leqslant 2C^{2n+2}C'\delta' + \frac{2^{2n}}{R^{2n-1}}\epsilon, \end{aligned} \tag{5.9}$$

for any $u \in [r,1]$. Hence on one hand, from (5.9) we get

$$\begin{aligned} \int_{D_{r,1}^{2n}} |f_\alpha - E_\alpha|^2 \, dg_{std}^{2n} &= \int_r^1 \int_{S_u^{2n-1}} |f_\alpha - E_\alpha|^2 \, dg_{std}^{2n-1} \, du \\ \leqslant (1-r) \left(2C^{2n+2}C'\delta' + \frac{2^{2n}}{R^{2n-1}}\epsilon \right) &\leqslant 2C^{2n+2}C'\delta' + \frac{2^{2n}}{R^{2n-1}}\epsilon. \end{aligned} \tag{5.10}$$

On the other hand, since (5.4) is true for $u = r$, and since we have (5.1), from Lemma 3.5 (section 3) we get

$$\int_{B_r^{2n}(0)} |f_\alpha - E_\alpha|^2 dg_{std}^{2n} \leqslant C''(r^2 + \epsilon r) \leqslant C''(R^2 + \epsilon R), \tag{5.11}$$

where $C'' = C''(n)$.

Adding (5.10) and (5.11), we obtain

$$\begin{aligned} \int_{B_1^{2n}(0)} |f_\alpha - E_\alpha|^2 dg_{std}^{2n} &= \int_{B_r^{2n}(0)} |f_\alpha - E_\alpha|^2 dg_{std}^{2n} + \int_{D_{r,1}^{2n}} |f_\alpha - E_\alpha|^2 \, dg_{std}^{2n} \\ &\leqslant C''(R^2 + \epsilon R) + 2C^{2n+2}C'\delta' + \frac{2^{2n}}{R^{2n-1}}\epsilon \\ &= C''R^2 + C''\epsilon R + 2C^{2n+2}C'\delta' + \frac{2^{2n}}{R^{2n-1}}\epsilon. \end{aligned} \tag{5.12}$$

Look now at (5.7) and (5.12). We can choose ϵ and δ' to be small enough, so that we will have

$$C''\epsilon R + 2C^{2n+2}C'\delta' + \frac{2^{2n}}{R^{2n-1}}\epsilon \leqslant C''R^2.$$

Hence if we choose ϵ and δ' small, then (5.7) for the case of $u = 1$, and (5.12), will give us

$$\int_{S^{2n-1}} |f_\alpha - E_\alpha|^2 \, dg_{std}^{2n-1} = \int_{S^{2n-1}} |f_\alpha(1,\theta) - E_\alpha|^2 \, d\theta \leqslant C'' R^2, \tag{5.13}$$

and

$$\int_{B_1^{2n}(0)} |f_\alpha - E_\alpha|^2 dg_{std}^{2n} \leqslant 2C'' R^2. \tag{5.14}$$

Returning to the manifold M, from (5.13) and (5.14) we get

$$\begin{aligned} &\int_{\partial\Delta_\alpha} |f - E_\alpha|^2 dg_0^{2n-1} = \int_{S^{2n-1}} |f_\alpha - E_\alpha|^2 dg_{0,\alpha}^{2n-1} \\ &\leqslant C^{2n-1} \int_{S^{2n-1}} |f_\alpha - E_\alpha|^2 dg_{std}^{2n-1} \leqslant C^{2n-1} C'' R^2, \end{aligned} \tag{5.15}$$

and

$$\begin{aligned} &\int_{\Delta_\alpha} |f - E_\alpha|^2 dg_0^{2n} = \int_{B_1^{2n}(0)} |f_\alpha - E_\alpha|^2 dg_{0,\alpha}^{2n} \\ &\leqslant C^{2n} \int_{B_1^{2n}(0)} |f_\alpha - E_\alpha|^2 dg_{std}^{2n} \leqslant 2C^{2n} C'' R^2. \end{aligned} \tag{5.16}$$

Now consider two adjacent simplices Δ_α and Δ_β, having a common face which we denote by $\Sigma \subset M$. Then (5.15) implies

$$\int_\Sigma |f - E_\alpha|^2 dg_0^{2n-1} \leqslant \int_{\partial\Delta_\alpha} |f - E_\alpha|^2 dg_0^{2n-1} \leqslant C^{2n-1} C'' R^2,$$

and

$$\int_\Sigma |f - E_\beta|^2 dg_0^{2n-1} \leqslant \int_{\partial\Delta_\beta} |f - E_\beta|^2 dg_0^{2n-1} \leqslant C^{2n-1} C'' R^2.$$

Therefore,

$$\begin{aligned} \mathrm{Vol}_{g_0}(\Sigma) |E_\alpha - E_\beta|^2 &= \int_\Sigma |E_\alpha - E_\beta|^2 dg_0^{2n-1} \\ \leqslant 2\Big(\int_\Sigma |f - E_\alpha|^2 dg_0^{2n-1} &+ \int_\Sigma |f - E_\beta|^2 dg_0^{2n-1} \Big) \leqslant 4C^{2n-1} C'' R^2, \end{aligned}$$

Since we have only finite number of faces of simplices of our triangulation, it follows that the minimum of a g_0-volume of such a face, is a positive real number. Denote it by $c > 0$. Hence we get the following: if Δ_α and Δ_β, where $\alpha, \beta \in I$, are adjacent simplices from our triangulation, then

$$|E_\alpha - E_\beta|^2 \leqslant \frac{4C^{2n-1}C''}{c} R^2.$$

Now, if we consider any two simplices Δ_α and Δ_β (not necessarily adjacent), then we can connect Δ_α with Δ_β via a sequence of distinct simplices from our triangulation, where any two consequent simplices in this sequence are adjacent, and hence by the triangle inequality we get

$$|E_\alpha - E_\beta|^2 \leqslant \frac{4|I|^2 C^{2n-1}C''}{c} R^2,$$

for any $\alpha, \beta \in I$. Therefore there exists some $E \in \mathbb{R}$ such that

$$|E_\alpha - E|^2 \leqslant \frac{4|I|^2 C^{2n-1}C''}{c} R^2, \tag{5.17}$$

for any $\alpha \in I$ (we can just take $E = E_\gamma$ for any $\gamma \in I$).

Therefore, from (5.16) and (5.17) we get

$$\begin{aligned} \int_{\Delta_\alpha} |f - E|^2 dg_0^{2n} &\leqslant 2\bigg(\int_{\Delta_\alpha} |f - E_\alpha|^2 dg_0^{2n} + |E_\alpha - E|^2 \mathrm{Vol}_{g_0}(\Delta_\alpha) \bigg) \\ &\leqslant 4C^{2n}C''R^2 + \frac{8|I|^2 C^{2n-1}C''\mathrm{Vol}_{g_0}(\Delta_\alpha)}{c} R^2 \end{aligned} \tag{5.18}$$

Summing (5.18) over all $\alpha \in I$, we get

$$\begin{aligned} \int_M |f - E|^2 dg_0^{2n} &= \sum_{\alpha \in I} \int_{\Delta_\alpha} |f - E|^2 dg_0^{2n} \\ &\leqslant 4|I|C^{2n}C''R^2 + \frac{8|I|^2 C^{2n-1}C''\mathrm{Vol}_{g_0}(M)}{c} R^2 \\ &= \left(4|I|C^{2n}C'' + \frac{8|I|^2 C^{2n-1}C''\mathrm{Vol}_{g_0}(M)}{c} \right) R^2 \end{aligned}$$

Note that

$$\int_M |f|^2 dg^{2n} \leqslant \int_M |f|^2 dg^{2n} + E^2 = \int_M |f-E|^2 dg^{2n} = \int_M |f-E|^2 dg_0^{2n}.$$

Therefore

$$\int_M |f|^2 dg^{2n} \leqslant C''' R^2,$$

where

$$C''' = 4|I|C^{2n}C'' + \frac{8|I|^2 C^{2n-1} C'' \mathrm{Vol}_{g_0}(M)}{c}.$$

Hence we have finally proved the following:
If $f : M \to \mathbb{R}$ is a smooth function with

$$\int_M f\, dg^{2n} = 0,$$

and

$$\int_M \|\nabla_g f\|_g^2 \, dg^{2n} \leqslant 1,$$

then

$$\int_M |f|^2 dg^{2n} \leqslant C''' R^2.$$

Therefore we immediately get a lower bound for the first eigenvalue:

$$\lambda_1(g) \geqslant \frac{1}{C''' R^2}.$$

Note that the constant C''' depends only on M, on the metric g_0 on M, on our triangulation of M to simplices Δ_α, and on the collection of maps $\Psi_\alpha : \overline{\Delta_\alpha} \to \overline{B}_1^{2n}(0)$. Therefore, since we have freedom to choose $R > 0$ to be arbitrarily small, this means that λ_1 associated with the metric g, can be arbitrarily large.

6 Further discussion

6.1 Comparison between approaches

In this section we would like to compare our approach with previous approaches [P, M]. At first, let us comment on our proof of the main result (Theorem 1.10). Section 5 explains how we deduce Theorem 1.10 from Proposition 1.11. In the beginning we make a triangulation of the symplectic manifold (M, ω), thus dividing it into a union of simplexes. Then we choose a Riemannian metric g_0 on M, such that g_0 is compatible with ω, and such that inside each open simplex of the triangulation there exists a Darboux chart in which the metric g_0 is standard euclidean. Then finally, we define the desired metric on M by starting with g_0, then inserting the special metric constructed in Proposition 1.11 into each one of the mentioned Darboux charts in each simplex, and then doing "compressing the neck" along a certain "radially look like" vector field in each simplex. However, we claim that in fact, instead of using a triangulation on several simplexes, we could use only one simplex. More precisely, choose a Riemannian metric g_0 on M, such that g_0 is compatible with ω. It is not very difficult to find a smooth embedding $\Phi : B \to M$ of the open unit ball

$$B = B_1^{2n}(0) = \{x \in \mathbb{R}^{2n} \mid |x| < 1\} \subset \mathbb{R}^{2n}$$

into M, such that the complement $M \setminus \Phi(B)$ is a null set (in the sense that it has measure 0), and such that Φ is a bi-Lipschitz map from B onto $\Phi(B)$, when we consider the standard euclidean metric on B, and the metric g_0 on $\Phi(B) \subset M$ (cf. Lemma 3.1). Theorem 1.10 is then a consequence of the following result (which can be proved based on Proposition 1.11, and by following the same ideas as in section 5):

Theorem 6.1. *Let ω be a symplectic structure on the open ball $B = B_1^{2n}(0) \subset \mathbb{R}^{2n}$, let g_0 be a Riemannian metric on B which is compatible with ω, and assume that g_0 is equivalent to the euclidean metric g_{std}, i.e. there exist constants $0 < c < C$ such that $c^2 g_{std} \leqslant g_0 \leqslant C^2 g_{std}$. Then for any $\epsilon > 0$ there exists a Riemannian metric g on B, such that g coincides with g_0 near the boundary of B, such that g is compatible with ω, and such that for any smooth function $f : B \to \mathbb{R}$ with zero mean (relative to the volume density induced by g, or equivalently, relative to the volume form ω^n), we have*

$$\int_B |f|^2 \, dg_{std}^{2n} \leqslant \epsilon \int_B \|\nabla_g f\|_g^2 \, dg_{std}^{2n}.$$

In other words, we are able to prove an analogue of the symplectic flexibility of the first eigenvalue of the Laplacian, in the case of an *open* ball (provided that the symplectic structure has good enough behaviour near the boundary of the ball). This is the advantage of our approach over previous approaches [P, M]. It would be interesting to understand if it is still possible to improve approaches in [P, M], in order to prove a statement in the spirit of Theorem 6.1 and thus to provide a different proof of Theorem 1.10.

6.2 Symplectic flexibility of the first Dirichlet and first nonzero Neumann eigenvalues

Let us remark that our approach allows us to prove the symplectic flexibility of the first Dirichlet and the first nonzero Neumann eigenvalues on a compact symplectic manifold with boundary, provided that the symplectic form behaves nicely enough near the boundary.

Theorem 6.2. *Let $U \subset \mathbb{R}^{2n}$ be a bounded domain with smooth boundary, let ω be a symplectic structure on U, such that there exists a Riemannian metric g_0 on U which is compatible with ω and which is equivalent to the euclidean metric g_{std} (i.e. for some constants $0 < c < C$ we have $c^2 g_{std} \leqslant g_0 \leqslant C^2 g_{std}$). Then*

1. *There exists a Riemannian metric g on U, which is compatible with ω, and which has arbitrarily large first nonzero Neumann eigenvalue.*

2. *There exists a Riemannian metric g on U, which is compatible with ω, and which has arbitrarily large first Dirichlet eigenvalue.*

In both cases 1 and 2, the metric g can be chosen to coincide with g_0 near the boundary of U.

It turns out that in Theorem 6.2, the case of the first nonzero Neumann eigenvalue is easier and basically follows from Theorem 6.1, while the case of the first Dirichlet eigenvalue requires Proposition 1.11 and we are currently not aware of a simpler approach. Below we discuss possible ways of proving each of the cases of Theorem 6.2. As it can be easily seen, Theorem 6.2 can also be extended to the case of a compact symplectic manifold with boundary, in which the symplectic form behaves nicely enough near the boundary.

6.2.1 The first nonzero Neumann eigenvalue

Recall that by the well-known variational characterisation, the first nonzero (i.e. the second) Neumann eigenvalue equals to the infimum of the Rayleigh quotient

$$\frac{\int_U \|\nabla_g f\|_g^2 \, dg^{2n}}{\int_U |f|^2 \, dg^{2n}},$$

when we run over all smooth functions $f : U \to \mathbb{R}$ which are L^2-orthogonal to the first Neumann eigenfunction (which equals to 1 identically), or in other words, when we run over all smooth functions $f : U \to \mathbb{R}$ having zero mean. Hence we can argue similarly as in section 6.1. Namely, first we can find a smooth embedding $\Phi : B \to U$ of the open unit ball $B = B_1^{2n}(0) \subset \mathbb{R}^{2n}$ into U, such that the complement $M \setminus \Phi(B)$ is a null set (in the sense that it has measure 0), and such that Φ is a bi-Lipschitz map from B onto $\Phi(B)$, when we consider the standard euclidean metric both on B and on $\Phi(B) \subseteq U$. Then having such a map Φ, we can apply Theorem 6.1 to conclude that on U there exists a Riemannian metric g, which is compatible with ω, and which has arbitrarily large first nonzero Neumann eigenvalue.

6.2.2 The first Dirichlet eigenvalue

In this case, the variation characterisation is as follows: the first Dirichlet eigenvalue equals to the infimum of the Rayleigh quotient

$$\frac{\int_U \|\nabla_g f\|_g^2 \, dg^{2n}}{\int_U |f|^2 \, dg^{2n}},$$

when we run over all smooth functions $f : U \to \mathbb{R}$ which are *compactly supported* in U (or in other words, which vanish near the boundary of U). To show the result, we first find a smooth embedding $\Phi : B \to U$ of the open unit ball $B = B_1^{2n}(0) \subset \mathbb{R}^{2n}$ into U, having the following properties:

1. The complement $M \setminus \Phi(B)$ is a null set (in the sense that it has measure 0).
2. Φ is a bi-Lipschitz map from B onto $\Phi(B)$, when we consider the standard euclidean metric both on B and on $\Phi(B) \subseteq U$.
3. Φ extends to a continuous map $\overline{\Phi} : \overline{B} \to \overline{U}$, such that for some relatively open subset $\Sigma \subseteq \partial B$, we have $\overline{\Phi}(\Sigma) \subseteq \partial U$.

Then consider the pullbacks $\Phi^*\omega$ and $\Phi^* g_0$, of the symplectic form ω and the Riemannian metric g_0 on U, to B, and denote them, by abuse of notation, by ω and g_0 as well. Now, Proposition 1.11 implies (similarly as in the case of the proof of Theorem 1.10 in section 5) the following refinement of Theorem 6.1:
For any $\epsilon > 0$ there exists a Riemannian metric g on B, such that g coincides with g_0 near the boundary of B, such that g is compatible with ω, and such that for any smooth function $h : B \to \mathbb{R}$ satisfying

$$\int_B \|\nabla_g h\|_g^2 \, dg_{std}^{2n} \leqslant 1,$$

there exists some $E \in \mathbb{R}$, such that

$$\int_B |h - E|^2 \, dg_{std}^{2n} \leqslant \epsilon,$$

and moreover for any $u \in (0, 1)$ sufficiently close to 1, we have

$$\int_{S_u^{2n-1}} |h - E|^2 \, dg_{std}^{2n-1} \leqslant \epsilon.$$

Take such a metric g on B, consider the push-forward $\Phi_* g$ of g from B to $\Phi(B)$ via the map Φ, and extend it to U by setting it to be equal to g_0 on $U \setminus \Phi(B)$. Denote, by abuse of notation, the resulting metric on U again by g. Then we claim that the first Dirichlet eigenvalue for the metric g on U is arbitrarily large, provided that ϵ is small enough. Indeed, if $f : U \to \mathbb{R}$ is a smooth function with compact support, which satisfies

$$\int_U \|\nabla_g f\|_g^2 \, dg_{std}^{2n} \leqslant 1,$$

then denoting $h = \Phi^* f$, we get

$$\int_B \|\nabla_g h\|_g^2 \, dg_{std}^{2n} \leqslant C',$$

(where C' depends only on the map Φ). Then, by the properties of the metric g, there exists some $E \in \mathbb{R}$, such that

$$\int_B |h - E|^2 \, dg_{std}^{2n} \leqslant C'\epsilon, \tag{6.2.1}$$

and moreover for any $u \in (0,1)$ sufficiently close to 1, we have

$$\int_{S_u^{2n-1}} |h - E|^2 \, dg_{std}^{2n-1} \leqslant C'\epsilon. \tag{6.2.2}$$

But for our relatively open subset $\Sigma \subseteq \partial B$ we have $\overline{\Phi}(\Sigma) \subseteq \partial U$, and since the function f is compactly supported in U, this means that for $u \in (0,1)$ sufficiently close to 1, we have that h equals to 0 on $u\Sigma = \{ux \,|\, x \in \Sigma\}$, and together with (6.2.2), this implies that E is small, which in turn, together with (6.2.1), implies that $\int_B |h|^2 \, dg_{std}^{2n}$ is small, and as a consequence, $\int_U |f|^2 \, dg_{std}^{2n}$ is small, provided that we took ϵ to be small enough. This shows that the first Dirichlet eigenvalue of g on U can be arbitrarily large.

Remark. *It is also possible to prove both of the cases (of the first nonzero Neumann eigenvalue and of the first Dirichlet eigenvalue) of Theorem 6.2, using smooth triangulation of the domain U as in the proof of Theorem 1.10 in section 5, instead of using a filling of the whole volume of U with help of a smooth embedding of an open ball as it was described above. However in such proofs, Proposition 1.11 will be needed in both cases.*

References

[C-D] B. Colbois and J. Dodziuk, *Riemannian metrics with large λ_1*, Proc. Amer. Math. Soc. **122** (1994), 905–906.

[E-I] A. El Soufi and S. Ilias, *Le volume conforme et ses applications d'après Li et Yau*, Séminaire de Théorie Spectrale et Géométrie, Année 1983–1984, Univ. Grenoble I, Saint-Martin-d'Hères, France, 1984, VII.1–VII.15.

[F-N] L. Friedlander and N. Nadirashvili, *A differential invariant related to the first eigenvalue of the Laplacian*, Internat. Math. Res. Notices **17** (1999), 939–952.

[He] J. Hersch, *Quatre propriétés isopérimétriques de membranes sphériques homogènes*, C. R. Acad. Sci. Paris Sér. A-B **270** (1970), A1645–A1648.

[Ho] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math. **199** (1967), 147–171.

[M] D. Mangoubi, *Spectral flexibility of symplectic manifolds $T^2 \times M$*, Math. Ann. **341** (2008), no. 1, 1–13.

[P] L. Polterovich, *Symplectic aspects of the first eigenvalue*, J. Reine Angew. Math. **502** (1998), 1–17.

[R-S] L. P. Rothschild and E. M. Stein, *Hypoelliptic differential operators and nilpotent groups*, Acta Math. **137** (1976), 247–320.

[Y-Y] P. C. Yang and S. T. Yau, *Eigenvalues of the Laplacian of compact Riemann surfaces and minimal submanifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **7** (1980), 55–63.

Lev Buhovski
School of Mathematical Sciences, Tel Aviv University
e-mail: levbuh@post.tau.ac.il