Towards the C^0 flux conjecture

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Abstract

In this note, we generalise a result of Lalonde, McDuff and Polterovich concerning the C^0 flux conjecture, thus confirming the conjecture in new cases of a symplectic manifold. Also, we prove the continuity of the flux homomorphism on the space of smooth symplectic isotopies endowed with the C^0 topology, which implies the C^0 rigidity of Hamiltonian paths, conjectured by Seyfaddini.

1 Introduction and main results

The celebrated Eliashberg-Gromov rigidity theorem [E1, E2, G] states that on any closed symplectic manifold (M, ω), the group $\mathrm{Symp}(M, \omega)$ of symplectomorphisms of M is C^0-closed inside the group $\mathrm{Diff}(M)$ of diffeomorphisms of M. A related natural conjecture (called the C^0 flux conjecture) was raised in Banyaga's foundational paper [B]: is the group $\mathrm{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms of M C^0-closed inside $Symp_0(M, \omega)$, the connected component of identity in $\mathrm{Symp}(M, \omega)$?

The reader may wonder why it is asked if $\mathrm{Ham}(M, \omega)$ is C^0-closed in $\mathrm{Symp}_0(M, \omega)$ rather than $\mathrm{Symp}(M, \omega)$. The difficulty in addressing the latter question is that, although the Eliashberg-Gromov rigidity theorem tells us that $\mathrm{Symp}(M, \omega)$ is C^0-closed in the group $\mathrm{Diff}(M)$ of diffeomorphisms of M, it is not known if $\mathrm{Symp}_0(M, \omega)$ is C^0-closed in $\mathrm{Symp}(M, \omega)$. To avoid this difficulty the C^0 flux conjecture is usually stated for $\mathrm{Symp}_0(M, \omega)$.

A weak form of the C^0 flux conjecture is the C^1 flux conjecture, which states that $\mathrm{Ham}(M, \omega)$ is C^1-closed in $\mathrm{Symp}_0(M, \omega)$. This statement is equivalent to that the flux group $\Gamma \subset H^1(M, \mathbb{R})$ is discrete. Some cases of the C^1 flux conjecture were proven in [B], [L-M-P], [M]; it was finally confirmed in full generality by Ono [O-1]. However, the C^0 flux conjecture still remains open in case of a general symplectic manifold.

[1]The author also uses the spelling "Buhovski" for his family name.

It has been confirmed by Lalonde, McDuff and Polterovich in certain cases [L-M-P] (these cases are described below). Also, Humilière and Vichery established more cases of the C^0 flux conjecture in their joint work [Hu-V].

A different weak form of the C^0 flux conjecture (the "C^0 rigidity of Hamiltonian paths") was proposed by Seyfaddini [Sey]: Is it true that on any closed symplectic manifold, the space of smooth Hamiltonian isotopies is C^0 close in the space of smooth symplectic isotopies? In [Sey], Seyfaddini showed that a symplectic isotopy which is a C^0 limit of a sequence of Hamiltonian isotopies is itself Hamiltonian, provided that the corresponding sequence of generating Hamiltonians is a Cauchy sequence in the $L^{(1,\infty)}$ topology.

The results of this note are concerned with the C^0 flux conjecture, and with the mentioned conjecture of Seyfaddini (the C^0 rigidity of Hamiltonian paths).

1.1 The C^0 flux conjecture

We denote $H = \mathrm{Ham}(M,\omega) \subseteq G = \mathrm{Symp}_0(M,\omega)$, by $\widetilde{G} = \widetilde{\mathrm{Symp}}_0(M,\omega)$ we denote the universal cover of $G = \mathrm{Symp}_0(M,\omega)$, and by $\widetilde{H} \subseteq \widetilde{G}$ we denote those elements of $\widetilde{G}$ whose endpoint belongs to H. Next, by $H_0 \subseteq G$ we denote the C^0 closure of H inside G, and by $\widetilde{H}_0 \subseteq \widetilde{G}$ we denote the lift of H_0 to $\widetilde{G}$. Also, we use the notation $\mathrm{Map}_0(M)$ for the connected component of the identity in the space of all smooth maps $M \to M$.

Denote by $\Gamma \subset H^1(M,\mathbb{R})$ the flux group, i.e. the image of $\widetilde{H}$ (or, equivalently, of $\pi_1(G)$) under the flux homomorphism, and by $\Gamma_0 \subset H^1(M,\mathbb{R})$ the image of $\widetilde{H}_0$ under the flux homomorphism. It is not hard to see that the C^0 flux conjecture is equivalent to the equality $\Gamma_0 = \Gamma$ (this follows from the well-known fact that for a path ϕ^t, $t \in [0,1]$ of symplectic diffeomorphisms, its endpoint ϕ^1 belongs to H if and only if its flux belongs to Γ). The restriction of the flux homomorphism to $\pi_1(G)$ admits a natural extension to a homomorphism (which we again call flux homomorphism) from $\pi_1(\mathrm{Map}_0(M))$ to the $H^1(M,\mathbb{R})$. Following [L-M-P], we denote by Γ_{top} the image of $\pi_1(\mathrm{Map}_0(M))$ under the flux homomorphism. Consider the evaluation homomorphism $ev : \pi_1(\mathrm{Map}_0(M)) \to \pi_1(M)$. For any $a \in \pi_1(M)$ we denote by $\Gamma^a_{top} \subseteq \Gamma$ the image of $ev^{-1}(a) \subseteq \pi_1(\mathrm{Map}_0(M))$ under the flux homomorphism.

The following result was proved in [L-M-P]:

Theorem 1.1. *If M is Lefschetz, then $\Gamma_0 \subseteq \Gamma_{top}$.*

As a consequence, Lalonde, McDuff and Polterovich conclude:

Corollary 1.2. *Assume that M is Lefschetz and that $\Gamma_{top} = \Gamma$. Then the C^0 flux conjecture holds for M.*

As an example, one can take M to be a closed Kähler manifold of nonpositive curvature such that its fundamental group has a trivial center. As another example, one can take the $2n$-dimensional torus with a translation invariant symplectic structure. See [L-M-P] for more details.

Now we turn to our results. Our main result is:

Theorem 1.3. *Let (M, ω) be a closed symplectic manifold. Then $\Gamma_0 \subseteq \Gamma_{top} + \overline{\Gamma^e_{top}}$, where $e \in \pi_1(M)$ is the identity, and $\overline{\Gamma^e_{top}} \subseteq H^1(M, \mathbb{R})$ is the closure of Γ^e_{top} inside $H^1(M, \mathbb{R})$.*

As a result, we obtain the following corollary:

Corollary 1.4. *Let (M, ω) be a closed symplectic manifold such that $\Gamma_{top} = \Gamma$. Then the C^0 flux conjecture holds for M.*

Indeed, if $\Gamma_{top} = \Gamma$, then since Γ is closed (the closeness of Γ is exactly the statement of the C^1 flux conjecture proven in [O-1]), it follows that $\Gamma_{top} + \overline{\Gamma^e_{top}} = \Gamma$ and hence $\Gamma_0 = \Gamma$ by Theorem 1.3.

In particular, the C^0 flux conjecture holds for a closed symplectically aspherical symplectic manifold (M, ω) such that the fundamental group $\pi_1(M)$ has a trivial center. Indeed, in this case, since the center of $\pi_1(M)$ is trivial, we get $ev(\pi_1(\mathrm{Map}_0(M))) = \{e\}$ and so $\Gamma_{top} = \Gamma^e_{top}$, and moreover, since M is symplectically aspherical, we conclude that $\Gamma^e_{top} = \{0\}$. Therefore $\Gamma_{top} = 0$ and hence $\Gamma_{top} = \Gamma = \{0\}$.

As another example, we get that the C^0 flux conjecture holds for any product $(M, \omega) = (\mathbb{T}^{2k} \times N, \sigma \oplus \tau)$, where $(\mathbb{T}^{2k}, \sigma)$ is a symplectic torus with a translation invariant σ, and (N, τ) is a closed symplectically atoroidal symplectic manifold. Indeed, since $\mathbb{T}^{2k}$ is symplectically aspherical, and N is symplectically atoroidal, it follows that for any $a = (b, c) \in \pi_1(M) \cong \pi_1(\mathbb{T}^{2k}) \times \pi_1(N)$ and any $\mathfrak{a} \in \pi_1(\mathrm{Map}_0(M))$ with $ev(\mathfrak{a}) = a$, the flux of $\mathfrak{a}$ is uniquely determined by b. Moreover, since translations of the torus $\mathbb{T}^{2k}$ generate a large enough subgroup of symplectomorphisms of $(\mathbb{T}^{2k}, \sigma)$, for any $b \in \pi_1(\mathbb{T}^{2k})$ we can find an element $\mathfrak{b} \in \pi_1(G) = \pi_1(\mathrm{Symp}_0(M, \omega))$ such that $ev(\mathfrak{b}) = (b, 0)$. Therefore we conclude $\Gamma_{top} = \Gamma$.

Remark 1.5. *The reader may ask if there exist examples of closed symplectic manifolds for which $\Gamma_{top} \neq \Gamma$. The following construction is due to Seidel [Se-1, Se-2]. Let (N, ω_N) be a closed symplectic manifold with $H^1(N, \mathbb{R}) = 0$, let $\psi : N \to N$ be a symplectic diffeomorphism which is smoothly isotopic to the identity, but which is not isotopic to the identity via a smooth path of symplectic diffeomorphisms. Look at the symplectic mapping torus $E = E_\psi$ of ψ, which is the total space of the fibration over the two-torus with fibre N and monodromy ψ in one direction, or explicitly,*

$$E = \mathbb{R}^2 \times N / (p, q, x) \sim (p - 1, q, x) \sim (p, q - 1, \psi(x)),$$

$$\omega_E = dp \wedge dq + \omega_N.$$

Because ψ is smoothly isotopic to the identity, the fibration $E \to \mathbb{T}^2$ is trivial as a smooth one, and it is easy to see that for E we have $\Gamma_{top} = H^1(E, \mathbb{Z})$. However, it is possible that for E we have $\Gamma \neq H^1(E, \mathbb{Z})$. The closed 1-form dp on E generates the symplectic vector field $\frac{\partial}{\partial q}$, whose time-1 map is $\phi(p, q, x) = (p, q+1, x) = (p, q, \psi(x))$. If ϕ turns out to be a non-Hamiltonian diffeomorphism, then we get that $[dp] \notin \Gamma$, so in particular $\Gamma \neq \Gamma_{top}$. One way of detecting this is by looking at the Floer cohomology $HF^*(\psi)$. That is, if we are in a situation when $HF^*(\psi)$ has total rank different from that of $H^*(N)$, then $HF^*(\phi) \cong H^*(\mathbb{T}^2) \otimes HF^*(\psi)$ is not isomorphic to $H^*(E)$, and hence in particular ϕ is a non-Hamiltonian diffeomorphism (here we consider cohomologies with coefficients in the corresponding Novikov ring).

1.2 The C^0 rigidity of Hamiltonian paths

Our next result is:

Theorem 1.6. *Let (M, ω) be a closed symplectic manifold. Fix a Riemannian metric g on M, which induces a distance function $d : M \times M \to \mathbb{R}$, which in turn, induces a distance d between maps $M \to M$: for any $f, h : M \to M$ we set $d(f, h) = \sup_{x \in M} d(f(x), h(x))$. Fix a norm $|\cdot|$ on $H^1(M, \mathbb{R})$. Then there exist constants $c = c(M, \omega, g), C = C(M, \omega, g, |\cdot|)$, such that for any path ϕ^t, $t \in [0,1]$ of symplectomorphisms of M, $\phi^0 = id_M$, $\phi^1 = \phi$, with $\max_{t \in [0,1]} d(id_M, \phi^t) < c$, we have $|Flux(\{\phi^t\})| \leqslant C d(id_M, \phi)$.*

Theorem 1.6 has a direct corollary:

Corollary 1.7. 1) *On any closed symplectic manifold, the flux homomorphism is continuous with respect to the C^0 distance between smooth paths of symplectomorphisms.*
2) *C^0 rigidity of Hamiltonian paths: on any closed symplectic manifold, the space of smooth Hamiltonian isotopies of M is C^0-closed in the space of smooth symplectic isotopies of M. This confirms the mentioned above conjecture of Seyfaddini.*

Let us remark, that there is another weak version of the C^0 flux conjecture, which is due to Seyfaddini, and it concerns with the topological (continuous) Hamiltonian dynamics initially introduced by Oh and Müller [Oh-M]: Is it true that any Hamiltonian homeomorphism (in the sense of [Oh-M]) which belongs to $\mathrm{Symp}_0(M, \omega)$, is in fact a Hamiltonian diffeomorphism?

Notations

Let $A > 0$. We denote by $B(A) \subset \mathbb{R}^2$ the open euclidean disc centered at the origin having area A, i.e. $B(A) = \{z \in \mathbb{R}^2 \,|\, \pi|z|^2 < A\}$. We denote $S(A) = \partial B(A) = \{z \in \mathbb{R}^2 \,|\, \pi|z|^2 = A\}$, the euclidean circle centered at the origin enclosing a disc of area A. Also, we use the notation $B'(A) = B(A) \setminus \{0\} \subset \mathbb{R}^2$ for the punctured disc. On T^*S^1 with canonical coordinates (q,p), where $q \in \mathbb{R}/\mathbb{Z}$, $p \in \mathbb{R}$, and with the standard symplectic form $dp \wedge dq$, we will use the notation $S^1 \subset T^*S^1$ for the zero-section, and we denote $W(A) = \{(q,p) \,|\, |p| < A\} \subset T^*S^1$, so that $W(A)$ is a neighbourhood of the zero-section in T^*S^1 having area $2A$.

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2 Proofs

Consider the evaluation homomorphism $ev : \pi_1(\mathrm{Map}_0(M)) \to \pi_1(M)$. In the next lemma we show that its restriction to $\pi_1(G)$ can be naturally extended to a homomorphism $ev' : \widetilde{H}_0 \to \pi_1(M)$:

Lemma 2.1. *The homomorphism $ev|_{\pi_1(G)} : \pi_1(G) \to \pi_1(M)$ admits a natural extension to a homomorphism $ev' : \widetilde{H}_0 \to \pi_1(M)$.*

Proof. Let us first present a construction of ev'. Fix a Riemannian metric g on M. Let $\tilde{\phi} \in \widetilde{H}$, let $\phi^t, t \in [0,1]$ be a path of symplectomorphisms representing $\tilde{\phi}$, and denote $\phi = \phi^1$. Consider a Hamiltonian diffeomorphism $\psi \in H$ such that ψ is sufficiently C^0-close to ϕ (it is possible to find such a Hamiltonian diffeomorphism, since by our assumption ϕ lies in the C^0 closure of H inside G). Let $\psi^t, t \in [0,1]$ be a Hamiltonian isotopy of M, such that $\psi^1 = \psi$. Define a continuous loop f^t, $t \in [0,3]$ in $\mathrm{Map}_0(M)$, such that $f^t = \phi^t$ for $t \in [0,1]$, such that for any $x \in M$, the path $f^t(x)$, $t \in [1,2]$ is the shortest g-geodesic connecting $\phi(x)$ and $\psi(x)$, and such that $f^t = \psi^{3-t}$ for $t \in [2,3]$. We now define $ev'(\phi)$ to be the value of the evaluation map ev at the loop f^t, $t \in [0,3]$.

Let us show that the definition does not depend on the choice of ψ and of the path ψ^t, $t \in [0,1]$. Let ψ_1^t, ψ_2^t, $t \in [0,1]$ be two Hamiltonian isotopies of M, such that $\psi_1 = \psi_1^1$ and $\psi_2 = \psi_2^1$ are sufficiently C^0-close to ϕ. Define the corresponding loops f_1^t, f_2^t, $t \in [0,3]$ as above. Define the loop h_1^t, $t \in [0,6]$ in $\mathrm{Map}_0(M)$ by $h_1^t = f_1^{3-t}$, $t \in [0,3]$ and $h_1^t = f_2^{t-3}$, $t \in [3,6]$. It is enough to show that the value of ev at the loop h_1^t, $t \in [0,6]$ equals $e \in \pi_1(M)$. Clearly, the loop h_1^t, $t \in [0,6]$ is homotopic to the loop h_2^t, $t \in [0,4]$, where $h_2^t = \psi_1^t$ for $t \in [0,1]$, where for any $x \in M$ the path $h_2^t(x)$, $t \in [1,2]$ is the shortest g-geodesic between $\psi_1(x)$ and $\phi(x)$, and the path $h_2^t(x)$, $t \in [2,3]$ is the shortest g-geodesic between $\phi(x)$ and $\psi_2(x)$, and finally $h_2^t = \psi_2^{4-t}$ for $t \in [3,4]$. Also, since ψ_1 and ψ_2 are C^0 close to ϕ, it follows that the loop h_2^t, $t \in [0,4]$ is homotopic to the loop h_3^t, $t \in [0,3]$, where $h_3^t = \psi_1^t$ for $t \in [0,1]$, where for any $x \in M$ the path $h_3^t(x)$, $t \in [1,2]$ is the shortest g-geodesic between $\psi_1(x)$ and $\psi_2(x)$, and where $h_3^t = \psi_2^{3-t}$ for $t \in [2,3]$. It is enough to show that the value of the evaluation map ev at h_3^t, $t \in [0,3]$ equals $e \in \pi_1(M)$. Now pick some increasing bijective smooth function $\nu : [0,1] \to [0,1]$, such that the derivatives of ν of all orders vanish at $1 \in [0,1]$, and look at the smooth Hamiltonian flow h^t, $t \in [0,2]$ defined by $h^t = \psi_1^{\nu(t)}$ for $t \in [0,1]$ and $h^t = \psi_2^{\nu(2-t)}$ for $t \in [1,2]$. Then by the solution of the Arnold's conjecture [C-Z, F1, F2, O-2, H-S, Li-T, Fu-O-1, R, Fu-O-2][1], the time-2 map h^2 of the Hamiltonian flow h^t, $t \in [0,2]$ has a fixed point $p \in M$, such that its trajectory under the flow is a contractible loop. Hence we get $\psi_1(p) = \psi_2(p)$, and as a result, the loop $t \mapsto h_3^t(p)$, $t \in [0,3]$ is contractible. Therefore the value of the evaluation map ev at the loop h_3^t, $t \in [0,3]$ equals $e \in \pi_1(M)$.

Finally, it is easy to see the independence of ev' of the choice of metric g, and that ev' is a homomorphism.

□

The main step in the proof of Theorem 1.3 is the following proposition:

Proposition 2.2. *Let (M,ω) be a closed symplectic manifold, and let $\tilde{\phi} \in \widetilde{H}_0$. Then $Flux(\tilde{\phi}) \in \overline{\Gamma^a_{top}}$, where $a = ev'(\tilde{\phi}) \in \pi_1(M)$.*

Proof. Choose a Riemannian metric g on M. Denote by $\phi \in G$ the endpoint of $\tilde{\phi}$, and let ϕ^t, $t \in [0,1]$ be a symplectic isotopy of M representing $\tilde{\phi}$, such that $\phi^0 = \mathrm{id}_M$, $\phi^1 = \phi$. Choose a smooth closed loop $\gamma : [0,1] \to M$, and define the loop $\alpha = \phi \circ \gamma$. There exists a neighbourhood U of $\alpha([0,1])$ which is symplectomorphic to the product of a neighbourhood of the zero-section in T^*S^1 and $n-1$ small 2-dimensional discs, i.e. there exists $\epsilon > 0$, such that for $W(\epsilon) = \{(q,p) \in T^*S^1 \,|\, |p| < \epsilon\} \subset T^*S^1$ (here $S^1 \cong \mathbb{R}/\mathbb{Z}$, so that the symplectic area of $W(\epsilon)$ in T^*S^1 is 2ϵ) and for the standard

[1]Strictly speaking, except for the case of a semi-positive symplectic manifold, the existing proofs of the Arnold's conjecture are conditional to the virtual cycle techniques which are not yet accepted by all the experts.

2-dimensional disc $B(\epsilon) \subset \mathbb{R}^2$ of area ϵ centered at the origin, we have a symplectic embedding $\iota : W(\epsilon) \times B(\epsilon)^{\times n-1} \to M$, such that $S^1 \times \{0\} \times ... \times \{0\}$ is mapped onto $\alpha([0,1])$, where $S^1 \subset T^*S^1$ is the zero-section. We set $U = \iota(W(\epsilon) \times B(\epsilon)^{\times n-1})$.

Now let $\psi \in H$ be sufficiently C^0-close to ϕ, and let ψ^t, $t \in [0,1]$ be a Hamiltonian isotopy on M such that $\psi^1 = \psi$. Define, as in the proof of Lemma 2.1, a continuous loop f^t, $t \in [0,3]$ in $\mathrm{Map}_0(M)$, such that $f^t = \phi^t$ for $t \in [0,1]$, such that for any $x \in M$, the path $f^t(x)$, $t \in [1,2]$ is the shortest g-geodesic connecting $\phi(x)$ and $\psi(x)$, and such that $f^t = \psi^{3-t}$ for $t \in [2,3]$. Then since ψ is sufficiently C^0-close to ϕ, Lemma 2.1 tells us that the value of ev at the loop f^t, $t \in [0,3]$ equals to $ev'(\tilde{\phi})$. Define smooth cylinders $w, u, v : [0,1] \times [0,1] \to M$ by $w(s,t) = f^t(\gamma(s)) = \phi^t(\gamma(s))$, $u(s,t) = f^{t+1}(\gamma(s))$, $v(s,t) = f^{t+2}(\gamma(s)) = \psi^{1-t}(\gamma(s))$, for $(s,t) \in [0,1] \times [0,1]$. The loop $\beta := \psi \circ \gamma$ is C^0-close to the loop $\alpha = \phi \circ \gamma$, hence the image $\beta([0,1])$ lies inside U, and moreover the image $u([0,1] \times [0,1])$ lies inside U. The union of the images of w, u, v is a torus, which is the isotopy of the loop γ via the path f^t, $t \in [0,3]$. We therefore have the equality $\omega(w) + \omega(u) + \omega(v) = \mathrm{Flux}(f^t)(\gamma)$. We have $\omega(w) = \mathrm{Flux}(\phi^t)(\gamma)$, and $\omega(v) = 0$ since the isotopy ψ^t, $t \in [0,1]$ is Hamiltonian. Thus we get $\mathrm{Flux}(f^t)(\gamma) - \mathrm{Flux}(\phi^t)(\gamma) = \omega(u)$. Hence it is enough to show that for any initially chosen loop $\gamma : [0,1] \to M$ as above, the symplectic area $\omega(u)$ is arbitrarily small, provided that ψ is sufficiently C^0-close to ϕ.

Let us show this by a contradiction. Assume the contrary, i.e. that there exists some $\epsilon' > 0$, such that one can find ψ arbitrarily C^0-close to ϕ for which we have $|\omega(u)| \geqslant \epsilon'$. WLOG we may assume that $\epsilon' < \epsilon$. Now pick $\psi \in H$ which is sufficiently C^0-close to ϕ and such that $|\omega(u)| \geqslant \epsilon'$. Recall that we have a symplectic embedding $\iota : W(\epsilon) \times B(\epsilon)^{\times n-1} \to M$, such that $S^1 \times \{0\} \times ... \times \{0\}$ is mapped onto $\alpha([0,1])$, and we have $U = \iota(W(\epsilon) \times B(\epsilon)^{\times n-1})$. Put $\delta = \epsilon'/2$, and consider the Lagrangian $L = S^1 \times S(\delta) \times ... \times S(\delta) \subset W(\epsilon') \times B(\epsilon')^{\times n-1} \subset W(\epsilon) \times B(\epsilon)^{\times n-1}$, where $S(\delta) = \{z \in \mathbb{R}^2 \,|\, \pi|z|^2 = \delta\}$ is the circle on $\mathbb{R}^2$ centered at the origin enclosing a disc of area δ. Since ψ is sufficiently C^0-close to ϕ, it follows that $\psi \circ \phi^{-1}$ is sufficiently C^0-close to id_M, and in particular $\psi \circ \phi^{-1}(\iota(W(\epsilon') \times B(\epsilon')^{\times n-1})) \subset U = \iota(W(\epsilon) \times B(\epsilon)^{\times n-1})$, and $\psi \circ \phi^{-1}(\iota(L)) \subset \iota(W(\epsilon'/4) \times B'(\epsilon)^{\times n-1})$ (recall that $B'(\epsilon) = B(\epsilon) \setminus \{0\} \subset \mathbb{R}^2$ is the open punctured euclidean disc centered at the origin having area ϵ). Now, if we denote $\tilde{L} := \iota^{-1}(\psi \circ \phi^{-1}(\iota(L))) \subset W(\epsilon'/4) \times B'(\epsilon)^{\times n-1}$, then $\tilde{L}$ is a Lagrangian which is C^0 close to L, and $\pi_1(\tilde{L})$ is generated by the loops $\tilde{\beta}_1, ..., \tilde{\beta}_n$, which are the push-forwards of the loops $\beta_1, ..., \beta_n$ on $L = S^1 \times S(\delta) \times ... \times S(\delta)$, such that the homotopy classes of $\beta_1, ..., \beta_n$ in $\pi_1(L)$ correspond to the factors $S^1, S(\delta), ..., S(\delta)$. Consider the 1-form $\lambda = p_0 dq_0 + \frac{1}{2}(x_1 dy_1 - y_1 dx_1) + ... + \frac{1}{2}(x_{n-1} dy_{n-1} - y_{n-1} dx_{n-1}) = p_0 dq_0 + \frac{1}{2} r_1^2 d\theta_1 + ... + \frac{1}{2} r_{n-1}^2 d\theta_{n-1}$ on $W(\epsilon) \times B(\epsilon)^{\times n-1}$. Then $d\lambda = dp_0 \wedge dq_0 + dx_1 \wedge dy_1 + ... + dx_{n-1} \wedge dy_{n-1}$ is the standard symplectic form on $W(\epsilon) \times B(\epsilon)^{\times n-1}$. Since the map $\iota^{-1} \circ \psi \circ \phi^{-1} \circ \iota$ is well defined on $W(\epsilon') \times B(\epsilon')^{\times n-1}$, and is symplectic, the 1-form $(\iota^{-1} \circ \psi \circ \phi^{-1} \circ \iota)^* \lambda - \lambda$ on $W(\epsilon') \times B(\epsilon')^{\times n-1}$ is closed, and its evaluation at the loop $S^1 \times \{0\} \times ... \times \{0\} \subset W(\epsilon') \times B(\epsilon')^{\times n-1}$ equals to the symplectic area $\omega(u)$. Hence at the level of cohomology we have $[(\iota^{-1} \circ \psi \circ \phi^{-1} \circ \iota)^* \lambda - \lambda] = \omega(u)[dq]$. In particular, we have $\lambda(\tilde{\beta}_1) = (\iota^{-1} \circ \psi \circ \phi^{-1} \circ \iota)^* \lambda(\beta_1) = \omega(u) + \lambda(\beta_1) = \omega(u)$, and for $2 \leqslant j \leqslant n$ we have $\lambda(\tilde{\beta}_j) = (\iota^{-1} \circ \psi \circ \phi^{-1} \circ \iota)^* \lambda(\beta_j) = \lambda(\beta_j) = \delta$. Now let us present two possible ways of finishing the proof via arriving to a contradiction. The second way is easier and it was suggested by Seyfaddini.

First way: Consider the 1-form $\lambda' = \frac{1}{2}(x_0 dy_0 - y_0 dx_0) + \frac{1}{2}(x_1 dy_1 - y_1 dx_1) + ... + \frac{1}{2}(x_{n-1} dy_{n-1} - y_{n-1} dx_{n-1}) = \frac{1}{2} r_0^2 d\theta_0 + \frac{1}{2} r_1^2 d\theta_1 + ... + \frac{1}{2} r_{n-1}^2 d\theta_{n-1}$ on $B(\epsilon'/2) \times B'(\epsilon)^{\times n-1}$ endowed with coordinates $(x_0, y_0, ..., x_{n-1}, y_{n-1})$, where $x_i = r_i \cos \theta_i$, $y_i = r_i \sin \theta_i$ for $i = 0, 1, ..., n-1$. We have that $d\lambda' = dx_0 \wedge dy_0 + dx_1 \wedge dy_1 + ... + dx_{n-1} \wedge dy_{n-1}$ is the standard symplectic form on $B(\epsilon'/2) \times B'(\epsilon)^{\times n-1}$. Consider the embedding $\iota' : W(\epsilon'/4) \times B'(\epsilon)^{\times n-1} \hookrightarrow B(\epsilon'/2) \times B'(\epsilon)^{\times n-1}$, given by $\iota'(q_0, p_0, x_1, y_1, ..., x_{n-1}, y_{n-1}) = (r_0, \theta_0, x_1, y_1, ..., x_{n-1}, y_{n-1})$, $\pi r_0^2 = p_0 + \epsilon'/4$, $\theta_0 = 2\pi q_0$. Then we have $(\iota')^* \lambda' = \lambda + \frac{1}{4} \epsilon' dq_0$. Hence for the Lagrangian $\hat{L} = \iota'(\tilde{L}) \subset B(\epsilon'/2) \times B'(\epsilon)^{\times n-1} \subset \mathbb{R}^2 \times B'(\epsilon)^{\times n-1}$ and the loops $\hat{\beta}_j := \iota' \circ \tilde{\beta}_j$, $j = 1, ..., n$, generating $\pi_1(\hat{L})$, we have $\lambda'(\hat{\beta}_1) = \omega(u) + \epsilon'/4$. Therefore, if we consider $\hat{L}$ as a Lagrangian inside $\mathbb{R}^2 \times B'(\epsilon)^{\times n-1}$ endowed with the standard symplectic form, then it follows that the symplectic area of any disc in $\mathbb{R}^2 \times B'(\epsilon)^{\times n-1}$ with boundary on $\hat{L}$, is an integer multiple of $\omega(u) + \epsilon'/4$, and hence its absolute value is $\geqslant |\omega(u)| - \epsilon'/4 \geqslant \epsilon' - \epsilon'/4 = 3\epsilon'/4 > \epsilon'/2$. By the Chekanov's theorem [Ch], the displacement energy $e(\hat{L})$ of $\hat{L}$ inside $\mathbb{R}^2 \times B'(\epsilon)^{\times n-1}$ is greater than or equal to the minimal area of a non-constant holomorphic disc with boundary on $\hat{L}$. Hence we conclude that $e(\hat{L}) > \epsilon'/2$. But on the other hand, since $\hat{L} \subset B(\epsilon'/2) \times B'(\epsilon)^{\times n-1}$, one can clearly displace $\hat{L}$ with a Hamiltonian isotopy of energy $\epsilon'/2$. Contradiction.

Second way: Consider the Liouville form $\lambda' = p_0 dq_0 + p_1 dq_1 + ... + p_{n-1} dq_{n-1}$ on $T^* \mathbb{T}^n$. Let $\iota' : W(\epsilon'/4) \times B'(\epsilon)^{\times n-1} \to T^* \mathbb{T}^n$ be the symplectic embedding given by

$$\iota'(q_0, p_0, r_1, \theta_1, ..., r_{n-1}, \theta_{n-1}) = (q_0, p_0, q_1, p_1, ..., q_{n-1}, p_{n-1}),$$

$\pi r_i^2 - \delta = p_i$, $\theta_i = 2\pi q_i$ for $i = 1, 2, ..., n-1$. The image of ι' lies inside $W(\epsilon'/4) \times (T^* S^1)^{\times n-1} = W(\epsilon'/4) \times T^* \mathbb{T}^{n-1}$. We have $(\iota')^* \lambda' = \lambda - \frac{\delta}{2\pi} d\theta_1 - ... - \frac{\delta}{2\pi} d\theta_{n-1}$. Hence for the Lagrangian $\hat{L} = \iota'(\tilde{L}) \subset W(\epsilon'/4) \times T^* \mathbb{T}^{n-1} \subset T^* \mathbb{T}^n$ and the loops $\hat{\beta}_j := \iota' \circ \tilde{\beta}_j$, $j = 1, ..., n$, generating $\pi_1(\hat{L})$, we have $\lambda'(\hat{\beta}_1) = \omega(u)$, and $\lambda'(\hat{\beta}_j) = 0$, $2 \leqslant j \leqslant n$. Now consider the symplectic shift $\Phi : T^* \mathbb{T}^n \to T^* \mathbb{T}^n$ given by $\Phi(q_0, p_0, q_1, p_1, ..., q_{n-1}, p_{n-1}) = (q_0, p_0 - \omega(u), q_1, p_1, ..., q_{n-1}, p_{n-1})$. Then it follows that the shifted Lagrangian $\Phi(\hat{L}) \subset T^* \mathbb{T}^n$ is exact, and moreover it does not intersect the zero-section, since $\hat{L} \subset W(\epsilon'/4) \times T^* \mathbb{T}^{n-1}$ and $|\omega(u)| \geqslant \epsilon' > \epsilon'/4$. However, by the theorem of Gromov [G] (see section 2.3.B$_4''$ in [G]), a closed exact Lagrangian submanifold of a cotangent bundle must intersect the zero-section. Contradiction. $\square$

Now, Theorem 1.3 is a straightforward consequence of Proposition 2.2:

Proof of Theorem 1.3. By Proposition 2.2, for any $\tilde{\phi} \in \widetilde{H}_0$ we have $\mathrm{Flux}(\tilde{\phi}) \in \overline{\Gamma^a_{top}} = \Gamma^a_{top} + \overline{\Gamma^e_{top}} \subseteq \Gamma_{top} + \overline{\Gamma^e_{top}}$, where $a = ev'(\tilde{\phi}) \in \pi_1(M)$. Hence $\Gamma_0 \subseteq \Gamma_{top} + \overline{\Gamma^e_{top}}$.

$\square$

Now we turn to the proof of Theorem 1.6.

Proof of Theorem 1.6. It is clearly enough to prove that for any smooth embedded loop $\gamma : [0,1] \to M$, there exist constants $c = c(M, \omega, g, \gamma), C = C(M, \omega, g, \gamma)$, such that for any path ϕ^t, $t \in [0,1]$ of symplectomorphisms of M, $\phi^0 = \mathrm{id}_M$, $\phi^1 = \phi$, with $\max_{t\in[0,1]} d(\mathrm{id}_M, \phi^t) < c$, we have $|\mathrm{Flux}(\{\phi^t\})(\gamma)| \leqslant C d(\mathrm{id}_M, \phi)$. Now fix a smooth embedded loop $\gamma : [0,1] \to M$. Then a neighbourhood of $\gamma([0,1])$ in M is standard, and hence for some $\epsilon > 0$ one can find a symplectic embedding $\iota : W(\epsilon) \times B(\epsilon)^{\times n-1} \to M$, such that $\iota(S^1 \times \{0\} \times ... \times \{0\}) = \gamma([0,1])$. We set

$$c_1 = d(\iota(W(\epsilon/2) \times S(\epsilon/3)^{\times n-1}), M \setminus \iota(W(\epsilon) \times B'(\epsilon)^{\times n-1}))$$

(recall that the notation $B'(\epsilon) = B(\epsilon) \setminus \{0\} \subset \mathbb{R}^2$ stands for the punctured disc). Now let ϕ^t, $t \in [0,1]$ be a path of symplectomorphisms of M, $\phi^0 = \mathrm{id}_M$, $\phi^1 = \phi$, with $\max_{t\in[0,1]} d(\mathrm{id}_M, \phi^t) < c_1$. Define the "flux function" $\kappa : [0,1] \to \mathbb{R}$ by $\kappa(t) = \mathrm{Flux}(\{\phi^s\}_{s\in[0,t]})(\gamma)$. We assume that $\kappa(1) = \mathrm{Flux}(\{\phi^t\})(\gamma) \neq 0$. If $\max_{t\in[0,1]} |\kappa(t)| > \epsilon/3$, then we define $T \in [0,1]$ to be minimal such that $|\kappa(T)| = \epsilon/3$, otherwise we set $T = 1$. We have $|\kappa(t)| \leqslant \epsilon/3$ for all $t \in [0,T]$. Now, on $\iota(W(\epsilon) \times B(\epsilon)^{\times n-1})$, consider the time dependent vector field $X^t_\kappa = \kappa'(t) \frac{\partial}{\partial p_0}$, where $(q_0, p_0, x_1, y_1, ..., x_{n-1}, y_{n-1})$ are the standard coordinates on $W(\epsilon) \times B(\epsilon)^{\times n-1}$. Denote by X^t the time dependent symplectic vector field of the flow ϕ^t. The difference $Y^t = X^t - X^t_\kappa$, $t \in [0,T]$, is a time dependent Hamiltonian vector field on $\iota(W(\epsilon) \times B(\epsilon)^{\times n-1})$. With help of a cut-off, we can find a time dependent Hamiltonian vector field Z^t, $t \in [0,T]$ on M, such that $Y^t(x) = Z^t(x)$ for any $x \in \iota(W(\epsilon/2) \times B(\epsilon/2)^{\times n-1})$ and $t \in [0,T]$. Now look at the time dependent symplectic vector field $\tilde{X}^t = X^t - Z^t$, $t \in [0,T]$, on M, and denote by $\tilde{\phi}^t$, $t \in [0,T]$ its symplectic flow on M. Then $\psi^t := (\phi^t)^{-1} \circ \tilde{\phi}^t$, $t \in [0,T]$ is a Hamiltonian flow on M since it has zero flux at all times. Consider the Lagrangian $L = \iota(S^1 \times S(\epsilon/3) \times ... \times S(\epsilon/3)) \subset \iota(W(\epsilon) \times B(\epsilon)^{\times n-1}) \subset M$. We have that $\tilde{X}^t(x) = X^t_\kappa(x)$ for any $x \in \iota(W(\epsilon/2) \times B(\epsilon/2)^{\times n-1})$ and $t \in [0,T]$, and hence $\tilde{\phi}^t(\iota(q_0, p_0, x_1, y_1, ..., x_{n-1}, y_{n-1})) = \iota(q_0, p_0 + \kappa(t), x_1, y_1, ..., x_{n-1}, y_{n-1})$ whenever $\iota(q_0, p_0, x_1, y_1, ..., x_{n-1}, y_{n-1}) \in L$ and $t \in [0,T]$, and so for any $t \in [0,T]$, $\tilde{\phi}^t(L)$ is obtained from L by shifting it by $\kappa(t)$ in the "p_0 direction". Clearly there exists a constant $c_2 = c_2(M, \omega, g, \gamma, \iota)$, such that the distance $d(\tilde{\phi}^T(L), L)$ is greater than or equal to $c_2|\kappa(T)|$. Also note that since $d(\mathrm{id}_M, \phi^t) < c_1$ for any t, we get that $\psi^t(L) = (\phi^t)^{-1} \circ \tilde{\phi}^t(L) \subset \iota(W(\epsilon) \times B'(\epsilon)^{\times n-1})$ for any $t \in [0,T]$. Now, L cannot be Hamiltonianly displaced *inside* $\iota(W(\epsilon) \times B'(\epsilon)^{\times n-1})$, and so we must have $\psi^T(L) \cap L = (\phi^T)^{-1} \circ \tilde{\phi}^T(L) \cap L \neq \emptyset$. Since in addition we have $d(\tilde{\phi}^T(L), L) \geqslant c_2|\kappa(T)|$, we conclude that $d(\mathrm{id}_M, \phi^T) \geqslant c_2|\kappa(T)|$.

We have shown that for any path ϕ^t, $t \in [0,1]$ of symplectomorphisms of M, $\phi^0 = \mathrm{id}_M$, $\phi^1 = \phi$, with $\max_{t\in[0,1]} d(\mathrm{id}_M, \phi^t) < c_1$, we must have $d(\mathrm{id}_M, \phi^T) \geqslant c_2|\kappa(T)|$. Thus, if we set $c = \min(c_1, c_2\epsilon/3)$ and $C = 1/c_2$, then for any path ϕ^t, $t \in [0,1]$ of symplectomorphisms of M, $\phi^0 = \mathrm{id}_M$, $\phi^1 = \phi$, with $\max_{t\in[0,1]} d(\mathrm{id}_M, \phi^t) < c$, we have $c_2\epsilon/3 \geqslant c > d(\mathrm{id}_M, \phi^T) \geqslant c_2|\kappa(T)|$, hence $|\kappa(T)| \neq \epsilon/3$, which means that $T = 1$, and we therefore get $|\mathrm{Flux}(\{\phi^t\})(\gamma)| = |\kappa(1)| = |\kappa(T)| \leqslant Cd(\mathrm{id}_M, \phi^T) = Cd(\mathrm{id}_M, \phi)$.

□

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