THE GAP BETWEEN NEAR COMMUTATIVITY AND ALMOST COMMUTATIVITY IN SYMPLECTIC CATEGORY

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Abstract. On any closed symplectic manifold of dimension greater than 2, we construct a pair of smooth functions, such that on the one hand, the uniform norm of their Poisson bracket equals to 1, but on the other hand, this pair cannot be reasonably approximated (in the uniform norm) by a pair of Poisson commuting smooth functions. This comes in contrast with the dimension 2 case, where by a partial case of a result of Zapolsky [Z-2], an opposite statement holds.

1. Introduction and results

During the last ten years, function theory on symplectic manifolds has attracted a great deal of attention [B, BEP, CV, EP-1, EP-2, EP-3, EPR, EPZ, P, Z-1, Z-2]. The C^0-rigidity of the Poisson bracket [EP-2] (cf. [B]) is one of the achievements of this theory, and it states that on a closed symplectic manifold (M, ω), the uniform norm of the Poisson bracket of a pair of smooth functions on M is a lower semi-continuous functional, when we consider the uniform (or the C^0) topology on $C^\infty(M) \times C^\infty(M)$. Informally speaking, this means that one cannot significantly reduce the C^0 norm of the Poisson bracket of two smooth functions by an arbitrarily small C^0 perturbation. The C^0-rigidity of the Poisson bracket holds also when the symplectic manifold (M, ω) is open, if we restrict to smooth compactly supported functions. In this context it is natural to ask, how strongly one should perturb a given pair of smooth functions in order to significantly reduce the C^0 norm of their Poisson bracket. The following question was asked privately by Polterovich in 2009 (later it also appeared in [Z-2]):

Question 1.1. *Let (M, ω) be a closed symplectic manifold. Does there exist a constant $C > 0$, such that for any pair of smooth functions $f, g : M \to \mathbb{R}$ satisfying $\|\{f, g\}\| =$*

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1, there exists a pair of smooth functions $F, G : M \to \mathbb{R}$, such that $\|F - f\|, \|G - g\| \leqslant C$ and such that $\{F, G\} = 0$ on M?

The 2-dimensional case of this question was answered affirmatively by Zapolsky, and in fact, it appeared as a particular case of a more general statement which applies to functions on a manifold with a volume form [Z-2].

The main result of this note is

Theorem 1.2. *Let (M, ω) be a symplectic manifold of dimension $2n > 2$. Then for any $0 < p < q$ there exists a pair of smooth compactly supported functions $f, g : M \to \mathbb{R}$, such that $\|f\| = \|g\| = 1$, $\|\{f, g\}\| = q$, and such that for any $s \in [0, q]$ we have*

$$\frac{1}{2} - \frac{1}{2p}s \leqslant \rho_{f,g}(s) \leqslant \frac{1}{2} - \frac{1}{2q}s \tag{1.1}$$

Here $\rho_{f,g}(s)$ is the *profile function* [BEP], which is defined as

$$\rho_{f,g}(s) = \inf\{\|F - f\| + \|G - g\| \mid F, G \in C_c^\infty(M) \text{ and } \|\{F, G\}\| \leqslant s\}.$$

Now let (M, ω) be a symplectic manifold of dimension $2n > 2$. Note that for a pair $f, g : M \to \mathbb{R}$ of smooth compactly supported functions, the value $\rho_{f,g}(0)$ is precisely the uniform distance from the pair f, g of functions to the set of pairs $F, G : M \to \mathbb{R}$ of smooth compactly supported Poisson commuting functions. Hence, if we take any $C > 0$, and set $q = 1/(64C^2)$, then Theorem 1.2 implies the existence of a pair $f, g : M \to \mathbb{R}$ of smooth compactly supported functions, such that $\|\{f, g\}\| = q = 1/(64C^2)$ and $\rho_{f,g}(0) = 1/2$, and hence for the functions $f_1 = 8Cf$, $g_1 = 8Cg$ we first of all get $\|\{f_1, g_1\}\| = 1$, and moreover we get $\rho_{f_1,g_1}(0) = 4C$, which implies that there does not exist a pair $F, G : M \to \mathbb{R}$ of smooth compactly supported Poisson commuting functions, such that $\|F - f_1\|, \|G - g_1\| \leqslant C$. Thus, we obtain

Corollary 1.3. *The answer to Question 1.1 is negative in dimension > 2.*

The proof of Theorem 1.2 relies on a certain computation of the pb_4 invariant, which is carried out similarly as an analogous computation in the proof of Proposition 1.21 in [BEP]. Some part of the proof of Theorem 1.2 is reminiscent of the proof of Theorem 1.4 (ii) in [BEP].

Remark 1.4. *Question 1.1 has an analogue in the setting of Hermitian matrices equipped with a commutator (see* [PeS, H]*).*

2. Proofs

Let us first remind the definition of the pb_4 invariant which was initially introduced in [BEP] and which participates in the proof of Theorem 1.2.

Definition 2.1. *Let (M, ω) be a symplectic manifold. Let $X_0, X_1, Y_0, Y_1 \subset M$ be a quadruple of compact sets. Then we define*

$$pb_4(X_0, X_1, Y_0, Y_1) = \inf \|\{F, G\}\|,$$

where the infimum is taken over the set of all pairs of smooth compactly supported functions $F, G : M \to \mathbb{R}$ such that $F \leqslant 0$ on X_0, $F \geqslant 1$ on X_1, $G \leqslant 0$ on Y_0, and $G \geqslant 1$ on Y_1. Note that such set of pairs of functions is non-empty whenever $X_0 \cap X_1 = Y_0 \cap Y_1 = \emptyset$. If the latter condition is violated, we put

$$pb_4(X_0, X_1, Y_0, Y_1) = +\infty.$$

The following result was proved in [BEP] (in [BEP] this is Proposition 1.21):

Proposition 2.2. *Let (M, ω) be a symplectic surface of area $B > 0$. Consider a curvilinear quadrilateral $\Pi \subset M$ of area $A > 0$ with sides denoted in the cyclic order by a_1, a_2, a_3, a_4 - that is Π is a topological disc bounded by the union of four smooth embedded curves a_1, a_2, a_3, a_4 connecting four distinct points in M in the cyclic order as listed here and (transversally) intersecting each other only at their common end-points. Let L be an exact section of T^*S^1. Assume that $M \neq S^2$ and that $2A \leqslant B$. Then in the symplectic manifold $M \times T^*S^1$ (with the split symplectic form) we have*

$$pb_4(a_1 \times L, a_3 \times L, a_2 \times L, a_4 \times L) = 1/A > 0.$$

The proof of Proposition 2.2, which is presented in [BEP], is divided into two parts. The first part proves the inequality $pb_4(a_1 \times L, a_3 \times L, a_2 \times L, a_4 \times L) \leqslant 1/A$ by providing a concrete example of a pair of functions F, G as in Definition 2.1. The second part proves the inequality $pb_4(a_1 \times L, a_3 \times L, a_2 \times L, a_4 \times L) \geqslant 1/A$, and it uses the Gromov's theory of pseudo-holomorphic curves.

Now we turn to the proofs of our results.

Lemma 2.3. *Consider $\mathbb{R}^{2n}$ endowed with the standard symplectic structure ω_{std}. Then given any open set $U \subseteq \mathbb{R}^{2n}$ and any $A > 0$, there exists a smooth symplectic embedding $(B^2(A), \sigma_{std}) \to (U, \omega_{std})$, where $B^2(A) \subset \mathbb{R}^2$ is the disc of area A centred at the origin, and σ_{std} is the standard area form on $\mathbb{R}^2$ (and on $B^2(A)$).*

Proof. It is enough to construct a smooth embedding of $B^2(\pi)$ into $B^{2n}(2\pi)$ (where $B^{2n}(2\pi)$ is the $2n$-dimensional ball of capacity 2π, or equivalently, radius $\sqrt{2}$, centred at the origin), such that its image is a symplectic curve having arbitrarily large symplectic area. An example of such an embedding is the map $u : B^2(\pi) \to \mathbb{C}^n$ given by $u(z) = (z^k, z, 0, 0, ..., 0)$, where $k \in \mathbb{N}$ is large enough. □

Lemma 2.4. *Assume that we have a curvilinear quadrilateral of area $A > 0$ on the plane $\mathbb{R}^2$, with sides a_1, a_2, a_3, a_4 (written in the cyclic order), and let $L \subset T^*\mathbb{T}^{n-1}$ be an exact section. Then in the symplectic manifold $\mathbb{R}^2 \times T^*\mathbb{T}^{n-1}$ (with the split symplectic form) we have*

$$pb_4(a_1 \times L, a_3 \times L, a_2 \times L, a_4 \times L) = 1/A > 0.$$

Proof. The proof goes similarly as the proof of Proposition 2.2 (which is Proposition 1.21 in [BEP]). $\square$

Proof of Theorem 1.2. Consider a Darboux neighborhood $U \subset M$. Denote $l = 1/\sqrt{q}$, $\epsilon = 1/p - 1/q$. Then $q = 1/l^2$, $p = 1/(l^2+\epsilon)$. By Lemma 2.3, there exists a symplectic embedding $u : (B^2(2l^2+3\epsilon), \sigma_{std}) \to (U, \omega)$. By the symplectic neighborhood theorem, a neighborhood of $u(B^2(2l^2+3\epsilon))$ contains an open subset which is symplectomorphic to the product $B^2(2l^2+2\epsilon) \times B^2(\epsilon')^{\times(n-1)}$ of a disc of area $2l^2+2\epsilon$ and $n-1$ copies of a disc of area ϵ', for some $\epsilon' > 0$. Let $\phi : (W = B^2(2l^2+2\epsilon) \times B^2(\epsilon')^{\times(n-1)}, \omega_{std}) \to (M, \omega)$ be a symplectic embedding whose image is this open subset. Choose a curvilinear quadrilateral inside $B^2(2l^2+2\epsilon)$ with sides a_1, a_2, a_3, a_4 (which are given in cyclic order), and area $l^2+\epsilon$.

Claim 2.5. *There exist smooth compactly supported functions $\tilde{f}_1, \tilde{g}_1 : B^2(2l^2+2\epsilon) \to \mathbb{R}$, such that $\|\{\tilde{f}_1, \tilde{g}_1\}\| = 1/l^2 = q$, such that $\tilde{f}_1 = 0$ on a_1, $\tilde{f}_1 = 1$ on a_3, $\tilde{g}_1 = 0$ on a_2, $\tilde{g}_1 = 1$ on a_4, such that $\tilde{f}_1, \tilde{g}_1 \geqslant 0$ on $B^2(2l^2+2\epsilon)$ and such that $\|\tilde{f}_1\| = \|\tilde{g}_1\| = 1$.*

Proof. Denote $\epsilon_1 = \epsilon/l > 0$, and consider $\epsilon_2 > 0$ such that $(2l+\epsilon_1+3\epsilon_2)(l+3\epsilon_2) = 2l^2+2\epsilon$. Denote $\epsilon_3 = \min(\epsilon_1, \epsilon_2)/2$. It is enough to find smooth compactly supported functions $\hat{f}_1, \hat{g}_1 : (-\epsilon_2, 2l+\epsilon_1+2\epsilon_2) \times (-\epsilon_2, l+2\epsilon_2) \to \mathbb{R}$, such that $\|\{\hat{f}_1, \hat{g}_1\}\| = 1/l^2$, such that $\hat{f}_1 = 0$ on $\{0\} \times [0, l]$, $\hat{f}_1 = 1$ on $\{l+\epsilon_1\} \times [0, l]$, $\hat{g}_1 = 0$ on $[0, l+\epsilon_1] \times \{0\}$, $\hat{g}_1 = 1$ on $[0, l+\epsilon_1] \times \{l\}$, such that $\hat{f}_1, \hat{g}_1 \geqslant 0$ on $(-\epsilon_2, 2l+\epsilon_1+2\epsilon_2) \times (-\epsilon_2, l+2\epsilon_2)$ and such that $\|\hat{f}_1\| = \|\hat{g}_1\| = 1$. We give an explicit construction. First, choose smooth functions $u_1, v_1, u_2, v_2 : \mathbb{R} \to [0, 1]$ with the following properties:

(a) $\mathrm{supp}(u_1) \subset (0, 2l+\epsilon_1+\epsilon_2)$, $u_1(l+\epsilon_1) = 1$, $\|u_1'\| = 1/(l+\epsilon_3)$.
(b) $\mathrm{supp}(v_1) \subset (-\epsilon_2, l+\epsilon_2)$, $v_1(y) = 1$ on $[0, l]$.
(c) $\mathrm{supp}(u_2) \subset (-\epsilon_2, 2l+\epsilon_1+2\epsilon_2)$, $u_2(x) = 1$ on $[0, 2l+\epsilon_1+\epsilon_2]$.
(d) $\mathrm{supp}(v_2) \subset (0, l+2\epsilon_2)$, $v_2(l) = 1$,
$\max_{[0,l]} |v_2'(y)| = \max_{[0,l+\epsilon_2]} |v_2'(y)| = (l+\epsilon_3)/l^2$.

Now define $\hat{f}_1, \hat{g}_1 : (-\epsilon_2, 2l+\epsilon_1+2\epsilon_2) \times (-\epsilon_2, l+2\epsilon_2) \to \mathbb{R}$ by $\hat{f}_1(x, y) = u_1(x)v_1(y)$, $\hat{g}_1(x, y) = u_2(x)v_2(y)$. Then

$$\begin{aligned}\{\hat{f}_1, \hat{g}_1\}(x, y) &= v_1(y)u_2(x)u_1'(x)v_2'(y) - u_1(x)v_2(y)u_2'(x)v_1'(y) \\ &= v_1(y)u_2(x)u_1'(x)v_2'(y) = u_1'(x)v_2'(y)v_1(y).\end{aligned}$$

We have $\|u_1'\| = 1/(l+\epsilon_3)$, and

$$\|v_2'v_1\| = \max |v_2'(y)v_1(y)| = \max_{[0,l+\epsilon_2]} |v_2'(y)v_1(y)| = (l+\epsilon_3)/l^2,$$

and hence $\|\{\hat{f}_1, \hat{g}_1\}\| = \|u_1'\| \cdot \|v_2'v_1\| = 1/l^2$. The rest of the claimed properties of $\hat{f}_1, \hat{g}_1$ follow immediately. $\square$

Consider functions $\tilde{f}_1, \tilde{g}_1$, as in Claim 2.5. Choose $0 < \rho_1 < \rho < \rho_2$ such that $\pi\rho_2^2 < \epsilon'$, and choose a smooth function $\tilde{h}_1 : B^2(\epsilon') \to \mathbb{R}$ such that its support lies inside the annulus $D^2_{\rho_1,\rho_2} = \{z \in B^2(\epsilon') \,|\, \rho_1 < |z| < \rho_2\}$, such that $0 \leqslant \tilde{h}_1 \leqslant 1$ on $B^2(\epsilon')$, and such that $\tilde{h}_1 = 1$ on $S^1_\rho = \{z \in B^2(\epsilon') \,|\, |z| = \rho\}$. Now define $f_1, g_1 : W \to \mathbb{R}$ by

$$f_1(z, w_1, ..., w_{n-1}) = \tilde{f}_1(z)\tilde{h}_1(w_1)...\tilde{h}_1(w_{n-1}),$$

$$g_1(z, w_1, ..., w_{n-1}) = \tilde{g}_1(z)\tilde{h}_1(w_1)...\tilde{h}_1(w_{n-1}),$$

for $(z, w_1, ..., w_{n-1}) \in W$, and then define $f, g : M \to \mathbb{R}$ by setting $f = g = 0$ on $M \setminus \phi(W)$ and setting $f(\phi(x)) = f_1(x)$, $g(\phi(x)) = g_1(x)$, for any $x \in W$. First note that $f_1 = 0$ on $a_1 \times (S^1_\rho)^{n-1}$, $f_1 = 1$ on $a_3 \times (S^1_\rho)^{n-1}$, $g_1 = 0$ on $a_2 \times (S^1_\rho)^{n-1}$, and $g_1 = 1$ on $a_4 \times (S^1_\rho)^{n-1}$. Secondly, we have

$$\{f, g\}(\phi(z, w_1, ..., w_{n-1})) = \{\tilde{f}_1, \tilde{g}_1\}(z)\tilde{h}_1^2(w_1)...\tilde{h}_1^2(w_{n-1})$$

for any $(z, w_1, ..., w_{n-1}) \in W$, and hence $\|\{f, g\}\| = q$. Also we have that $f, g \geqslant 0$ on M, and that $\|f\| = \|g\| = 1$. Let us show that the constructed functions f, g satisfy (1.1). For showing the upper bound, choose a smooth compactly supported function $h : M \to [0, 1]$ such that $h = 1$ on the union of supports $\mathrm{supp}(f) \cup \mathrm{supp}(g)$, and define functions $F, G : M \to \mathbb{R}$ by $F = (\frac{1}{2} - \frac{s}{2q} + \frac{s}{q} f)h$, $G = g$ (the upper bound in (1.1) is essentially the statement of Theorem 1.4 (ii), inequality (8) in [BEP]). Let us show the lower bound, for a given value of s. Since the profile function $\rho_{f,g}$ is non-negative and non-increasing, it is sufficient to prove the lower bound for $s \in (0, p)$. Take $s \in (0, p)$ and denote $t = \frac{1}{2} - \frac{1}{2p}s > 0$. Now assume that we have a pair of smooth compactly supported functions $F, G \in C^\infty_c(M)$, such that $\|F - f\| + \|G - g\| \leqslant t$. Denote $\alpha = \|F - f\|$, $\beta = \|G - g\|$. Then $\alpha, \beta \geqslant 0$ and $\alpha + \beta \leqslant t$. Now choose $\delta > 0$ small, and pick smooth functions $u, v : \mathbb{R} \to \mathbb{R}$, such that the function u satisfies $u(x) = 0$ for $x \in [-\alpha, \alpha]$, $|u(x) - x| \leqslant (1 + \delta)\alpha$ for $x \in \mathbb{R}$, and $|u'(x)| \leqslant 1$ for $x \in \mathbb{R}$, and such that the function v satisfies $v(x) = 0$ for $x \in [-\beta, \beta]$, $|v(x) - x| \leqslant (1 + \delta)\beta$ for $x \in \mathbb{R}$, and $|v'(x)| \leqslant 1$ for $x \in \mathbb{R}$. Then the functions $F' = u \circ F$, $G' = v \circ G$ satisfy $\|\{F', G'\}\| \leqslant \|\{F, G\}\|$. But moreover, we have that $F'(x) = 0$ whenever $f(x) = 0$, $F'(x) \geqslant 1 - (2 + \delta)\alpha$ whenever $f(x) = 1$, $G'(x) = 0$ whenever $g(x) = 0$, and $G'(x) \geqslant 1 - (2 + \delta)\beta$ whenever $g(x) = 1$, for any $x \in M$. Hence if we denote $W' = B^2(2l^2 + 2\epsilon) \times (D^2_{\rho_1,\rho_2})^{\times(n-1)} \subset W$, then the supports of F', G' lie inside $\phi(W')$, so in particular the supports of ϕ^*F', ϕ^*G' lie inside W', and moreover we have that $\phi^*F' = 0$ on $a_1 \times (S^1_\rho)^{n-1}$, $\phi^*F' \geqslant 1 - (2+\delta)\alpha$ on $a_3 \times (S^1_\rho)^{n-1}$, $\phi^*G' = 0$ on $a_2 \times (S^1_\rho)^{n-1}$, and $\phi^*G' \geqslant 1 - (2 + \delta)\beta$ on $a_4 \times (S^1_\rho)^{n-1}$. Consider a symplectic embedding ψ of $W' = B^2(2l^2 + 2\epsilon) \times (D^2_{\rho_1,\rho_2})^{\times(n-1)}$ into $\mathbb{R}^2 \times (T^*S^1)^{\times(n-1)} = \mathbb{R}^2 \times T^*\mathbb{T}^{n-1}$ (where $\mathbb{T}^{n-1} = (S^1)^{n-1}$ is the $(n-1)$-dimensional torus), given as a product of maps, such that at the factor $B^2(2l^2 + 2\epsilon)$ we have the standard embedding into $\mathbb{R}^2$ (given by the identity map), and such that at each factor $D^2_{\rho_1,\rho_2}$, the circle S^1_ρ is mapped onto the zero section of T^*S^1. Denote the push-forwards $F_2 = \psi_*\phi^*F' = F' \circ \phi \circ \psi^{-1}$, $G_2 = \psi_*\phi^*G' = G' \circ \phi \circ \psi^{-1}$, which are a priori defined on $\psi(W')$, and extend them by 0 to obtain functions on the whole $\mathbb{R}^2 \times T^*\mathbb{T}^{n-1}$. We have $F_2 = 0$ on $a_1 \times L_0$, $F_2 \geqslant 1 - (2 + \delta)\alpha$ on $a_3 \times L_0$, $G_2 = 0$ on $a_2 \times L_0$, and $G_2 \geqslant 1 - (2 + \delta)\beta$ on $a_4 \times L_0$, where $L_0 \subset T^*\mathbb{T}^{n-1}$ is the zero section. Therefore in view of Lemma 2.4, we get

$$\|\{F,G\}\| \geqslant \|\{F',G'\}\| = \|\{F_2,G_2\}\|$$
$$\geqslant (1-(2+\delta)\alpha)(1-(2+\delta)\beta)\cdot pb_4(a_1\times L_0, a_3\times L_0, a_2\times L_0, a_4\times L_0)$$
$$= \frac{(1-(2+\delta)\alpha)(1-(2+\delta)\beta)}{l^2+\epsilon} \geqslant \frac{1-(2+\delta)(\alpha+\beta)}{l^2+\epsilon} \geqslant \frac{1-(2+\delta)t}{l^2+\epsilon} = (1-(2+\delta)t)p.$$

Since we can choose $\delta > 0$ to be arbitrarily small, we in fact get

$$\|\{F,G\}\| \geqslant (1-2t)p = s.$$

Thus we have shown that for any $F, G \in C_c^\infty(M)$ with $\|F-f\|+\|G-g\| \leqslant t$ we have $\|\{F,G\}\| \geqslant s$. This immediately implies

$$\rho_{f,g}(s) \geqslant t = \frac{1}{2} - \frac{1}{2p}s.$$

□

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