# SOME POSITIVE RESULTS AND COUNTEREXAMPLES IN COMONOTONE APPROXIMATION II

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Abstract. Let $f$ be a continuous function on $[-1,1]$, which changes its monotonicity finitely many times in the interval, say $s$ times. In the first part of this paper we have discussed the validity of Jackson type estimates for the approximation of $f$ by algebraic polynomials that are comonotone with it. We have proved the validity of a Jackson type estimate involving the Ditzian –Totik (first) modulus of continuity and a constant which depends only on $s$, and we have shown by counterexamples that in many cases the Jackson estimates involving the D–T moduli, do not hold when there are certain relations between $s$, the number of changes of monotonicity and $r$, the number of derivatives of the approximated function. Here we deal with all other cases and we obtain Jackson type estimates involving modified D–T moduli. We also provide counterexamples to complete the picture. Our technique for the positive results involves a two-tier approach. We first approximate the given function by comonotone piecewise polynomials which yield good approximation and then we replace the latter by polynomials

## §1. Introduction

Let $f \in \mathbb{C}[-1,1]$ change monotonicity finitely many times, say $s \geq 1$, in the interval, and we wish to approximate $f$ by polynomials $p_n \in \mathcal{P}_n$, the space of polynomials of degree not exceeding $n$, which are comonotone with $f$. To be specific, let $s \geq 1$ and let $\mathbb{Y}_s$ be the set of all collections $Y := \{y_i\}_{i=1}^s$ of points, $-1 < y_s < \dots < y_1 < 1$. For $Y \in \mathbb{Y}_s$ we set

$$\Pi(x,Y) := \prod_{i=1}^{s}(x-y_i) \ ,$$

and denote by $\Delta^{(1)}(Y)$ the set of functions $f \in \mathbb{C}[-1,1]$, which change monotonicity at the points $y_i$, and which are nondecreasing in $(y_1,1)$, that is, $f$ is nondecreasing in

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the intervals $(y_{2j+1}, y_{2j})$ and it is nonincreasing in $(y_{2j}, y_{2j-1})$. In particular, if $f$ is continuously differentiable in $(-1,1)$, then

$$f \in \Delta^{(1)}(Y), \quad \text{iff} \quad f'(x)\Pi(x,Y) \ge 0, \quad -1 < x < 1.$$

Put

$$\mathbb{Y} := \cup_s \mathbb{Y}_s \ ,$$

and recall that we call a collection $Y \in \mathbb{Y}$, $s$-admissible for $f$ and write $Y \in A_s(f)$, if $Y \in \mathbb{Y}_s$ and $f \in \Delta^{(1)}(Y)$. We write $f \in \Delta^{(1,s)}$, if $A_s(f) \ne \emptyset$. Note that a function may belong at the same time to different classes $\Delta^{(1,s_1)}$ and $\Delta^{(1,s_2)}$ (that is, with $s_1 \ne s_2$).

For $Y \in \mathbb{Y}$ and $f \in \mathbb{C}[-1,1]$ we denote

$$E_n^{(1)}(f,Y) := \inf\{\|f - p_{n-1}\| : p_{n-1} \in \Delta^{(1)}(Y) \cap \mathcal{P}_{n-1}\}, \tag{1.1}$$

and for $f \in \Delta^{(1,s)}$ we set

$$E_n^{(1,s)}(f) := \sup_{Y \in A_s(f)} E_n^{(1)}(f,Y). \tag{1.2}$$

As usual in $\mathbb{C}[-1,1]$, we denote by $\|\cdot\|$, the sup-norm over the interval. We will also have the notation $\|\cdot\|_J$ for the sup-norm over the interval $J$.

The first Jackson type estimates for truly comonotone polynomial approximation were obtained by Newman [11] (see also Iliev [5] for some relevant work) who proved that for $f \in \Delta^{(1,s)}$

$$E_n^{(1,s)}(f) \le c(s)\omega(f, 1/n), \qquad n \ge 1. \tag{1.3}$$

In the first part of this work [9], we proved that if $f \in \Delta^{(1,s)}$, then

$$E_n^{(1,s)}(f) \le c(s)\omega^{\varphi}(f, 1/n), \qquad n \ge 1, \tag{1.4}$$

where $c(s)$ is a constant depending only on $s$, $\omega^{\varphi}(f,t)$ is the D–T modulus of continuity and $\varphi(x) := \sqrt{1-x^2}$ [2], (see also (2.1) with $k=1$). Note that (1.4) immediately implies (it also follows by (2.5) below),

$$E_n^{(1,s)}(f) \le c(s)\frac{\|\varphi f'\|}{n}, \qquad n \ge 1, \tag{1.5}$$

provided $f \in \mathbb{B}^1$. Here $\mathbb{B}^r$, $r \geq 1$, is the space of functions $f \in \mathbb{C}[-1,1]$, which possess a locally absolutely continuous $(r-1)$st derivative in $(-1,1)$, such that

$$\|\varphi^r f^{(r)}\| < \infty.$$

We have also provided counterexamples showing that the Jackson estimates fail to hold in many other cases. Namely, we proved [9] that for an arbitrary constant $A > 0$ and $s \geq 1$, if $2 \leq r \leq 2s+2$, excluding the case $s=1$, $r=3$, then for any $n$, there exists a function $f = f_{s,r,n,A} \in \Delta^{(1,s)} \cap \mathbb{B}^r$, for which

$$E_n^{(1,s)}(f) \geq e_n^{(1,s)}(f) \geq A\|\varphi^r f^{(r)}\| > 0, \tag{1.6}$$

where

$$e_n^{(1,s)}(f) := \inf_{Y \in A_s(f)} E_n^{(1)}(f,Y).$$

In this article we shall prove that for all other cases we have an estimate similar to (1.5). Namely, we prove

**Theorem 1.** *Let $f \in \Delta^{(1,s)} \cap \mathbb{B}^r$, with either $s=1$ and $r=3$, or $r > 2s+2$. Then we have*

$$E_n^{(1,s)}(f) \leq c(r)\frac{\|\varphi^r f^{(r)}\|}{n^r}, \qquad n \geq r\ , \tag{1.7}$$

Clearly (1.6) excludes the possibility of extending (1.7) to other pairs of $s$ and $r \geq 2$. However, if we allow the constant in (1.7) to depend on the set $Y$, then we can still salvage the estimates. Namely,

**Theorem 2.** *Let $f \in \Delta^{(1,s)} \cap \mathbb{B}^r$, $s, r \geq 1$, and $Y \in A_s(f)$. Then*

$$E_n^{(1)}(f,Y) \leq C(r,Y)\frac{\|\varphi^r f^{(r)}\|}{n^r}, \qquad n \geq r, \tag{1.8}$$

*and*

$$E_n^{(1)}(f,Y) \leq c(r,s)\frac{\|\varphi^r f^{(r)}\|}{n^r} \qquad n \geq N(r,Y), \tag{1.9}$$

*hold, with $c(r,s)$, a constant depending only on $r$ and $s$; and $C(r,Y)$ and $N(r,Y)$ constants which depend only on $r$ and $Y$.*

*Remark.* The case $r=1$ is (1.5) and for $r=2$, Theorem 2 follows from [6, Theorem 1]. Theorem 1 for $r>2s+3$ follows from Theorem 6 below and all cases $r>3$ in Theorem 2, are consequences of Theorem 8 below. But note that Theorem 1 is also valid when $r=2s+3$ and in the particular case $s=1$ and $r=3$ and that Theorem 2 asserts the validity of (1.8) and (1.9) also for $r=3$.

In fact we will prove stronger results, Propositions 1 and 2, which we state at the end of Section 2, but first we need some definitions and properties of modified moduli of smoothness and this is the content of most of Section 2. In Section 3 we construct piecewise polynomials which are comonotone with $f$ and approximate it well. The key results are Lemma 9 and 12 which are summarized in Proposition 3. Their proofs and especially that of Lemma 12 (which is the subject of a major construction in [10]) are complicated and any fresh ideas for simpler proofs would be most welcome. Then in Section 4 we replace the piecewise polynomials by the appropriate polynomials. We prove the Propositions 1 and 2, as well as have some counterexamples in Sections 5 and 6.

While we have to postpone the statements of Propositions 1 and 2, if we combine them with the well-known estimates of unconstrained polynomial approximation (see, e.g., [2] or [12]), then we obtain certain relations between the degrees of unconstrained and comonotone approximation which we can easily state at this stage. Denote as usual,

$$E_n(f) := \inf\{\|f-p_{n-1}\| : p_{n-1}\in\mathcal{P}_{n-1}\}$$

the error of the best uniform approximation of $f$. It follows that for each $Y\in A_s(f)$,

$$E_n(f) \le E_n^{(1)}(f,Y) \le E_n^{(1,s)}(f). \tag{1.10}$$

Proposition 1 and (1.4) yield a partial inverse to (1.10), namely,

**Theorem 3.** *Let $f\in\Delta^{(1,s)}$ and assume that either $0<\alpha<1$, or $s=1$ and $2<\alpha<3$, or $\alpha>2s+2$. Then, if*

$$E_n(f) < \frac{1}{n^\alpha}\,,\quad \forall n>\alpha, \tag{1.11}$$

*then*

$$E_n^{(1,s)}(f) < \frac{C(\alpha,s)}{n^\alpha}\ , \quad \forall n > \alpha, \tag{1.12}$$

*where $C(\alpha,s)$ is a constant depending only on $s$ and $\alpha$.*

Similarly we have a partial inverse of (1.10) related to Proposition 2. Namely,

**Theorem 4.** *Let $f \in \Delta^{(1)}(Y)$, where $Y \in \mathbb{Y}_s$. and assume that $\alpha > 0$, $\alpha \neq 2$. Then, if* (1.11) *holds, then*

$$E_n^{(1)}(f,Y) < \frac{C(\alpha,Y)}{n^\alpha}\ , \quad \forall n > \alpha, \tag{1.13}$$

*and*

$$E_n^{(1)}(f,Y) < \frac{C(\alpha,s)}{n^\alpha}\ , \quad \forall n > N(\alpha,Y), \tag{1.14}$$

*where $C(\alpha,s)$ is a constant depending only on $s$ and $\alpha$, and $C(\alpha,Y)$ and $N(\alpha,Y)$ are constants which depend only on $\alpha$ and $Y$.*

*Remark.* By virtue of (1.6) it follows that whenever $\alpha \in [1, 2s+2]$, except for the case $s = 1$ and $2 < \alpha < 3$, it is impossible to replace $N(\alpha,Y)$ in (1.14) by $N(\alpha,s)$, hence it is also impossible to replace $C(\alpha,Y)$ in (1.13) by $C(\alpha,s)$, whenever $\alpha \in [1, 2s+2]$, except for the case $s = 1$ and $2 < \alpha < 3$.

Indeed we prove the following in Section 6.

**Theorem 5.** *Let $s \geq 1$, and let $1 \leq \alpha \leq 2s+2$, excluding the case $s = 1$, $2 < \alpha < 3$. Then for any constant $B > 0$ and each $n > \alpha$, a function $g := g_{s,\alpha,n,B} \in \Delta^{(1,s)}$ exists, for which we simultaneously have,*

$$E_m(g) \leq \frac{1}{m^\alpha}, \quad \forall m > \alpha \tag{1.15}$$

*and*

$$E_n^{(1,s)}(g) \geq e_n^{(1,s)}(g) \geq B. \tag{1.16}$$

*Remark.* Obviously, if $f$ is monotone, i.e., $s=0$, then there can be no dependence on $Y$ in (1.8), (1.9), (1.13) and (1.14) and indeed the corresponding estimates are well-known. They have been proved by the authors and by Dzyubenko, Kopotun and Listopad. In particular Kopotun [6,8] has shown that the monotone analogue of Theorem 4 fails for $\alpha=2$. In a forthcoming paper, we will prove that if $s\neq 0$, then the case $\alpha=2$ is not an exceptional one, that is, Theorem 4 holds as well for $\alpha=2$.

In the sequel we will have constants $c$ which depend on $r$, $s$ and $k$; or on one or two of them. We will also have constants $C$ which may depend on other parameters. However, we will use the notation $c$ and $C$ for such constants which are of no significance to us and may differ on different occurrences, even in the same line; and we will have constants with indices $c_1, c_2, \dots$ and $C_1, C_2, \dots$ when we have a reason to keep trace of them in the computations that we have to carry in the proofs.

## §2. Definitions and statement of the main results

Put $I := [-1,1]$. Given $f \in \mathbb{C}(I)$, and $k \geq 1$, let

$$\Delta_h^k f(x) := \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x - \frac{k}{2}h + ih),$$

be the symmetric difference of order $k$, defined for all $x$ and $h \geq 0$, such that $x \pm \frac{k}{2}h \in I$.

The Ditzian–Totik (DT-)moduli of smoothness [DT] are defined by

$$\omega_k^\varphi(f,t) := \sup_{0\leq h\leq t} \sup_{x} |\Delta_{h\varphi(x)}^k f(x)|, \quad t \geq 0, \tag{2.1}$$

where the inner supremum is taken over all $x$ such that

$$x \pm \frac{k}{2} h\varphi(x) \in (-1,1). \tag{2.2}$$

Also, set

$$\varphi_\delta(x) := \sqrt{(1 - x - \frac{\delta}{2}\varphi(x))(1 + x - \frac{\delta}{2}\varphi(x))}, \qquad x \pm \frac{\delta}{2}\varphi(x) \in I,$$

and for a function $g$ defined in $(-1,1)$, and $r \geq 1$, denote

$$\omega_{k,r}^\varphi(g,t) := \sup_{0\leq h\leq t} \sup_{x} |\varphi_{kh}^r(x) \Delta_{h\varphi(x)}^k g(x)|, \quad t \geq 0, \tag{2.3}$$

where again the inner supremum is taken over all $x$ so that (2.2) holds. We will apply the moduli in (2.3) for derivatives of $f$, which is going to be differentiable in $(-1,1)$ up to the appropriate order. Then, in order for the definition of $\omega^{\varphi}_{k,r}(f^{(r)},t)$, to make sense, we have to assume that $\lim_{x\to\pm1}\varphi^r(x)f^{(r)}(x)$, exists, moreover, in order that $\omega^{\varphi}_{k,r}(f^{(r)},t)\to 0$, as $t\to 0$, we have to assume that

$$\lim_{x\to\pm1}\varphi^r(x)f^{(r)}(x)=0,$$

which we denote by $f\in\mathbb{C}^r_{\varphi}$. Clearly, $\mathbb{C}^r_{\varphi}\subset\mathbb{B}^r$, $r\ge 1$. For $r=0$, we put

$$\mathbb{C}^0_{\varphi}:=\mathbb{C}(I),\quad \omega^{\varphi}_{k,0}(f,t):=\omega^{\varphi}_{k}(f,t).$$

We recall some properties of the DT-moduli of smoothness (2.1) and (2.3), which are similar to properties of the ordinary moduli of smoothness (see, e.g., [DT] and [S], pages 165–167). For $f\in\mathbb{C}^r_{\varphi}$ and $0\le p<r$, we have

$$\omega^{\varphi}_{k+r-p,p}(f^{(p)},t)\le ct^{r-p}\omega^{\varphi}_{k,r}(f^{(r)},t),\quad t\ge 0; \tag{2.4}$$

while if $f\in\mathbb{B}^r$ and $0\le p<r$, then $f\in\mathbb{C}^p_{\varphi}$ and

$$\omega^{\varphi}_{r-p,p}(f^{(p)},t)\le ct^{r-p}\|\varphi^r f^{(r)}\|,\quad t\ge 0. \tag{2.5}$$

In fact, we note although we make no use of it, that the converse of (2.5) is valid too, namely, if $f\in\mathbb{C}^p_{\varphi}$, $0\le p<r$ and $\omega^{\varphi}_{r-p,p}(f^{(p)},t)\le t^{r-p}$, $t\ge 0$, then $f\in\mathbb{B}^r$, and $\|\varphi^r f^{(r)}\|\le c$. Finally, for each $f\in\mathbb{C}^r_{\varphi}$ we have

$$t^{-k}\omega^{\varphi}_{k,r}(f^{(r)},t)\le c\tau^{-k}\omega^{\varphi}_{k,r}(f^{(r)},\tau),\quad 0<\tau\le t. \tag{2.6}$$

Let $\phi\in\Phi^k$, i.e., $\phi(0+)=0$, $\phi(t)$ is nondecreasing and $t^{-k}\phi(t)$ is nonincreasing in $(0,\infty)$. While in general for $f\in\mathbb{C}^r_{\varphi}$, $\omega^{\varphi}_{k,r}(f^{(r)},t)$ is not necessarily in $\Phi^k$, it satisfies (2.6). Hence, following ideas of Stechkin and Timan (see e.g., [12], (2.33)), the function

$$\tilde\phi(t):=t^k\sup_{u\ge t}u^{-k}\omega^{\varphi}_{k,r}(f^{(r)},u),\quad t>0,\quad \tilde\phi(0):=0, \tag{2.7}$$

is in $\Phi^k$ and satisfies

$$c\tilde{\phi}(t) \leq \omega^{\varphi}_{k,r}(f^{(r)}, t) \leq \tilde{\phi}(t). \tag{2.8}$$

For $\phi \in \Phi^k$, denote by $B^r H^{\phi}_k$, $r \geq 0$, the set of functions $f \in \mathbb{C}^r_{\varphi}$, satisfying

$$\omega^{\varphi}_{k,r}(f^{(r)}, t) \leq \phi(t), \quad t \geq 0.$$

Note that in view of (2.8), if $f \in \mathbb{C}^r_{\varphi}$, then we always have $f \in B^r H^{\tilde{\phi}}_k$. Also it follows from (2.4) that

$$B^r H^{\phi}_k \subseteq B^p H^{\phi_{r,p}}_{k+r-p}, \quad 0 \leq p \leq r, \tag{2.9}$$

where $\phi_{r,p}(t) := ct^{r-p}\phi(t)$.

Denote by $\Phi^k_*$ the subset of functions $\phi \in \Phi^k$, satisfying

$$\int_0^1 \frac{\phi(t)}{t} dt < \infty,$$

and for each $\phi \in \Phi^k_*$ put

$$\phi_*(t) := \int_0^t \frac{\phi(u)}{u} du.$$

It follows that if $\phi \in \Phi^k_*$, then

$$\phi(t) = k\frac{\phi(t)}{t^k} \int_0^t u^{k-1} du \leq k \int_0^t \frac{\phi(u)}{u} du = k\phi_*(t), \tag{2.10}$$

hence

$$\left(t^{-k}\phi_*(t)\right)' = t^{-k-1}\left(\phi(t) - k\phi_*(t)\right) \leq 0,$$

which implies

$$\phi_* \in \Phi^k. \tag{2.11}$$

Note that if $\phi \in \Phi^k$ and $l \geq 1$, then $\phi_l(t) := t^l \phi \in \Phi^{k+l}_*$ and the inverse inequality to (2.10) holds, namely

$$(\phi_l)_*(t) \leq \phi_l(t).$$

In particular for $f \in \mathbb{C}^r_{\varphi}$ we have,

$$(\tilde{\phi}_l)_*(t) \leq ct^l \omega^{\varphi}_{k,r}(f^{(r)}, t). \tag{2.12}$$

We are ready to state our main results.

**Proposition 1.** *Suppose that either $k = s = 1$ and $r = 2$, or $r = 2s + 2$. Let $\phi \in \Phi_*^k$ and $f \in \Delta^{(1.s)} \cap B^r H_k^\phi$. Then*

$$(2.13) \qquad E_n^{(1,s)}(f) \le \frac{c}{n^r} \phi_*(\frac{1}{n}), \quad n \ge k + r,$$

*where $c = c(r, k)$.*

In general, for $r = 2$ we have

**Proposition 2.** *Let $\phi \in \Phi_*^k$ and $f \in \Delta^{(1)}(Y) \cap B^2 H_k^\phi$. Then*

$$(2.14) \qquad E_n^{(1)}(f, Y) \le \frac{C(k, Y)}{n^2} \phi_*(\frac{1}{n}), \quad n \ge k + 2,$$

*and*

$$(2.15) \qquad E_n^{(1)}(f, Y) \le \frac{c}{n^2} \phi_*(\frac{1}{n}), \quad n \ge N(k, Y)$$

*where $C(k, Y)$ and $N(k, Y)$ are constants depending only on $k$ and $Y$, and $c = c(k, s)$.*

Propositions 1 and 2 augmented by (1.4), together with (2.5), immediately yield Theorems 1 and 2. Moreover (we elaborate in Section 5), they provide us with the following result which strengthen (1.7) for $r > 2s + 3$. Namely,

**Theorem 6.** *Let $f \in \Delta^{(1,s)} \cap \mathbb{C}_\varphi^r$, with $r > 2s + 2$. Then we have*

$$(2.16) \qquad E_n^{(1,s)}(f) \le \frac{c(r, k)}{n^r} \omega_{k,r}^\varphi(f^{(r)}, \frac{1}{n}), \qquad n \ge k + r,$$

*where $c(r, k)$ is a constant depending only on $r$ and $k$.*

We wish to emphasize that (2.16) does not imply (1.7) for $r = 2s + 3$. Also note that (2.16) is not valid in the case $s = 1$, $r = 3$ as this would imply (1.7) for $s = 1$, $r = 4$, which violates (1.6).

We will show in Section 6 that (2.16) fails for all $r \le 2s + 2$, except for the case $r = 0$ and $k = 1$, which is (1.4). Namely, we will prove,

**Theorem 7.** *Let $k, s \geq 1$ and $r \leq 2s+2$, excluding the case $r = 0$ and $k = 1$. Then for any constant $A > 0$ and every $n \geq 1$, a function $f := f_{k,s,r,n,A} \in \Delta^{(1,s)} \cap \mathbb{C}^r_\varphi$ exists, for which*

$$E_n^{(1,s)}(f) \geq e_n^{(1,s)}(f) > A\omega^\varphi_{k,r}(f^{(r)}, 1). \tag{2.17}$$

However, if we allow the constant in (2.16) to depend on the set $Y$, then by Proposition 2 (again we elaborate in Section 5), we can salvage the estimates. Namely,

**Theorem 8.** *Let $f \in \Delta^{(1,s)} \cap \mathbb{C}^r_\varphi$, $s \geq 1$, $r > 2$ and assume $Y \in A_s(f)$. Then*

$$E_n^{(1)}(f, Y) \leq \frac{C(k,r,Y)}{n^r}\omega^\varphi_{k,r}(f^{(r)}, \frac{1}{n}), \quad n \geq k+r, \tag{2.18}$$

*and*

$$E_n^{(1)}(f, Y) \leq \frac{c}{n^r}\omega^\varphi_{k,r}(f^{(r)}, \frac{1}{n}), \quad n \geq N(k,r,Y) \tag{2.19}$$

*where $C(k,r,Y)$ and $N(k,r,Y)$ are constants depending only on $k$, $r$ and $Y$, and $c = c(k,r,s)$ depends only on $k$, $r$ and $s$.*

*Remark.* We do not know whether or not Theorem 8 is valid for $r \leq 2$ except for the some special cases. It is valid for $r = 0$ and $k = 1$, which follows from (1.4); for $r = 0$ and $k = 2$, which is proved in [7] and which in turn implies the case, $r = k = 1$. We know that Theorem 8 is false when $r = 0$ and $k > 2$, this follows from [13]; and that it is false when $r = 1$ and $k > 3$ (see [3, Examples 3.1 and 3.2]). Finally, for $r = 2$ and $k > s$, one can modify Kopotun's counterexample [8] to obtain one for Theorem 8. But when this paper was written we had no answer as to what happens for the outstanding cases.

We can now settle most of the outstanding cases, (see our forthcoming paper). Namely, surprisingly (comparing to the purely monotone case, see table below), Theorem 8 is valid for $r = 1$ and $k = 2$, and in turn for $r = 2$ and $k = 1$. Theorem 8 is false for $r = 2$ and $k \geq 3$ and therefore also for $r = 1$ and $k \geq 4$. We still have no clue about the last two cases $r = 1$, $k = 3$, and $r = 2$, $k = 2$.

We illustrate the results in the following table. The sign "+" in entry $(r, k)$ means that (2.16) is valid for this pair of numbers (and the proper $s$), the sign "$\oplus$" means that (2.18) and (2.19) are valid for this pair of numbers, and the sign "$-$" means that none of these inequalities is valid.

$$
\begin{array}{cccccccc}
r & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
2s+4 & + & + & + & + & + & \cdots \\
2s+3 & + & + & + & + & + & \cdots \\
2s+2 & \oplus & \oplus & \oplus & \oplus & \oplus & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
3 & \oplus & \oplus & \oplus & \oplus & \oplus & \cdots \\
2 & \oplus & ? & - & - & - & \cdots \\
1 & \oplus & \oplus & ? & - & - & \cdots \\
0 & + & \oplus & - & - & - & \cdots \\
 & 1 & 2 & 3 & 4 & 5 & k
\end{array}
$$

Fig. 1. $s \geq 1$

It is interesting to compare this with the purely monotone case where we have (see [6]),

$$
\begin{array}{ccccccc}
r & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
4 & + & + & + & + & + & \cdots \\
3 & + & + & + & + & + & \cdots \\
2 & - & - & - & - & - & \cdots \\
1 & + & - & - & - & - & \cdots \\
0 & + & + & - & - & - & \cdots \\
 & 1 & 2 & 3 & 4 & 5 & k
\end{array}
$$

Fig. 2. $s = 0$

We conclude this section with the concept of the length of the interval $J = [a, b] \subseteq I$, relative to its position in $I$. To this end, we define

$$/J/ := \frac{|J|}{\varphi(\frac{a+b}{2})},$$

where $|J| := b - a$ is the length of $J$, and we observe that whenever $J_1 \subseteq J$, then

$$/J_1/ \leq /J/. \tag{2.20}$$

Indeed, let $x$ be the midpoint of $J_1$, then (2.20) follows from the following inequalities. If $x \leq \frac{a+b}{2}$, then

$$\frac{|J_1|}{2} \leq x - a = \frac{b-a}{2} - \left| x - \frac{a+b}{2} \right| \leq \frac{/J/}{2} \varphi(x),$$

while if $x > \frac{a+b}{2}$, then

$$\frac{|J_1|}{2} \le b - x = \frac{b-a}{2} - \left|x - \frac{a+b}{2}\right| \le \frac{/J/}{2}\varphi(x).$$

To see this we observe that $\frac{/J/}{2}\varphi$ is a nonnegative concave function which assumes at $x = \frac{a+b}{2}$, the same value $\frac{b-a}{2}$, as the piecewise linear function $\frac{b-a}{2} - \left|x - \frac{a+b}{2}\right|$, which vanishes at $x = a$ and $x = b$.

Recall the definition of ordinary modulus of smoothness restricted to $J$,

$$\omega_k(f, t; J) := \sup_x \sup_{0 \le h \le t} |\Delta_h^k f(x)|, \quad t \ge 0,$$

where the inner supremum is taken over all $h$ such that $x \pm \frac{k}{2}h \in J$.

Observe, if $x \pm \frac{k}{2}h \in J \subseteq I$, then (2.20) yields $h \le \frac{/J/}{k}\varphi(x)$, which in turn implies,

$$\begin{aligned} \omega_k(f, |J|; J) &\le \sup_x \sup_{h \le \frac{/J/}{k}\varphi(x)} |\Delta_h^k f(x)| \\ &\le \sup_x \sup_{h \le \frac{/J/}{k}} |\Delta_{h\varphi(x)}^k f(x)| \\ &\le \omega_k^\varphi(f, \frac{/J/}{k}) \le \omega_k^\varphi(f, /J/). \end{aligned} \tag{2.21}$$

Similarly, for $J \subset (-1,1)$ and $f \in \mathbb{C}_\varphi^r$ we have

$$\omega_k(f^{(r)}, |J|; J) \le \frac{1}{w^r(a,b)} \omega_{k,r}^\varphi(f^{(r)}, /J/), \tag{2.22}$$

where $w(a,b) := \sqrt{(1+a)(1-b)}$.

By Whitney's theorem

$$\|f - L_{k-1}(f, \cdot; J)\|_J \le c\omega_k(f, |J|; J),$$

where $L_{k-1}(f, \cdot; J)$ is the Lagrange polynomial of degree $\le k-1$ interpolating $f$ at the points $a + i\frac{b-a}{k-1}$, $0 \le i \le k-1$. Hence by virtue of (2.21) and (2.22),

$$\|f - L_{k-1}(f, \cdot; J)\|_J \le c\omega_k^\varphi(f, /J/), \quad J \subseteq I, \tag{2.23}$$

and for $f \in \mathbb{C}_\varphi^r$,

$$\|f^{(r)} - L_{k-1}(f^{(r)}, \cdot; J)\|_J \le \frac{c}{w^r(a,b)} \omega_{k,r}^\varphi(f^{(r)}, /J/), \quad J \subset (-1,1). \tag{2.24}$$

Finally, note that (2.2) implies $h \le \frac{2}{k}\varphi(x)$, whence $h^2 \le h\varphi(x)$, $k > 1$ ($h^2 \le 2h\varphi(x)$, $k = 1$). Therefore,

$$\omega_k(f, t^2) \le \omega_k^\varphi(f, t), \quad t \ge 0, \quad k > 1, \tag{2.25}$$

$$(\omega_k(f, t^2) \le 2\omega_k^\varphi(f, t), \quad t \ge 0 \quad k = 1).$$

**Lemma 1.** *If $f \in B^{2r}H_k^\phi$ and $\phi \in \Phi_*^k$, then $f \in \mathbb{C}^r(I)$, and*

$$\omega_{k+r}(f^{(r)}, t, I) \le c\phi_*(\sqrt{t}), \quad t \ge 0. \tag{2.26}$$

*Proof.* It follows from (2.25) that

$$\omega(t) := \omega_{k+2r}(f, t) \le \omega_{k+2r}^\varphi(f, \sqrt{t}).$$

Since $f \in B^{2r}H_k^\phi$, then by (2.4)

$$\omega_{k+2r}^\varphi(f, t) \le ct^{2r}\phi(t).$$

Hence

$$\int_0^t \frac{\omega(u)}{u^{r+1}}\, du \le c\int_0^t \frac{\phi(\sqrt{u})}{u}\, du = 2c\phi_*(\sqrt{t}). \tag{2.27}$$

In particular we have

$$\int_0^1 \frac{\omega(u)}{u^{r+1}}\, du < \infty,$$

which by Brudnyi and Gopengaus [1] (see also [12, Theorem 3.5]) implies $f \in \mathbb{C}^r(I)$. In turn by [1], (2.27) now yields (2.26). $\square$

## §3. Suitable splines

Throughout this Section we assume that $f \in B^rH_k^\phi \cap \Delta^{(1)}(Y)$, with $r, k \ge 1$, $\phi \in \Phi^k$ and $Y \in A_s(f)$.

We begin with the Chebyshev partition of $I$, namely, we fix $n \ge 1$ and for each $j = 0, \dots, n$ we set $x_j := x_{j;n} := \cos(j\pi/n)$. Denote $I_j := I_{j;n} := [x_j, x_{j-1}]$, $j = 1, \dots, n$. Then it is readily seen that

$$\frac{2}{n} \le /I_j/ = \frac{|I_j|}{\varphi(\frac{x_j + x_{j-1}}{2})} \le \frac{\pi}{n}. \tag{3.1}$$

Our first lemma in this section is

**Lemma 2.** *Let $1 < j < n$, and assume that $f'(x) \ge 0$, $x \in I_j$. Then a nondecreasing polynomial $p_j$ of degree $\le k+r-1$ exists, such that it interpolates $f$ at $x_j$ and $x_{j-1}$, and*

$$\|f - p_j\|_{I_j} \le c n^{-r} \phi(n^{-1}). \tag{3.2}$$

*Proof.* Since $f \in B^r H_k^\phi$, then $f'$ is by definition, continuous in $I_j$, $1 < j < n$. Then it follows by Lemma 2 of [10], that such a polynomial $p_j$ exists, satisfying

$$\begin{aligned} \|f - p_j\|_{I_j} &\le c|I_j| \omega_{k+r-1}(f', |I_j|; I_j) \\ &\le c|I_j|^r \omega_k(f^{(r)}, |I_j|; I_j) \\ &\le c \left( \frac{|I_j|}{w(x_j, x_{j-1})} \right)^r \omega_{k,r}^\varphi(f^{(r)}, /I_j/) \\ &\le c \left( \frac{|I_j|}{w(x_j, x_{j-1})} \right)^r \omega_{k,r}^\varphi(f^{(r)}, n^{-1}), \end{aligned} \tag{3.3}$$

where we have applied (2.22) and (3.1). Hence we conclude the proof of (3.2) by observing that for $I_j$ such that $1 < j < n$ we have

$$|I_j| \le c w(x_j, x_{j-1}) n^{-1}. \quad \square$$

For a given $Y$, let

$$O_i := O_{i;n}(Y) := (x_{j+1}, x_{j-2}), \quad \text{if} \quad y_i \in [x_j, x_{j-1}),$$

where $x_{-1} := 1$, $x_{n+1} := -1$, and set

$$O := O(Y; n) := \bigcup_{i=1}^{s} O_i,$$

and

$$O^* := O^*(Y; n) := O(Y; n) \cup I_1 \cup I_n.$$

Next let $\tilde{O}_q =: [a_q, b_q]$, $q \le s$, be the connected components of the closure $\overline{O}$ of $O$, indexed so that $b_{q+1} < a_q$. We show

**Lemma 3.** *Let $r > s$. Then, for each $q$, a polynomial $P_q$ of degree $\le k+r-1$, exists such that*

$$P_q(a_q) = f(a_q), \tag{3.4}$$

$$P_q'(x)\Pi(x,Y) \ge 0, \qquad x \in \tilde{O}_q \cap [x_{n-1}, x_1], \tag{3.5}$$

*and*

$$\|f - P_q\|_{\tilde{O}_q \cap [x_{n-1},x_1]} \le cn^{-r}\phi(n^{-1}). \tag{3.6}$$

*Proof.* Since $f \in B^r H_k^\phi$, then by defintion $f \in C^r[x_{n-1}, x_1]$. Now $r > s$, hence it follows by Lemma 3.3 of [4] applied to $f'$, that there exist polynomials $P$ of degree $\le k+r-1$ satisfying (3.5), such that

$$\|f' - P'\|_{\tilde{O}_q \cap [x_{n-1},x_1]} \le c|\tilde{O}_q \cap [x_{n-1}, x_1]|^{r-1}\omega_k(f^{(r)}, |\tilde{O}_q \cap [x_{n-1}, x_1]|; \tilde{O}_q \cap [x_{n-1}, x_1]).$$

We take $P_q$ so that (3.4) holds and $P_q'(x) = P'(x)$, which in turn implies

$$\begin{aligned} \|f - P_q\|_{\tilde{O}_q \cap [x_{n-1},x_1]} &\le c|\tilde{O}_q \cap [x_{n-1}, x_1]|^{r}\omega_k(f^{(r)}, |\tilde{O}_q|; \tilde{O}_q \cap [x_{n-1}, x_1]) \\ &\le c/\tilde{O}_q/^r\phi(/\tilde{O}_q/). \end{aligned} \tag{3.7}$$

The last inequality follows as in the proof of Lemma 2. Finally, we have

$$/\tilde{O}_q/ \le \frac{c}{n}, \tag{3.8}$$

and the proof of (3.6) is complete. □

As is readily seen from its proof, Lemma 3 can be formulated for $r > 1$, provided we take $n > N(Y)$ where $N(Y)$ is taken so that for each $n > N(Y)$ every $\tilde{O}_q$ contains exactly one point $y_q \in Y$, and $Y \subset [x_{n-2}, x_2)$. Namely,

**Lemma 3'.** *Let $r > 1$, and suppose that $n > N(Y)$. Then for each $q$, a polynomial $P_q$ of degree $\le k+r-1$, exists such that* (3.4), (3.5) *and* (3.6) *hold.*

Denote by $\Sigma_m := \Sigma_{m;n}$ the collection of continuous splines on the Chebyshev partition, with polynomial pieces of degree $< m$, and denote by $\Sigma_{m,O} := \Sigma_{m,O(Y;n)}$ the subcollection of splines $S \in \Sigma_m$, which are polynomials on each $\tilde{O}_q$. Note that $S'$ exists except perhaps at the Chebyshev nodes so we will use it freely without mentioning the finitely many excluded points. We note that we can combine Lemmas 2, 3 and 3' to yield results which are interesting by themselves, namely,

**Lemma 4.** *Let $r > s$. Then there exists a spline $S \in \Sigma_{k+r,O}$ such that*

$$S'(x)\Pi(x,Y) \geq 0 \qquad x \in [x_{n-1}, x_1], \tag{3.9}$$

*and*

$$\|f - S\|_{[x_{n-1},x_1]} \leq cn^{-r}\phi(n^{-1}). \tag{3.10}$$

And

**Lemma 4'.** *Let $r > 1$, and $n > N(Y)$. Then there exists a spline $S \in \Sigma_{k+r,O}$ such that* (3.9) *and* (3.10) *hold.*

*Proof.* We will prove Lemma 4, the proof of Lemma 4' being similar. Evidently, the only discontinuities that $S$ might have when we put together the polynomial pieces constructed in Lemmas 2 and 3(and 3'), occur at the points $b_q$, whose number is at most $s$. To rectify that we add a piecewise constant function with the proper jumps at those $b_q$'s which are in $[x_{n-1}, x_1]$. Since by (3.6), the size of each jump is bounded by

$$|f(b_q) - P_q(b_q)| \leq cn^{-r}\phi(n^{-1}),$$

and there are at most $s$ such points, the proof of (3.10) is complete. □

Thus, had we been satisfied with the construction of a continuous spline which is comonotone with $f$ and approximates it well on $[x_{n-1}, x_1]$, then we would have been satisfied with Lemma 4 and we would only have to limit ourselves to $r > s$, or we would have been satisfied with Lemma 4' and would have to limit ourselves to $n > N(Y)$. In order to extend the comonotonicity and preserve the good approximation on the whole interval $[-1, 1]$ we have to further restrict $r$. We first extend Lemma 2 to the intervals containing the endpoints.

**Lemma 5.** *Take $r = 2$ and assume that $\phi \in \Phi_*^k$. If $f$ is monotone in $I_1$, respectively $I_n$, then a monotone polynomial $p_1$, respectively $p_n$, of degree $\leq k+1$ exists, such that*

$$\|f - p_j\|_{I_j} \leq cn^{-2}\phi_*(n^{-1}), \quad j = 1, n, \ \textit{respectively.}$$

*Proof.* We prove the case $j = 1$, the other is similar. First we observe that Lemma 1 implies that $f \in \mathbb{C}^1(I)$. Hence, as in the proof of Lemma 2 such a polynomial $p_1$ exists, satisfying

$$\tag{3.11} \begin{aligned} \|f - p_1\|_{I_1} &\le c|I_1|\omega_{k+1}(f', |I_1|; I_1) \\ &\le c|I_1|\phi_*(\sqrt{|I_1|}) \le cn^{-2}\phi_*(n^{-1}), \end{aligned}$$

where we applied Lemma 1 again, and the fact that $|I_1| \le \frac{c}{n^2}$. $\square$

An obvious consequence of Lemma 5 and (2.9) is

**Corollary 1.** *Take $r > 2$. If $f$ is monotone in $I_1$, respectively $I_n$, then a monotone polynomial $p_1$ respectively $p_n$, of degree $\le k + r - 1$ exists, such that*

$$\|f - p_j\|_{I_j} \le cn^{-r}\phi(n^{-1}), \quad j = 1, n, \ \textit{respectively.}$$

Finally we extend Lemma 3 all the way to the endpoints and for this we have to restrict $r$ even further. Namely,

**Lemma 6.** *Suppose either $s = k = 1$ and $r = 2$, or $r = 2s + 2$ and assume that $\phi \in \Phi_*^k$. If $1 \in \tilde{O}_1$, then there exists a polynomial $P_1$ of degree $\le k + r - 1$, such that*

$$P_1'(x)\Pi(x, Y) \ge 0, \qquad x \in \tilde{O}_1,$$

*and*

$$\tag{3.12} \|f - P_1\|_{\tilde{O}_1} \le cn^{-r}\phi_*(n^{-1}).$$

*Proof.* It follows from the assumptions on $r$ and Lemma 1 that $f \in C^{r/2}(I)$, and

$$\omega_{k+r/2}(f^{(r/2)}, t) \le c\phi_*(\sqrt{t}).$$

As in (3.7) we apply Lemma 3.3 or 3.2 of [4], and conclude that such a polynomial exists, satisfying

$$\begin{aligned} \|f - P_1\|_{\tilde{O}_1} &\le c|\tilde{O}_1|^{r/2}\omega_{k+r/2}(f^{(r/2)}, |\tilde{O}_1|; \tilde{O}_1) \\ &\le c|\tilde{O}_1|^{r/2}\phi_*(\sqrt{|\tilde{O}_1|}) \le cn^{-r}\phi_*(n^{-1}), \end{aligned}$$

where we used the observation that $|\tilde{O}_1| \le \frac{c}{n^2}$. $\square$

In particular for $n = 1$, we get

**Corollary 2.** *Under the assumptions of lemma 6,*

$$E^{(1,s)}_{k+r}(f) \le c\phi_*(1).$$

Thus, Lemmas 4 and 4' may be extended to the whole interval. The proofs are the same so they will not be repeated. Namely,

**Lemma 7.** *Let either $s = k = 1$ and $r = 2$, or $r = 2s+2$, and assume that $\phi \in \Phi^k_*$. Then there exists a spline $S \in \Sigma_{k+r,O} \cap \Delta^{(1)}(Y)$, such that*

$$\|f - S\| \le cn^{-r}\phi_*(n^{-1}). \tag{3.13}$$

And

**Lemma 7'.** *Let $r = 2$ and assume that $\phi \in \Phi^k_*$. Then for $n > N(Y)$, there exists a spline $S \in \Sigma_{k+r,O} \cap \Delta^{(1)}(Y)$, such that* (3.13) *holds.*

The next results are needed for Proposition 2 when $n$ is "small".

**Lemma 8.** *Let $T := \{t_1, \dots, t_{k+1}\}$ be a collection of $k+1$ points $t_j \in (-1,1)$. If $g \in B^2H^\phi_k$ and $\phi \in \Phi^k_*$, then, for each $x \in I$,*

$$|g'(x) - L_k(g', x; t_1, \dots, t_{k+1})| \le C(T)|\prod_{j=1}^{k+1}(x - t_j)|\phi_*(1),$$

*where $L_k(x, g', t_1, \dots, t_{k+1})$ is the Lagrange polynomial of degree $\le k$, which interpolates $g'$ at the points $t_j$.*

*Proof.* In the proof $C$ stand for different constants, that depend only on $T$. Put $Q_k(x) := L_k(g', x; t_1, \dots, t_{k+1})$, and $L_k(x) := L_k(g', x; I)$. By Whitney's inequality and Lemma 1,

$$\begin{aligned} \|g' - Q_k\| &\le \|g' - L_k\| + \|L_k - Q_k\| \\ &\le \|g' - L_k\|(1 + |I|^k \max_{1\le i\le k} |t_{i+1} - t_i|^{-k}) \\ &\le C\omega_{k+1}(g', |I|, I) \\ &\le C\phi_*(\sqrt{|I|}) \le C\phi_*(1). \end{aligned} \tag{3.14}$$

Now let $J=[a,b]\subset(-1,1)$ be an interval such that $T\subset(a,b)$ and $|J|>1$. Put

$$L_{k-1}:=L_{k-1}(g'',x,J),\quad l_k(x):=g'((a+b)/2)+\int_{(a+b)/2}^{x}L_{k-1}(u)du.$$

Then by (2.24) and (2.10)

$$\|g''-L_{k-1}\|_J\le c\frac{1}{w^2(a,b)}\omega^{\varphi}_{k,2}(g'',/J/)\le C\phi(1)\le C\phi_*(1),$$

which together with (3.14) implies,

$$\begin{aligned}\|Q_k'-L_{k-1}\|&\le c\|Q_k-l_k\|_J\\&\le c(\|Q_k-g'\|+\|l_k-g'\|_J)\\&\le C\phi_*(1)+c\|L_{k-1}-g''\|_J\le C\phi_*(1).\end{aligned}$$

Hence

$$\|Q_k'-g''\|_J\le\|L_{k-1}-g''\|_J+\|L_{k-1}-Q_k'\|\le C\phi_*(1).$$

Now, given $x\in J$, let $t_j\in T$ be the closest to $x$. Then

$$\begin{aligned}|g'(x)-Q_k(x)|&=|\int_{t_j}^{x}(g''(u)-Q_k'(u))\,du|\\&\le C|x-t_j|\phi_*(1)\le C(T)\prod_{i=1}^{k+1}|x-t_i|\phi_*(1).\end{aligned}$$

This completes the proof. □

And

**Lemma 8'.** *If $r=2$ and $\phi\in\Phi_*^k$, then*

$$E^{(1)}_{k+2}(f,Y)\le C(Y,k)\phi_*(1).$$

*Proof.* First assume that $s\ge k+1$. Since evidently

$$L_k(f',x;y_1,\dots,y_{k+1})\equiv 0,$$

we may put

$$P(x) := f(0) = f(0) + \int_0^x L_k(f', u; y_1, \dots, y_{k+1})\, du.$$

Then it readily follows that $P \in \Delta^{(1)}(Y)$, and by Lemma 8,

$$\|f - P\| \le \|f' - L_k(\cdot; f', y_1, \dots, y_{k+1})\| \le C(Y, k)\phi_*(1).$$

Thus Lemma 8' is proved for the case $s \ge k+1$, and it remains to deal with $s \le k$. In this case we write $t_i := y_i$, $i = 1, \dots, s$, and we fix $k - s + 1$ arbitrary additional points $t_{s+1}, \dots, t_{k+1}$, say in the interval $(-1, y_s)$. Let

$$\pi(x) := \prod_{i=s+1}^{k+1} (x - t_i).$$

Then by Lemma 8 there exists a constant $C_0 = C_0(Y, k)$, such that

$$|f'(x) - L_k(f', x; t_1, \dots, t_{k+1})| \le C_0 |\Pi(x, Y)\pi(x)| \phi_*(1), \quad x \in I.$$

Put $C_1 := C_0\|\pi\|$, and let

$$P'(x) := L_k(x, f', t_1, \dots, t_{k+1}) + C_1 \Pi(x, Y)\phi_*(1),$$

and

$$P(x) := f(0) + \int_0^x P'(u)\, du.$$

Then we have

$$\|f - P\| \le \|f' - P'\| \le 2C_1\|\Pi(\cdot, Y)\|\phi_*(1) =: C(Y, k)\phi_*(1).$$

And at the same time,

$$\begin{aligned}\Pi(x, Y)P'(x) &= \Pi(x, Y)(L_k(x, f', t_1, \dots, t_{k+1}) - f'(x)) \\ &\qquad + \Pi(x, Y)f'(x) + C_1\Pi^2(x, Y)\phi_*(1) \\ &\ge \Pi(x, Y)f'(x) \ge 0.\end{aligned}$$

This completes the proof. $\square$

Our last result relates the polynomial pieces of the spline $S \in \Sigma_m$ to one another. First we need some notation. We let $I_{i,j} := I_{i,j;n}$ denote the smallest interval containing $I_i$ and $I_j$. For $S \in \Sigma_m$ we put

$$a_m(S) := a_{m;n}(S) := \max_{i,j} \left( \frac{|I_j|}{|I_{i,j}|} \right)^m \|p_i - p_j\|_{I_i},$$

where $p_i$ denotes the polynomial defined by $p_i|_{I_i} := S|_{I_i}$. Observe, that, for each $i$ and $j$, if $\nu$ is between $i$ and $j$, then

$$\frac{|I_{i,\nu}|}{|I_\nu|} < \frac{|I_{i,j}|}{|I_j|}. \tag{3.15}$$

We are ready to state our result (compare with Lemma 6 of [10])

**Lemma 9.** *For each $S \in \Sigma_k$ we have*

$$a_k(S) \le c\omega_k^\varphi(S, n^{-1}) \le c a_k(S). \tag{3.16}$$

*Proof.* Write $\omega := \omega_k^\varphi(S, n^{-1})$. We begin with the proof of the left-hand estimate, that is, we have to prove that for each $i$ and $j$,

$$\|p_i - p_j\|_{I_i} \le c \left( \frac{|I_{i,j}|}{|I_j|} \right)^k \omega. \tag{3.17}$$

To this end let us first assume that $j = i \pm 1$ and set $L(x) := L_{k-1}(S, x; I_{i,j})$. Then (2.23) implies

$$\|p_i - L\|_{I_i} = \|S - L\|_{I_i} \le \|S - L\|_{I_{i,j}} \le c\omega_k^\varphi(S, /I_{i,j}/) \le c\omega,$$

where in the right-hand inequality we applied (3.1). Observing that $p_j - L$ is a polynomial of degree $\le k-1$, we have

$$\|p_j - L\|_{I_i} \le \|p_j - L\|_{I_{i,j}} \le c\|p_j - L\|_{I_j} \le c\omega.$$

Hence

$$\|p_i - p_j\|_{I_i} \le c\omega, \tag{3.18}$$

which implies (3.17) for $j = i \pm 1$. Otherwise, assume $i < j$. Then for each $\nu$, $i < \nu < j$, it follows by (3.15) and (3.18) that

$$\begin{aligned}\|p_\nu - p_{\nu\pm1}\|_{I_i} &\le \|p_\nu - p_{\nu\pm1}\|_{I_{i,\nu}} \le c\left(\frac{|I_{i,\nu}|}{|I_\nu|}\right)^{k-1}\|p_\nu - p_{\nu\pm1}\|_{I_\nu}\\ &\le c\left(\frac{|I_{i,\nu}|}{|I_\nu|}\right)^{k-1}\omega \le c\left(\frac{|I_{i,j}|}{|I_j|}\right)^{k-1}\omega.\end{aligned}$$

Therefore,

$$\begin{aligned}\|p_j - p_i\|_{I_i} &\le \|p_j - p_{j-1}\|_{I_i} + \cdots + \|p_{i+1} - p_i\|_{I_i}\\ &\le c|j-i|\left(\frac{|I_{i,j}|}{|I_j|}\right)^{k-1}\omega \le c\left(\frac{|I_{i,j}|}{|I_j|}\right)^{k}\omega.\end{aligned}$$

Thus (3.17) is proved. We turn to the proof of the right-hand estimate in (3.16). We take $x$ and $h \le \frac{1}{n}$ satisfying (2.2), and for each $i = 0, \dots, k$ we let $\nu_i$, be so that $x + (i - \frac{k}{2})h\varphi(x) \in I_{\nu_i}$. Then

$$\begin{aligned}|\Delta^k_{h\varphi(x)}S(x)| &= |\Delta^k_{h\varphi(x)}(S - p_{\nu_0})| \le 2^k \max_{i=1,\dots,k}\|p_{\nu_i} - p_{\nu_0}\|_{I_{\nu_i}}\\ &\le c\max_{i=1,\dots,k}\left(\frac{|I_{\nu_0}|}{|I_{\nu_0,\nu_i}|}\right)^k\|p_{\nu_i} - p_{\nu_0}\|_{I_{\nu_i}} \le c a_k(S),\end{aligned}$$

where we used the inequality $|I_{\nu_0,\nu_i}| \le c|I_{\nu_0}|$, $i = 1, \dots, k$, which follows from (3.1). □

## §4 Comonotone polynomials

We begin with a lemma which is a trivial consequence of (1.4) and (3.1). Namely,

**Lemma 10.** *Let $f \in C(I) \cap \Delta^{(1)}(Y)$, be locally absolutely continuous and such that*

$$\|f'\|_{I_j} \le \frac{1}{|I_j|}, \quad j = 1, \dots, n. \tag{4.1}$$

*Then a polynomial $V_n \in \Delta^{(1)}(Y)$ of degree $\le n$ exists, such that*

$$\|f - V_n\| \le c. \tag{4.2}$$

Next we need a partition of unity of the type we had in [10] but we require a few more properties. We quote Lemma 5.4 of [4], namely

**Lemma GS.** *For each fixed integer $l$, there exists a collection $\{\tau_{j,n}\}_{j=1}^{n}$, of polynomials of degree $\le cln$, with the following properties.*

$$\sum_{j=1}^{n} \tau_{j,n}(x) \equiv 1; \tag{4.3}$$

$$|\tau_{j,n}^{(\lambda)}(x)| \le C \frac{h_j}{\rho_n^{\lambda+1}(x)} \left( \frac{\rho_n(x)}{|x-x_j| + \rho_n(x)} \right)^{l+1}, \quad x \in I, \quad \lambda = 0,1,2,\dots; \tag{4.4}$$

*where $\rho_n(x) := n^{-1}\sqrt{1-x^2} + n^{-2}$ and $C = C(s,l,\lambda)$,*

$$\tau'_{j,n}(y_i) = 0, \qquad i = 1,\dots,s, \quad j = 1,\dots,n; \tag{4.5}$$

*and*

$$\tau_{j,n}(y_i) = 0, \qquad \forall i,j \text{ such that } y_i \in \tilde{O}_q \text{ and } I_j \not\subset \tilde{O}_q. \tag{4.6}$$

We are ready to state the analogue of [10] Lemma 7, namely,

**Lemma 11.** *Let $l \ge 3k$, and assume that $S \in \Sigma_{k,O}$ and $S'(y_i) = 0$, $i = 1,\dots,s$. Then for $n_1 \ge n$ with $n_1$ divisible by $n$, the polynomial*

$$D_{n_1}(x) := \sum_{i=1}^{n} p_i(x) \sum_{\nu : I_{\nu,n_1} \subseteq I_i} \tau_{\nu,n_1}(x), \tag{4.7}$$

*satisfies*

$$D'_{n_1}(y_i) = 0, \quad 1 \le i \le s, \tag{4.8}$$

*and for each $\lambda = 0,\dots,s+1$,*

$$|S^{(\lambda)}(x) - D_{n_1}^{(\lambda)}(x)| \le C_0 a_k(S) \frac{1}{\rho_n^{\lambda}(x)}, \quad x \in I, \tag{4.9}$$

*where $C_0 = C_0(s,k,l)$. Moreover, for each $0 < \delta < 1$,*

$$\begin{aligned} |S'(x) - D'_{n_1}(x)| &\le C_1 a_k(S,(x-\delta,x+\delta)) \frac{1}{\rho_n(x)} \\ &\quad + C_2 a_k(S) \left( \frac{\rho_{n_1}(x)}{\rho_{n_1}(x)+\delta} \right)^{l+1-3k} \frac{1}{\rho_n(x)}, \quad x \in I, \end{aligned} \tag{4.10}$$

*where $C_1 = C_1(k,s,l)$ and $C_2 = C_2(k,s,l)$. We have used the notation*

$$a_k(S,(x-\delta,x+\delta)) := \max_{i,j} \left(\frac{h_i}{h_{i,j}}\right)^k \|p_i - p_j\|_{I_i},$$

*where the maximum is taken over all $i$ and $j$ such that $I_i \cap (x-\delta, x+\delta) \neq \emptyset$ and $I_j \cap (x-\delta, x+\delta) \neq \emptyset$.*

*Proof.* Evidently (4.8) follows by (4.5) and (4.6), by virtue of the fact that $S \in \Sigma_{k,O}$. Here is where we need the assumption that $S$ is a single polynomial in each connected component of $O$. The rest of the proof follows verbatim, the proof of [10] Lemma 7, where one has to replace $\varphi(\cdot)$ by 1. $\square$

Finally, we can construct a good polynomial approximation to $S$.

**Lemma 12.** *If $S \in \Sigma_{k,O} \cap \Delta^{(1)}(Y)$ is such that*

$$a_k(S) \leq 1, \tag{4.11}$$

*then there is a polynomial $P_n \in \Delta^{(1)}(Y)$ of degree $\leq cn$, satisfying*

$$\|S - P_n\| \leq c. \tag{4.12}$$

*Proof.* The first part of the proof is a repetition of the proof of [10] Lemma 9, almost word for word. In fact, the main difference is that one has to replace each $\varphi(h_j)$ and $\varphi(\rho)$, by 1, whence simplifying some of the computations. However, one other modification is required in the construction of $S_4$ in that proof. Namely, we require that whenever a connected component $\tilde{O}_q \cap F^e \neq \emptyset$, then $\tilde{O}_q \subseteq F^e$. Since $\tilde{O}_q$ consists of at most $3s$ intervals $I_j$, this makes no difference in our construction and implies that $S_4 \in \Sigma_{k,O}$. Thus, with the polynomial $V_n$ from Lemma 10 of this paper, which is associated with $S_3$ of the abovementioned proof, and the polynomial $D_{n_1}$ from Lemma 11 of this paper which is associated with $S_4$, we end up with a polynomial,

$$\tilde{R}_n := V_n + R_n := V_n + D_{n_1} + cQ_n + cM_n,$$

which satisfies

$$\|\tilde{R}_n - S\| \le c, \tag{4.13}$$

and

$$\tilde{R}'_n(x)\Pi(x,Y) \ge 0, \qquad x \in I \setminus O. \tag{4.14}$$

It remains to show that

$$\tilde{R}'_n(x)\Pi(x,Y) \ge 0, \qquad x \in O. \tag{4.15}$$

Observe that by virtue of Lemma 10, the polynomial $V_n$ satisfies (4.15), and that by [10] (4.21) and (4.25), the same is true for $Q_n$ and $M_n$, respectively. Hence, we only have to adjust $D_{n_1}$ to satisfy (4.15) without hurting (4.14). To this end, recall that $D_{n_1}$ satisfies (4.8) and (from (4.9)) for each $\lambda = 2, \dots, s+1$,

$$|S_4^{(\lambda)}(x) - D_{n_1}^{(\lambda)}(x)| \le C_0 \frac{1}{\rho_n^\lambda(x)}, \quad x \in I. \tag{4.16}$$

So we take $\tilde{O}_q$, a connected component of the closure $\bar{O}$ of $O$ and we let $y_{i_q+1}, \dots, y_{i_q+\nu} \in \tilde{O}_q$, ($\nu \le s$). Further, we put $x_{j_q}$ to be the closest to $\tilde{O}_q$, such that $I_{j_q} \cap O = \emptyset$. Then (4.8) and (4.16) together with (4.11) yield for $x \in \tilde{O}_q$,

$$\begin{aligned} |D'_{n_1}(x) - S'_4(x)| &= |\Pi_{\mu=1}^{\nu}(x - y_{i_q+\mu})[x, y_{i_q+1}, \dots, y_{i_q+\nu}; D'_{n_1} - S'_4]| \\ &= |\Pi_{\mu=1}^{\nu}(x - y_{i_q+\mu})(D_{n_1}^{(\nu+1)}(\theta) - S_4^{(\nu+1)}(\theta))| \\ &\le c\rho_n^{-\nu-1}(\theta)|\Pi_{\mu=1}^{\nu}(x - y_{i_q+\mu})|, \end{aligned} \tag{4.17}$$

where we recall that the square brackets denote the divided difference of order $\nu$ of the function $D'_{n_1} - S'_4$ and $\theta \in \tilde{O}_q$. Hence

$$|D'_{n_1}(x) - S'_4(x)| \le \frac{c}{\rho_n(x_{j_q})} \frac{\Pi_{\mu=1}^{\nu}|x - y_{i_q+\mu}|}{\Pi_{\mu=1}^{\nu}|x_{j_q} - y_{i_q+\mu}|} \le \frac{c_3}{\rho_n(x_{j_q})} \frac{|\Pi(x,Y)|}{|\Pi(x_{j_q},Y)|}. \tag{4.18}$$

For the first inequality in (4.18), we used the fact that $\tilde{O}_q$ is connected and contains at most $3s$ intervals, so that

$$|\tilde{O}_q| \sim \rho_n(\theta) \sim \rho_n(x_{j_q}) \sim (x_{j_q} - y_{i_q+\mu}), \quad 1 \le \mu \le \nu;$$

and for the second inequality we used the above together with the fact that for any $y_i \notin \tilde{O}_q$,

$$\frac{|x-y_i|}{|x_{j_q}-y_i|} \geq \frac{|x-y_i|}{|x-x_{j_q}|+|x-y_i|} \geq \frac{\rho_n(x)}{|\tilde{O}_q|+\rho_n(x)} \geq c > 0.$$

Now, recall the polynomials $T_j(x) = T_j(x, 6s, Y)$ that were introduced in [3, the definition above Lemma 5.3]. Evidently,

$$\|T_j\| \leq c_4, \tag{4.19}$$

and by [3, (6.15)]

$$T_j'(x)\Pi(x,Y)\operatorname{sgn}\Pi(x_j,Y) \geq 0, \qquad x \in I. \tag{4.20}$$

Finally, by [3, (5.22)], we have

$$|T_{j_q}'(x)| \geq \frac{c_5}{\rho_n(x_{j_q})} \frac{|\Pi(x,Y)|}{|\Pi(x_{j_q},Y)|}, \qquad x \in \tilde{O}_q. \tag{4.21}$$

Thus if we set

$$\tilde{T}_{j_q} := \frac{c_3}{c_5} T_{j_q} \operatorname{sgn}\Pi(x_{j_q}, Y),$$

and

$$\tilde{D}_{n_1} := D_{n_1} + \sum_q \tilde{T}_{j_q},$$

where the sum is taken on all the connected components $\tilde{O}_q$, then it follows by (4.18), (4.21) and (4.20), together with the fact that $S_4$ is comonotone with $S$, that for $x \in \tilde{O}_{q_0}$,

$$\begin{aligned} \tilde{D}_{n_1}'(x)\Pi(x,Y) &= (\tilde{D}_{n_1}'(x) - S_4'(x))\Pi(x,Y) + \tilde{T}_{j_{q_0}}(x)\Pi(x,Y) \\ &\quad + \sum_{q \neq q_0} \tilde{T}_{j_q}(x)\Pi(x,Y) + S_4'(x))\Pi(x,Y) \geq 0. \end{aligned} \tag{4.22}$$

Hence, for the polynomial

$$P_n := \tilde{R}_n + \sum_q \tilde{T}_{j_q},$$

we have

$$P_n'(x)\Pi(x,Y) \geq 0. \qquad x \in I.$$

We conclude the proof by observing that (4.13) and (4.19) imply that

$$\|P_n - S\| \leq c + (\sum_q \frac{c_3 c_4}{c_5}) = c + c_6 s. \quad \square$$

Combining Lemmas 9 and 12 we conclude that

**Proposition 3.** *If $S \in \Sigma_{m,O(Y;n)} \cap \Delta^{(1)}(Y)$, then*

$$E^{(1)}_{c_7 n}(S, Y) \le c_8 \omega^{\varphi}_m (S, \frac{1}{n}), \tag{4.23}$$

*where $c_7 = c_7(m, s)$ and $c_8 = c_8(m, s)$.*

## §5 Proofs of the positive results

We begin with

*Proof of Proposition 1.* It follows by Lemma 7 and Proposition 3 that for $n > c$, there exist appropriate $P_n$ and $S$ such that,

$$\begin{aligned} \|f - P_n\| &\le \|f - S\| + \|S - P_n\| \\ &\le cn^{-r}\phi_*(n^{-1}) + c\omega^{\varphi}_{k+r}(S, n^{-1}). \end{aligned}$$

By virtue of (3.13) we have

$$\begin{aligned} \omega^{\varphi}_{k+r}(S, n^{-1}) &\le c(n^{-r}\phi_*(n^{-1}) + c\omega^{\varphi}_{k+r}(f, n^{-1})) \\ &\le cn^{-r}(\phi_*(n^{-1}) + \omega^{\varphi}_{k,r}(f^{(r)}, n^{-1})) \\ &\le cn^{-r}(\phi_*(n^{-1}) + \phi(n^{-1})) \\ &\le cn^{-r}\phi_*(n^{-1}), \end{aligned}$$

where in the second inequality we have applied (2.4) and in the last, we have used (2.10). This proves Proposition 1 for $n > c$ while for $k + r \le n \le c$, it readily follows from Corollary 2. This completes our proof. $\square$

Similarly

*Proof of Proposition 2.* It is easy to see that (2.14) follows from (2.15) with Lemma 8' playing the roll of Corollary 2, so we only need to prove the latter. Now (2.15) follows by Lemma 7' and Proposition 3, in the same manner in which we proved Proposition 1. $\square$

We are ready to give the proofs of the theorems.

*Proof of Theorem 6.* Since $f \in B^r H_k^{\tilde{\phi}}$ where $\tilde{\phi}$ is defined in (2.7), and $r > 2s+2$, then we obtain by (2.9) that $f \in B^{2s+2} H_{k+r-2s-2}^{\tilde{\phi}_{r-2s-2}}$, with $r - 2s - 2 \geq 1$. The proof is now concluded by Proposition 1 and (2.12). $\square$

*Proof of Theorem 8.* We follow the proof of Theorem 6 (artificially) replacing $s$ by 0 everywhere and replacing Proposition 1 by Proposition 2. $\square$

*Proof of Theorem 1.* As we already mentioned we could deduce most cases of Theorem 1 from Theorem 6. We have however two outstanding cases, and the proof is not any simpler if we restrict ourselves to the two special ones. Thus we give an independent proof of all cases. We take $f \in \mathbb{B}^r$ such that $\|\varphi^r f^{(r)}\| \leq 1$. First we let $r > 2s+2$, then we observe that (2.5) implies that $f \in B^{2s+2} H_{r-2s-2}^{\hat{\phi}}$, where $\hat{\phi}(t) := ct^{r-2s-2}$. Evidently

$$\hat{\phi}_* = \frac{1}{r-2s-2} \hat{\phi} \in \Phi_*^{r-2s-2},$$

and Theorem 1 follows from Proposition 1. The remaining case is $s = 1$ and $r = 3$. For this case we repeat the above proof (artificially) replacing $s$ by 0 everywhere and applying Proposistion 1. $\square$

*Proof of Theorem 2.* As we have already mentioned the case $r = 1$ is (1.5) and for $r = 2$, Theorem 2 follows from [6, Theorem 1]. Most other cases can be derived from Theorem 8, but we have the outstanding case $r = 3$ and as in the proof of Theorem 1, the proof of the general case is as simple. Actually we only need to repeat the proof of Theorem 1 (artificially) replacing $s$ by 0 everywhere and finally replacing Proposition 1 by Proposition 2. $\square$

We have two theorems left to prove.

*Proof of Theorem 3.* The case $0 < \alpha < 1$, readily follows from the fact that (1.11) implies that $\omega^\varphi(f,t) = O(t^\alpha)$ and then from (1.4). All other cases follow from Proposition 1. In fact, if $\alpha > 2s+3$ we can derive the result from Theorem 6, but there are two uncovered intervals for which we anyway need to call upon Proposition 1, so we will apply it for all cases. Indeed, if $\alpha > 2s+2$, then let $m := [\alpha] + 1$. Then (1.11) implies

$$E_n(f) \leq n^{-2s-2} \hat{\phi}(n^{-1}), \qquad n \geq m,$$

where $\hat{\phi}(t) := t^{\alpha-2s-2}$. Hence by [12, Theorem 18.2], we obtain that $f \in \mathbb{C}_\varphi^{2s+2}$ and $\omega_{m-2s-2,2s+2}^\varphi(f^{(2s+2)}, t) \le c(\alpha)t^{\alpha-2s-2}$. As in the proof of Theorem 1 we evidently have,

$$\hat{\phi}_* = \frac{1}{\alpha - 2s - 2}\hat{\phi} \in \Phi_*^{m-2s-2},$$

so that (1.12) readily follows by Proposition 1. Finally, when $s = 1$ and $2 < \alpha < 3$, we let $m = 3$ and (artificially) replace $s$ by 0 everywhere in the proof. Again (1.12) follows by Proposition 1. $\square$

*Proof of Theorem 4.* For $0 < \alpha < 2$, Theorem 4 follows from [7]. All other cases are proved in the same manner as Theorem 3 where we replace Proposition 1 by Proposition 2. Again for $\alpha > 3$ we could use Theorem 8. $\square$

## §6 Counterexamples

*Proof of Theorem 5.* We first recall that the direct theorem for nonconstrained approximation yields the estimate

$$E_m(f) \le \frac{c_1}{m^r}\|\varphi^r f^{(r)}\|, \qquad m \ge r. \tag{6.1}$$

where $c_1 = c_1(r)$. Now, for $\alpha = 2$, we take $r = 2$, for $\alpha = 2s + 2$, we take $r := 2s + 2$, and for all other $\alpha$, we take $r := [\alpha] + 1$. Put $A := Bc_1$, $f := f_{s,r,n,A}$ and

$$g := (c_1\|\varphi^r f^{(r)}\|)^{-1} f.$$

Since $g \in \mathbb{B}^r$ and $\|\varphi^r g^{(r)}\| = c_1^{-1}$, then (6.1) implies (1.15), where we used the inequality $\alpha \le r \le [\alpha] + 1$, while (1.16) follows from (1.6). $\square$

*Proof of Theorem 7.* We first observe that for $f \in \mathbb{C}_\varphi^r$, $r \ge 0$, we have

$$\omega_{k,r}^\varphi(f^{(r)}, t) \le c_0(k, r)\omega_{1,r}^\varphi(f^{(r)}, t), \quad k \ge 1, \tag{6.2}$$

and

$$\omega_{k,r}^\varphi(f^{(r)}, t) \le c_1(k, r)\omega_{2,r}^\varphi(f^{(r)}, t), \quad k \ge 2, \tag{6.3}$$

which is going to reduce the proof to the cases of $k=1$ and $k=2$ only. Next we note that if $f\in\mathbb{B}^{r+1}$, then by (2.5) $f\in\mathbb{C}^r_\varphi$ and

$$\omega^\varphi_{1,r}(f^{(r)},t)\le c_2(r)t\|\varphi^{r+1}f^{(r+1)}\|, \tag{6.4}$$

and

$$\omega^\varphi_{2,r-1}(f^{(r-1)},t)\le c_3(r)t^2\|\varphi^{r+1}f^{(r+1)}\|,\quad r>0. \tag{6.5}$$

Recall that in view of (1.6), if $2\le r\le 2s+2$ excluding the case $s=1$ and $r=3$, then for any constant $B>0$, there exists a function $g:=g_{s,r,n,B}\in\mathbb{B}^r\cap\Delta^{(1)}(Y_s)$, (for some $Y_s$, such that

$$\|\varphi^r g^{(r)}\|=1, \tag{6.6}$$

while

$$E_n^{(1,s)}(g)\ge e_n^{(1,s)}(g)>B. \tag{6.7}$$

We are ready to prove (2.17). To this end, let $A>0$ be given. If $1\le r<2s+2$ except for $r=2$ and $s=1$, then taking $B:=c_0(k,r)c_2(r)A$ and setting $f:=g_{s,r+1,n,B}$, we obtain by (6.2), (6.4) and (6.6),

$$\omega^\varphi_{k,r}(f^{(r)},1)\le c_0\omega^\varphi_{1,r}(f^{(r)},1)\le c_0c_2\|\varphi^{r+1}f^{(r+1)}\|=c_0c_2.$$

Hence by (6.7),

$$e_n^{(1,s)}(f)=e_n^{(1,s)}(g_{s,r+1,n,B})>B\ge\frac{B}{c_0c_2}\omega^\varphi_{k,r}(f^{(r)},1)=A\omega^\varphi_{k,r}(f^{(r)},1),$$

and (2.17) is proved. Now let $k\ge 2$ and assume $r=0$ or $r=2$ and $s=1$. Then we take $B:=c_1(k,r)c_2(r+1)A$ and we set $f:=g_{s,r+2,n,B}$. It follows by (6.3), (6.5) and (6.6) that

$$\omega^\varphi_{k,r}(f^{(r)},1)\le c_1\omega^\varphi_{2,r}(f^{(r)},1)\le c_1c_3\|\varphi^{r+2}f^{(r+2)}\|=c_1c_3,$$

whence (6.7) implies,

$$e_n^{1,s}(f) = e_n^{1,s}(g_{s,r+2,n,B}) > B \geq \frac{B}{c_1 c_3}\omega_{k,r}^{\varphi}(f^{(r)},1) = A\omega_{k,r}^{\varphi}(f^{(r)},1),$$

and (2.17) is proved.

Two cases remain. $r = 2$ and $k = s = 1$, and $r = 2s+2$ for all $s$ and $k$. By (6.2) the latter is proved once we prove it for $k = 1$, so we may assume that. In these cases we need to look more carefully at the function yielding (6.7) and even to modify it somewhat. The following straightforward inequality is the key to proving the remaining cases for $k = 1$, namely,

$$\omega_{1,r}^{\varphi}(f^{(r)},t) \leq 2\|\varphi^r f^{(r)}\|. \tag{6.8}$$

First for the case $r = 2$ and $s = 1$, we begin with

$$g_2(x) := -\frac{1}{4}(x+1)\log(x+1), \quad M_2 := \|g_2\|,$$

and let $x_0 \in (-1,1)$ be such that

$$|g_2'(x_0)| > n^2(A + M_2 + 1).$$

Similar to [8, p. 205], we denote by $T_2(x)$ the Taylor polynomial of degree 2 at $x_0$ of the function $g_2$ and set

$$f(x) := \begin{cases} g_2(x) - T_2(x), & \text{if } x \geq x_0, \\ 0, & \text{if } x < x_0, \end{cases}$$

then evidently $f \in \mathbb{C}_\varphi^r$. Observe that $T_2''(x) \equiv g_2''(x_0)$. Hence if we let $\tilde{f}(x) := f(x) + T_2(x)$, then by (6.8)

$$\omega_{1,2}^{\varphi}(f'',t) = \omega_{1,2}^{\varphi}(\tilde{f}'',t) \leq 2\|\varphi^2\tilde{f}''\| \leq 2\|\varphi^2 g_2''\| = 1.$$

This in turn implies, following the arguments we used in [9, p.205-206], that

$$e_n^{(1,1)}(f) > A \geq A\omega_{1,2}^{\varphi}(f^{(2)},1),$$

which is (2.17). For the case $r = 2s+2$ we have to modify the function $g$ yielding (6.7), in order that our $f$ be not only in $\mathbb{B}^r$ but actually in $\mathbb{C}_\varphi^r$. To this end we take $B := 4A$ and we let

$$g(x) := g_{s,r,n,B}(x) := -(1+x)^{s+1}\log(1+x) - L_{s+2}(x),$$

where $L_{s+2}$ is a polynomial of degree $s+2$, whose derivative has $s+1$ prescribed zeros, none at $x=-1$ and it is so chosen that $g \in \mathbb{B}^r \cap \Delta^{(1)}(Y_s)$, and satisfying (6.6) and (6.7). (See [9, p. 204] for details.) We observe that

$$g^{(r)}(x) = a_r(1+x)^{-s-1}, \tag{6.9}$$

where $a_r$ is a constant depending only on $r$, and

$$g'(-1) \neq 0. \tag{6.10}$$

Now, for each $0 < \epsilon < 1$, denote by $T_\epsilon(x)$ the Taylor polynomial of degree $r$ at the point $-1+\epsilon$, and put

$$g_\epsilon(x) := \begin{cases} g(x), & \text{if } \ x \geq -1+\epsilon, \\ T_\epsilon(x), & \text{otherwise.} \end{cases}$$

Evidently,

$$g_\epsilon \in \mathbb{C}_\varphi^r,$$

and

$$\|\varphi^r g_\epsilon^{(r)}\| \leq \|\varphi^r g^{(r)}\| = 1.$$

Hence by (6.8),

$$\omega_{1,r}^\varphi(g_\epsilon^{(r)}, t) \leq 2.$$

Since

$$g(x) - T_\epsilon(x) = (s+1)\frac{1}{r!}\int_{-1+\epsilon}^{x} (x-t)^r g^{(r+1)}(t)dt,$$

it follows by (6.9) that for $x \in [-1, -1+\epsilon]$,

$$|g(x) - T_\epsilon(x)| \leq \frac{|a_r|}{r!}\int_{-1}^{-1+\epsilon} (1+t)^r(1+t)^{-s-2}\, dt < c\epsilon^{r-s-1} < c\epsilon,$$

and

$$|g'(x) - T_\epsilon'(x)| < c\epsilon^{r-s-2} < c\epsilon.$$

In view of (6.10), it is thus possible to select $\epsilon_0 > 0$, so that

$$\|g - g_{\epsilon_0}\| < \frac{B}{2},$$

and that $g_{\epsilon_0}$ is comonotone with $g$. We conclude that

$$e_n^{(1,s)}(g_{\epsilon_0}) \geq e_n^{(1,s)}(g) - \|g - g_{\epsilon_0}\| > B - \frac{B}{2} = 2A \geq A\omega_{1,r}^\varphi(g_{\epsilon_0}, 1).$$

This completes our proof. $\square$

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