SOME ESTIMATES FOR CONVEX POLYNOMIAL APPROXIMATION IN L_p

Y. K. Hu[1], D. Leviatan and X. M. Yu

Abstract. We prove a direct theorem for convex polynomial L_p-approximation, $0 < p < \infty$, in terms of the classical modulus of smoothness $\omega_3(f,t)_p$. This theorem may be regarded as an extension to L_p of the well-known pointwise estimates of the Timan type and their variants of R. DeVore, Y. K. Hu and D. Leviatan for convex approximation in L_p. It leads to a characterization of convex functions in Lipschitz classes (and more general Besov spaces) in terms of their approximation by algebraic polynomials.

1. Introduction

Let $f \in L_p[-1,\,1]$, $0 < p < \infty$, and Π_n be the set of all algebraic polynomials of degree $\le n$. We are interested in approximating a convex function f in L_p by convex polynomials $P_n \in \Pi_n$ and estimating the degree of approximation.

In 1981, Švedov [Š] proved that for a given convex $f \in L_p[-1,\,1]$, $0 < p \le \infty$ and $n \ge 1$, convex polynomials $P_n \in \Pi_n$ exist, such that

$$\|f - P_n\|_p \le \; C\omega_2(f, \frac{1}{n})_p \; , \tag{1.1}$$

where C depends only on p and $\omega_2(f,\cdot)_p$ is the ordinary second order modulus of smoothness of f in the L_p (quasi-) norm. Švedov also showed that in (1.1), ω_2 cannot be replaced by ω_4 while keeping the constant C independent of f and n.

Several improvements of Švedov's result have since been obtained in a series of papers by DeVore, Leviatan and Yu (see [DeL], [L], [LY] and [Y]). The improvements have been in the direction of obtaining estimates that yield characterization of smoothness classes of convex functions in L_p by means of their degree of convex approximation, or (in $\mathbf{C}[-1,\,1]$) also by means of pointwise estimates. Namely, for a convex $f \in L_p[-1,\,1]$, $0 < p \le \infty$ and $n \ge 1$, convex polynomials $P_n \in \Pi_n$ exist, such that

$$\|f - P_n\|_p \le \; C\omega_2^{\varphi}(f, \frac{1}{n})_p \; ,$$

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where $\omega_2^\varphi(f,\cdot)_p$ is the Ditzian-Totik [DiT] second modulus of smoothness of f in L_p. For a convex $f \in \mathbf{C}[-1,1]$ and $n \geq 1$, convex polynomials $P_n \in \Pi_n$ exist, such that

$$|f(x) - P_n(x)| \leq C\omega_2(f, \rho_n(x))_\infty ,$$

where C is an absolute constant and $\rho_n(x) := n^{-2} + n^{-1}\sqrt{1-x^2}$, $-1 \leq x \leq 1$, $n \geq 1$. Over the years since Švedov's result attempts have been made to close the gap between ω_2 and ω_4, namely, the possibility of estimating convex polynomial approximation in terms of ω_3. First, for $p = \infty$, Hu, Leviatan and Yu [HLY] verified that for a convex function $f \in \mathbf{C}[-1,1]$ and $n \geq 2$, convex polynomials $P_n \in \Pi_n$ exist, such that

$$\|f - P_n\|_\infty \leq C\omega_3(f, \frac{1}{n})_\infty .$$

Then Kopotun [K1] has obtained stronger estimates by replacing the ordinary third modulus in the above inequality by the third Ditzian-Totik modulus of smoothness $\omega_3^\varphi(f,\cdot)_\infty$. He has also obtained pointwise estimates involving the third modulus of smoothness, namely, for such a convex $f \in \mathbf{C}[-1,1]$ and $n \geq 2$, convex polynomials $P_n \in \Pi_n$ exist, such that

$$|f(x) - P_n(x)| \leq C\omega_3(f, \rho_n(x))_\infty . \tag{1.2}$$

Finally, the following result by DeVore, Hu and Leviatan [DeHL] shows that similar estimates in terms of $\omega_3^\varphi(f,\cdot)_p$ are valid also for $0 < p < \infty$, thus completely closing the gap left by Švedov.

Theorem A. *Let $f \in L_p[-1,1]$, $0 < p < \infty$, be convex. Then for each $n \geq 2$, a convex polynomial $P_n \in \Pi_n$ exists, such that*

$$\|f - P_n\|_p \leq C\omega_3^\varphi(f, \frac{1}{n})_p ,$$

where C is a constant which depends only on p when $p \to 0$.

In this paper, we consider the case $0 < p < \infty$, and establish a theorem that may be regarded as an extension of (1.2) to L_p.

We follow some ideas of Leviatan and Operstein [LO]. We use the usual notation for the (quasi-)norms of a sequence $\{a_k\}$ of numbers and a sequence $\{g_k\}$ of functions

$$\|a_k\|_{\ell_p} := (\sum_{k=1}^\infty |a_k|^p)^{1/p}, \qquad \|g_k\|_{\ell_p(L_p)} := \left(\sum_{k=1}^\infty \|g_k\|_{L_p}^p\right)^{1/p} .$$

The following was proved in [LO].

Theorem B. *For every function $f \in L_p[-1,1]$, $0 < p < \infty$, a sequence of algebraic polynomials $P_k \in \Pi_{2^k}$, which preserve monotonicity and convexity exists, such that*

$$\left\| \frac{f - P_k}{\omega(\rho_{2^k})} \right\|_{\ell_p(L_p)} \le C \left\| \frac{\omega_2(f, 2^{-k})_p}{\omega(2^{-k})} \right\|_{\ell_p},$$

for every majorant $\omega : \mathbf{R}_+ \to \mathbf{R}_+$ satisfying

$$M^{-1}\omega(t_2) \le \omega(t_1) \le M\omega(t_2), \qquad 0 < t_1 \le t_2 \le 2t_1\ , \tag{1.3}$$

for some $M \ge 1$. The constant $C = C(M)$ depends on M, and on p, if $p \to 0$.

For nonconstrained approximation, this theorem was proved by E. Dyn'kin [Dy] for $\omega(t) = t^\alpha$, $\alpha > 0$, and by Operstein [O-1] and [O-2] for a general ω. In this case the theorem is valid with $\omega_r(f, 2^{-k})_p$ for any $r \ge 1$, replacing $\omega_2(f, 2^{-k})_p$, and the theorem leads to characterization of some classes of functions which are described by smoothness conditions.

While Theorem B gives best estimates of this type for monotone approximation, in view of Theorem A, it is natural to expect that for convex approximation Theorem B would hold with $\omega_3(f,t)_p$. Indeed, the main result of this paper is that such an estimate is valid, namely, we prove the following.

Theorem 1.1. *For every convex function $f \in L_p[-1,1]$, $0 < p \le \infty$, a sequence of convex polynomials $P_k \in \Pi_{2^k}$ exists, such that*

$$\left\| \frac{f - P_k}{\omega(\rho_{2^k})} \right\|_{\ell_p(L_p)} \le C \left\| \frac{\omega_3(f, 2^{-k})_p}{\omega(2^{-k})} \right\|_{\ell_p}, \tag{1.4}$$

for every majorant $\omega : \mathbf{R}_+ \to \mathbf{R}_+$ satisfying (1.3) for some $M \ge 1$. The constant $C = C(M,p)$ depends on M, and on p, if $p \to 0$.

Note that if we take $\omega(\cdot) = \omega_3(f, \cdot)_\infty$, which evidently satisfies (1.3), then for $p = \infty$, (1.4) becomes (1.2) and we have nothing new. Thus, we concentrate in this paper on proving the case $0 < p < \infty$.

Denote by

$$\mathrm{Lip}^*(\alpha, p) := \{f \in L_p[-1,1] : \omega_3(f,t)_p = O(t^\alpha)\}, \quad 0 < \alpha < 3, \quad 0 < p < \infty\ ,$$

the extension of the classical Lipschitz classes to the L_p metric. Then, from Theorem 1.1, we obtain the characterization of convex functions $f \in \mathrm{Lip}^*(\alpha, p)$.

Corollary 1.2. *A convex function f is in $Lip^*(\alpha, p)$, if and only if, a sequence of convex polynomials $P_k \in \Pi_{2^k}$ exists, such that*

$$\left\| (f - P_k) \min(1, t2^k)^\gamma \right\|_{\ell_p(L_p)} = O(t^\alpha),$$

for some $\alpha < \gamma < 3$.

Also, let $\psi : \mathbf{R}_+ \to \mathbf{R}_+$, be a nondecreasing function satisfying the condition

$$\psi(ct) \le C_0 c^s \psi(t), \qquad \forall c \ge 1,\ t > 0, \quad \text{and } 0 < s = s(\psi) < \infty\ .$$

The generalized Besov spaces B^{ψ}_{pq}, $0 < p, q \le \infty$, are defined by

$$B^{\psi}_{pq}[-1,\,1] := \{f \in L_p[-1,\,1] : \int_0^1 \left(\frac{\omega_r(f,t)_p}{\psi(t)}\right)^q \frac{dt}{t} < \infty\}.$$

In the special case that $\psi(t) = t^s$, $0 < s < \infty$, the classes B^{ψ}_{pq} coincide with the classical Besov spaces B^s_{pq}. From Theorem 1.1, we have the following corollary.

Corollary 1.3. *A convex* f *belongs to* $B^{\psi}_{pq}[-1,\,1]$, $0 < s(\psi) < 3$, *if and only if, a sequence of convex polynomials* $P_k \in \Pi_{2^k}$ *exists, such that*

$$\left\{ \left\| \frac{f - P_k}{\psi_i(\rho_{2^k})} \right\|_p \right\}_{i,k=0}^{\infty} \in \ell_q(\ell_p),$$

where $\psi_i(u) := \psi(u)\left(2^i u + (2^i u)^{-1}\right)^{\gamma}$, $s(\psi) < \gamma < 3$.

To prove Theorem 1.1, we shall follow the methods employed in [LO] and [DeHL]. In §2, we give several auxiliary results including the convex continuous piecewise quadratic approximation in L_p and polynomial approximation to truncated power functions. The proof of Theorem 1.1 then follows in §3. In the sequel as above the constant C is an absolute constant which depends only on the parameter that are specified and we take the liberty to use it to denote different constants on different occurrences even in the same line.

2. Auxiliary results

Let $x_{n,j} := \cos \frac{n-j}{n}\pi$, $j = 0, 1, \ldots, n$, $x_{n,j} := -1$, $j < 0$ and $x_{n,j} := 1$, $j > n$, and set $d_{n,j}(x) := 1 + |x - x_{n,j}| / |I_{n,j}|$, where $|I_{n,j}| = |x_{n,j} - x_{n,j-1}|$, $j = 1, 2, \ldots, n$. Then we have the following estimates for $|I_{n,j}|$ and $d_{n,j}(x)$.

Lemma 2.1. *For* $n \ge 1$,

$$|I_{n,j}| \sim \rho_n(x), \qquad x \in I_{n,j}, \tag{2.1}$$

and

$$\frac{|I_{n,j}|}{|I_{n,i}|} \le\ C d_{n,j}(x), \qquad x \in I_{n,i}, \quad i, j = 1, \ldots, n\ . \tag{2.2}$$

Proof. Relation(2.1) is self-evident. In order to prove (2.2) we note that the following inequalities were proved in [K2, (26)] (our notation is somewhat different).

$$C_1 d^{-2}_{n,j}(x)\rho_n(x) \le \rho_n(x_{n,j}) \le C_2 d_{n,j}(x)\rho_n(x).$$

Now (2.2) follows by combining this with (2.1). $\square$

Direct computations yield (see [LO]),

Lemma 2.2. *Let $\beta \geq 2$. Then there is a constant $C = C(\beta)$ such that*

$$\int_{-1}^{1} d_{n,j}(x)^{-\beta}\, dx \leq\; C|I_{n,j}|, \qquad j = 1, \ldots, n, \tag{2.3}$$

and

$$\sum_{j=1}^{n} d_{n,j}^{-\beta}(x) \leq\; C, \qquad x \in [-1,\, 1]. \tag{2.4}$$

Finally, we need the following result (see [LO, Lemma 2.3])

Lemma 2.3. *Let $B_l := \{I_{n,j} : 2^{-l} < |I_{n,j}| \leq 2^{-l+1},\; j = 1, \ldots, n,\; n = 2^k,\; k = 0, 1, \ldots\}$, and let $\tilde{I}_{n,j} \subset [-1, 1]$ be intervals such that $\tilde{I}_{n,j}$ contains $I_{n,j}$ with $|\tilde{I}_{n,j}| \leq A|I_{n,j}|$ for some absolute constant A. Then there exists an absolute constant K such that*

$$\sum_{I_{n,j} \in B_l} \chi_{\tilde{I}_{n,j}}(x) \leq\; K, \qquad -1 \leq x \leq 1.$$

We also need some results of Kopotun on the approximation of truncated powers [K1], namely

Lemma 2.4. *For $n \geq 1$, there exist polynomials $\sigma_{n,j}$, $R_{n,j}$ and $\bar{R}_{n,j}$, $j = 1, \ldots, n-1$, of degrees not exceeding $\frac{25}{8}sn$, satisfying*

$$|(x - x_{n,j})_+ - \sigma_{n,j}(x)| \leq\; Cd_{n,j}(x)^{-s}|I_{n,j}|, \qquad x \in [-1,\, 1], \tag{2.5}$$

$$\left|(x - x_{n,j})_+^2 - R_{n,j}(x)\right| \leq\; Cd_{n,j}(x)^{-s}|I_{n,j}|^2, \qquad x \in [-1,\, 1] \tag{2.6}$$

and

$$\left|(x - x_{n,j})_+^2 - \bar{R}_{n,j}(x)\right| \leq\; Cd_{n,j}(x)^{-s}|I_{n,j}|^2, \qquad x \in [-1,\, 1], \tag{2.7}$$

where $s \geq 1 + log_2 M + 2\max(1, \frac{1}{p})$ with the M in (1.3). Moreover, if we denote $\chi_{n,j} := \chi_{[x_{n,j},1]}$, then there is an integer L such that for all $n \geq 1$ and $j = 1, \ldots, n-1$,

$$(x_{n,j+1} - x_{n,j})\sigma''_{Ln,Lj}(x) - R''_{Ln,Lj}(x) \geq -2\chi_{n,j}(x), \qquad x \in [-1,\, 1] \tag{2.8}$$

and

$$(x_{n,j} - x_{n,j-1})\sigma''_{Ln,Lj}(x) + \bar{R}''_{Ln,Lj}(x) \geq 2\chi_{n,j}(x), \qquad x \in [-1,\, 1]. \tag{2.9}$$

Our last lemma is a recent result on convex spline approximation due to DeVore, Hu and Leviatan [DeHL].

Lemma 2.5. *Let $f \in L_p[-1, 1]$, $0 < p < \infty$, be convex. Then, there exists a continuous convex piecewise quadratic $S_n(f)$ with knots at $x_{n,j}$, $j = 0, 1, \ldots, n$, such that*

$$\|f - S_n(f)\|_{L_p(I_{n,j})} \leq\; C\omega_3(f, |\tilde{I}_{n,j}|, \tilde{I}_{n,j})_p, \qquad j = 1, \ldots, n, \tag{2.10}$$

where each interval $\tilde{I}_{n,j}$ contains $I_{n,j}$ with $|\tilde{I}_{n,j}| \leq A|I_{n,j}|$ (where A is independent of f, n, j and p).

3. Proof of Theorem 1.1

Let $f \in L_p[-1, 1]$, $0 < p < \infty$, be convex. From Lemma 2.5, there exists a continuous convex piecewise quadratic $S_n(f)$ with knots at $x_{n,j}$, $j = 0, 1, \dots, n$, which satisfies (2.10). Now (1.3) and (2.1) imply $\omega(\rho_n) \sim \omega(|I_{n,j}|)$ for $x \in I_{n,j}$. Hence,

$$\begin{aligned} \left\| \frac{f - S_n(f)}{\omega(\rho_n)} \right\|_p^p &= \sum_{j=1}^{n} \left\| \frac{f - S_n(f)}{\omega(\rho_n)} \right\|_{L_p(I_{n,j})}^p \\ &\le C \sum_{j=1}^{n} \frac{\|f - S_n(f)\|_{L_p(I_{n,j})}^p}{\omega(|I_{n,j}|)^p} \\ &\le C \sum_{j=1}^{n} \frac{\omega_3(f, |\tilde{I}_{n,j}|, \tilde{I}_{n,j})_p^p}{\omega(|I_{n,j}|)^p} . \end{aligned} \tag{3.1}$$

Next we construct a sequence of convex polynomials $P_n(f)$ which are close to $S_n(f)$ in the sense of (3.1). The construction is originally due to Kopotun [K1] and was used for the L_p case and in the way we need it here, by DeVore, Hu and Leviatan [DeHL], §3.

First we classify the knots $x_{n,j}$ according to four types depending on the second order divided differences of $S_n(f)$,

$$a_{n,j} := [x_{n,j-1}, x_{n,j}, x_{n,j+1}]S_n(f), \qquad j = 1, \dots, n-1.$$

We define $x_{n,j}$ to be of type I, if $a_{n,j+1} < a_{n,j} \le a_{n,j-1}$. We define $x_{n,j}$ to be of type II, if $a_{n,j-1} < a_{n,j} \le a_{n,j+1}$. If $\max\{a_{n,j-1}, a_{n,j+1}\} < a_{n,j}$, we say that $x_{n,j}$ is of type III and all other $x'_{n,j}s$ are of type IV.

Let

$$A_{n,0} := [x_{n,0}, x_{n,1}]S_n(f) - [x_{n,1}, x_{n,2}]S_n(f) + [x_{n,0}, x_{n,2}]S_n(f),$$
$$A_{n,j} := a_{n,j} - a_{n,j+1}, \qquad j = 1, \dots, n-2,$$

and

$$B_{n,j} := -A_{n,j-1}, \qquad j = 2, \dots, n-1.$$

Then the continuous convex piecewise quadratic $S_n(f)$ obtained in Lemma 2.5 has the following representation on $[-1, 1]$.

$$\begin{aligned} S_n(f)(x) =& S_n(f)(-1) + A_{n,0}(x+1) + a_{n,1}(x+1)^2 \\ &+ \sum_{x_{n,j} \in I \cup III} A_{n,j} \left((x_{n,j+1} - x_{n,j})(x - x_{n,j})_+ - (x - x_{n,j})_+^2 \right) \\ &+ \sum_{x_{n,j} \in II \cup III} B_{n,j} \left((x_{n,j} - x_{n,j-1})(x - x_{n,j})_+ + (x - x_{n,j})_+^2 \right), \end{aligned} \tag{3.2}$$

where by the definition of the various types it follows that $A_{n,j} > 0$ for $x_{n,j} \in I \cup III$ and $B_{n,j} > 0$ for $x_{n,j} \in II \cup III$. Also, we have the following estimates for $|A_{n,j}|$ and $|B_{n,j}|$, (see [DeHL], (3.12)).

$$|A_{n,j}|, \ |B_{n,j}| \le C \ |I_{n,j}|^{-2-\frac{1}{p}} \omega_3(f, |\tilde{I}_{n,j}|, \tilde{I}_{n,j})_p. \tag{3.3}$$

By Lemma 2.4, the polynomials

$$\begin{aligned} P_n(f)(x) =& S_n(f)(-1) + A_{n,0}(x+1) + a_{n,1}(x+1)^2 \\ &+ \sum_{x_{n,j}\in I\cup III} A_{n,j}\left((x_{n,j+1}-x_{n,j})\sigma_{Ln,Lj}(x) - R_{Ln,Lj}(x)\right) \\ &+ \sum_{x_{n,j}\in II\cup III} B_{n,j}\left((x_{n,j}-x_{n,j-1})\sigma_{Ln,Lj}(x) + \bar{R}_{Ln,Lj}(x)\right), \end{aligned} \tag{3.4}$$

satisfy

$$P_n(f)''(x) \geq S_n(f)''(x) \geq 0$$

for all $x \in [-1, 1]$, $x \neq x_{n,j}$, $1 \leq j \leq n-1$, so that they are convex in $[-1, 1]$.

Denote

$$\begin{aligned} J_n^{(1)}(x) := \sum_{x_{n,j}\in I\cup III} A_{n,j}&\left[(x_{n,j+1}-x_{n,j})\left((x-x_{n,j})_+ - \sigma_{Ln,Lj}(x)\right)\right. \\ &\left. - \left((x-x_{n,j})_+^2 - R_{Ln,Lj}(x)\right)\right], \end{aligned}$$

and

$$\begin{aligned} J_n^{(2)}(x) := \sum_{x_{n,j}\in II\cup III} B_{n,j}&\left[(x_{n,j}-x_{n,j-1})\left((x-x_{n,j})_+ - \sigma_{Ln,Lj}(x)\right)\right. \\ &\left. + \left((x-x_{n,j})_+^2 - \bar{R}_{Ln,Lj}(x)\right)\right]. \end{aligned}$$

Then, it follows from (3.2) and (3.4) that

$$S_n(f)(x) - P_n(f)(x) = J_n^{(1)}(x) + J_n^{(2)}(x). \tag{3.5}$$

Now, by (2.5)–(2.7) and (3.3), we have

$$\begin{aligned} \left|J_n^{(k)}(x)\right| &\leq C\sum_{j=1}^{n} |I_{n,j}|^{-2-\frac{1}{p}}\omega_3(f, |\tilde{I}_{n,j}|, \tilde{I}_{n,j})_p |I_{n,j}|^2 d_{n,j}^{-s}(x) \\ &= C\sum_{j=1}^{n} |I_{n,j}|^{-\frac{1}{p}}\omega_3(f, |\tilde{I}_{n,j}|, \tilde{I}_{n,j})_p d_{n,j}^{-s}(x), \qquad k = 1, 2, \end{aligned} \tag{3.6}$$

where $s > log_2 M + 2\max(1, \frac{1}{p})$. Observe that (1.3) and (2.2) imply that for $x \in I_{n,i}$, $1 \leq i \leq n$,

$$\begin{aligned} \frac{\omega(|I_{n,j}|)}{\omega(|I_{n,i}|)} &\leq \left(1 + \frac{|I_{n,j}|}{|I_{n,i}|}\right)^\alpha \\ &\leq C\left(1 + d_{n,j}(x)\right)^\alpha \leq C d_{n,j}^\alpha(x), \qquad j = 1, \ldots, n, \end{aligned} \tag{3.7}$$

where $\alpha = log_2 M$ with M given in (1.3). Then, by virtue of (2.3), (3.5), (3.6) and (3.7), we have

$$\begin{aligned}
\left\| \frac{S_n(f) - P_n(f)}{\omega(\rho_n)} \right\|_p^p &= \sum_{i=1}^n \left\| \frac{S_n(f) - P_n(f)}{\omega(\rho_n)} \right\|_{L_p(I_{n,i})}^p \\
&\leq C \sum_{i=1}^n \frac{\|S_n(f) - P_n(f)\|_{L_p(I_{n,i})}^p}{\omega(|I_{n,i}|)^p} \\
&\leq C \sum_{i=1}^n \int_{I_{n,i}} \frac{\left| J_n^{(1)}(x) + J_n^{(2)}(x) \right|^p}{\omega(|I_{n,i}|)^p} dx \\
&\leq C \sum_{i=1}^n \int_{I_{n,i}} \left(\sum_{j=1}^n \frac{|I_{n,j}|^{-\frac{1}{p}} \omega_3(f, |\tilde{I}_{n,j}|, \tilde{I}_{n,j})_p d_{n,j}^{-s}(x)}{\omega(|I_{n,j}|)} \frac{\omega(|I_{n,j}|)}{\omega(|I_{n,i}|)} \right)^p dx \\
&\leq C \sum_{i=1}^n \int_{I_{n,i}} \left(\sum_{j=1}^n \frac{|I_{n,j}|^{-\frac{1}{p}} \omega_3(f, |\tilde{I}_{n,j}|, \tilde{I}_{n,j})_p d_{n,j}^{-s+\alpha}(x)}{\omega(|I_{n,j}|)} \right)^p dx.
\end{aligned} \tag{3.8}$$

When $1 \leq p < \infty$, we apply Jensen's inequality to the p-th power of the inner sum (the sum on j) on the right-hand side of (3.8). By (2.4), we obtain

$$\begin{aligned}
\left\| \frac{S_n(f) - P_n(f)}{\omega(\rho_n)} \right\|_p^p &\leq C \sum_{i=1}^n \int_{I_{n,i}} \sum_{j=1}^n \left(\frac{|I_{n,j}|^{-\frac{1}{p}} \omega_3(f, |\tilde{I}_{n,j}|, \tilde{I}_{n,j})_p}{\omega(|I_{n,j}|)} \right)^p d_{n,j}^{-s+\alpha}(x) dx \\
&\leq C \sum_{j=1}^n \frac{|I_{n,j}|^{-1} \omega_3(f, |\tilde{I}_{n,j}|, \tilde{I}_{n,j})_p^p}{\omega(|I_{n,j}|)^p} \int_{-1}^1 d_{n,j}^{-s+\alpha}(x) dx \\
&\leq C \sum_{j=1}^n \frac{\omega_3(f, |\tilde{I}_{n,j}|, \tilde{I}_{n,j})_p^p}{\omega(|I_{n,j}|)^p},
\end{aligned} \tag{3.9}$$

where for the last inequality we used (2.3).

When $0 < p < 1$, we have by virtue of the subadditivity of $\|\cdot\|_p^p$ and (2.3),

$$\begin{aligned}
\left\| \frac{S_n(f) - P_n(f)}{\omega(\rho_n)} \right\|_p^p &\leq C \sum_{i=1}^n \sum_{j=1}^n \int_{I_{n,i}} \frac{|I_{n,j}|^{-1} \omega_3(f, |\tilde{I}_{n,j}|, \tilde{I}_{n,j})_p^p}{\omega(|I_{n,j}|)^p} d_{n,j}^{(-s+\alpha)p}(x) dx \\
&\leq C \sum_{j=1}^n \frac{|I_{n,j}|^{-1} \omega_3(f, |\tilde{I}_{n,j}|, \tilde{I}_{n,j})_p^p}{\omega(|I_{n,j}|)^p} \int_{-1}^1 d_{n,j}^{(-s+\alpha)p}(x) dx \\
&\leq C \sum_{j=1}^n \frac{\omega_3(f, |\tilde{I}_{n,j}|, \tilde{I}_{n,j})_p^p}{\omega(|I_{n,j}|)^p}.
\end{aligned} \tag{3.10}$$

Now, it follows from (3.1), (3.9) and (3.10) that

$$\left\| \frac{f - P_n(f)}{\omega(\rho_n)} \right\|_p^p \le C \sum_{j=1}^{n} \frac{\omega_3(f, |\tilde{I}_{n,j}|, \tilde{I}_{n,j})_p^p}{\omega(|I_{n,j}|)^p}.$$

Finally, by Lemma 2.3 we have (see [LO] for a similar approach),

$$\begin{aligned} \sum_{k=1}^{\infty} \left\| \frac{f - P_{2^k}(f)}{\omega(\rho_{2^k})} \right\|_p^p &\le C \sum_{k=1}^{\infty} \sum_{j=1}^{2^k} \frac{\omega_3(f, |\tilde{I}_{2^k,j}|, \tilde{I}_{2^k,j})_p^p}{\omega(|I_{2^k,j}|)^p} \\ &\le C \sum_{l=1}^{\infty} \sum_{I_{2^k,j} \in B_l} \frac{\omega_3(f, |\tilde{I}_{2^k,j}|, \tilde{I}_{2^k,j})_p^p}{\omega(|I_{2^k,j}|)^p} \\ &\le C \sum_{l=1}^{\infty} \frac{\omega_3(f, 2^{-l})_p^p}{\omega(2^{-l})^p} \sup_{-1 \le x \le 1} \sum_{I_{2^k,j} \in B_l} \chi_{\tilde{I}_{2^k,j}}(x) \\ &\le C \sum_{l=1}^{\infty} \frac{\omega_3(f, 2^{-l})_p^p}{\omega(2^{-l})^p}. \end{aligned}$$

This completes the proof of Theorem 1.1. $\square$

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