Approximation of Monotone Functions: A Counter Example

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Abstract. When we approximate a continuous nondecreasing function f in $[-1,1]$, we wish sometimes that also the approximating polynomials be nondecreasing. However, this constraint restricts very much the degree of approximation that the polynomials can achieve, namely, only the rate of $\omega_2(f,1/n)$. It turns out as we will prove somewhere else that relaxing the monotonicity requirement in intervals of length $1/n^2$ near the endpoints allows the polynomials to achieve the rate of ω_3. On the other hand, we show in this paper, that even when we relax the requirement of monotonicity of the polynomials on sets of measures approaching 0, (no matter how slowly or how fast), ω_4 is not reachable.

§1. Introduction

Let $f \in C[-1,1]$ be nondecreasing on $I := [-1,1]$. Then DeVore [1] proved that there exist nondecreasing polynomials such that

$$\|f - P_n\|_{C(I)} \le c\,\omega_2(f,1/n), \tag{1.1}$$

where c is an absolute constant and $\omega_k(f;\cdot)$ denotes the modulus of smoothness of order k, of f. (See also pointwise estimates by DeVore and Yu [2] and similar estimates involving the second Ditzian- Totik modulus of smoothness by Leviatan [3].)

On the other hand it is known (see Shvedov [4]) that in (1.1) one cannot replace ω_2 by ω_k with any $k \ge 3$.

It is quite natural to ask whether one can strengthen (1.1) in the sense of being able to replace ω_2 by moduli of smoothness of higher order, if one is willing to allow P_n not to be monotone on a rather "small" subset of I. This indeed is the case. If we allow the polynomials not to be monotone in intervals of length $1/n^2$ near the end points, then it is possible to achieve the estimates

$$\|f - p_n\| \le c\,\omega_3(f,1/n). \tag{1.2}$$

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We will prove that as a special case of a more general result (pointwise estimates for comonotone approximation) in another paper. However, even this improvement comes to a halt; it cannot be extended to ω_4, and thus not to ω_k for any $k > 3$.

In order to state our theorem we need some notation. Given $\epsilon > 0$ and a nondecreasing function $f \in C[-1,1]$, we denote

$$E_n^{(1)}(f;\epsilon) := \inf_{P_n} \|f - P_n\|_{C(I)},$$

where the infimum is taken over all polynomials P_n of degree not exceeding n satisfying

$$\operatorname{meas}(\{x \,:\, P_n'(x) \geq 0\} \cap I) \geq 2 - \epsilon.$$

Theorem. *For each sequence $\overline{\epsilon} = \{\epsilon_n\}_{n=1}^{\infty}$, of nonnegative numbers tending to 0, there exists a nondecreasing function $f := f_{\overline{\epsilon}} \in C[-1,1]$ such that*

$$\limsup_{n\to\infty} \frac{E_n^{(1)}(f;\epsilon_n)}{\omega_4(f,1/n)} = \infty. \tag{1.3}$$

Remarks 1. *A weaker version of our theorem, where P_n is required to be monotone in I, that is, under the stronger condition that $\epsilon_n = 0$, $n = 1, 2, \ldots$, is due to Wu and Zhou* [5]. One should note that Wu and Zhou have a single function while Shvedov [4] obtained for an arbitrarily large prescribed c, and for each n, a different function $f_{n,c}$, which violates (1.1).

2. *Note that we allow the relaxing of monotonicity (on small sets) anywhere in I, not necessarily near the endpoints, and ω_4 cannot be had.*

§2. A Counter Example (Proof of the Theorem)

While above we have used c as an absolute constant which may differ on different occurrences, in this section we will have to keep track of the constants, therefore we denote them by $C_1, C_2, \ldots$. We begin by recalling some simple properties of the Chebyshev polynomials for the interval $[-2,2]$. For $\nu > 1$, let

$$t_\nu(x) := \cos \nu \arccos \frac{x}{2}, \quad x \in [-2,2],$$

denote the Chebyshev polynomial and let $z_j := 2\cos\frac{j\pi}{\nu}$, $j = 0, \ldots, \nu$, be its extrema. Given $0 < b < \frac{1}{2}$, we take two points on both sides of z_j, $j = 1, \ldots, \nu - 1$, namely, we set $z_{j,l} := 2\cos(\frac{(j+b)\pi}{\nu})$ and $z_{j,r} := 2\cos(\frac{(j-b)\pi}{\nu})$. Note that

$$|t_\nu(z_{j,l})| = |t_\nu(z_{j,r})| = \cos \pi b,$$

and

$$z_{j,r} - z_{j,l} = 4 \sin\frac{j\pi}{\nu} \sin\frac{\pi b}{\nu} < 4\pi\frac{b}{\nu}. \tag{2.1}$$

We truncate the Chebyshev polynomial by setting

$$t_\nu^*(x) := t_{\nu,b}^*(x) := \begin{cases} \cos\pi b, & t_\nu(x) > \cos\pi b \\ -\cos\pi b, & t_\nu(x) < -\cos\pi b \\ t_\nu(x), & \text{otherwise.} \end{cases}$$

Since

$$1 - \cos\pi b = 2\sin^2\frac{\pi b}{2} < 5b^2, \tag{2.2}$$

for any $x \in I$, it follows by the monotonicity of the areas as we go away from the origin, and the alternation in sign of these areas, that

$$\Big|\int_0^x (t_\nu(u) - t_\nu^*(u))\,du\Big| \le \Big|\int_{z_{[\frac{\nu}{2}],l}}^{z_{[\frac{\nu}{2}],r}} (t_\nu(u) - t_\nu^*(u))\,du\Big| < 4\pi\frac{b}{\nu}5b^2 = C_1\frac{b^3}{\nu}, \tag{2.3}$$

where we have applied also (2.1).

Now, given $n \ge 1$ and $0 < b < \frac{1}{2}$, let $\nu := [b^{\frac{3}{4}}n] + 2$, where $[a]$ denotes the largest integer not exceeding a. Put

$$t_{\nu,b} := t_\nu + \cos\pi b, \quad \text{and} \quad \tilde{t}_{\nu,b} := t_{\nu,b}^* + \cos\pi b.$$

Finally,

$$T_{\nu,b}(x) := \int_0^x t_{\nu,b}(u)\,du \quad \text{and} \quad f_{n,b}(x) := \int_0^x \tilde{t}_{\nu,b}(u)\,du, \quad x \in I.$$

Obviously $f_{n,b}$ is a nondecreasing function on I and it readily follows by (2.3) that

$$\|f_{n,b} - T_{\nu,b}\| \le C_1\frac{b^3}{\nu} \le C_1\frac{b^{\frac{9}{4}}}{n}, \tag{2.4}$$

where we denote by $\|\cdot\|_J$ the max-norm taken on the interval J, and when the norm is on I, we suppress the subscript.

If we set $\tilde{z}_{j,l} := 2\cos\big(\frac{(j+b/2)\pi}{\nu}\big)$, and $\tilde{z}_{j,r} := 2\cos\big(\frac{(j-b/2)\pi}{\nu}\big)$, then we have for all j for which $z_j \in I$,

$$\tilde{z}_{j,r} - z_j = 4\sin\frac{(j-\frac{b}{4})\pi}{\nu}\sin\frac{b\pi}{4\nu} > \frac{b}{\nu}, \tag{2.5}$$

and similarly,

$$z_j - \tilde{z}_{j,l} > \frac{b}{\nu}. \tag{2.5'}$$

Let j be odd. Since $\sin b\pi/4 > 3b/4$ for b satisfying $b\pi/4 < \pi/6$, we have

$$\begin{aligned} T'_{\nu,b}(x) = t_{\nu,b}(x) &\le -\cos b\pi/2 + \cos b\pi = -2\sin b\pi/4\sin 3b\pi/4 \\ &< -2\frac{3b}{4}\frac{3b}{2} = -\frac{9b^2}{4}, \quad x \in [\tilde{z}_{j,l}, \tilde{z}_{j,r}]. \end{aligned} \tag{2.6}$$

Then since $I \subset [-2,2]$, it follows by the Bernstein inequality that

$$\|T_{\nu,b}^{(4)}\| = \|t_{\nu,b}^{(3)}\| = \|t_\nu^{(3)}\| \le \frac{C_2\nu^3}{(1-(\frac{1}{2})^2)^{3/2}}\|t_\nu\|_{[-2,2]} = C_3\nu^3.$$

Hence, by (2.4),

$$\begin{aligned} \omega_4(f_{n,b},\frac{1}{n}) &\le \omega_4(f_{n,b}-T_{\nu,b},\frac{1}{n}) + \omega_4(T_{\nu,b},\frac{1}{n}) \\ &\le 2^4\|f_{n,b}-T_{\nu,b}\| + \frac{1}{n^4}\|T_{\nu,b}^{(4)}\| \\ &\le 2^4C_1\frac{b^{9/4}}{n} + \frac{C_3\nu^3}{n^4} \le C_4\frac{b^{9/4}}{n}, \end{aligned} \tag{2.7}$$

by the relation between ν and n.

Next we need a simple lemma.

Lemma 1. *There exists a constant C_5 such that for any interval $J \subseteq I$, we have the following. For any measurable sets $E \subseteq I$, if*

$$P_n'(x) \ge 0, \qquad x \in J\setminus E, \tag{2.8}$$

then

$$\|f_{n,b}-P_n\|_J \ge \frac{b^2|J|}{n} - \frac{C_5}{n}(b^{9/4} + b|E| + \frac{b^{5/4}}{n}). \tag{2.9}$$

Proof: Let J_0 denote the middle third of J. We consider two cases. First we assume that J_0 contains at most one of the z_j's. Then by the definition of ν we get

$$|J| < C_6\frac{1}{\nu} < C_6\frac{b^{-3/4}}{n}.$$

Hence

$$\|f_{n,b}-P_n\|_J \ge 0 > \frac{b^2|J|}{n} - C_6\frac{b^{5/4}}{n^2}. \tag{2.10}$$

On the other hand, if J_0 contains at least two extrema z_j, then it contains at least $2C_7\nu|J|$ extrema, for some constant C_7. These extrema satisfy (2.5) and (2.5′), and about half of them (and at least one) have odd indices, then together with (2.6) we conclude that

$$\operatorname{meas}(J_0 \cap \{x \,:\, T_{\nu,b}'(x) < -\frac{9b^2}{4}\}) \ge \frac{1}{2}\frac{2b}{\nu}C_7\nu|J| = C_7b|J|. \tag{2.11}$$

Now, if $C_7b|J| \le |E|$, then

$$\|f_{n,b}-P_n\|_J \ge 0 \ge \frac{b^2|J|}{n} - \frac{b|E|}{nC_7}. \tag{2.12}$$

Otherwise, $C_7 b|J| > |E|$. Then by (2.11) there is a point $x_0 \in J_0 \setminus E$, for which

$$T'_{\nu,b}(x_0) < -\frac{9b^2}{4}.$$

Hence, (2.8) yields,

$$\frac{9b^2}{4} \le P'_n(x_0) - T'_{\nu,b}(x_0) \le \frac{2}{|J|}\frac{n}{\sqrt{1-(1/3)^2}}\|P_n - T_{\nu,b}\|_J,$$

where we have used the Bernstein inequality. Therefore from (2.4),

$$\begin{aligned}\frac{b^2|J|}{n} \le \frac{3\sqrt{2}}{4}\frac{b^2|J|}{n} &\le \|P_n - T_{\nu,b}\|_J \\ &\le \|P_n - f_{n,b}\|_J + \|f_{n,b} - T_{\nu,b}\|_J \\ &\le \|P_n - f_{n,b}\|_J + C_1\frac{b^{9/4}}{n}.\end{aligned} \tag{2.13}$$

Taking $C_5 := \max\{C_6, \frac{1}{C_7}, C_1\}$, (2.9) now follows by combining (2.10), (2.12) and (2.13). ■

We are now in a position to define $f_{\overline{\epsilon}} := f$, for a given sequence $\overline{\epsilon} = \{\epsilon_n\}$. Let $b_n := (\max\{\epsilon_n^2, \frac{1}{n}\})^{2/5}$, and set $d_0 := 1$, and

$$d_j := \frac{b_{n_j}^{9/4}}{n_j} d_{j-1} = \prod_{\nu=1}^{j} \frac{b_{n_\nu}^{9/4}}{n_\nu}, \quad j \ge 1,$$

where the sequence $\{n_\nu\}$ is defined by induction as follows. First, we choose n_1 so large that $b_{n_1}^{1/8} < \frac{1}{12}$ (as needed in (2.15) below) and $J_0 := I$. Suppose that $\{n_1, \dots, n_{\sigma-1}\}$ and $J_{\sigma-2} \subseteq J_{\sigma-3} \subseteq \cdots \subseteq J_0$, $\sigma \ge 2$, have been defined. Then put

$$F_{\sigma-1} := \sum_{j=1}^{\sigma-1} d_{j-1} f_{n_j, b_{n_j}},$$

and let $J_{\sigma-1}$ be an interval such that $J_{\sigma-1} \subseteq J_{\sigma-2}$ and

$$F'_{\sigma-1}(x) = 0, \quad x \in J_{\sigma-1}. \tag{2.14}$$

(The induction process will guarantee the existence of such intervals.) Let $N_{1,\sigma}$ be such that

$$|J_{\sigma-1}| \ge b_n^{1/8}, \quad n \ge N_{1,\sigma}, \tag{2.15}$$

and let

$$N_{2,\sigma} := \left(\frac{\|F_{\sigma-1}^{(2)}\|}{d_{\sigma-1}}\right)^{10}. \tag{2.16}$$

Finally, we take

$$n_\sigma > \max\{n_{\sigma-1}, N_{1,\sigma}, N_{2,\sigma}\}$$

so big that the function $f'_{n_\sigma,b_{n_\sigma}}$ oscillates a few times inside the interval $J_{\sigma-1}$ and since it vanishes on some interval in each oscillation, that is, inside $J_{\sigma-1}$, there exists an interval $J_\sigma \subset J_{\sigma-1}$ as required in (2.14).

Now denote

$$\Phi_\sigma := \sum_{j=\sigma}^{\infty} d_{j-1} f_{n_j,b_{n_j}},$$

where the convergence of the series is justified by the definition of the d_j's and the fact that $\|f_{n,b_n}\| \le 2$, for all n. In fact

$$\begin{aligned} \|\Phi_\sigma\| &\le 2d_{\sigma-1}\Big(1 + \frac{b_{n_\sigma}^{9/4}}{n_\sigma} + \frac{b_{n_\sigma}^{9/4} b_{n_{\sigma+1}}^{9/4}}{n_\sigma n_{\sigma+1}} + \ldots\Big) \\ &\le 2d_{\sigma-1} \sum_{j=0}^{\infty} 2^{-j} = 4d_{\sigma-1}. \end{aligned} \tag{2.17}$$

So we define

$$f := f_{\bar\epsilon} := \sum_{j=1}^{\infty} d_{j-1} f_{n_j,b_{n_j}},$$

and we prove

Lemma 2. *For each $\sigma \ge 1$ we have*

$$\omega_4(f, 1/n_\sigma) \le C_8 d_\sigma. \tag{2.18}$$

Proof: First, by (2.17)

$$\omega_4(\Phi_{\sigma+1}, 1/n_\sigma) \le 2^4 \|\Phi_{\sigma+1}\| \le 2^6 d_\sigma. \tag{2.19}$$

At the same time, (2.7) yields

$$\omega_4(d_{\sigma-1} f_{n_\sigma,b_{n_\sigma}}, 1/n_\sigma) \le d_{\sigma-1} C_4 \frac{b_{n_\sigma}^{9/4}}{n_\sigma} = C_4 d_\sigma. \tag{2.20}$$

Finally,

$$\begin{aligned} \omega_4(F_{\sigma-1}, 1/n_\sigma) &\le 4\omega_2(F_{\sigma-1}, 1/n_\sigma) \\ &\le \frac{4}{n_\sigma^2} \|F_{\sigma-1}^{(2)}\| \\ &= 4\frac{\|F_{\sigma-1}^{(2)}\|}{d_{\sigma-1}} n_\sigma^{-1/10} \left(\frac{1}{n_\sigma^{2/5} b_{n_\sigma}}\right)^{9/4} d_\sigma \\ &\le 4d_\sigma, \end{aligned} \tag{2.21}$$

by virtue of (2.16) and the definitions of b_{n_σ}, d_σ and n_σ. Lemma 2 follows by combining (2.19), (2.20) and (2.21). ∎

The last lemma that we need is

Lemma 3. *There is an absolute constant C_9 such that whenever $E \subset I$ is a measurable set satisfying*

$$|E| \leq \epsilon_{n_\sigma}, \tag{2.22}$$

and P_{n_σ} is a polynomial satisfying

$$P'_{n_\sigma}(x) \geq 0, \quad x \in I \setminus E, \tag{2.23}$$

then

$$\|f - P_{n_\sigma}\| \geq (b_{n_\sigma}^{-1/8} - C_9) d_\sigma. \tag{2.24}$$

Proof: Since $F_{\sigma-1}$ is constant on $J_{\sigma-1}$, we may write

$$f(x) = d_{\sigma-1} f_{n_\sigma, b_{n_\sigma}}(x) + \Phi_{\sigma+1}(x) + M, \quad x \in J_{\sigma-1}. \tag{2.25}$$

Let

$$Q_{n_\sigma} := \frac{1}{d_{\sigma-1}}(P_{n_\sigma} - M).$$

Then it follows from (2.23),

$$Q'_{n_\sigma}(x) \geq 0, \quad x \in J_{\sigma-1} \setminus E.$$

Thus by virtue of Lemma 1,

$$\|Q_{n_\sigma} - f_{n_\sigma, b_{n_\sigma}}\|_{J_{\sigma-1}} \geq \frac{b_{n_\sigma}^2 |J_{\sigma-1}|}{n_\sigma} - \frac{C_5}{n_\sigma}(b_{n_\sigma}^{9/4} + b_{n_\sigma}|E| + \frac{b_{n_\sigma}^{5/4}}{n_\sigma}). \tag{2.26}$$

The definition of n_σ and (2.15) yield,

$$b_{n_\sigma}^2 |J_{\sigma-1}| = b_{n_\sigma}^{17/8} \left(\frac{|J_{\sigma-1}|}{b_{n_\sigma}^{1/8}} \right) \geq b_{n_\sigma}^{17/8}.$$

On the other hand, (2.22) and the definition of b_{n_σ} imply

$$b_{n_\sigma}|E| \leq b_{n_\sigma}\epsilon_{n_\sigma} \leq b_{n_\sigma}^{9/4},$$

and

$$\frac{b_{n_\sigma}^{5/4}}{n_\sigma} \leq b_{n_\sigma}^{15/4} < b_{n_\sigma}^{9/4}.$$

Hence (2.26) implies

$$\|Q_{n_\sigma} - f_{n_\sigma, b_{n_\sigma}}\|_{J_{\sigma-1}} \geq \frac{1}{n_\sigma}(b_{n_\sigma}^{17/8} - 3C_5 b_{n_\sigma}^{9/4}) = \frac{b_{n_\sigma}^{9/4}}{n_\sigma}(b_{n_\sigma}^{-1/8} - 3C_5).$$

In other words,

$$\|P_{n_\sigma} - M - d_{\sigma-1} f_{n_\sigma, b_{n_\sigma}}\|_{J_{\sigma-1}} \geq d_{\sigma-1} \frac{b_{n_\sigma}^{9/4}}{n_\sigma}(b_{n_\sigma}^{-1/8} - 3C_5) = d_\sigma(b_{n_\sigma}^{-1/8} - 3C_5).$$

In view of (2.25), it follows from (2.17) that,

$$\begin{aligned}\|f - P_{n_\sigma}\| \ge \|f - P_{n_\sigma}\|_{J_{\sigma-1}} &\ge \|P_{n_\sigma} - M - d_{\sigma-1} f_{n_\sigma, b_{n_\sigma}}\|_{J_{\sigma-1}} - \|\Phi_{\sigma+1}\| \\ &\ge (b_{n_\sigma}^{-1/8} - (3C_5 + 4))d_\sigma,\end{aligned}$$

and Lemma 3 is proved with $C_9 := 3C_5 + 4$. ■

The proof of (1.3) now follows from Lemmas 2 and 3 since

$$\begin{aligned}\limsup_{n\to\infty} \frac{E_n^{(1)}(f;\epsilon_n)}{\omega_4(f,1/n)} &\ge \limsup_{\sigma\to\infty} \frac{E_{n_\sigma}^{(1)}(f;\epsilon_{n_\sigma})}{\omega_4(f,1/n_\sigma)} \\ &\ge \limsup_{\sigma\to\infty} \frac{1}{C_8}(b_{n_\sigma}^{-1/8} - C_9) = \infty. \qquad ■\end{aligned}$$

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