CONSTANTS IN COMONOTONE POLYNOMIAL APPROXIMATION– A SURVEY

D. Leviatan and I. A. Shevchuk

Abstract. We survey the Jackson and Jackson type estimates for comonotone polynomial approximation of continuous and r-times continuously differentiable functions which change their monotonicity finitely many times in a finite interval, say $[-1,1]$, with special attention to the constants involved in the estimates. We describe four possibilities ranging from the existence of estimates involving absolute constants and which are valid for polynomials of all degrees except the very few first ones, through estimates where either the constants or the degrees of the polynomials depend on the location of the points of change of monotonicity, then through estimates where either the constants or the degrees of the polynomials depend on the specific function, and finally cases where there are no Jackson type estimates possible.

§1. Introduction

We are going to survey what is known about the degree of approximation of a continuous function f on a finite interval, say $[-1,1]$, which changes its monotonicity finitely many time, say at s fixed points, in that interval, by polynomials that keep the shape of the function in the sense that they change monotonicity exactly at these points. Such approximants are said to be comonotone with the function. We will concentrate on the constants in the Jackson-type estimates which present new and interesting phenomena. We are seeking estimates with constants C and N which are independent of the degree n of the polynomials. The constants may depend on parameters like the number r of derivatives that f possess, the number s of changes of monotonicity and their locations, and the order k of the various moduli of smoothness of the function f, or its derivative as the case may be. We will indicate which parameters the constants depend on and by default this will mean independence of all others.

We will illustrate the results by truth tables. For completeness we present also the truth tables for the corresponding estimates in the case of monotone approximation, most of which are well known and follow from results by Lorentz, Zeller, DeVore, Shvedov, Yu, Wu, Zhou, Kopotun, Listopad, Dzyubenko, and the authors, (see [LS–LS3] and the references therein). We will mention the monotone case only briefly, in Remarks 1–4, especially in relation to some recent new results.

Since shape constraints limit the degree of approximation, we will relax the constraints of shape in small parts of the interval and see that we can improve the guaranteed degree

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of approximation. However, this can be done only up to a limit as we demonstrate in Section 3.

§2. Truly comonotone approximation

Suppose that $f \in \mathbb{C}[-1,1]$ changes monotonicity at $Y_s : -1 < y_s < \cdots < y_1 < 1$ and assume that in $[y_1, 1]$, the function is nondecreasing. We will denote by $\Delta^{(1)}(Y_s)$, the collection of all such functions. The degree of comonotone approximation by polynomials p_{n-1} of degree not exceeding $n-1$ will be denoted by

$$E_n^{(1)}(f, Y_s) := \inf_{p_{n-1} \in \Delta^{(1)}(Y_s)} \|f - p_{n-1}\|.$$

Here and below $\|\cdot\|$ denotes the uniform norm over $[-1,1]$, or if needed, the L_∞-norm.

The first Jackson type estimates for comonotone polynomial approximation were obtained by Newman [N] (see also Iliev [I]) who proved that

$$E_n^{(1)}(f, Y_s) \le C(s)\omega\left(f, \frac{1}{n}\right), \qquad n \ge 1,$$

where $\omega(f,t)$ is the modulus of continuity of f, that is, $\omega(f,t) := \omega_1(f,t)$, where we write

$$\omega_k(f,t) := \sup_{h \in [0,t]} \max_{x \in [-1,1-kh]} \left| \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x+jh) \right|,$$

for the moduli of smoothness of f.

If $f \in \mathbb{C}^1[-1,1]$, then Beatson and Leviatan [BL] showed that

$$E_n^{(1)}(f, Y_s) \le C(s)\frac{1}{n}\omega\left(f', \frac{1}{n}\right), \qquad n \ge 2.$$

At the same time Shvedov [S] showed that it is not possible to obtain the Jackson estimate with higher moduli of smoothness when we just assume that $f \in \mathbb{C}[-1,1]$ unless the constant depends on Y_s. Namely, he proved that given $s \ge 1$, for any $A > 0$ and each $n \ge 2$, a function $g = g_{A,n}$ which changes monotonicity s times exists, such that if p_n is a polynomial comonotone with g, then

$$\|g - p_n\| > A\omega_2(g, 1).$$

On the other hand he showed that

$$E_n^{(1)}(f, Y_s) \le C(Y_s)\omega_2\left(f, \frac{1}{n}\right), \qquad n \ge 1.$$

Thus, Gilewicz and Shevchuk [GS] have investigated the question when does one have for each $g \in \Delta^{(1)}(Y_s) \cap \mathbb{C}^r[-1,1]$, the estimates,

$$E_n^{(1)}(g, Y_s) \le \frac{C}{n^r}\omega_k\left(g^{(r)}, \frac{1}{n}\right) \quad n \ge N. \tag{1}$$

They proved that (1) holds with $C = C(k,r,s)$ and $N = N(k,r,s)$, only when either $k = 1$, or $r > s$, or in the particular case $k = 2$ and $r = s$, moreover in these cases one can always take $N = k + r$. If $k = 2$ and $0 \le r < s$, or $k = 3$ and $1 \le r \le s$, or if $k > 3$ and $2 \le r \le s$, then the estimates hold either with $C = C(k, r, Y_s)$ and $N = k + r$, or with $C = C(k,r,s)$ and $N = N(k, r, Y_s)$, and they fail to hold with $C = C(k,r,s)$ and $N = N(k,r,s)$.

In all other cases, namely, when either $r = 0$ and $k \ge 3$, or $r = 1$ and $k > 3$, it is impossible to achieve any such estimate even with constants C and N which depend on g. In fact, Wu and Zhou [WZ] proved that there is a function $g \in \mathbb{C}[-1,1]$, which changes monotonicity exactly at the y_i's, satisfying

$$\limsup_{n\to\infty} \frac{E_n^{(1)}(g, Y_s)}{\omega_3(g, 1/n)} = \infty,$$

which shows that for $r = 0$ and $k \ge 3$, we cannot have an estimate of type (1), even with C and N depending on g. Dzyubenko, Gilewicz and Shevchuk [DGS] proved the same for $r = 1$ and $k \ge 4$.

We illustrate the results in the following tables. The sign "$+$" in entry (k,r) means that (1) is valid for this pair of numbers (and the proper s for which the table is given) with $C = C(k,r,s)$ and $N = k + r$. The sign "$\oplus$" means that (1) is valid with some of the constants depending on the actual location of the points of change of monotonicity, and the estimates are invalid with $C = C(k,r,s)$ and $N = N(k,r,s)$. The sign "$-$" means that no such estimate exists, even with constants C and N which depend on g. Finally, in rare cases (1) holds with constants C and N, one of which at least depends on g (see Remark 1 below). We designate them with the sign "$\ominus$".

$$\begin{array}{ccccccc}
r & \vdots & \vdots & \vdots & \vdots & \vdots & \cdot^{\cdot^{\cdot}} \\
2 & + & + & + & + & + & \cdots \\
1 & + & + & + & + & + & \cdots \\
0 & + & + & \ominus & - & - & \cdots \\
 & 1 & 2 & 3 & 4 & 5 & k
\end{array}$$

Fig. 1, $s = 0$ (the monotone case).

$$\begin{array}{ccccccc}
r & \vdots & \vdots & \vdots & \vdots & \vdots & \cdot^{\cdot^{\cdot}} \\
3 & + & + & + & + & + & \cdots \\
2 & + & + & + & + & + & \cdots \\
1 & + & + & \oplus & - & - & \cdots \\
0 & + & \oplus & - & - & - & \cdots \\
 & 1 & 2 & 3 & 4 & 5 & k
\end{array}$$

Fig. 2, $s = 1$.

$$\begin{array}{ccccccc}
r & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
4 & + & + & + & + & + & \cdots \\
3 & + & + & + & + & + & \cdots \\
2 & + & + & \oplus & \oplus & \oplus & \cdots \\
1 & + & \oplus & \oplus & - & - & \cdots \\
0 & + & \oplus & - & - & - & \cdots \\
 & 1 & 2 & 3 & 4 & 5 & k
\end{array}$$

Fig. 3, $s = 2$.

$$\begin{array}{ccccccc}
r & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
s+2 & + & + & + & + & + & \cdots \\
s+1 & + & + & + & + & + & \cdots \\
s & + & + & \oplus & \oplus & \oplus & \cdots \\
s-1 & + & \oplus & \oplus & \oplus & \oplus & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
2 & + & \oplus & \oplus & \oplus & \oplus & \cdots \\
1 & + & \oplus & \oplus & - & - & \cdots \\
0 & + & \oplus & - & - & - & \cdots \\
 & 1 & 2 & 3 & 4 & 5 & k
\end{array}$$

Fig. 4, $s > 2$.

Remark 1. Obviously, in the case of monotone approximation there can be no difference between the signs "$+$" and "$\oplus$" ($Y_s = \emptyset$). It turns out that there is one more advantage to the monotone case, namely, in Fig. 1, we have a case where "$\ominus$" occurs. That (1) is invalid in that case, if both C and N are absolute constants is due to Shvedov [S]. However, we have recently shown [LS3] that in that special case, (1) is valid with $C = C(g)$ and $N = 3$, or with an absolute constant C and $N = N(g)$. (See similar phenomena in Fig. 8 below).

For later reference we note that it follows from the above that if $f \in W_\infty^r$, then

$$E_n^{(1)}(f, Y_s) \le C(r,s)\frac{\|f^{(r)}\|}{n^r}, \quad n \ge r. \tag{2}$$

It is possible to extend some of these estimates to pointwise estimates. With $\rho_n(x) := \sqrt{1-x^2}/n + 1/n^2$, Dzyubenko, Gilewicz and Shevchuk [DGS] investigated the possibility of having a polynomial p_{n-1}, which is comonotone with $g \in \mathbb{C}^r[-1,1]$, such that

$$|g(x) - p_{n-1}(x)| \le C[\rho_n(x)]^r \omega_k(g^{(r)}, \rho_n(x)), \qquad n \ge N. \tag{3}$$

Again we summarize these results in tables and we compare them to the table of the monotone case. Here too we have a new result, the "–" sign in the case $r = 0$ and $k = 3$ in Fig. 5 (see [LS3]).

$$\begin{array}{ccccccc}
r & \vdots & \vdots & \vdots & \vdots & \vdots & \cdot^{\cdot^{\cdot}} \\
2 & + & + & + & + & + & \cdots \\
1 & + & + & + & + & + & \cdots \\
0 & + & + & - & - & - & \cdots \\
 & 1 & 2 & 3 & 4 & 5 & k
\end{array}$$

Fig. 5, $s = 0$ (the monotone case).

$$\begin{array}{ccccccc}
r & \vdots & \vdots & \vdots & \vdots & \vdots & \cdot^{\cdot^{\cdot}} \\
3 & + & + & + & + & + & \cdots \\
2 & + & + & + & + & + & \cdots \\
1 & + & + & \oplus & - & - & \cdots \\
0 & + & \oplus & - & - & - & \cdots \\
 & 1 & 2 & 3 & 4 & 5 & k
\end{array}$$

Fig. 6, $s = 1$.

$$\begin{array}{ccccccc}
r & \vdots & \vdots & \vdots & \vdots & \vdots & \cdot^{\cdot^{\cdot}} \\
3 & \oplus & \oplus & \oplus & \oplus & \oplus & \cdots \\
2 & \oplus & \oplus & \oplus & \oplus & \oplus & \cdots \\
1 & \oplus & \oplus & \oplus & - & - & \cdots \\
0 & \oplus & \oplus & - & - & - & \cdots \\
 & 1 & 2 & 3 & 4 & 5 & k
\end{array}$$

Fig. 7, $s > 1$.

(Note that Fig. 6 is the same table as in Fig. 2, and note the major difference between the case $s = 1$ and $s > 1$.)

Another way to take into account the different behavior of polynomials near the endpoints of the interval is the Ditzian–Totik moduli of smoothness which are defined in the following way. With $\varphi(x) := \sqrt{1-x^2}$ and $k \geq 1$, we let

$$\Delta_{h\varphi}^k f(x)$$
$$:= \begin{cases} \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} f(x + (i - \frac{k}{2})h\varphi(x)), & x \pm \frac{k}{2}h\varphi(x) \in [-1,1] \\ 0, & \text{otherwise.} \end{cases}$$

We define

$$\omega_k^\varphi(f,t) := \sup_{0<h\leq t} \|\Delta_{h\varphi}^k f\|, \qquad \omega^\varphi(f,t) := \omega_1^\varphi(f,t).$$

Recall that the advantage of these moduli over the ordinary ones is two-fold. First, they are smaller (sometimes by order of magnitude) and second, they lend themselves to inverse results, see (10) below.

Recently we proved,

Theorem 1. ([LS]) We have the inequality

$$E_n^{(1)}(f,Y_s) \le C(s)\omega^\varphi\left(f,\frac{1}{n}\right), \qquad n\ge 1.$$

Let $\mathbb{B}^r$, $r\ge 1$, be the space of functions $g\in\mathbb{C}[-1,1]$, which possess a locally absolutely continuous $(r-1)$st derivative in $(-1,1)$, such that

$$\|\varphi^r g^{(r)}\| < \infty.$$

Then Theorem 1 implies

$$E_n^{(1)}(f,Y_s) \le C(s)\frac{\|\varphi f'\|}{n}, \qquad n\ge 1,$$

provided $f\in\mathbb{B}^1$.

In view of (2), we begin with the seemingly easier case and prove

Theorem 2. ([LS2]) If $f\in\mathbb{B}^r$, with either $s=1$ and $r=3$, or $r>2s+2$, then

$$E_n^{(1)}(f,Y_s) \le C(r)\frac{\|\varphi^r f^{(r)}\|}{n^r}, \qquad n\ge r. \tag{4}$$

Unlike (2), in all other cases estimates of type (4) are invalid. We have

Theorem 3. ([LS]) Let the constant $A>0$ be arbitrary and let $2\le r\le 2s+2$, excluding the case $r=3$ and $s=1$. Then for any n, there exists a function $g=g_{r,s,n}\in\mathbb{B}^r$, which changes monotonicity s times in $[-1,1]$, for which

$$e_n^{(1,s)}(g) \ge A\|\varphi^r g^{(r)}\|.$$

Here we use the notation

$$e_n^{(1,s)}(g) := \inf E_n^{(1)}(g,Y_s^*),$$

where the infimum is taken over all collections Y_s^*, such that g changes monotonicity at Y_s^*. Since g may be constant in some subinterval (and this is indeed the case for some of the functions $g_{r,s,n}$ of Theorem 3), then there may be many admissible Y_s^*'s. **Thus we encounter a new phenomenon.**

Still we can salvage something, namely

Theorem 4. ([LS2]) If $f\in\mathbb{B}^r$, $r\ge 1$, then

$$E_n^{(1)}(f,Y_s) \le C(r,Y_s)\frac{\|\varphi^r f^{(r)}\|}{n^r}, \qquad n\ge r,$$

and

$$E_n^{(1)}(f,Y_s) \le C(r,s)\frac{\|\varphi^r f^{(r)}\|}{n^r}, \qquad n\ge N(r,Y_s).$$

In an effort to explain the new phenomenon, we go back to estimates involving the Ditzian–Totik moduli. Thus, we are interested in the truth tables for the estimates

$$E_n^{(1)}(g,Y_s) \le \frac{C}{n^r}\omega_k^\varphi\left(g^{(r)},\frac{1}{n}\right), \qquad n\ge N, \tag{5}$$

for each $g\in\Delta^{(1)}\cap\mathbb{C}^r[-1,1]$. First Kopotun and Leviatan proved,

Theorem 5. ([KLe]) The inequalities

$$E_n^{(1)}(f,Y_s) \le C(Y_s)\omega_2^{\varphi}\left(f,\frac{1}{n}\right), \qquad n \ge 2,$$

and

$$E_n^{(1)}(f,Y_s) \le C(s)\omega_2^{\varphi}\left(f,\frac{1}{n}\right), \qquad n \ge N(Y_s),$$

hold.

At the same time since for all pairs (k,r), $\omega_k^{\varphi}(g^{(r)},t) \le \omega_k(g^{(r)},t)$, so that any "$-$" sign in tables 1–4, implies a similar one for the D–T tables. Furthermore repeating the arguments in [LS2] one can prove that the truth tables of (5), coincides with the truth tables of the estimate (1), (see Figs. 1– 4).

On the other hand such truth tables imply Theorem 2 for $r=1$ and Theorem 4 for $r=1$ and $r=2$, but they do not imply any other case.

Therefore we need a modification of the D–T moduli. We set

$$\varphi_{\delta}(x) := \sqrt{(1-x-\frac{\delta}{2}\varphi(x))(1+x-\frac{\delta}{2}\varphi(x))}, \qquad x \pm \frac{\delta}{2}\varphi(x) \in [-1,1],$$

and for $g \in \mathbb{C}(-1,1)$, and $r \ge 1$, denote

$$\omega_{k,r}^{\varphi}(g,t) := \sup_{0\le h\le t}\sup_{x}|\varphi_{kh}^{r}(x)\Delta_{h\varphi(x)}^{k}g(x)|, \quad t \ge 0,$$

where the inner supremum is taken over all x so that

$$x \pm \frac{k}{2}h\varphi(x) \in (-1,1).$$

Note that for $g \in \mathbb{C}[-1,1]$,

$$\omega_{k,0}^{\varphi}(g,t) := \omega_k^{\varphi}(g,t).$$

We will have estimates involving $\omega_{k,r}^{\varphi}(g^{(r)},t)$, $r>0$, and in order to guarantee that $\omega_{k,r}^{\varphi}(g^{(r)},t) \to 0$, as $t \to 0$, we need to restrict our functions to the space $\mathbb{C}_{\varphi}^{r}$, namely, those functions in $\mathbb{C}^{r}(-1,1)$ for which $\lim_{x\to\pm1}\varphi^{r}(x)g^{(r)}(x)=0$. For completion we put $\mathbb{C}_{\varphi}^{0} := \mathbb{C}[-1,1]$.

For $g \in \mathbb{C}_{\varphi}^{r}$ and $0 \le p < r$, it can be shown that

$$\omega_{k+r-p,p}^{\varphi}(g^{(p)},t) \le C(k,r)t^{r-p}\omega_{k,r}^{\varphi}(g^{(r)},t), \quad t \ge 0,$$

and, if $g \in \mathbb{B}^{r}$ and $0 \le p < r$, then $g \in \mathbb{C}_{\varphi}^{p}$, and

$$\omega_{r-p,p}^{\varphi}(g^{(p)},t) \le C(r)t^{r-p}\|\varphi^{r}g^{(r)}\|, \quad t \ge 0.$$

While conversely, if $g \in \mathbb{C}_{\varphi}^{p}$ and $\omega_{r-p,p}^{\varphi}(g^{(p)},t) \le t^{r-p}$, then $g \in \mathbb{B}^{r}$ and

$$\|\varphi^{r}g^{(r)}\| \le C(r);$$

or if say $g \in \mathbb{C}[-1,1]$, $p < \alpha < k$ and $\omega_k^{\varphi}(g,t) \le t^{\alpha}$, then $g \in \mathbb{C}_{\varphi}^{p}$ and

$$\omega_{k-p,p}^{\varphi}(g^{(p)},t) \le C(\alpha,k)t^{\alpha-p}, \quad t \ge 0. \tag{6}$$

The results we have for these moduli are

Theorem 6. ([LS2]) If $f \in \mathbb{C}^r_\varphi$, with $r > 2s+2$, then

$$E_n^{(1)}(f, Y_s) \le \frac{C(k,r,s)}{n^r} \omega^\varphi_{k,r}\left(f^{(r)}, \frac{1}{n}\right), \qquad n \ge k+r.$$

To the contrary we have,

Theorem 7. ([LS2]) Let $r \le 2s+2$, excluding the case $r = 0$ and $k = 1$. Then for any constant $A > 0$ and every $n \ge 1$, there is function $g := g_{k,r,s,n,A} \in \mathbb{C}^r_\varphi$ which changes monotonicity s times in $[-1,1]$, for which

$$e_n^{(1,s)}(g) > A\omega^\varphi_{k,r}(g^{(r)}, 1).$$

Again we can salvage something, namely,

Theorem 8. ([LS2]) If $f \in \mathbb{C}^r_\varphi$, with $r > 2$, then

$$E_n^{(1)}(f, Y_s) \le \frac{C(k,r,Y_s)}{n^r} \omega^\varphi_{k,r}\left(f^{(r)}, \frac{1}{n}\right), \quad n \ge k+r,$$

and

$$E_n^{(1)}(f, Y_s) \le \frac{C(k,r,s)}{n^r} \omega^\varphi_{k,r}\left(f^{(r)}, \frac{1}{n}\right), \quad n \ge N(k,r,Y_s).$$

We wish to point out that Theorem 8 is also valid for $r = 2$ and $k = 1$, and for $r = 1$ and $k = 2$, (see [LS4]). On the other hand, the following negative result holds true.

Theorem 9. ([LS4]) There is a function $g \in \mathbb{C}^2_\varphi$, which changes monotonicity at the y_i's, satisfying

$$\limsup_{n\to\infty} \frac{n^2 E_n^{(1)}(g, Y_s)}{\omega^\varphi_{3,2}(g'', 1/n)} = \infty.$$

Remark 2. There is even a function $g \in \mathbb{C}^2[-1,1]$, which changes monotonicity at the y_i's, for which Theorem 9 is true. Moreover, the proof in [LS4] provides a corresponding result for monotone approximation as well.

We are now in a position to summarize in truth tables what is known about the estimates

$$E_n^{(1)}(g, Y_s) \le \frac{C}{n^r} \omega^\varphi_{k,r}\left(g^{(r)}, \frac{1}{n}\right), \quad n \ge N, \tag{7}$$

for every $g \in \Delta^{(1)}(Y_s) \cap \mathbb{C}^r_\varphi$. It is natural to combine these tables with the truth tables for the estimates

$$E_n^{(1)}(g, Y_s) \le \frac{C}{n^r} \|\varphi^r g^{(r)}\|, \quad n \ge N, \tag{8}$$

for every $g \in \Delta^{(1)}(Y_s) \cap \mathbb{B}^r$. Thus, in the tables below the column $k = 0$ is related to (8), whereas the other columns are related to (7).

$$
\begin{array}{ccccccccc}
r & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
4 & + & + & + & + & + & + & \cdots \\
3 & + & + & + & + & + & + & \cdots \\
2 & + & \ominus & ? & - & - & - & \cdots \\
1 & + & + & \ominus & ? & - & - & \cdots \\
0 & & + & + & \ominus & - & - & \cdots \\
 & 0 & 1 & 2 & 3 & 4 & 5 & k
\end{array}
$$

Fig. 8, $s = 0$ (the monotone case).

$$
\begin{array}{ccccccccc}
r & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
6 & + & + & + & + & + & + & \cdots \\
5 & + & + & + & + & + & + & \cdots \\
4 & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \cdots \\
3 & + & \oplus & \oplus & \oplus & \oplus & \oplus & \cdots \\
2 & \oplus & \oplus & ? & - & - & - & \cdots \\
1 & + & \oplus & \oplus & ? & - & - & \cdots \\
0 & & + & \oplus & - & - & - & \cdots \\
 & 0 & 1 & 2 & 3 & 4 & 5 & k
\end{array}
$$

Fig. 9, $s = 1$.

$$
\begin{array}{ccccccccc}
r & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
2s+4 & + & + & + & + & + & + & \cdots \\
2s+3 & + & + & + & + & + & + & \cdots \\
2s+2 & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
3 & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \cdots \\
2 & \oplus & \oplus & ? & - & - & - & \cdots \\
1 & + & \oplus & \oplus & ? & - & - & \cdots \\
0 & & + & \oplus & - & - & - & \cdots \\
 & 0 & 1 & 2 & 3 & 4 & 5 & k
\end{array}
$$

Fig. 10, $s > 1$.

Remark 3. As we aluded to in Remark 1, in Fig. 8 we have several cases of "$\ominus$". That (7) is invalid for these cases with absolute constants C and N, is due to Kopotun and Listopad [KLi]. We [LS3] have shown that these are "$\ominus$" cases. Note that in all Figs. 8–10 we still have outstanding cases for which it is not known if we should write "$-$" or "$\ominus$" (in Figs. 8–9) and "$-$", "$\ominus$" or "$\oplus$" (in Fig. 10). That is, we do not know whether Theorem 9 is valid when we replace $\omega^{\varphi}_{3,2}$ by $\omega^{\varphi}_{2,2}$.

If we denote, as usual,

$$E_n(f) := \inf_{p_{n-1}} \|f - p_{n-1}\|$$

the degree of the best uniform approximation of f, then obviously,

$$E_n(f) \le E_n^{(1)}(f, Y_s). \tag{9}$$

Cases "+" and "$\oplus$" together with (6) and the Ditzian-Totik [DT] inverse theorem

$$\omega_k^{\varphi}(f,t) \le C(k) t^k \sum_{n=k}^{[1/t]} n^{k-1} E_n(f), \quad 0 < t \le \frac{1}{k}, \tag{10}$$

provide a partial inverse to (9), analogous to Theorems 2 and 4, (except for the case $2s+2 < \alpha \le 2s+3$ and the case $s=1$ and $2 < \alpha < 3$ in Theorem 10 and $\alpha = 3$ in Theorem 11, which we have to prove separately), namely,

Theorem 10. ([LS2]) Assume that either $0 < \alpha < 1$, or $s = 1$ and $2 < \alpha < 3$, or $\alpha > 2s+2$. Then, if

$$E_n(f) < \frac{1}{n^\alpha}\ , \quad \forall n > \alpha, \tag{11}$$

then

$$E_n^{(1)}(f, Y_s) < \frac{C(\alpha, s)}{n^\alpha}\ , \quad \forall n > \alpha.$$

and

Theorem 11. ([LS2]) Assume that $\alpha > 0$. Then, if (11) holds, then

$$E_n^{(1)}(f, Y_s) < \frac{C(\alpha, Y_s)}{n^\alpha}\ , \quad \forall n > \alpha,$$

and

$$E_n^{(1)}(f, Y_s) < \frac{C(\alpha, s)}{n^\alpha}\ , \quad \forall n > N(\alpha, Y_s).$$

We note that it is impossible to make these constants independent of Y_s. Indeed we have

Theorem 12. (LS2]) Let $1 \le \alpha \le 2s+2$, excluding the case $s=1$ and $2 < \alpha < 3$. Then for any constant $B > 0$ and each $n > \alpha$,there exists a function $g := g_{s,\alpha,n,B}$ which changes monotonicity s times in $[-1,1]$ and for which we simultaneously have,

$$E_m(g) \le \frac{1}{m^\alpha}, \quad \forall m > \alpha$$

and

$$e_n^{(1,s)}(g) \ge B.$$

Remark 4. A corresponding result to Theorem 10 in monotone approximation is valid for all $\alpha > 0$, except for the exceptional case $\alpha = 2$. The latter is of the type "$\ominus$" (see [KLi] and [LS3]).

§3. Nearly comonotone approximation

We are going to give up the monotonicity constraint in the neighborhood of the points Y_s and near the endpoints of the interval. (Recall that $f \in \Delta^{(1)}(Y_s)$.) This will result in improved order of approximation, but only up to a certain extent. To this end, we write the Chebyshev nodes $x_j := x_{n,j} := \cos \pi j/n$, $j = 0, \dots, n$ and put $x_{-1} := x_0$, $x_{n+1} := x_n$. Given Y_s, if $x_{j+1} \le y_k < x_j$, then we delete the neighborhood (x_{j+2}, x_{j-1}). In some cases we may have to delete also small neighborhoods of the end points. Namely, we set

$$O(n, Y_s) := \bigcup_{j:[x_{j+1}, x_j) \cap Y_s \neq \emptyset} (x_{j+2}, x_{j-1}),$$

and

$$O^*(n, Y_s) := O(n, Y_s) \cup [-1, -1 + 1/n^2] \cup [1 - 1/n^2, 1].$$

First we have

Theorem 13. ([LS1]) There are constants $c = c(s,k)$ and $C = C(s,k)$ for which, if $f \in \mathbb{C}^1[-1,1]$, then for each $n \ge k$, a polynomial p_n of degree not exceeding n, which is comonotone with f on $[-1,1] \setminus O([n/c], Y)$ (i.e.,

$$p_n'(x) \prod_{i=1}^{s} (x - y_i) \ge 0, \quad x \in [-1,1] \setminus O([n/c], Y)),$$

exists such that

$$|f(x) - p_n(x)| \le C(s,k)\, \rho_n(x) \omega_k(f', \rho_n(x)), \quad x \in [-1,1].$$

Also,

Theorem 14. ([LS1]) There are constants $c = c(s,k)$ and $C(s,k)$ such that for every $n \ge 2$, a polynomial p_n of degree not exceeding n, which is comonotone with f on $[-1,1] \setminus O^*([n/c], Y_s)$ exists, such that

$$|f(x) - p_n(x)| \le C(s)\, \omega_3(f, \rho_n(x)).$$

Finally, the party is over when we try to get ω_4. Indeed, in [DLS] we prove that this is impossible. In order to state our theorem we need some notation. Given $\epsilon > 0$ and a nondecreasing function $g \in \mathbb{C}[-1,1]$, we denote

$$E_n^{(1)}(g;\epsilon) := \inf_{p_{n-1}} \|g - p_{n-1}\|,$$

where the infimum is taken over all polynomials p_{n-1} of degree not exceeding $n-1$ satisfying

$$\operatorname{meas}(\{x \,:\, p_{n-1}'(x) \ge 0\} \cap [-1,1]) \ge 2 - \epsilon.$$

Then

Theorem 15. ([DLS]) For each sequence $\overline{\epsilon} = \{\epsilon_n\}_{n=1}^{\infty}$, of nonnegative numbers tending to 0, there exists a monotone function $g := g_{\overline{\epsilon}} \in \mathbb{C}[-1,1]$, such that

$$\limsup_{n\to\infty} \frac{E_n^{(1)}(g;\epsilon_n)}{\omega_4(g,1/n)} = \infty.$$

This result is extended to the comonotone case in [LS5]. Thus we can summarize these results in a truth table, which is the same for all $s \geq 0$, namely,

$$\begin{array}{ccccccc}
r & \vdots & \vdots & \vdots & \vdots & \vdots & \cdot^{\cdot^{\cdot}} \\
2 & + & + & + & + & + & \cdots \\
1 & + & + & + & + & + & \cdots \\
0 & + & + & + & - & - & \cdots \\
 & 1 & 2 & 3 & 4 & 5 & k
\end{array}$$

Fig. 11, All s and all types of estimates.

Remark 5. Theorems 13 and 14 give pointwise estimates for nearly comonotone approximation, involving the ordinary moduli of smoothness. However, we have the same truth tables for estimates involving all other types of moduli considered in this paper if we relax the comonotonicity requirements near the points of monotonicity change and near the end points. One should emphasize though, that sometimes in the analogs of Theorem 13 we have to replace O with O^*. It should also be noted, that the constant c in Theorems 13 and 14 cannot be too small (see [LS2]), whereas in their analogs this is not essential [LS5].

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School of Mathematical Sciences, Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel

Institute of Mathematics, National Academy of Sciences of Ukraine, Kyiv 252601, Ukraine