INTERPOLATORY POINTWISE ESTIMATES FOR POLYNOMIAL APPROXIMATION

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Dedicated to the memory of Vladislav Kirilovich Dzjadyk, 18.02.1919–26.10.1998. His influence on this work and on our previous research is invaluable.

Abstract. We discuss whether or not it is possible to have interpolatory pointwise estimates in the approximation of a function $f \in C[0,1]$, by polynomials. For the sake of completeness as well as in order to strengthen some existing results, we discuss briefly the situation in unconstrained approximation. Then we deal with positive and monotone constraints where we show exactly when such interpolatory estimates are achievable by proving affirmative results and by providing the necessary counterexamples in all other cases.

§1. Introduction and main results

The effect of the endpoints of the finite interval on the quality of approximation of continuous functions by algebraic polynomials, was first observed by Nikolskiĭ [Nik46]. Later pointwise estimates of this phenomenon were given by Timan [Tim51] ($k = 1$), Dzjadyk [Dzj58, Dzj77] ($k = 2$), Freud [Fre59] ($k = 2$), and Brudnyĭ [Bru63] ($k \ge 2$), who proved that if $f \in C^r[0,1]$, then for each $n \ge N = r + k - 1$, a polynomial $p_n \in \Pi_n$ exists, such that

$$|f(x) - p_n(x)| \le c(r,k)\rho_n^r(x)\omega_k(f^{(r)}, \rho_n(x)), \quad 0 \le x \le 1. \tag{1.1}$$

where $\rho_n(x) := \frac{1}{n^2} + \frac{1}{n}\varphi(x)$, $\varphi(x) := \sqrt{x(1-x)}$, and where ω_k, $k \ge 1$, is the ordinary kth modulus of smoothness, and $c(r,k)$ is a constant which depends only on r and k, and which is independent of f and n. Note that for $n = r + k - 1$, (1.1) is in fact a Whitney inequality [Whi57].

Lorentz at a conference in Oberwolfach [Lor64], raised the question whether $\rho_n(x)$ can be replaced by the smaller quantity $\delta_n(x) := \frac{1}{n}\varphi(x)$, that is, whether

$$|f(x) - p_n(x)| \le c(r,k)\delta_n^r(x)\omega_k(f^{(r)}, \delta_n(x)), \quad 0 \le x \le 1, \tag{1.2}$$

holds. Such estimates we call interpolatory pointwise estimates.

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Indeed, Teljakovskiĭ [Tel66] ($k = 1$, $r = 0$), Gopengauz [Gop67] ($k = 1$, $r \geq 1$), DeVore [DeV76, DeV77] ($k = 2, r = 0$), and Gonska and Hinnemann [HiGo83, GoHi85] ($r \geq 0$, $k \leq r + 2$), proved that for $f \in C^r[0,1]$, if $k \leq r + 2$, there are polynomials $p_n \in \Pi_n$ for which (1.2) holds, for all $n \geq N = 4r + 4$.

When $r + k = 2$, then it is known that (1.2) is valid for all $n \geq 1 (= r + k - 1)$, i.e., we have an interpolatory Whitney-type estimate and we don't need to exclude all $n < 4r + 4$. So, although in this paper we intend to concentrate mostly on constrained approximation, for the sake of completeness we give in Section 2 a short proof of the interpolatory Whitney-type estimate for unconstrained approximation. Namely,

Proposition 1. *Let $k \leq r + 2$ and assume that $f \in C^r[0,1]$. Then there is a polynomial $p_{2[(r+k+1)/2]-1} \in \Pi_{2[(r+k+1)/2]-1}$ for which*

$$(1.3) \qquad |f(x) - p_{2[(r+k+1)/2]-1}(x)| \leq c(r)\varphi^r(x)\omega_k(f^{(r)}, \varphi(x)), \quad 0 \leq x \leq 1,$$

and if $k \leq r$, then also

$$(1.4) \qquad |f(x) - p_{2[(r+k+1)/2]-1}(x)| \leq c(r)\varphi^{2[(r+k+1)/2]}(x)\omega_k(f^{(r)}, 1), \quad 0 \leq x \leq 1,$$

where $c(r)$ depends only on r.

Note that the polynomial degree depends on whether $r + k$ is even or odd.

Combining (1.2) and (1.3) we may now conclude the validity of

Corollary 1. *Let $k \leq r + 2$ and assume that $f \in C^r[0,1]$. Then for each $n \geq 2[(r + k + 1)/2] - 1$, there is a polynomial $p_n \in \Pi_n$, such that (1.2) holds with a constant $c(r)$ depending only on r.*

Remark 1. (i) It should be noted that it is impossible in Corollary 1, to replace $2[(r + k + 1)/2] - 1$ by any lower figure as is manifested in the approximation of $f(x) := x^{2[(r+k+1)/2]-1}$.
(ii) Proposition 1 is a simple consequence of inequality (2.3) below, which is proved in [She84a, She84b] (see also Galan [Gal91] and [She92, Theorem 4.3, Lemma 6.4]), and which is a realization of an idea of Dzjadyk to combine the Whitney and Marchaud estimates in one inequality. Inequality (1.3) also follows from some inequalities of Gopengauz [Gop92, Gop94] (which also follow from (2.3) below).

Conversely, it has been shown by Yu [Yu85a], that (1.2) is not valid if $k > r+2$. Namely, she proved for all $r \geq 0$, that if $k > r + 2$, then for each n and every constant $A > 0$, a function $f = f_{r,k,n,A} \in C^r[0,1]$ exists, such that for any polynomial $p_n \in \Pi_n$, there is a point $x \in [0,1]$ for which

$$|f(x) - p_n(x)| > A\delta_n^r(x)\omega_k(f^{(r)}, \delta_n(x)),$$

holds.

We strengthen this negative result by proving in Section 3,

Theorem 1. *Given $r \ge 0$, a function $f = f_r \in C^r[0,1]$ exists, such that for any algebraic polynomial p,*

$$\limsup_{x\to 0} \frac{|f(x) - p(x)|}{\varphi^r(x)\omega_{r+3}(f^{(r)}, \varphi(x))} = \infty. \tag{1.5}$$

Remark 2. (*i*) Evidently (1.5) implies that for the same function f_r and any polynomial $p_n \in \Pi_n$, for arbitrary n, we have

$$\limsup_{x\to 0} \frac{|f(x) - p_n(x)|}{\delta_n^r(x)\omega_k(f^{(r)}, \delta_n(x))} = \infty,$$

for any $k \ge r+3$.
(*ii*) Obviously by symmetry, we have a similar phenomenon at $x = 1$. In fact it is easy to construct a function which exhibits this phenomenon at both endpoints. Indeed one may take $f_r^*(x) := f_r(x) + \chi_r(x)(f_r(1-x) - f_r(x))$, where $\chi \in C^{2r+3}$ is such that $0 \le \chi(x) \le 1$, $0 \le x \le 1$, and $\chi(x) = 0$, $0 \le x \le 1/4$, $\chi(x) = 1$, $3/4 \le x \le 1$.

The main purpose of this paper is to discuss interpolatory pointwise estimates for constrained approximation. We will investigate positive and monotone approximation. We denote by Δ^0 the set of positive functions on $[0,1]$, and by Δ^1 the collection of monotone nondecreasing functions on that interval.

The problem of positive approximation is to estimate the pointwise degree of approximation of a function $f \in C^r[0,1] \cap \Delta^0$ by polynomials $p_n \in \Pi_n \cap \Delta^0$. Estimates of the form (1.1) for positive approximation are known. In fact, Timan [Tim51] proved his result for unconstrained approximation in the case $r = 0$ and $k = 1$, using positive linear operators. Other positive linear operators which provide (1.1) for $r = 0$ and $k = 1$, were constructed by, among others, Lehnhoff [Leh83] and Cao and Gonska [CaGo94]. It is worthwhile noting that for unconstrained approximation, linear operators which yield etimates of the form (1.1) for $r = 0$ and any $k \ge 1$, were first constructed by Brudnyĭ [Bru68]. Recently, Dzyubenko [Dzy96] has given an affirmative answer to all other cases of (1.1) for positive approximation, applying nonlinear methods. To the best of our knowledge the difficult question whether positive **linear** operators can provide an affirmative answer to (1.1) in the case $r = 0$ and $k = 2$ has been settled only recently by Gavrea [Gav96].

Gavrea [Gav95] was also the first to construct positive linear operators which yield the interpolatory estimates (1.2) in the case $r = 0$ and $k = 1$, and recently [Gav96], he has constructed such operators which yield (1.2) for $r = 0$ and $k = 2$. It is interesting to note that these operators also preserve monotonicity and convexity of the function, see Cottin et al. [CoAl98]. Thus (1.2) is valid for positive approximation provided $r + k \le 2$.

It remains to investigate the cases where $r + k > 2$. To this end we have the following affirmative answer which we prove in Section 4.

Theorem 2. *Let $k \le r$. If $f \in C^r[0,1] \cap \Delta^0$, then for each $n \ge N := 2[(r+k+1)/2]$, there is a polynomial $p_n \in \Pi_n \cap \Delta^0$, such that*

$$|f(x) - p_n(x)| \le c(r)\delta_n^r(x)\omega_k(f^{(r)}, \delta_n(x)), \quad 0 \le x \le 1. \tag{1.6}$$

A special case of Theorem 2 is an interpolatory Whitney-type estimate which we need in the proof of the theorem and which we prove in Section 2. Namely,

Proposition 2. *Let $r \geq 2$ and $k \leq r$. For any $f \in C^r[0,1] \cap \Delta^0$, a polynomial $p_{2[(r+k+1)/2]} \in \Pi_{2[(r+k+1)/2]} \cap \Delta^0$ exists, such that*

$$|f(x) - p_{2[(r+k+1)/2]}(x)| \leq c(r)\varphi^{2[(r+k+1)/2]}(x)\omega_k(f^{(r)}, 1), \quad 0 \leq x \leq 1, \tag{1.7}$$

which in turn implies

$$|f(x) - p_{2[(r+k+1)/2]}(x)| \leq c(r)\varphi^r(x)\omega_k(f^{(r)}, \varphi(x)), \quad 0 \leq x \leq 1. \tag{1.8}$$

Remark 3. One should emphasize that in case $r > 1$, it is impossible in general to have Theorem 2 for a polynomial of lower degree, say degree $r + k - 1$ when $r + k$ is even, or $r + k$ when it is odd. We will give counterexamples in Section 3.

Together with the affirmative cases $r + k \leq 2$, Theorem 2 covers all occurrences where (1.8) holds. For all other cases we have a negative result of the nature of Yu's result [Yu85a]. Namely, we prove in Section 3,

Theorem 3. *Let $k > r \geq 1$. Then for each $n \geq 1$ and every constant $A > 0$, a function $f = f_{k,r,n,A} \in C^r[0,1] \cap \Delta^0$ exists, such that for any polynomial $p_n \in \Pi_n \cap \Delta^0$, there is a point $x \in [0,1]$, for which*

$$|f(x) - p_n(x)| > A\frac{x^{r/2}}{n^r}\omega_k\big(f^{(r)}, \frac{\sqrt{x}}{n}\big), \tag{1.9}$$

holds.

Remark 4. (*i*) By symmetry, we have a similar phenomenon at $x = 1$.
(*ii*) A quick glance at (1.9) shows that for the function $f_{k,r,n,A}$, one may take the function $f_{r+1,r,n,2^{r+1-k}A}$.
(*iii*) Theorem 1 implies the validity of Theorem 3, also for the case $r = 0$ and $k > 2$, since the corresponding function in Theorem 1 is nonnegative. One should also note that the arguments used in the proof of Theorem 6 below, are applicable to the proof of Theorem 3 for the case $k > \max\{2, r\}$. One must only replace in the proof of Theorem 6, Δ^1 by Δ^0.

The problem of monotone approximation is that of estimating the degree of approximation of a monotone nondecreasing function by monotone nondecreasing polynomials. Unlike positive approximation, this problem is interesting also when estimating the norm, but we deal here with pointwise estimates. The latter are due to Lorentz and Zeller [LoZe68] ($r = 0$, $k = 1$), DeVore and Yu [DeYu85] ($r + k = 2$), and Shevchuk [She92, Theorem 17.3] ($r > 0$), who proved that if $r = 0$ and $k \leq 2$, or if $r > 0$, if $f \in C^r[0,1] \cap \Delta^1$, then for each $n \geq N := r + k - 1$, a polynomial $p_n \in \Pi_n \cap \Delta^1$ exists such that (1.1) holds. Shvedov [Shv79] has shown that (1.1) is not valid in case of monotone approximation, for $r = 0$ and $k > 2$, if we insist on having $c = c(r,k)$ and $N = N(r,k)$. Furthermore, recently it has been proved by Wu and Zhou [WuZh92] for $r = 0$ and $k > 3$, and by Leviatan and Shevchuk [LeSh99] for $r = 0$ and $k = 3$, that (1.1) fails in case of monotone approximation also when we allow $c = c(f)$ and $N = N(f)$.

DeVore and Yu [DeYu85] were first to establish interpolatory pointwise estimates for monotone approximation, showing that for $r+k\le 2$, if $f\in C^r[0,1]\cap\Delta^1$, then for each $n\ge 1$, a polynomial $p_n\in\Pi_n\cap\Delta^1$ exists, such that

$$|f(x)-p_n(x)|\le c\delta_n^r(x)\omega_k(f^{(r)},\delta_n(x)),\quad 0\le x\le 1. \tag{1.10}$$

We just mention that similar interpolatory estimates for convex approximation for $r+k\le 2$ are due independently to Yu [Yu85b] and Leviatan [Lev86], and that linear approximation operators which preserve monotonicity and convexity also of higher order and which yield (1.10) for $r+k\le 2$ have been constructed by Cao and Gonska [CaGo94].

DeVore and Yu naturally have raised the question whether (1.10) might be valid for other pairs of r and k. We show here that this is not so, even for the Sobolev classes $W^r[0,1]$ of functions f, having an $(r-1)$st absolutely continuous derivative and such that $|f^{(r)}(x)|\le 1$ a.e. in $[0,1]$. In Section 3, we prove

Theorem 4. *Let $r>2$. Then for each n there is a function $f=f_{r,n}\in W^r[0,1]\cap\Delta^1$, such that for every polynomial $p_n\in\Pi_n\cap\Delta^1$, either*

$$\limsup_{x\to 0}\frac{|f(x)-p_n(x)|}{\varphi^r(x)}=\infty, \tag{1.11}$$

or

$$\limsup_{x\to 1}\frac{|f(x)-p_n(x)|}{\varphi^r(x)}=\infty. \tag{1.12}$$

Remark 5. Actually we prove that if (1.11) fails, then $f(1)\ne p_n(1)$. Of course, by symmetry, if (1.12) fails then $f(0)\ne p_n(0)$.

An immediate consequence of Theorem 4 is

Corollary 2. *Let $r+k>2$. Then for each n there exists a function $f=f_{r,k,n}\in C^r[0,1]\cap\Delta^1$, such that for every polynomial $p_n\in\Pi_n\cap\Delta^1$, either*

$$\limsup_{x\to 0}\frac{|f(x)-p_n(x)|}{\varphi^r(x)\omega_k(f^{(r)},\varphi(x))}=\infty,$$

or

$$\limsup_{x\to 1}\frac{|f(x)-p_n(x)|}{\varphi^r(x)\omega_k(f^{(r)},\varphi(x))}=\infty.$$

Nevertheless all is not lost, and it is possible to salvage something by insisting only on interpolation at one endpoint rather than at both. Namely, we prove in Section 4,

Theorem 5. *Let $k\le\max\{r,2\}$. If $f\in C^r[0,1]\cap\Delta^1$, then for each $n\ge N:=k+r-1$, there is a polynomial $p_n\in\Pi_n\cap\Delta^1$, such that*

$$|f(x)-p_n(x)|\le c(r)\rho_n^r(x)\omega_k(f^{(r)},\rho_n(x)),\quad 0\le x\le 1, \tag{1.13}$$

and

$$|f(x)-p_n(x)|\le c(r)\frac{x^{r/2}}{n^r}\omega_k\big(f^{(r)},\frac{\sqrt{x}}{n}\big),\quad 0\le x\le 1. \tag{1.14}$$

A special case of Theorem 5 is an interpolatory Whitney-type estimate which we need in the proof of the theorem and which we prove in Section 2. Namely,

Proposition 3. *Let $k \le \max\{r,2\}$. If $f \in C^r[0,1] \cap \Delta^1$, then there is a polynomial $p_{k+r-1} \in \Pi_{k+r-1} \cap \Delta^1$, such that*

$$|f(x) - p_{r+k-1}(x)| \le c(r)x^{r/2}\omega_k(f^{(r)}, \sqrt{x}), \quad 0 \le x \le 1. \tag{1.18}$$

and if $k \le r$, then

$$|f(x) - p_{r+k-1}(x)| \le c(r)x^{r}\omega_k(f^{(r)}, 1), \quad 0 \le x \le 1. \tag{1.19}$$

Theorem 5 is the best one can achieve since we show in Section 3 that,

Theorem 6. *Let $k > \max\{r,2\}$. Then for each $n \ge 1$ and every constant $A > 0$, a function $f = f_{r,k,n,A} \in C^r[0,1] \cap \Delta^1$ exists, such that for any polynomial $p_n \in \Pi_n \cap \Delta^1$, there is a point $x \in [0,1]$ for which (1.9) holds.*

Remark 6. Again by symmetry, we have a similar phenomenon at $x = 1$.

In order to have an overview, we summarize the results in the following array, where the $+$ sign in the (k,r) entry means that the interpolatory pointwise estimates hold for unconstrained approximation as well as for positive and monotone approximation (in the latter in the sense of Theorem 5); the $-$ sign means that interpolatory estimates fail in all cases (consult the variations in failure in the various cases); and the $\oplus$ sign means that the interpolatory estimates are valid for the unconstrained case, while they fail for positive and monotone approximation, except for the one $\bigoplus$ sign where the interpolatory estimate is valid also for monotone approximation (in the sense of Theorem 5), that is, it fails only for positive approximation.

$$\begin{array}{cccccccc}
r & \vdots & \vdots & \vdots & \iddots & \iddots & \iddots & \iddots \\
3 & + & + & + & \oplus & \oplus & - & \cdots \\
2 & + & + & \oplus & \oplus & - & - & \cdots \\
1 & + & \bigoplus & \oplus & - & - & - & \cdots \\
0 & + & + & - & - & - & - & \cdots \\
 & 1 & 2 & 3 & 4 & 5 & 6 & k
\end{array}$$

Fig.

Remark 6. Pointwise estimates for unconstrained approximation whether of Timan type or interpolatory ones, have in many cases included also estimates of simulataneous approximation of the function f and its derivatives by the polynomials and their respective derivatives. We have not discussed such questions here. However, we should point out that if we had analogous results of interpolatory type for simultaneous approximation of f and its first derivative, with $k = r = 2$ in Theorem 5 (1.14) (i.e., monotone approximation), then the estimates for f', would contradict (1.9) of Theorem 3 (positive approximation), for $k = 2$, $r = 1$. Thus the situation is more complicated in constrained approximation.

In the proofs we will have absolute constants and constants which depend only on r and k. Since eventually proofs of the affirmative results will end up with the latter, we will not distinguish between them and use the notation c for both types. These constants may differ from one another on different occurrences even when they appear on the same line. Sometimes however, we will need to carry to later point in a proof specific such constants, so these we will denote by c_1, c_2 etc. Constants which may depend on other parameters will be denoted by A, A_1, A_2. Again they may differ from one another on different occurrences. Finally, $\|\cdot\|$ will always denote the usual sup-norm, that is, $\|f\| = \max\{|f(x)|, x \in [0,1]\}$.

§2 Whitney-type interpolatory estimates

Given $r \ge 0$ and $k \ge 1$, set $m := r+k$ and let $0 \le x_0 \le x_1 \le \cdots \le x_m \le 1$ be so that $x_{i+r+1} > x_i$ for all $i = 0, \dots, m-r-1$. Then the generalized divided difference of order m at the knots $x_0, \dots, x_m$, $[x_0, \dots, x_m; f]$, is well defined for $f \in C^r[0,1]$.

For each pair of integers (p,q), with $0 \le p \le k-1$ and $p+r+1 \le q \le m$, we write $d(p,q) := \min\{x_{q+1} - x_p, x_q - x_{p-1}\}$, where $x_{-1} = -1$ and $x_{m+1} = 2$. Now, given a continuous function $\omega : [0,2] \to [0,\infty)$, denote

$$\Lambda_{p,q,r}(x_0, \dots, x_m; \omega) := \frac{\int_{x_q - x_p}^{d(p,q)} u^{r+p-q-1} \omega(u)\, du}{\prod_{i=0}^{p-1}(x_q - x_i) \prod_{j=q+1}^{m}(x_j - x_p)}, \tag{2.1}$$

and

$$\Lambda_r(x_0, \dots, x_m; \omega) := \max_{0 \le p \le k-1} \max_{p+r+1 \le q \le m} \Lambda_{p,q,r}(x_0, \dots, x_m; \omega). \tag{2.2}$$

Then by virtue of [She92, Theorem 4.3 and Lemma 6.4] we have the following

Lemma 1. *There is a constant $c(r,k)$ such that for every $f \in C^r[0,1]$,*

$$|[x_0, \dots, x_m; f]| \le c(r,k) \Lambda_r(x_0, \dots, x_m; \omega_k(f^{(r)}, \cdot)). \tag{2.3}$$

We are ready for the

Proof of Proposition 1. We have to deal separately with $m = r+k$ even and odd. When m is odd, we replace k by $k+1$ and m by $m+1$, in the construction below, and this accounts for the different degrees of the polynomials. It should be noted that when m is odd, then $k \le r+2$ implies $k+1 \le r+2$ so that we work exactly with the same assumptions and we may even replace in (1.3) ω_k by ω_{k+1}. Thus we will assume m is even.

Put

$$x_0 = x_1 = \cdots = x_{\frac{m}{2}-1} = 0, \quad x_{\frac{m}{2}} = x, \quad \text{and} \quad x_{\frac{m}{2}+1} = \cdots = x_m = 1, \tag{2.4}$$

where $x \in (0,1)$ is a fixed arbitrary point. Given $f \in C^r[0,1]$, we write $\omega(\cdot) := \omega_k(f^{(r)}, \cdot)$, and we will show that

$$\Lambda_r := \Lambda_r(x_0, \dots, x_m; \omega) \le c \frac{\omega(\varphi(x))}{\varphi^k(x)}. \tag{2.5}$$

Moreover, if $k \le r$, then also

$$\Lambda_r \le c\omega(1). \tag{2.6}$$

To this end observe that since $q - p \ge r + 1 \ge m/2$, then the denominator in (2.1) is always equal to 1. Hence, for all admissible pairs (p, q) we have

$$\Lambda_{p,q,r}(x_0, \dots, x_m; \omega) = \int_{x_q - x_p}^{d(p,q)} u^{r+p-q-1}\omega(u)\, du. \tag{2.7}$$

If $r \ge m/2$, then for all pairs (p, q), $x_q - x_p = 1$, while $d(0, m) = 2$. Hence,

$$\begin{aligned} \Lambda_r &= \int_1^2 u^{-k-1}\omega(u)\, du \\ &\le c\omega(1) \\ &\le c\frac{\omega(\varphi(x))}{\varphi^k(x)}. \end{aligned} \tag{2.8}$$

Thus (2.6) is proved as well as (2.5) for $r \ge m/2$. If $r + 1 = m/2$, then $k - 1 = m/2 \ge 1$, and $x_{r+1} - x_0 = x$ and $x_m - x_{k-1} = 1 - x$. Then by virtue of (2.7),

$$\Lambda_r = \max\{\int_x^1 u^{-2}\omega(u)\, du, \int_{1-x}^1 u^{-2}\omega(u)\, du, c\omega(1)\}.$$

If $x \le 1/2$ say, (in the case $x > 1/2$, x is replaced by $1 - x$), then

$$\begin{aligned} \int_{1-x}^1 u^{-2}\omega(u)\, du &\le \int_x^1 u^{-2}\omega(u)\, du \\ &= \int_x^{\sqrt{x}} u^{-2}\omega(u)\, du + \int_{\sqrt{x}}^1 u^{-2}\omega(u)\, du \\ &\le \omega(\sqrt{x}) \int_x^\infty \frac{du}{u^2} + c\frac{\omega(\sqrt{x})}{(\sqrt{x})^k} \int_{\sqrt{x}}^1 u^{k-2}\, du \\ &\le c\frac{\omega(\sqrt{x})}{(\sqrt{x})^k} \le c\frac{\omega(\varphi(x))}{\varphi^k(x)}, \end{aligned}$$

where in the second term in the third row we used the fact that the function $t^{-k}\omega(t)$ is essentialy decreasing in $(0, 1]$ (i.e., $t_1^{-k}\omega(t_1) \le ct_2^{-k}\omega(t_2)$, $0 < t_2 \le t_1 \le 1$) and in the left inequality on the last row we used the fact that in this case $k \ge 2$. This completes the proof of (2.5).

Now let ℓ_{r+k-1} denote the Lagrange–Hermite polynomial which interpolates f and its derivatives up to the order $\frac{m}{2} - 1$ at both 0 and 1. Then it is well known that

$$f(x) - \ell_{r+k-1}(x) = \varphi^m(x)[x_0, \dots, x_m; f].$$

Hence (1.3) follows by (2.3) and (2.5) and (1.4) follows by (2.3) and (2.6). This concludes our proof. $\square$

Next is the

Proof of Proposition 2. As in the proof of Proposition 1, we distinguish between the cases $m = r + k$ even and odd. For odd m, we replace throughout the proof k by $k + 1$ and m by $m + 1$, and we note again that in this case $k + 1 \le r$, so that again we work exactly with the same assumptions and we may even replace in (1.7) and (1.8), ω_k by ω_{k+1}. Thus we assume that m is even.

With $x_0, \dots, x_m$ as in (2.4), and ℓ_{r+k-1} denoting the Lagrange–Hermite polynomial interpolating f and its derivatives up to order $\frac{m}{2} - 1$ at both 0 and 1, we have by (2.8)

$$|f(x) - \ell_{r+k-1}(x)| \le c_1 \varphi^{r+k}(x)\omega(1) =: q_{r+k}(x).$$

Since $r + k$ is even, we observe that q_{r+k} is a polynomial (of exact degree $r + k$). Now $p_{r+k} := \ell_{r+k-1} + q_{r+k}$ is a polynomial of degree $r + k$ satisfying

$$|f(x) - p_{r+k}(x)| \le c\varphi^{r+k}(x)\omega(1),$$

while evidently, p_{r+k} is nonnegative. This completes the proof of (1.7) for $r + k$ even. $\square$

The main part of the proof of Proposition 3 is quite similar, but we have an additional case to consider.

Proof of Proposition 3. We begin with the special case $r = 1$ and $k = 2$. It is well known that ℓ_1, the linear interpolant of f' at 0 and 1, satisfies

$$|f'(x) - \ell_1(x)| \le c\omega_2(f', \sqrt{x(1-x)}),$$

and obviously it is nonnegative. Hence putting $p_2(x) := f(0) + \int_0^x \ell_1(u)\, du$, we have a nondecreasing polynomial which yields

$$\begin{aligned} |f(x) - p_2(x)| &\le \int_0^x |f'(u) - \ell_1(u)|\, du \\ &\le cx\omega_2(f', \sqrt{x}) \\ &\le \sqrt{x}\omega_2(f', \sqrt{x}). \end{aligned}$$

The rest of the proof is again similar to that of Proposition 1, we take $k \le r$ but we make no distinction between odd and even $r + k$. We put

$$x_0 = \cdots = x_{r-2} = 0, \quad x_{r-1} = x, \quad \text{and} \quad x_r = \cdots = x_{r+k-1} = 1,$$

where $x \in (0, 1)$ is arbitrary. Now we show that

$$\Lambda_{r-1} := \Lambda_{r-1}(x_0, \dots, x_{r+k-1}; \omega) \le c\frac{\omega(1)}{1-x}. \tag{2.9}$$

Indeed, as before, if $k < r$, then we have

$$\Lambda_{r-1} \le c\omega(1),$$

and if $k = r$, then

$$\Lambda_{r-1} \le \max\{\int_{1-x}^{1} \frac{\omega(u)}{u^2}\,du, c\omega(1)\} \le c\frac{\omega(1)}{1-x}.$$

Let ℓ_{r+k-2} be the Lagrange–Hermite polynomial for f', interpolating $f^{(j)}(0)$, $j = 1, \dots, r-1$ and $f^{(j)}(1)$, $j = 1, \dots, k$. Then

$$f'(x) - \ell_{r+k-2}(x) = x^{r-1}(1-x)^k[x_0, \dots, x_{r+k-1}; f'].$$

Thus by (2.3) and (2.9),

$$|f'(x) - \ell_{r+k-2}(x)| \le c_1 x^{r-1}\omega(1). \tag{2.10}$$

Put

$$p_{r+k-1}(x) := f(0) + \int_0^x \ell_{r+k-2}(u)\,du + \frac{c_1}{r}x^r\omega(1).$$

Then by (2.10) we obtain

$$|f(x) - p_{r+k-1}(x)| \le \int_0^x c_1 u^{r-1}\omega(1)\,du + \frac{c_1}{r}x^r\omega(1) \le cx^r\omega(1) \le cx^{r/2}\omega(\sqrt{x}),$$

which is (1.19) and (1.18), while at the same time by virtue of (2.10), we have

$$p'_{r+k-1}(x) = f'(x) + (\ell_{r+k-2}(x) - f'(x)) + c_1 x^{r-1}\omega(1) \ge f'(x) \ge 0.$$

This completes our proof. $\square$

§3 Negative results and counterexamples

We begin with unconstrained approximation.

Proof of Theorem 1. First we recall the general convention to denote by Φ^k, the set of continuous nondecreasing functions $\omega : [0, \infty) \to [0, \infty)$, such that $\omega(0) = 0$ and $t^{-k}\omega(t)$ is nonincreasing in $[0, \infty)$.

By [She92,Theorem 2.2], if $k > 1$ and $\omega \in \Phi^k$, then the function

$$F(x, \omega, k) := \frac{1}{(k-2)!}\int_x^1 \frac{x(u-x)^{k-2}}{u^k}\omega(u)\;du, \quad 0 \le x \le 1,$$

is continuous and satisfies

$$\omega(t) \le \omega_k(F, t) \le k\omega(t). \tag{3.1}$$

Given $r \geq 0$, we successively define a sequence of points $t_i \in (0,1]$, as follows. Set $t_1 := 1/2$ and assume that t_{2i-1}, $i \geq 1$, has been defined. Then put $t_{2i} := t_{2i-1}^{r+3}$ and $t_{2i+1} := t_{2i-1} t_{2i}^{r+2}$. Obviously, $t_i \to 0$ as $i \to \infty$. Now let

$$\omega(t) := \begin{cases} 1/2, & t \geq 1/2, \\ t_{2i-1} t^{r+3}/t_{2i}, & t_{2i} \leq t \leq t_{2i-1}, \\ t_{2i+1}, & t_{2i+1} \leq t \leq t_{2i}, \\ 0, & t = 0. \end{cases}$$

Then it is readily seen that $\omega \in \Phi^{r+3}$. Since

$$(r+2)\frac{t_{2i}^{r+2}}{\omega(t_{2i})} \int_{t_{2i}}^{t_{2i-1}} \frac{\omega(u)}{u^2}\, du = \frac{1}{t_{2i-1}} - t_{2i}^{r+1},$$

we see that

$$\limsup_{x \to 0} \frac{x^{r+2}}{\omega(x)} \int_x^1 \frac{\omega(u)}{u^2}\, du = \infty. \tag{3.2}$$

Also

$$\int_{t_{2i+1}}^{t_{2i}} \frac{\omega(u)}{u^2}\, du = 1 - \frac{t_{2i+1}}{t_{2i}},$$

which implies

$$\int_0^1 \frac{\omega(u)}{u^2}\, du = \infty, \tag{3.3}$$

and finally since $\omega(x) \leq x$, $0 \leq x \leq 1$, then

$$\int_{t_{2i}}^1 \frac{\omega(u)}{u^2}\, du \leq \int_{t_{2i}}^1 \frac{1}{u}\, du = \log t_{2i}^{-1}. \tag{3.4}$$

We are ready to define the function f_r. We write

$$f_r(x) := \frac{1}{(r-1)!} \int_0^x (x-t)^{r-1} F(t, \omega, r+3)\, dt,$$

whence

$$f_r^{(r)}(x) = F(x, \omega, r+3).$$

Observe that since $\omega \in \Phi^{r+3}$, then by (3.1)

$$\omega(t) \leq \omega_{r+3}(f_r^{(r)}, t) \leq (r+3)\omega(t).$$

Taking the highest power of x and of u in the double integral which defines f_r gives the dominant term in the computation of the function, and it is not hard to see that with a suitable constant c_1 we have

$$|f(x) - c_1 x^{r+1} \int_x^1 \frac{\omega(u)}{u^2}\, du| \le c x^{r/2} \omega(\sqrt{x}),$$

which in turn implies

$$|f(x) - c_1 x^{r+1} \int_{\sqrt{x}}^1 \frac{\omega(u)}{u^2}\, du| \le c x^{r/2} \omega(\sqrt{x}). \tag{3.5}$$

Now fix n and assume to the contrary that (1.5) is not valid. Then for some constant A, we have

$$|f(x) - p_n(x)| \le A x^{r/2} \omega(\sqrt{x}). \tag{3.6}$$

Hence, it follows by (3.5) that

$$|p_n(x) - c_1 x^{r+1} \int_{\sqrt{x}}^1 \frac{\omega(u)}{u^2}\, du| \le A x^{r/2} \omega(\sqrt{x}),$$

and by change of variable

$$|q_{2n}(x) - c_1 x^{2r+2} \int_x^1 \frac{\omega(u)}{u^2}\, du| \le A_1 x^r \omega(x). \tag{3.7}$$

This together with (3.4) and the definition of ω yields

$$\begin{aligned} |q_{2n}(t_{2i})| &\le c_1 t_{2i}^{2r+2} \log t_{2i}^{-1} + A_1 t_{2i}^{2r+2} t_{2i-1} \\ &\le A t_{2i}^{2r+2} \log t_{2i}^{-1}, \end{aligned}$$

which in turn implies that $q_{2n}(x) = x^{2r+2} Q(x)$, where $Q \in \Pi_{2n-2r-2}$. By virtue of (3.3) there is a $\delta > 0$ so that

$$\|Q\| \le \frac{1}{2c_1} \int_x^1 \frac{\omega(u)}{u^2}\, du, \quad 0 < x \le \delta.$$

Hence it follows from (3.7) that

$$x^{2r+2} \int_x^1 \frac{\omega(u)}{u^2}\, du \le \frac{2A_1}{c_1} x^r \omega(x), \quad 0 < x \le \delta,$$

which contradicts (3.2). Thus (3.6) is not valid and (1.5) is proven. This completes our proof. □

We now prove the assertions made in Remark 3 by constructing counterexamples.

Counterexamples for Remark 3. For $m = r + k$ even, consider the function

$$f(x) := x^{m/2+2}(1-x)^{m/2-2} \in \Pi_m \cap \Delta^0.$$

Assume to the contrary, that there exists a nonnegative polynomial $p_{m-1} \in \Pi_{m-1}$ satisfying,

$$|f(x) - p_{m-1}(x)| \le c\varphi^r(x)\omega_k(f^{(r)}, \varphi(x)).$$

Then, $f - p_{m-1} \in \Pi_m$ and must have the form

$$f(x) - p_{m-1}(x) = A \cdot x^{m/2}(1-x)^{m/2},$$

Comparing the coefficients of x^m on both sides we have $A = 1$. This yields

$$\begin{aligned} p_{m-1}(x) &= x^{m/2+2}(1-x)^{m/2-2} - x^{m/2}(1-x)^{m/2} \\ &= x^{m/2}(1-x)^{m/2-2}(x^2 - (1-x)^2), \end{aligned}$$

whence p_{m-1} changes sign in $[0,1]$, contradicting our assumption. The minimal degree is thus m.

For m odd, take

$$f(x) := x^{(m+1)/2+2}(1-x)^{(m+1)/2-2} \in \Pi_{m+1} \cap \Delta^0,$$

and the same proof shows that we cannot approximate it well by polynomials of degree $m+1$. $\square$

Next is the

Proof of Theorem 3. Without loss of generality assume $n \ge k = r + 1$. For $b \in (0,1)$, fix $S_{b,r} \in C^{2r+1}$ which is nonnegative and so that $S_{b,r}(x) = 1$, if $x \le b/2$, $S_{b,r}(x) = 0$, if $x \ge b$ and $\|S_{b,r}^{(j)}\| \le cb^{-j}$, $j = 0, \dots, 2r+1$. Set $g_{b,r}(x) = x(x-b)^r$, if r is even, and $g_{b,r}(x) = (x-b)^{r+1}$, if r is odd. Finally, put

$$f_{b,r} := g_{b,r}S_{b,r}.$$

Evidently $f_{b,r} \in C^{2r+1} \cap \Delta^0$ and we have the inequalities

$$\|f_{b,r}\| \le c_1 b^{r+1}, \quad \|f_{b,r}^{(r)}\| \le c_2 b. \tag{3.8}$$

For given $r \ge 1$, $n \ge r+1$ and $A > 0$, we take $b \in (0,1)$ such that

$$c_1(2bn^2)^{r+1} + c_2 A 4^{r+1} b n^{r+2} < 1, \tag{3.9}$$

and put $f = f_{r+1,r,n,A} := f_{b,r}$. Suppose to the contrary that for some polynomial $p_n \in \Pi_n \cap \Delta^0$, we have

$$|f(x) - p_n(x)| \le A\frac{x^{r/2}}{n^r}\omega_{r+1}(f^{(r)}, \frac{\sqrt{x}}{n}), \quad 0 \le x \le 1. \tag{3.10}$$

Then it follows that

$$|f(x) - p_n(x)| \le A\|f^{(2r+1)}\|\frac{x^{r+1/2}}{n^{2r+1}}, \quad 0 \le x \le 1,$$

which implies

$$p_n^{(j)}(0) = f^{(j)}(0) = g_{b,r}^{(j)}(0), \quad j = 0, \dots, r.$$

Hence, for some $0 < \theta < 1$,

$$0 \le p_n(b) = p_n(b) - g_{b,r}(b) = \frac{b^{r+1}}{(r+1)!}(p_n^{(r+1)}(\theta b) - g_{b,r}^{(r+1)}(\theta b)),$$

so that

$$p_n^{(r+1)}(\theta b) \ge g_{b,r}^{(r+1)}(\theta b) = (r+1)!. \tag{3.11}$$

On the other hand, (3.8) implies

$$\omega_{r+1}(f^{(r)}, t) \le c_2 2^{r+1} b,$$

which combined with (3.10) yields

$$\|p_n\| \le \|f\| + \|p_n - f\| \le c_1 b^{r+1} + c_2 A 2^{r+1}\frac{b}{n^r}.$$

Therefore by virtue of Markov's inequality,

$$\|p_n^{(r+1)}\| \le c_1(2bn^2)^{r+1} + c_2 A 4^{r+1} b n^{r+2} < 1,$$

by the choice of b. This contradicts (3.11) and completes our proof. $\square$

Next we have the

Proof of Theorem 4. Given n, let $b = n^{-2}$ and let

$$f(x) := \begin{cases} (b^r - (b-x)^r)/r!, & 0 \le x \le b, \\ b^r/r!, & b < x \le 1. \end{cases}$$

Then $f \in W^r[0,1] \cap \Delta^1$. Suppose to the contrary, that there is a nondecreasing polynomial p_n for which (1.11) fails. Then for that polynomial and some constant A, we have

$$|f(x) - p_n(x)| \le A\varphi^r(x) \le Ax\sqrt{x}, \quad 0 \le x \le b,$$

since $r>2$. Hence $p_n(0)=f(0)=0$ and $p_n'(0)=f'(0)=b^{r-1}/(r-1)!$. Since $p_n\in\Delta^1$, we have $\|p_n\|=p_n(1)$, so that applying Markov's inequality we end up with

$$\frac{b^{r-1}}{(r-1)!}=p_n'(0)\le 2n^2\|p_n\|=2n^2p_n(1),$$

or

$$p_n(1)\ge\frac{b^{r-1}}{2(r-1)!n^2}.$$

On the other hand

$$f(1)=\frac{b^r}{r!}=\frac{b^{r-1}}{2(r-1)!n^2}\frac{2bn^2}{r}<\frac{b^{r-1}}{2(r-1)!n^2}.$$

Thus $f(1)\ne p_n(1)$, and (1.12) is satisfied. $\square$

And finally we have the

Proof of Theorem 6. Let $\sigma_n(x):=1/n^2+\sqrt{x}/n$. Then $\sigma_n(x)\le 3/(2n^2)+x/2$, and for every $q_n\in\Pi_n$, we readily have

$$|q_n(x)|\le\|q_n/\sigma_n^r\|(3/(2n^2)+x/2)^r,$$

so that by a Dzjadyk-type inequality (see, e.g., [She92, (14.9)]) we obtain,

$$|q_n^{(r+1)}(0)|\le c_rn^2\|q_n/\sigma_n^r\|. \tag{3.12}$$

Obviously we may take in (3.12), $c_r\ge 1$.

Now for $b\in(0,1)$, let $S_{b,r}\in C^{2r+1}[0,1]\cap\Delta^1$ be so that $S_{b,r}(x)=0$, $0\le x\le b$, $S_{b,r}(x)=1$, $x\ge 2b$, and $\|S_{b.r}^{(j)}\|\le cb^{-j}$, $j=1,\dots,2r+1$. Finally denote $Q_{b,r}(x):=(x-b)^{r+2}$ and put

$$f_{b,r}:=Q_{b,r}S_{b,r},$$

so that $f_{b,r}\in C^{2r+1}[0,1]\cap\Delta^1$.

We first show that for every $n\ge r+2$, and $b<(c_rn^2)^{-1}$, if $p_n\in\Pi_n$ satisfies $p_n^{(r+1)}(0)\ge 0$, then

$$\left\|\frac{f_{b,r}-p_n}{\sigma_n^r}\right\|\ge\frac{b}{c_rn^2}. \tag{3.13}$$

Indeed, if we assume to the contrary that

$$|f_{b,r}(x)-p_n(x)|\le\sigma_n^r(x)\frac{b}{c_rn^2},$$

then the polynomial $q_n:=p_n-Q_{b,r}\in\Pi_n$, satisfies

$$|q_n(x)|\le|p_n(x)-f_{b,r}(x)|+|f_{b,r}(x)-(x-b)^{r+2}|<2\sigma_n^r(x)\frac{b}{c_rn^2},\quad 0\le x\le 1,$$

since by the choice of b,

$$\|f_{b,r} - Q_{b,r}\| = b^{r+2} < \sigma_n^r(x)\frac{b}{c_r n^2}.$$

Thus by (3.12)

$$|q_n^{(r+1)}(0)| < c_r n^2 \frac{2b}{c_r n^2} = 2b.$$

On the other hand, our assumption $p_n^{(r+1)} \geq 0$ yields

$$q_n^{(r+1)}(0) = p_n^{(r+1)}(0) - Q_{b,r}^{(r+1)}(0) \geq -Q_{b,r}^{(r+1)}(0) = (r+2)!\, b \geq 2b,$$

a contradiction. This completes the proof of (3.13).

We also need the estimate

$$\omega_3(f_{r,b}^{(r)}, 1) = \omega_3(f_{r,b}^{(r)} - Q_{b,r}^{(r)}, 1) \leq 2^3 \|f_{r,b}^{(r)} - Q_{b,r}^{(r)}\| \leq c_r^* b^2, \tag{3.14}$$

for some constant c_r^*. Now choose $b \in (0,1)$ so that

$$b < \min\{\frac{1}{c_r n^2}, \frac{2^{3-k}}{c_r^* A c_r n^2}\}$$

and take $f := f_{r,b}$. If to the contrary $p_n \in \Pi_n \cap \Delta^1$, $n \geq r+2$, satisfies

$$\begin{aligned} |f(x) - p_n(x)| &\leq A \left(\frac{\sqrt{x}}{n}\right)^r \omega_k(f^{(r)}, \frac{\sqrt{x}}{n}) \\ &\leq A 2^{k-r-1} \left(\frac{\sqrt{x}}{n}\right)^{2r+1} \|f^{(2r+1)}\|, \quad 0 \leq x \leq 1, \end{aligned} \tag{3.15}$$

then, since f vanishes on $[0,b]$, we have

$$|p_n(x)| \leq 2^{k-r-1} A \left(\frac{\sqrt{x}}{n}\right)^{2r+1} \|f^{(2r+1)}\|, \quad 0 \leq x \leq b.$$

Hence $p_n^{(j)}(0) = 0$, $j = 0, \ldots, r$, and since we assumed that $p_n \in \Delta^1$, it follows that $p_n^{(r+1)}(0) \geq 0$. Thus by (3.13), there is a point $x \in [0,1]$ for which

$$|f(x) - p_n(x)| \geq \frac{b}{c_r n^2}\sigma_n^r(x) \geq \frac{b}{c_r n^2}\left(\frac{\sqrt{x}}{n}\right)^r.$$

Finally, by (3.14)

$$\begin{aligned} \frac{|f(x) - p_n(x)|}{(\sqrt{x}/n)^r \omega_k(f^{(r)}, \sqrt{x}/n)} &\geq \frac{b}{c_r n^2} \frac{2^{3-k}}{\omega_3(f^{(r)}, \sqrt{x}/n)} \\ &\geq \frac{b}{c_r n^2} \frac{2^{3-k}}{c_r^* b^2} > A, \end{aligned}$$

which contradicts (3.15) and thus completes our proof. $\square$

§4 Interpolatory estimates for positive and monotone approximation

We begin with some lemmas. The first lemma is well-known in simultaneous approximation. It readily follows from the work of Dzjadyk, Trigub, Lebed, Brudnyĭ and Gopengauz (see, e.g., [GoHi85, Satz 5.2])

Lemma 2. *Let $g \in C^r[0,1]$ and assume that the polynomial $Q_n \in \Pi_n$ satisfies*

$$|g(x) - Q_n(x)| \le A\rho_n^r(x)\omega_k(g^{(r)}, \rho_n(x)), \quad 0 \le x \le 1, \tag{4.1}$$

for some $A > 0$. Then for all $0 \le j \le r$, we have

$$|g^{(j)}(x) - Q_n^{(j)}(x)| \le cA\rho_n^{r-j}(x)\omega_k(g^{(r)}, \rho_n(x)), \quad 0 \le x \le 1. \tag{4.2}$$

Following Dzjadyk [Dzj77, p.274] (see also [She92, p.149]), we construct algebraic polynomials,

$$\tilde{q}_n(y) := \frac{\cos^2 n \arccos y}{(y-y_n)^2} + \frac{\sin^2 n \arccos y}{(y+1)^2}, \quad y \in [-1,1],$$

where $y_n := -\cos(\pi/2n)$, which are of degree $2n-2$ and which are readily seen to satisfy

$$\begin{gathered} c_1 n^4 \le \tilde{q}_n(y) \le cn^4, \quad y \in [-1, -1+2/n^2], \\ c_1 \le (1+y)^2 \tilde{q}_n(y) \le c, \quad y \in [-1+2/n^2, 1]. \end{gathered}$$

Thus for

$$q_n(x) := \frac{4}{c_1 n^4} \cdot \tilde{q}_n(2x-1),$$

we have

$$\begin{gathered} 1 \le q_n(x) \le c, \quad x \in [0, 1/n^2], \\ 1 \le (xn^2)^2 q_n(x) \le c, \quad x \in [1/n^2, 1]. \end{gathered} \tag{4.3}$$

Set

$$W_n(x) := \begin{cases} \varphi^{2r}(x)(q_n(x) + q_n(1-x))^{r/2}, & \text{if } r \text{ is even}, \\ \varphi^{2r}(x)(q_n(x) + q_n(1-x))^{(r+1)/2}(1+n^2\varphi^2(x)), & \text{if } r \text{ is odd}, \end{cases} \tag{4.4}$$

which is a polynomial of degree $r(n+1)$ for r even, and of degree $(r+1)(n+1)$ for r odd. By virtue of (4.3), it follows immediately that

$$\begin{gathered} \varphi^{2r}(x) \le W_n(x) \le c\varphi^{2r}(x), \quad x \in [0, 1/n^2] \cup [1-1/n^2, 1], \\ 2^{-r} \le n^{2r} W_n(x) \le c, \quad x \in [1/n^2, 1-1/n^2]. \end{gathered} \tag{4.5}$$

Finally let the polynomials τ_n (of degree $2nr-1$) be defined by,

$$\tau_n(x) := \left(\int_0^1 \varphi^{2r-2}(u) q_n^r(u)\, du\right)^{-1} \int_x^1 \varphi^{2r-2}(u) q_n^r(u)\, du.$$

Then we summarize their properties in the following lemma.

Lemma 3. *For all n, $0 \le \tau_n(x) \le 1$, $0 \le x \le 1$, and $\tau_n(0) = 1$. Also the polynomials satisfy*

$$\begin{aligned} \tau_n^{(j)}(0) &= 0, \quad j = 1, \ldots, r-1, \\ \tau_n^{(j)}(1) &= 0, \quad j = 0, \ldots, r-1, \end{aligned} \tag{4.6}$$

and

$$\tau_n(x) \le c(n^2 x)^{-r}, \quad \frac{1}{n^2} \le x \le 1. \tag{4.7}$$

Proof. Except for (4.7) all the assertions are straightforward. In order to prove (4.7), observe that by (4.3)

$$\begin{aligned} \int_0^1 \varphi^{2r-2}(u) q_n^r(u)\, du &\ge \int_0^{1/n^2} \varphi^{2r-2}(u)\, du \\ &\ge 2^{1-r} \int_0^{1/n^2} u^{r-1}\, du = cn^{-2r}, \end{aligned}$$

so that

$$(xn^2)^r \tau_n(x) \le cx^r n^{4r} \int_x^\infty u^{r-1}(un^2)^{-2r}\, du = cx^r \int_x^\infty u^{-r-1}\, du = c.$$

This completes the proof. $\square$

The main tool in the proof of Theorem 2 is

Lemma 4. *Let $g \in C^r$, and suppose that $Q_n \in \Pi_n$ satisfies (4.1) with $k = r$. Denote by T_{r-1} and T^*_{r-1}, the Taylor polynomials of degree $r-1$ of $g - Q_n$, about $x = 0$, and about $x = 1$, respectively. Then for the polynomial*

$$V_n = V_n(g, Q_n) := Q_n + T_{r-1}\tau_n + T^*_{r-1}\tau_n(1 - \cdot), \tag{4.8}$$

we have

$$|V_n(x) - Q_n(x)| \le c_2 A n^{-2r} \omega_r(g^{(r)}, 1/n^2), \quad 0 \le x \le 1, \tag{4.9}$$

$$|g(x) - V_n(x)| \le cA\delta_n^r(x) \omega_r(g^{(r)}, \delta_n(x)), \quad 0 \le x \le 1, \tag{4.10}$$

and

$$|g(x) - V_n(x)| \le c_3 A \varphi^{2r}(x) \omega_r(g^{(r)}, 1/n^2), \quad x \in [0, 1/n^2] \cup [1/n^2, 1 - 1/n^2]. \tag{4.11}$$

Proof. By Lemma 2, Q_n satisfies (4.2) and in particular, if we write $\omega(t) := A\omega_r(g^{(r)}, t)$, then

$$n^{2(r-j)}|g^{(j)}(0) - Q_n^{(j)}(0)| \le c\omega(1/n^2), \quad j = 0, \dots, r-1.$$

Hence

$$n^{2r}|T_{r-1}(x)| \le c(1 + n^2 x)^{r-1}\omega(1/n^2), \quad 0 \le x \le 1,$$

which in turn implies by virtue of Lemma 3,

$$n^{2r}\|T_{r-1}\tau_n\| \le c\omega(1/n^2).$$

Similarly,

$$n^{2r}\|T^*_{r-1}\tau_n(1-\cdot)\| \le c\omega(1/n^2),$$

and (4.9) follows. Now by (4.1) and (4.9) we readily have,

$$|g(x) - V_n(x)| \le c\rho_n^r(x)\omega(\rho_n(x)), \quad 0 \le x \le 1, \tag{4.12}$$

which in turn implies (4.10) for $x \in [1/n^2, 1 - 1/n^2]$.

For $x \in [0, 1/n^2] \cup [1 - 1/n^2, 1]$, (4.10) follows from (4.11), therefore we are done once we prove the latter. To this end assume $x \in [0, 1/n^2]$, the other case is analogous, and apply Lemma 2 again to (4.12). Then

$$|g^{(r)}(x) - V_n^{(r)}(x)| \le c\omega(1/n^2), \quad x \in [0, 1/n^2], \tag{4.13}$$

where we used the fact that $\rho_n(x) \le 2/n^2$ in that interval. Writing

$$g - V_n = (g - Q_n - T_{r-1}) - (T_{r-1}(\tau_n - 1) + T^*_{r-1}\tau_n(1-\cdot)) =: (g - Q_n - T_{r-1}) - \tilde{T}_{r-1},$$

we note that (4.6) implies $\tilde{T}_{r-1}^{(j)}(0) = 0$, $j = 0, \dots, r-1$. Also $g^{(j)}(0) - Q_n^{(j)}(0) - T_{r-1}^{(j)}(0) = 0$, $j = 0, \dots, r-1$. Therefore,

$$g^{(j)}(0) - V_n^{(j)}(0) = 0, \quad j = 0, \dots, r-1,$$

whence, applying (4.13) we have for $x \in [0, 1/n^2]$,

$$\begin{aligned} |g(x) - V_n(x)| &= \frac{1}{(r-1)!}\left|\int_0^x (x-u)^{r-1}(g^{(r)}(u) - V_n^{(r)}(u))\,du\right| \\ &\le c\omega(1/n^2)\int_0^x (x-u)^{r-1}\,du = cx^r\omega(1/n^2) \\ &\le c\varphi^{2r}(x)\omega(1/n^2). \end{aligned}$$

This proves (4.11) and completes our proof. $\square$

We are ready for the

Proof of Theorem 2. We will prove the theorem for all $n > 6r$. Then we will combine this with Proposition 2 and the proof will be complete. Note that the polynomials V_n and W_n above are of degree not exceeding $3rn$. Obviously, it suffices to prove Theorem 2 for $k = r$.

Write $n_1 := [n/(3r)]$. Since $n_1 > 2r - 1$, then by Dzyubenko [Dzy96], there is a polynomial $Q_{n_1} \in \Pi_{n_1}$, satisfying $Q_{n_1}(x) \geq 0$, $0 \leq x \leq 1$, and

$$|f(x) - Q_{n_1}(x)| \leq c_4 \rho_{n_1}^r(x) \omega(\rho_{n_1}^r(x)), \quad 0 \leq x \leq 1,$$

where again $\omega(t) := \omega_r(f^{(r)}, t)$.

Now put $P_n := V_{n_1} + c_5 \omega(1/n_1^2) W_{n_1} \in \Pi_n$, where

$$c_5 := c_4 \max\{c_3, 2^r c_2\}.$$

By virtue of Lemma 4 and (4.5), we get

$$\begin{aligned} |f(x) - P_n(x)| &\leq c\delta_{n_1}^r(x)\omega(\delta_{n_1}(x)) + c\omega(1/n_1^2)\min\{1/n_1^{2r}, \varphi^{2r}(x)\} \\ &\leq c\delta_{n_1}^r(x)\omega(\delta_{n_1}(x)) \leq c\delta_n^r(x)\omega(\delta_n(x)). \end{aligned}$$

Now Q_n is nonnegative so that by (4.9) (with $A = c_4$),

$$V_{n_1}(x) \geq -c_2 c_4 n_1^{-2r} \omega(1/n_1^2), \quad 0 \leq x \leq 1.$$

Hence for $x \in [1/n_1^2, 1 - 1/n_1^2]$, we have by (4.5),

$$P_n(x) \geq -c_2 c_4 n_1^{-2r} \omega(1/n_1^2) + c_5 2^{-r} n_1^{-2r} \omega(1/n_1^2) \geq 0.$$

On the other hand, for $x \in [0, 1/n_1^2] \cup [1 - 1/n_1^2, 1]$, by (4.11) and (4.5), and observing that $f(x) \geq 0$, $x \in [0,1]$ and $c_5 \geq c_3 c_4$, we conclude that

$$\begin{aligned} P_n(x) &= f(x) + (V_{n_1} - f(x)) + c_5\omega(1/n_1^2)W_{n_1}(x) \\ &\geq -c_3 c_4 \varphi^{2r}(x)\omega(1/n_1^2) + c_5\varphi^{2r}(x)\omega(1/n_1^2) \geq 0. \end{aligned}$$

This completes our proof. $\square$

For the proof of Theorem 5 we also need some notation. For $l = r, 2r$, denote the sequence of polynomials t_l (of degrees $(2n-1)l$),

$$t_l(x) = t_{n,l}(x) := \frac{\int_0^x u^{l-1} q_n^l(u)\, du}{\int_0^1 u^{l-1} q_n^l(u)\, du}, \tag{4.14}$$

where the polynomials q_n are defined prior to (4.3). Then it is easy to show the following

Lemma 5. *t_l is a nondecreasing polynomial thus, $t_l(x) \le t_l(1) = 1$, and $t_l^{(j)}(0) = 0$, $j = 0, \dots, l-1$. Also*

$$t_l(x) \le c(n^2x)^l, \quad 0 \le x \le n^{-2}, \tag{4.15}$$

and

$$cn^2 \le t_l'(x)(n^2x)^{1-l}(1+n^2x)^{2l} \le cn^2. \tag{4.16}$$

Finally

$$1 - t_l(x) \le c(1+n^2x)^{-l}, \quad 0 \le x \le 1. \tag{4.17}$$

Proof of Theorem 5. As in the proof of Theorem 2, we prove Theorem 5 for all n sufficiently large. Then we combine this with Proposition 3 and the proof is complete. We first prove the theorem under the assumption that $k \le r$. The outstanding case, i.e., $r = 1$ and $k = 2$, has to be treated separately and this will be done after the general proof.

As was mentioned in the introduction for our $f \in C^r \cap \Delta^1$, there exist polynomials $Q_n \in \Pi_n \cap \Delta^1$ which satisfy (4.1), with a constant A which depends only on r and k. Thus, if (once again) $\omega(t) := \omega_k(f^{(r)}, t)$, then it follows by Lemma 2 that

$$|f^{(j)}(0) - Q^{(j)}(0)| \le cn^{2j-2r}\omega(1/n^2).$$

Therefore, if T_{r-1} is the Taylor polynomial of $f - Q_n$ about $x = 0$, then

$$\begin{aligned} n^{2r}|T_{r-1}(x)| &\le c\omega(1/n^2)(1+n^2x)^{r-1}, \quad 0 \le x \le 1, \\ n^{2r-2}|T_{r-1}'(x)| &\le c\omega(1/n^2)(1+n^2x)^{r-2}, \quad 0 \le x \le 1, \end{aligned} \tag{4.18}$$

We will show that for a suitably prescribed constant c_r the polynomial

$$P_n := Q_n + T_{r-1}(1 - t_{2r}) + c_r n^{-2r}\omega(1/n^2)t_r,$$

fulfills the requirements of Theorem 5, except that it is of degree $4nr - r - 1$. Since in the proof of Theorem 2 we have explained how we modify the degree to end up with degree n, we will not repeat that here. First, we prove that

$$P_n'(x) \ge 0, \quad 0 \le x \le 1. \tag{4.19}$$

We begin with $\frac{1}{n^2} \le x \le 1$. Since Q_n is nondecreasing, we need to evaluate the derivative of the remaining terms. It follows by (4.18) that

$$\begin{aligned} |T_{r-1}(x)| &\le \frac{cx^{r-1}}{n^2}\omega(1/n^2), \quad \frac{1}{n^2} \le x \le 1, \\ |T_{r-1}'(x)| &\le \frac{cx^{r-2}}{n^2}\omega(1/n^2), \\ &\le cx^{r-1}\omega(1/n^2), \quad \frac{1}{n^2} \le x \le 1. \end{aligned}$$

Hence combining with (4.16) and (4.17),

$$\begin{aligned}|T'_{r-1}(x)(1-t_{2r}(x))-T_{r-1}(x)t'_{2r}(x)| &\le cx^{r-1}\omega(1/n^2)(1+n^2x)^{-2r}\\ &\le c_r n^{-2r}\omega(1/n^2)t'_r(x),\end{aligned}$$

for a suitable c_r. Thus (4.19) is proved for $\frac{1}{n^2}\le x\le 1$. In order to establish (4.19) for $0\le x\le \frac{1}{n^2}$ we write

$$W_n := Q_n + T_{r-1}(1-t_{2r}),$$

and we first prove

$$|f^{(r)}(x)-W_n^{(r)}(x)|\le c\omega(1/n^2),\quad 0\le x\le 1/n^2. \tag{4.20}$$

Indeed, by (4.17) and (4.18),

$$|T_{r-1}(x)(1-t_{2r}(x))|\le \frac{c}{n^{2r}}\omega(1/n^2)(1+n^2x)^{r-1}(1+n^2x)^{-2r}\le \frac{c}{n^{2r}}\omega(1/n^2),\quad 0\le x\le 1.$$

Hence by (4.1),

$$|f(x)-W_n(x)|\le c\rho_n^r(x)\omega(\rho_n(x)),\quad 0\le x\le 1. \tag{4.21}$$

Therefore Lemma 2 yields,

$$|f^{(r)}(x)-W_n^{(r)}(x)|\le c\omega(\rho_n(x)),\quad 0\le x\le 1,$$

and (4.20) is proven. Next by Lemma 5, we readily see that

$$W_n^{(j)}(0)=f^{(j)}(0),\quad j=0,\dots,r-1.$$

Then by Taylor formula and (4.20),

$$\begin{aligned}|f'(x)-W'_n(x)| &\le c\left|\int_0^x (x-u)^{r-2}\omega(1/n^2)\,du\right|\\ &= cx^{r-1}\omega(1/n^2),\quad 0\le x\le 1/n^2.\end{aligned} \tag{4.22}$$

Since by (4.16)

$$t'_r(x)\ge cn^{2r}x^{r-1},$$

we obtain from (4.22),

$$P'_n(x)=W'_n(x)+c_r\frac{1}{n^{2r}}\omega(1/n^2)t'_r(x)\ge f'(x)+(c_r-c)x^{r-1}\omega(1/n^2)\ge 0,\quad 0\le x\le 1/n^2,$$

for a suitably large c_r, where we also used the monotonicity of f. This proves (4.19) for the remaining interval, $0\le x\le 1/n^2$.

By virtue of (4.21) and the immediate inequality

$$\frac{1}{n^{2r}}\omega(1/n^2)t_r(x) \le \rho_n^r(x)\omega(\rho_n(x)), \quad 0 \le x \le 1,$$

we obtain that P_n satisfies (1.13). Thus it remains to prove that it satisfies (1.14). For $\frac{1}{n^2} \le x \le 1$, we have $\rho_n(x) \le \frac{2\sqrt{x}}{n}$ and (1.14) follows by (1.13). For $0 \le x \le 1/n^2$, just as we proved (4.22), we obtain

$$\begin{aligned} |f(x) - W_n(x)| &\le c\left|\int_0^x (x-u)^{r-1}\omega(1/n^2)\,du\right| \\ &= cx^r\omega(1/n^2) \\ &\le \frac{cx^r}{(n\sqrt{x})^k}\omega(\sqrt{x}/n) \\ &\le \frac{cx^{r/2}}{n^r}\omega(\sqrt{x}/n), \quad 0 \le x \le 1/n^2. \end{aligned}$$

Finally by (4.15)

$$\frac{c_r}{n^{2r}}\omega(1/n^2)t_r(x) \le c_r x^r \omega(1/n^2) \le \frac{cx^{r/2}}{n^r}\omega(\sqrt{x}/n), \quad 0 \le x \le 1/n^2.$$

Just as above. This concludes the proof of (1.14) when $k \le r$.

We turn now to the special case $r = 1$, $k = 2$, which requires a special proof. Let $n \ge n_0$ and write $\omega(t) := \omega_2(f', t)$. Without loss of generality we may assume that $f(0) = 0$. By Gavrea [Gav96], there is a polynomial $\tilde{V}_{n-1} \in \Pi_{n-1} \cap \Delta^0$ satisfying

$$|f'(x) - \tilde{V}_{n-1}(x)| \le c\omega(\sqrt{x}/n), \quad x \in [0,1]. \tag{4.23}$$

Therefore for

$$V_n(x) := \int_0^x \tilde{V}_{n-1}(u)\,du,$$

we have

$$|f(x) - V_n(x)| \le cx\omega(\sqrt{x}/n), \quad x \in [0,1]. \tag{4.24}$$

By Shevchuk [She92, Theorem 17.3] there is a polynomial $U_n \in \Pi_n \cap \Delta^1$, such that

$$|f(x) - U_n(x)| \le c\rho_n(x)\omega(\rho_n(x)), \quad 0 \le x \le 1, \tag{4.25}$$

so that by Lemma 2,

$$|f'(x) - U_n'(x)| \le c\omega(\rho_n(x)), \quad 0 \le x \le 1. \tag{4.26}$$

Consider the polynomial

$$
\begin{aligned}
R_n := V_n + (U_n - V_n)t_4 &= U_n + (1 - t_4)(V_n - U_n) \\
&= (1 - t_4)V_n + t_4 U_n, \qquad (4.27)
\end{aligned}
$$

where $t_4 \in \Pi_{8n-4}$ is defined by (4.14). In order to estimate $R_n - f$ and R_n', we observe that

$$
(nx)^2\omega(1/n^2) \le 4x\omega(\sqrt{x}/n) \le 4(\sqrt{x}/n)\omega(\sqrt{x}/n), \quad x \in [0, 1/n^2],
$$

and

$$
\begin{aligned}
\omega(\sqrt{x}/n) &\le n^2 x\omega(1/n^2), \quad x \in [1/n^2, 1], \\
\rho_n(x) &\le 2\sqrt{x}/n \le 2x/n^2, \quad x \in [1/n^2, 1].
\end{aligned}
$$

Thus, applying Lemma 5 and (4.23) through (4.27), we get for $x \in [0, 1/n^2]$,

$$
\begin{aligned}
(1 - t_4(x))|f(x) - V_n(x)| &\le |f(x) - V_n(x)| \le c(\sqrt{x}/n)\omega(\sqrt{x}/n), \\
t_4(x)|f(x) - U_n(x)| &\le c(n^2x)^4 n^{-2}\omega(1/n^2) \\
&\le c(nx)^2\omega(1/n^2) \\
&\le c(\sqrt{x}/n)\omega(\sqrt{x}/n),
\end{aligned}
$$

and

$$
\begin{aligned}
t_4'(x)|V_n(x) - U_n(x)| &\le cn^2(n^2x)^3 n^{-2}\omega(1/n^2) \\
&\le cn^2 x\omega(1/n^2), \\
t_4(x)|V_n'(x) - U_n'(x)| &\le c(n^2x)^4\omega(1/n^2) \\
&\le cn^2 x\omega(1/n^2).
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
|f(x) - R_n(x)| &\le (1 - t_4(x))|f(x) - V_n(x)| + t_4(x)|f(x) - U_n(x)| \\
&\le c(\sqrt{x}/n)\omega(\sqrt{x}/n), \quad x \in [0, 1/n^2], \qquad (4.28)
\end{aligned}
$$

and

$$
\begin{aligned}
R_n'(x) &\ge V_n'(x) - t_4'(x)|V_n(x) - U_n(x)| - t_4(x)|V_n'(x) - U_n'(x)| \\
&\ge -cxn^2\omega(1/n^2), \quad x \in [0, 1/n^2]. \qquad (4.29)
\end{aligned}
$$

We turn now to $x \in [1/n^2, 1]$. Here we have

$$
\begin{aligned}
|f(x) - U_n(x)| &\le c\rho_n(x)\omega(\rho_n(x)), \\
(1 - t_4(x))|U_n(x) - V_n(x)| &\le (1 - t_4(x))|f(x) - V_n(x)| + (1 - t_4(x))|f(x) - U_n(x)| \\
&\le cx(xn^2)^{-4}\omega(\sqrt{x}/n) \le cx(xn^2)^{-3}\omega(1/n^2) \\
&\le cn^{-2}\omega(1/n^2) \le c\rho_n(x)\omega(\rho_n(x)),
\end{aligned}
$$

and

$$t_4'(x)|V_n(x) - U_n(x)| \le n^2(xn^2)^{-5}x\omega(\sqrt{x}/n) \le c(xn^2)^{-3}\omega(1/n^2),$$
$$(1 - t_4(x))|V_n'(x) - U_n'(x)| \le c(xn^2)^{-4}\omega(\rho_n(x)) \le c(xn^2)^{-3}\omega(1/n^2).$$

Therefore,

$$|f(x) - R_n(x)| \le c\rho_n(x)\omega(\rho_n(x)), \quad x \in [1/n^2, 1], \tag{4.30}$$

and

$$\begin{aligned} R_n'(x) &\ge U_n'(x) - t_4'(x)|V_n(x) - U_n(x)| - (1 - t_4(x))|V_n'(x) - U_n'(x)| \\ &\ge -c(xn^2)^{-3}\omega(1/n^2), \quad x \in [1/n^2, 1]. \end{aligned} \tag{4.31}$$

Moreover, estimates (4.29) and (4.31) imply

$$R_n'(x) \ge -c_1(xn^2)(1 + xn^2)^{-4}\omega(1/n^2). \tag{4.32}$$

Now for the polynomial t_2, we have by Lemma 5,

$$\begin{aligned} 0 \le t_2(x) &\le \min\{1, c(n^2x)^2\}, \quad x \in [0, 1], \\ t_2'(x) &\ge c_2n^2(xn^2)(1 + xn^2)^{-4}, \quad x \in [0, 1], \end{aligned} \tag{4.33}$$

so we put,

$$\tilde{P}_n := R_n + (c_1/c_2)n^{-2}\omega(1/n^2)t_2 \in \Pi_{9n-4}.$$

Then (4.31) and (4.32) imply that $\tilde{P}_n'(x) \ge 0$, $x \in [0, 1]$. And since by (4.33),

$$\begin{cases} \frac{1}{n^2}\omega(1/n^2)t_2(x) & \le c(xn)^2\omega(1/n^2) \le c(\sqrt{x}/n)\omega(\sqrt{x}/n), \quad 0 \le x \le 1/n^2, \\ \frac{1}{n^2}\omega(1/n^2)t_2(x) & \le (c/n^2)\omega(1/n^2) \le c(\rho_n(x))\omega(\rho_n(x)), \quad 1/n^2 \le x \le 1, \end{cases}$$

then together with (4.28) and (4.30), we obtained the desired inequalities, except that the polynomial $\tilde{P}_n \in \Pi_{9n-4}$. A standard argument now yields a polynomial $P_n \in \Pi_n$ such that (1.13) and (1.14) are satisfied. This completes the proof. □

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