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ON MODULI OF SMOOTHNESS WITH JACOBI WEIGHTS*

ПРО МОДУЛІ ГЛАДКОСТІ З ВАГАМИ ЯКОБІ

We introduce the moduli of smoothness with Jacobi weights $(1-x)^\alpha(1+x)^\beta$ for functions in the Jacobi weighted spaces $L_p[-1,1]$, $0<p\le\infty$. These moduli are used to characterize the smoothness of (the derivatives of) functions in the weighted spaces L_p. If $1\le p\le\infty$, then these moduli are equivalent to certain weighted K-functionals (and so they are equivalent to certain weighted Ditzian–Totik moduli of smoothness for these p), while for $0<p<1$ they are equivalent to certain "Realization functionals".

Введено модулі гладкості з вагами Якобі $(1-x)^\alpha(1+x)^\beta$ для функцій, що належать ваговим просторам Якобі $L_p[-1,1]$, $0<p\le\infty$. Ці модулі використовуються, щоб охарактеризувати гладкість функцій та їх похідних у вагових просторах L_p. При $1\le p\le\infty$ ці модулі еквівалентні деяким ваговим K-функціоналам (таким чином, еквівалентні деяким ваговим модулям гладкості Діціана–Тотіка для цих p). Водночас при $0<p<1$ ці модулі еквівалентні деяким „функціоналам реалізацій".

1. Introduction and main results. The main purpose of this paper is to introduce moduli of smoothness with Jacobi weights $(1-x)^\alpha(1+x)^\beta$ for functions in the Jacobi weighted $L_p[-1,1]$, $0<p\le\infty$, spaces. These moduli generalize the moduli that were recently introduced by the authors in [9, 10] in order to characterize the smoothness of (the derivatives of) functions in the ordinary (unweighted) L_p spaces.

For a measurable function $f\colon[-1,1]\mapsto\mathbb{R}$ and an interval $I\subseteq[-1,1]$, we use the usual notation $\|f\|_{L_p(I)}:=\left(\int_I|f(x)|^p\,dx\right)^{1/p}$, $0<p<\infty$, and $\|f\|_{L_\infty(I)}:=\operatorname{ess\,sup}_{x\in I}|f(x)|$. For a weight function w, we let $L_{w,p}(I):=\{f\mid \|wf\|_{L_p(I)}<\infty\}$, and, for $f\in L_{w,p}(I)$, we denote by $E_n(f,I)_{w,p}:=\inf_{p_n\in\mathbb{P}_n}\|w(f-p_n)\|_{L_p(I)}$, the error of best weighted approximation of f by polynomials in $\mathbb{P}_n$, the set of algebraic polynomials of degree strictly less than n. For $I=[-1,1]$, we denote $\|\cdot\|_p:=\|\cdot\|_{L_p[-1,1]}$, $L_{w,p}:=L_{w,p}[-1,1]$, $E_n(f)_{w,p}:=E_n(f,[-1,1])_{w,p}$, etc. Finally, denote

$$\varphi(x):=\sqrt{1-x^2}.$$

Definition 1.1. *For $r\in\mathbb{N}_0$ and $0<p\le\infty$, denote $\mathbb{B}_p^0(w):=L_{w,p}$ and*

$$\mathbb{B}_p^r(w):=\left\{f\mid f^{(r-1)}\in AC_{\mathrm{loc}}(-1,1)\ \text{ and }\ \varphi^rf^{(r)}\in L_{w,p}\right\},\qquad r\ge1,$$

where $AC_{\mathrm{loc}}(-1,1)$ denotes the set of functions which are locally absolutely continuous in $(-1,1)$.

Now, define

$$J_p:=\begin{cases}(-1/p,\infty), & \text{if}\quad p<\infty,\\ [0,\infty), & \text{if}\quad p=\infty,\end{cases}$$

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let

$$w_{\alpha,\beta}(x) := (1-x)^\alpha(1+x)^\beta, \qquad \alpha,\beta \in J_p,$$

be the Jacobi weights, and denote $L_p^{\alpha,\beta} := L_{w_{\alpha,\beta},p}$.

Also denote

$$\mathcal{W}_\delta^{\xi,\zeta}(x) := (1-x-\delta\varphi(x)/2)^\xi(1+x-\delta\varphi(x)/2)^\zeta.$$

Note that $\mathcal{W}_0^{\alpha,\beta}(x) = w_{\alpha,\beta}(x)$, $\mathcal{W}_0^{1/2,1/2}(x) = \varphi(x)$ and, if $\xi,\zeta \geq 0$, $\mathcal{W}_\delta^{\xi,\zeta}(x) \leq w_{\xi,\zeta}(x)$.

For $k \in \mathbb{N}$ and $h \geq 0$, let

$$\Delta_h^k(f,x;J) := \begin{cases} \sum_{i=0}^k \binom{k}{i}(-1)^{k-i} f\left(x - \dfrac{kh}{2} + ih\right), & \text{if } \left[x - \dfrac{kh}{2}, x + \dfrac{kh}{2}\right] \subseteq J, \\ 0, & \text{otherwise}, \end{cases}$$

be the kth symmetric difference, and $\Delta_h^k(f,x) := \Delta_h^k(f,x;[-1,1])$.

We introduce the following definition, which for $\alpha,\beta=0$, was given in [10] (Definition 2.2) (for $\alpha,\beta=0$ and $p=\infty$ see the earlier [2] (Chapter 3.10)).

Definition 1.2. *For $k,r \in \mathbb{N}$ and $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$, $0 < p \leq \infty$, define*

$$\omega_{k,r}^\varphi(f^{(r)},t)_{\alpha,\beta,p} := \sup_{0 \leq h \leq t} \left\| \mathcal{W}_{kh}^{r/2+\alpha,r/2+\beta}(\cdot)\Delta_{h\varphi(\cdot)}^k(f^{(r)},\cdot)\right\|_p. \tag{1.1}$$

For $\delta > 0$, denote (see [10])

$$\mathfrak{D}_\delta := \{x \mid 1-\delta\varphi(x)/2 \geq |x|\} \setminus \{\pm 1\} = \left\{x \mid |x| \leq \frac{4-\delta^2}{4+\delta^2}\right\} = [-1+\mu(\delta), 1-\mu(\delta)],$$

where

$$\mu(\delta) := 2\delta^2/(4+\delta^2).$$

Observe that $\mathfrak{D}_{\delta_1} \subset \mathfrak{D}_{\delta_2}$ if $\delta_2 < \delta_1 \leq 2$, and that $\mathfrak{D}_\delta = \varnothing$ if $\delta > 2$. Also note that $\Delta_{h\varphi(x)}^k(f,x)$ is defined to be identically 0 if $x \notin \mathfrak{D}_{kh}$ and that $\mathcal{W}_\delta^{r/2+\alpha,r/2+\beta}$ is well defined on $\mathfrak{D}_\delta$ (except perhaps at the endpoints where it may be infinite).

Hence,

$$\omega_{k,r}^\varphi(f^{(r)},t)_{\alpha,\beta,p} = \sup_{0<h\leq t} \left\| \mathcal{W}_{kh}^{r/2+\alpha,r/2+\beta}(\cdot)\Delta_{h\varphi(\cdot)}^k(f^{(r)},\cdot)\right\|_{L_p(\mathfrak{D}_{kh})} \tag{1.2}$$

and

$$\omega_{k,r}^\varphi(f^{(r)},t)_{\alpha,\beta,p} = \omega_{k,r}^\varphi(f^{(r)},2/k)_{\alpha,\beta,p} \qquad \text{for} \qquad t \geq 2/k. \tag{1.3}$$

In a forthcoming paper [11], we will prove Whitney-, Jackson- and Bernstein-type theorems for the Jacobi weighted approximation of functions in the above spaces by algebraic polynomials. Thus, we get a constructive characterization of the smoothness classes with respect to these moduli by means of the degrees of approximation. This implies, in particular, that these moduli are the right measure of smoothness to be used while investigating constrained weighted approximation (see, e. g., [3, 7, 8]).

We will show that, for $r/2+\alpha, r/2+\beta \geq 0$, our moduli are equivalent to the following weighted averaged moduli.

Definition 1.3. *For $k \in \mathbb{N}$, $r \in \mathbb{N}_0$ and $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$, $0 < p < \infty$, the kth weighted averaged modulus of smoothness of f is defined as*

$$\omega_{k,r}^{*\varphi}(f^{(r)}, t)_{\alpha,\beta,p} := \left(\frac{1}{t} \int_0^t \int_{\mathfrak{D}_{k\tau}} |\mathcal{W}_{k\tau}^{r/2+\alpha,r/2+\beta}(x) \Delta_{\tau\varphi(x)}^k (f^{(r)}, x)|^p \, dx \, d\tau \right)^{1/p}.$$

If $p = \infty$ and $f \in \mathbb{B}_\infty^r(w_{\alpha,\beta})$, we write

$$\omega_{k,r}^{*\varphi}(f^{(r)}, t)_{\alpha,\beta,\infty} := \omega_{k,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,\infty}.$$

Clearly,

$$\omega_{k,r}^{*\varphi}(f^{(r)}, t)_{\alpha,\beta,p} \le \omega_{k,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,p}, \qquad t > 0, \quad 0 < p \le \infty. \tag{1.4}$$

We now define the weighted K-functional as well as the "Realization functional" as follows.

Definition 1.4. *For $k \in \mathbb{N}$, $r \in \mathbb{N}_0$ and $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$, $0 < p \le \infty$, define*

$$K_{k,r}^{\varphi}(f^{(r)}, t^k)_{\alpha,\beta,p} := \inf_{g \in \mathbb{B}_p^{k+r}(w_{\alpha,\beta})} \left\{ \left\| w_{\alpha,\beta}\varphi^r (f^{(r)} - g^{(r)}) \right\|_p + t^k \left\| w_{\alpha,\beta}\varphi^{k+r} g^{(k+r)} \right\|_p \right\}$$

and

$$R_{k,r}^{\varphi}(f^{(r)}, n^{-k})_{\alpha,\beta,p} := \inf_{P_n \in \mathbb{P}_n} \left\{ \left\| w_{\alpha,\beta}\varphi^r (f^{(r)} - P_n^{(r)}) \right\|_p + n^{-k} \left\| w_{\alpha,\beta}\varphi^{k+r} P_n^{(k+r)} \right\|_p \right\}.$$

Clearly, $K_{k,r}^{\varphi}(f^{(r)}, n^{-k})_{\alpha,\beta,p} \le R_{k,r}^{\varphi}(f^{(r)}, n^{-k})_{\alpha,\beta,p}$, $n \in \mathbb{N}$. Note that, as is rather well known, K-functionals are not the right measure of smoothness if $0 < p < 1$, since they may become identically zero.

Throughout this paper, all constants c may depend only on k, r, p, α and β, but are independent of the function as well as the important parameters t and n. The constants c may be different even if they appear in the same line.

Our first main result in this paper is the following theorem. It is a corollary of Lemma 3.2 and the sequence of estimates (4.3).

Theorem 1.1. *If $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $r/2 + \alpha \ge 0$, $r/2 + \beta \ge 0$, $1 \le p \le \infty$, and $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$, then there exists $N \in \mathbb{N}$ depending on k, r, p, α and β, such that for all $0 < t \le 2/k$ and $n \in \mathbb{N}$ satisfying $\max\{N, c_1/t\} \le n \le c_2/t$,*

$$\begin{aligned} K_{k,r}^{\varphi}(f^{(r)}, t^k)_{\alpha,\beta,p} &\le c R_{k,r}^{\varphi}(f^{(r)}, n^{-k})_{\alpha,\beta,p} \le c\,\omega_{k,r}^{*\varphi}(f^{(r)}, t)_{\alpha,\beta,p} \le \\ &\le c\,\omega_{k,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,p} \le c K_{k,r}^{\varphi}(f^{(r)}, t^k)_{\alpha,\beta,p}, \end{aligned} \tag{1.5}$$

where constants c may depend only on k, r, p, α, β as well as c_1 and c_2.

***Remark* 1.1.** Clearly, $K_{k,r}^{\varphi}(f^{(r)}, t^k)_{\alpha,\beta,p} \le \left\| w_{\alpha,\beta}\varphi^r f^{(r)} \right\|_p < \infty$, for all $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$, and it follows from Theorem 2.1 that, if $r/2 + \alpha < 0$ or/and $r/2 + \beta < 0$, then there exists a function $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$ such that $\omega_{k,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,p} = \infty$, for all $t > 0$. Hence, Theorem 1.1 is not valid if $r/2 + \alpha < 0$ or/and $r/2 + \beta < 0$.

We can somewhat simplify the statement of Theorem 1.1 if we remove the realization functional $R^{\varphi}_{k,r}$ from (1.5).

***Corollary* 1.1.** *If* $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $r/2+\alpha \ge 0$, $r/2+\beta \ge 0$, $1 \le p \le \infty$, *and* $f \in \mathbb{B}^r_p(w_{\alpha,\beta})$, *then, for all* $0 < t \le 2/k$,

$$K^{\varphi}_{k,r}(f^{(r)}, t^k)_{\alpha,\beta,p} \le c\omega^{*\varphi}_{k,r}(f^{(r)}, t)_{\alpha,\beta,p} \le c\omega^{\varphi}_{k,r}(f^{(r)}, t)_{\alpha,\beta,p} \le cK^{\varphi}_{k,r}(f^{(r)}, t^k)_{\alpha,\beta,p}.$$

In the case $0 < p < 1$, we have the following result on the equivalence of the moduli and Realization functionals. It is a corollary of Theorem 4.3 that will be proved in Section 4.

Theorem 1.2. *Let* $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $0 < p < 1$, $r/2+\alpha \ge 0$, $r/2+\beta \ge 0$, *and* $f \in \mathbb{B}^r_p(w_{\alpha,\beta})$. *Then there exist* $N \in \mathbb{N}$ *and* $\vartheta > 0$ *depending on* k, p, α *and* β, *such that, for any* $\vartheta_1 \in (0, \vartheta]$, $n \ge N$, $\vartheta_1/n \le t \le \vartheta/n$, *we have*

$$R^{\varphi}_{k,r}(f, n^{-k})_{\alpha,\beta,p} \sim \omega^{*\varphi}_{k,r}(f^{(r)}, t)_{\alpha,\beta,p} \sim \omega^{\varphi}_{k,r}(f^{(r)}, t)_{\alpha,\beta,p}.$$

Here, as usual, by $a(t) \sim b(t)$, $t \in T$, we mean that there exists a positive constant c_0 such that $c_0^{-1}a(t) \le b(t) \le c_0 a(t)$, for all $t \in T$.

Note that it follows from Theorem 1.2 that, for sufficiently small $t_1, t_2 > 0$ such that $t_1 \sim t_2$,

$$\omega^{*\varphi}_{k,r}(f^{(r)}, t_1)_{\alpha,\beta,p} \sim \omega^{\varphi}_{k,r}(f^{(r)}, t_1)_{\alpha,\beta,p} \sim \omega^{*\varphi}_{k,r}(f^{(r)}, t_2)_{\alpha,\beta,p} \sim \omega^{\varphi}_{k,r}(f^{(r)}, t_2)_{\alpha,\beta,p}.$$

If $1 \le p \le \infty$, we can say a bit more. Theorem 1.1 and the (obvious) monotonicity of $\omega^{\varphi}_{k,r}(f^{(r)}, t)_{\alpha,\beta,p}$, with respect to t, immediately yield the following quite useful property which is not easily seen from Definition 1.2.

***Corollary* 1.2.** *Let* $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $r/2+\alpha \ge 0$, $r/2+\beta \ge 0$, $1 \le p \le \infty$, $f \in \mathbb{B}^r_p(w_{\alpha,\beta})$ *and* $\lambda \ge 1$. *Then, for all* $t > 0$,

$$\omega^{\varphi}_{k,r}(f^{(r)}, \lambda t)_{\alpha,\beta,p} \le c\lambda^k \omega^{\varphi}_{k,r}(f^{(r)}, t)_{\alpha,\beta,p}. \tag{1.6}$$

By virtue of (5.2) the following result is an immediate consequence of Corollary 1.1.

Theorem 1.3. *Let* $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $r/2+\alpha \ge 0$, $r/2+\beta \ge 0$, *and* $1 \le p \le \infty$. *If* $f \in \mathbb{B}^r_p(w_{\alpha,\beta})$, *then, for some* $t_0 > 0$ *independent of* f,

$$\omega^{\varphi}_{k,r}(f^{(r)}, t)_{\alpha,\beta,p} \sim \omega^{k}_{\varphi}(f^{(r)}, t)_{w_{\alpha,\beta}\varphi^r,p}, \quad 0 < t \le t_0, \tag{1.7}$$

where the weighted DT moduli $\omega^k_\varphi(g, \cdot)_{w,p}$ *are defined in* (5.1).

It was shown in [9] (Theorem 5.1) that, for $\xi, \zeta \ge 0$ and $g \in B^1_p(w_{\xi,\zeta})$,

$$\omega^{k+1}_{\varphi}(g, t)_{w_{\xi,\zeta},p} \le ct\omega^{k}_{\varphi}(g', t)_{w_{\xi,\zeta}\varphi,p}, \quad t > 0.$$

Letting $\xi := r/2+\alpha$, $\zeta := r/2+\beta$, $g := f^{(r)}$, using the fact that $f^{(r)} \in B^1_p(w_{r/2+\alpha,r/2+\beta})$ if and only if $f \in B^{r+1}_p(w_{\alpha,\beta})$, by virtue of (1.7), as well as (1.6) if t is "large" (i.e., if $t > t_0$), we immediately get the following result.

Lemma 1.1. *Let* $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $r/2+\alpha \ge 0$, $r/2+\beta \ge 0$, *and* $1 \le p \le \infty$. *If* $f \in \mathbb{B}^{r+1}_p(w_{\alpha,\beta})$, *then*

$$\omega^{\varphi}_{k+1,r}(f^{(r)}, t)_{\alpha,\beta,p} \le ct\omega^{\varphi}_{k,r+1}(f^{(r+1)}, t)_{\alpha,\beta,p}, \quad t > 0.$$

Finally, the following lemma follows from [1] (Theorem 6.1.4) using (1.7).

Lemma 1.2. *Let $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $r/2+\alpha \geq 0$, $r/2+\beta \geq 0$, and $1 \leq p \leq \infty$. If $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$, then*

$$\omega_{k+1,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,p} \leq c\omega_{k,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,p}, \quad t > 0.$$

2. Hierarchy of $B_p^r(w_{\alpha,\beta})$, (un)boundedness of the moduli and their convergence to 0. Without special references we use the following evident inequalities:

$$(1-x) \leq 2(1-u) \quad \text{and} \quad (1+x) \leq 2(1+u), \quad \text{if} \quad u \in [\min\{0,x\}, \max\{0,x\}],$$

and

$$\varphi(x) \leq \varphi(u), \quad \text{if} \quad |u| \leq |x| \leq 1.$$

Also (see [10], Proposition 3.1(iv)),

$$|\varphi'(x)| \leq 1/\delta \quad \text{for} \quad x \in \mathfrak{D}_\delta. \tag{2.1}$$

First we show the hierarchy between the $\mathbb{B}_p^r(w_{\alpha,\beta})$, $r \geq 0$, spaces. Namely, the following lemma holds.

Lemma 2.1. *Let $r \in \mathbb{N}_0$, $1 \leq p \leq \infty$ and $r/2+\alpha, r/2+\beta \in J_p$. Then*

$$\mathbb{B}_p^{r+1}(w_{\alpha,\beta}) \subsetneq \mathbb{B}_p^r(w_{\alpha,\beta}). \tag{2.2}$$

Moreover, in the case $p=\infty$, if $r/2+\alpha > 0$ and $r/2+\beta > 0$, then, additionally,

$$f \in \mathbb{B}_\infty^{r+1}(w_{\alpha,\beta}) \quad \Longrightarrow \quad \lim_{x \to \pm 1} w_{\alpha,\beta}(x)\varphi^r(x) f^{(r)}(x) = 0. \tag{2.3}$$

***Remark* 2.1.** Note that we may not relax the condition $r/2+\alpha, r/2+\beta > 0$ in order to guarantee (2.3). Indeed, if $\alpha = -r/2$, for example, then the function $g(x) := x^r$ is certainly in $\mathbb{B}_\infty^{r+1}(w_{\alpha,\beta})$ but $\lim_{x\to 1} w_{\alpha,\beta}(x)\varphi^r(x) g^{(r)}(x) \neq 0$.

The same example shows that we may not relax the condition $r/2+\alpha, r/2+\beta \in J_p$ in order to guarantee (2.2), since $\left\|w_{\alpha,\beta}\varphi^r g^{(r)}\right\|_p = \infty$ if this condition is not satisfied, so that $g \notin \mathbb{B}_p^r(w_{\alpha,\beta})$.

***Remark* 2.2.** For any $r \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{R}$, (2.2) is not valid if $0 < p < 1$. For example, suppose that f is such that

$$f^{(r)}(x) = \sum_{n=1}^{\infty} g_n(x),$$

where, for each $n \in \mathbb{N}$,

$$g_n(x) := \begin{cases} \dfrac{H_n}{\varepsilon_n}\left(x+1-\dfrac{1}{n+1}\right), & \text{if } \dfrac{1}{n+1} < x+1 \leq \dfrac{1}{n+1}+\varepsilon_n, \\ H_n, & \text{if } \dfrac{1}{n+1}+\varepsilon_n < x+1 \leq \dfrac{1}{n}-\varepsilon_n, \\ \dfrac{H_n}{\varepsilon_n}\left(\dfrac{1}{n}-x-1\right), & \text{if } \dfrac{1}{n}-\varepsilon_n < x+1 \leq \dfrac{1}{n}, \\ 0, & \text{otherwise,} \end{cases}$$

$H_n := n^{r/2+\beta+1/p}$, $\varepsilon_n := c_0 n^{-2/(1-p)}$, and $c_0 > 0$ is a constant depending only on p that guarantees that $4\varepsilon_n n(n+1) < 1$, for all $n \in \mathbb{N}$. Then $f^{(r)} \in \mathrm{AC}_{\mathrm{loc}}(-1,1)$ and

$$\left\|w_{\alpha,\beta}\varphi^r f^{(r)}\right\|_p^p = \sum_{n=1}^{\infty} \|w_{\alpha,\beta}\varphi^r g_n\|_p^p \geq c \sum_{n=1}^{\infty} \frac{1}{n^{(r/2+\beta)p}} H_n^p n^{-2} = c \sum_{n=1}^{\infty} n^{-1} = \infty.$$

Hence, $f \notin B_p^r(w_{\alpha,\beta})$. At the same time,

$$\left\|w_{\alpha,\beta}\varphi^{r+1} f^{(r+1)}\right\|_p^p = \sum_{n=1}^{\infty} \|w_{\alpha,\beta}\varphi^{r+1} g_n'\|_p^p \leq c \sum_{n=1}^{\infty} \frac{1}{n^{((r+1)/2+\beta)p}} \left(H_n \varepsilon_n^{-1}\right)^p \varepsilon_n =$$

$$= c \sum_{n=1}^{\infty} n^{1-p/2} \varepsilon_n^{1-p} = c \sum_{n=1}^{\infty} n^{-1-p/2} < \infty,$$

so that $f \in B_p^{r+1}(w_{\alpha,\beta})$.

***Proof of Lemma* 2.1.** The proof follows along the lines of [10] (Lemma 3.4) with some modifications, we bring it here for the sake of completeness. Let $g \in \mathbb{B}_p^{r+1}(w_{\alpha,\beta})$, and assume, without loss of generality, that $g^{(r)}(0) = 0$ and that $\beta \geq \alpha$. For convenience, denote $A_p := \left\|w_{\alpha,\beta}\varphi^{r+1} g^{(r+1)}\right\|_p$.

First, if $p = \infty$, then $A_\infty < \infty$ and

$$w_{\alpha,\beta}(x)\varphi^r(x)\left|g^{(r)}(x)\right| = w_{\alpha,\beta}(x)\varphi^r(x)\left|\int_0^x g^{(r+1)}(u)\,du\right| \leq$$

$$\leq A_\infty w_{\alpha,\beta}(x)\varphi^r(x)\left|\int_0^x w_{\alpha,\beta}^{-1}(u)\varphi^{-r-1}(u)\,du\right| \leq$$

$$\leq 2^{\beta-\alpha} A_\infty \varphi^{r+2\alpha}(x)\left|\int_0^x \varphi^{-r-1-2\alpha}(u)\,du\right| =$$

$$= 2^{\beta-\alpha} A_\infty \varphi^{r+2\alpha}(x) \int_0^{|x|} \varphi^{-r-1-2\alpha}(u)\,du \leq$$

$$\leq 2^{\beta-\alpha} A_\infty \int_0^{|x|} \varphi^{-1}(u)\,du \leq$$

$$\leq 2^{\beta-\alpha} A_\infty \int_0^1 \varphi^{-1}(u)\,du = \pi 2^{\beta-\alpha-1} A_\infty.$$

Hence, $g \in \mathbb{B}_\infty^r(w_{\alpha,\beta})$, and (2.2) is proved if $p = \infty$.

In order to prove (2.3) we need to show that, if $r/2+\alpha, r/2+\beta > 0$, then

$$\lim_{x\to\pm 1} w_{\alpha,\beta}(x)\varphi^r(x)g^{(r)}(x) = 0. \tag{2.4}$$

(Note that we are still not losing generality by assuming that $g^{(r)}(0) = 0$.) We put $\varepsilon := \min\{r + 2\alpha, 1\} > 0$ and note that

$$\int_0^x \frac{1}{\varphi^2(u)}\,du = \frac{1}{2}\ln\frac{1+x}{1-x}.$$

Therefore,

$$w_{\alpha,\beta}(x)\varphi^r(x)|g^{(r)}(x)| \le 2^{\beta-\alpha}A_\infty\varphi^\varepsilon(x)\int_0^{|x|}\frac{1}{\varphi^{1+\varepsilon}(u)}\,du \le$$

$$\le 2^{\beta-\alpha}A_\infty\varphi^\varepsilon(x)\int_0^{|x|}\frac{1}{\varphi^2(u)}\,du =$$

$$= 2^{\beta-\alpha}A_\infty\varphi^\varepsilon(|x|)\ln\frac{1+|x|}{1-|x|}\to 0, \quad |x|\to 1,$$

and (2.4) is proved.

Now let $1 \le p < \infty$ and $q := p/(p-1)$. Then, denoting

$$\left|\int_0^x |G(u)|^q du\right|^{1/q} := \sup_{u\in[\min\{0,x\},\max\{0,x\}]}|G(u)|$$

if $q=\infty$, we have by Hölder's inequality

$$\left\|w_{\alpha,\beta}\varphi^r g^{(r)}\right\|_p^p = \int_{-1}^1 w_{\alpha,\beta}^p(x)\varphi^{rp}(x)\left|\int_0^x g^{(r+1)}(u)\,du\right|^p dx \le$$

$$\le \int_{-1}^1 w_{\alpha,\beta}^p(x)\varphi^{rp}(x)\left|\int_0^x w_{\alpha,\beta}^{-q}(u)\varphi^{-(r+1)q}(u)\,du\right|^{p/q}\times$$

$$\times\left|\int_0^x |w_{\alpha,\beta}(u)\varphi^{r+1}(u)g^{(r+1)}(u)|^p\,du\right| dx \le$$

$$\le A_p^p\int_{-1}^1 w_{\alpha,\beta}^p(x)\varphi^{rp}(x)\left|\int_0^x w_{\alpha,\beta}^{-q}(u)\varphi^{-(r+1)q}(u)du\right|^{p/q} dx \le$$

$$\le 2^{(\beta-\alpha)p}A_p^p\int_{-1}^1 \varphi^{rp+2\alpha p}(x)\left|\int_0^x \varphi^{-(r+1)q-2\alpha q}(u)du\right|^{p/q} dx =:$$

$$=: 2^{(\beta-\alpha)p} A_p^p \Theta(\alpha, p).$$

Note that

$$\Theta(\alpha,1) = \int_{-1}^{1} \varphi^{r+2\alpha}(x) \left(\sup_{u\in[\min\{0,x\},\max\{0,x\}]} \varphi^{-r-1-2\alpha}(u) \right) dx.$$

Recall that $r/2+\alpha \in J_p$ so that $rp+2\alpha p > -2$. We consider two cases.

Case 1. Suppose that $rp+2\alpha p \geq -1$. If $p=1$, then $r+2\alpha+1 \geq 0$ implies that $\Theta(\alpha,1) = = \int_{-1}^{1} \varphi^{-1}(x)dx = \pi$, and if $1<p<\infty$, then $((r+1)q-1+2\alpha q)p/q = rp+2\alpha p+1 \geq 0$, and hence

$$\Theta(\alpha,p) = 2\int_0^1 \frac{1}{\varphi(x)} \left(\int_0^x \frac{\varphi^{(r+1)q-1+2\alpha q}(x)}{\varphi^{(r+1)q+2\alpha q}(u)}\, du \right)^{p/q} dx \leq$$

$$\leq 2\int_0^1 \frac{1}{\varphi(x)} \left(\int_0^x \frac{1}{\varphi(u)}\, du \right)^{p/q} dx \leq 2\int_0^1 \frac{dx}{\varphi(x)} \left(\int_0^1 \frac{du}{\varphi(u)} \right)^{p/q} =$$

$$= 2(\pi/2)^p.$$

Case 2. Suppose now that $-2 < rp+2\alpha p < -1$. If $p=1$, then

$$\Theta(\alpha,1) = \int_{-1}^{1} \varphi^{r+2\alpha}(x)dx < \infty.$$

If $1<p<\infty$, then $(r+1)q+2\alpha q < 1$. Hence

$$\int_0^1 \varphi^{-(r+1)q-2\alpha q}(u)du < \int_0^1 \varphi^{-1}(u)du = \pi/2,$$

and so

$$\Theta(\alpha,p) \leq 2(\pi/2)^{p/q} \int_0^1 \varphi^{rp+2\alpha p}(x)dx < \infty.$$

Lemma 2.1 is proved.

We now show that, for a function $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$, if $r/2+\alpha \geq 0$ and $r/2+\beta \geq 0$, then the modulus $\omega_{k,r}^{\varphi}(f^{(r)},t)_{\alpha,\beta,p}$ is bounded.

Lemma 2.2. *Let* $k\in\mathbb{N}$, $r\in\mathbb{N}_0$, $r/2+\alpha\geq 0$, $r/2+\beta\geq 0$, *and* $0<p\leq\infty$. *If* $f\in\mathbb{B}_p^r(w_{\alpha,\beta})$, *then*

$$\omega_{k,r}^{\varphi}(f^{(r)},t)_{\alpha,\beta,p} \leq c \left\| w_{\alpha,\beta}\varphi^r f^{(r)} \right\|_p, \quad t>0, \tag{2.5}$$

where c *depends only on* k *and* p.

Proof. In view of (1.3), we may limit ourselves to $t \le 2/k$, and so $\mathfrak{D}_{kh} \ne \varnothing$ if $0 < h \le t$. We set

$$u_i(x) := x + (i - k/2)h\varphi(x), \quad 0 \le i \le k,$$

and note that, for $x \in \mathfrak{D}_{kh}$,

$$B_r(x) := \frac{\mathcal{W}_{kh}^{r/2+\alpha,r/2+\beta}(x)}{w_{\alpha,\beta}(u_i(x))\varphi^r(u_i(x))} =$$

$$= \left(\frac{1 - u_i(x) - (k-i)h\varphi(x)}{1 - u_i(x)}\right)^{r/2+\alpha} \left(\frac{1 + u_i(x) - ih\varphi(x)}{1 + u_i(x)}\right)^{r/2+\beta} \le 1.$$

Therefore,

$$\left\|\mathcal{W}_{kh}^{r/2+\alpha,r/2+\beta}(\cdot)f^{(r)}(u_i(\cdot))\right\|_{L_\infty(\mathfrak{D}_{kh})} =$$

$$= \left\|B_r(\cdot)w_{\alpha,\beta}(u_i(\cdot))\varphi^r(u_i(\cdot))f^{(r)}(u_i(\cdot))\right\|_{L_\infty(\mathfrak{D}_{kh})} \le$$

$$\le \left\|w_{\alpha,\beta}(u_i(\cdot))\varphi^r(u_i(\cdot))f^{(r)}(u_i(\cdot))\right\|_{L_\infty(\mathfrak{D}_{kh})} \le \left\|w_{\alpha,\beta}\varphi^r f^{(r)}\right\|_\infty,$$

that yields (2.5) for $p = \infty$.

To apply the same arguments to the case $0 < p < \infty$ we note that (2.1) yields $|\varphi'(x)| \le 1/(kh)$ for $x \in \mathfrak{D}_{kh}$, so that

$$u_i'(x) \ge 1 - |i - k/2|h|\varphi'(x)| \ge 1 - kh|\varphi'(x)|/2 \ge 1/2, \quad x \in \mathfrak{D}_{kh},$$

which implies

$$\int\limits_{\mathfrak{D}_{kh}} |F(u_i(x))|dx \le 2\int\limits_{-1}^{1} |F(u)|du$$

for each $F \in L_1[-1,1]$.

Hence,

$$\left\|\mathcal{W}_{kh}^{r/2+\alpha,r/2+\beta}(\cdot)f^{(r)}(u_i(\cdot))\right\|_{L_p(\mathfrak{D}_{kh})}^p \le \left\|w_{\alpha,\beta}(u_i(\cdot))\varphi^r(u_i(\cdot))f^{(r)}(u_i(\cdot))\right\|_{L_p(\mathfrak{D}_{kh})}^p \le$$

$$\le 2\int\limits_{-1}^{1} |w_{\alpha,\beta}(x)\varphi^r(x)f^{(r)}(x)|^p dx = 2\left\|w_{\alpha,\beta}\varphi^r f^{(r)}\right\|_p^p.$$

Thus,

$$\omega_{k,r}^{\varphi}(f^{(r)},t)_{\alpha,\beta,p} \le c \max_{0\le i\le k}\left\|\mathcal{W}_{kh}^{r/2+\alpha,r/2+\beta}(\cdot)f^{(r)}(u_i(\cdot))\right\|_{L_p(\mathfrak{D}_{kh})} \le c\left\|w_{\alpha,\beta}\varphi^r f^{(r)}\right\|_p.$$

Lemma 2.2 is proved.

***Remark* 2.3.** The same proof yields a local version of (2.5) as well. Namely, for each $h > 0$ and $[a, b] \subseteq \mathfrak{D}_{kh}$,

$$\left\| \mathcal{W}_{kh}^{r/2+\alpha, r/2+\beta}(\cdot)\Delta_{h\varphi(\cdot)}^{k}(f^{(r)}, \cdot) \right\|_{L_p[a,b]} \leq c \left\| w_{\alpha,\beta}\varphi^r f^{(r)} \right\|_{L_p(S)},$$

where $S := [a - kh\varphi(a)/2, b + kh\varphi(b)/2]$.

We now show that the modulus $\omega_{k,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,p}$ may be infinite for a function $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$ if either $r/2 + \alpha < 0$ or $r/2 + \beta < 0$.

When $p = \infty$, this is obvious. Indeed, suppose that $r/2 + \beta \geq 0$ and $-k \leq r/2 + \alpha < < 0$, and let $f(x) := (x - 1)^{k+r}$. Then $f \in \mathbb{B}_\infty^r(w_{\alpha,\beta})$ and $\Delta_{h\varphi(x)}^k(f^{(r)}, x) \equiv ch^k\varphi^k(x)$. Hence, $\mathcal{W}_{kh}^{r/2+\alpha,r/2+\beta}(x)\Delta_{h\varphi(x)}^k(f^{(r)}, x) \to \infty$ for x such that $1 - x - kh\varphi(x)/2 \to 0$. This implies that $\omega_{k,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,\infty} = \infty$ for all $t > 0$. Note also that, by considering $f \in C^r[-1, 1]$ such that $f(x) = (1 - |x|)^{k+r}$, $x \notin [-1/2, 1/2]$, one can easily see that the same conclusion holds if both $r/2 + \alpha$ and $/2 + \beta$ are in $[-k, 0)$.

When $p < \infty$, the arguments are not so obvious, but the conclusion is the same. The following theorem is valid.

Theorem 2.1. *Suppose that $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $\alpha \in \mathbb{R}$, $0 < p < \infty$, and $r/2 + \beta < 0$. If $0 < p < 1$ and $r \geq 1$, we additionally assume that $r/2 + \beta < 1 - 1/p$. Then there exists a function $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$, such that, for all $t > 0$,*

$$\omega_{k,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,p} = \infty.$$

Proof. Let $\{\varepsilon_n\}_{n=0}^\infty$ be a decreasing sequence of positive numbers, tending to zero, such that $\varepsilon_0 < 1/(2k)$ and

$$(2 + k)\varepsilon_n < \varepsilon_{n-1}, \quad n \in \mathbb{N}.$$

Define

$$J_n := [-1 + \varepsilon_n, -1 + \varepsilon_n(1 + 2^{-n})].$$

Now, let f be such that

$$f^{(r)}(x) := \begin{cases} (x + 1 - \varepsilon_n)^{-r/2-\beta-1/p}, & \text{if} \quad x \in J_n \quad \text{for some} \quad n \in \mathbb{N}, \\ 0, & \text{otherwise}. \end{cases}$$

Note that, in the case $r \geq 1$, since $-r/2 - \beta - 1/p + 1 > 0$, the function $f^{(r-1)}(x) = \int_0^x f^{(r)}(u)du$ is locally absolutely continuous on $(-1, 1)$.

Now,

$$2^{-|r/2+\alpha|p} \left\| w_{\alpha,\beta}\varphi^r f^{(r)} \right\|_p^p \leq \sum_{n=1}^\infty \int_{J_n} |(1 + x)^{r/2+\beta} f^{(r)}(x)|^p dx \leq$$

$$\leq \sum_{n=1}^\infty \varepsilon_n^{(r/2+\beta)p} \int_{J_n} |f^{(r)}(x)|^p dx =$$

$$= \sum_{n=1}^{\infty} \varepsilon_n^{(r/2+\beta)p} \int\limits_0^{\varepsilon_n 2^{-r}} t^{-(r/2+\beta)p-1} dt \leq$$

$$\leq c \sum_{n=1}^{\infty} 2^{(r/2+\beta)np} < \infty.$$

Hence, $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$.

We now let

$$x_n := -1 + \frac{k}{2}\varepsilon_n, \qquad h_n := \frac{\varepsilon_n}{\varphi(x_n)}, \qquad \text{and} \qquad I_{k,n} := [x_n, x_n + \varepsilon_n],$$

so that

$$\mathfrak{D}_{kh_n} = [x_n, -x_n] \qquad \text{and} \qquad h_n < \sqrt{2\varepsilon_n} \to 0, \quad n \to \infty.$$

Since $\varphi(x) \geq \varphi(x_n)$, $|x| \leq |x_n|$, we conclude that, for any $x \in I_{k,n} \subset [x_n, -x_n]$,

$$x - \left(\frac{k}{2} - 2\right) h_n\varphi(x) = x - \frac{k}{2}h_n\varphi(x) + 2h_n\varphi(x) \geq -1 + 2h_n\varphi(x) \geq$$

$$\geq -1 + 2h_n\varphi(x_n) = -1 + 2\varepsilon_n > -1 + \varepsilon_n(1 + 2^{-n}).$$

Now, since φ is concave and $\varphi(-1) = 0$, we have

$$\varphi(x_n + \varepsilon_n) < \frac{x_n + \varepsilon_n + 1}{x_n + 1}\varphi(x_n) = \left(1 + \frac{2}{k}\right)\varphi(x_n),$$

and so, for all $x \in I_{k,n}$,

$$x + \frac{k}{2}h_n\varphi(x) \leq x_n + \varepsilon_n + \frac{k}{2}h_n\varphi(x_n + \varepsilon_n) \leq x_n + \varepsilon_n + \left(1 + \frac{k}{2}\right) h_n\varphi(x_n) =$$

$$= -1 + (2 + k)\varepsilon_n < -1 + \varepsilon_{n-1}.$$

If $k \geq 2$, this implies that, for all $2 \leq i \leq k$ and $x \in I_{k,n}$,

$$f^{(r)}(x + (i - k/2)h_n\varphi(x)) = 0.$$

Now, denote

$$y(x) := x + (1 - k/2)h_n\varphi(x)$$

and observe that

$$\frac{1}{2} < y'(x) < \frac{3}{2}, \quad x \in [x_n, -x_n], \tag{2.6}$$

since, if $|x| \leq |x_n|$, then it follows from (2.1) that

$$h_n|\varphi'(x)| < 1/k \tag{2.7}$$

and so

$$|y'(x) - 1| \le \frac{k}{2} h_n |\varphi'(x)| < \frac{1}{2}.$$

For all $k \in \mathbb{N}$, using $\|f_1 + f_2\|_p \le \max\{1, 2^{1/p-1}\} \left(\|f_1\|_p + \|f_2\|_p\right)$, we obtain

$$2^{|\alpha+r/2|} \left\| \mathcal{W}_{kh_n}^{r/2+\alpha, r/2+\beta}(\cdot) \Delta_{h_n\varphi}^k (f^{(r)}, \cdot) \right\|_p \ge$$

$$\ge 2^{|\alpha+r/2|} \left\| \mathcal{W}_{kh_n}^{r/2+\alpha, r/2+\beta}(\cdot) \Delta_{h_n\varphi}^k (f^{(r)}, \cdot) \right\|_{L_p(I_{k,n})} \ge$$

$$\ge \left\| (1 + y(\cdot) - h_n\varphi(\cdot))^{r/2+\beta} \left(f^{(r)}(y(\cdot) - h_n\varphi(\cdot)) - k f^{(r)}(y(\cdot)) \right) \right\|_{L_p(I_{k,n})} \ge$$

$$\ge k \min\{1, 2^{1-1/p}\} \left\| (1 + y(\cdot) - h_n\varphi(\cdot))^{r/2+\beta} f^{(r)}(y(\cdot)) \right\|_{L_p(I_{k,n})} - c \left\| w_{\alpha,\beta} \varphi^r f^{(r)} \right\|_p \ge$$

$$\ge k \min\{1, 2^{1-1/p}\} \left\| (1 + y(\cdot) - \varepsilon_n)^{r/2+\beta} f^{(r)}(y(\cdot)) \right\|_{L_p(I_{k,n})} - c \left\| w_{\alpha,\beta} \varphi^r f^{(r)} \right\|_p,$$

where, in the second last inequality, we used the fact that $y'(x) - h_n\varphi'(x) = 1 - kh_n\varphi'(x)/2 \sim 1$ that follows from (2.7), and in the last inequality, we used that $r/2 + \beta < 0$ and that $\varepsilon_n \le h_n\varphi(x)$ for all $x \in [x_n, -x_n]$.

In order to complete the proof, we show that

$$H := \left\| (1 + y(\cdot) - \varepsilon_n)^{r/2+\beta} f^{(r)}(y(\cdot)) \right\|_{L_p(I_{k,n})} = \infty.$$

Assume to the contrary that $H < \infty$. Since

$$y(x_n) = -1 + \varepsilon_n \le y(x) < -1 + \varepsilon_{n-1}, \quad x \in I_{k,n},$$

there is a positive number $a_n \le \varepsilon_n$, such that

$$f^{(r)}(y(x)) = (1 - \varepsilon_n + y(x))^{-r/2-\beta-1/p}, \quad x \in [x_n, x_n + a_n].$$

Therefore,

$$H^p \ge \int_{x_n}^{x_n+a_n} (1 - \varepsilon_n + y(x))^{-1} dx.$$

Using the change of variable $v = u(x) := 1 - \varepsilon_n + y(x)$ and (2.6) we get

$$H^p \ge \frac{2}{3} \int_{x_n}^{x_n+a_n} (u(x))^{-1} u'(x) dx = \frac{2}{3} \int_{0}^{u(x_n+a_n)} \frac{dv}{v} = \infty,$$

that contradicts our assumption $H < \infty$.

Thus, we have found a sequence $\{h_n\}_{n=0}^{\infty}$ of positive numbers, tending to zero, such that $\left\| \mathcal{W}_{kh_n}^{r/2+\alpha, r/2+\beta} \Delta_{h_n\varphi}^k (f^{(r)}, \cdot) \right\|_p = \infty$ for all $n \in \mathbb{N}$. This means that $\omega_{k,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,p} = \infty$ for all $t > 0$.

Theorem 2.1 is proved.

We now state some properties of the Jacobi weights that we need in several proofs below.

Proposition 2.1. *For any $\alpha, \beta \in \mathbb{R}$, $x \in \mathfrak{D}_{2\delta}$ and $u \in [x - \delta\varphi(x)/2, x + \delta\varphi(x)/2]$,*

$$2^{-|\alpha|-|\beta|} w_{\alpha,\beta}(u) \le w_{\alpha,\beta}(x) \le 2^{|\alpha|+|\beta|} w_{\alpha,\beta}(u), \tag{2.8}$$

in particular,

$$\varphi(u)/2 \le \varphi(x) \le 2\varphi(u). \tag{2.9}$$

Also,

$$2^{-|\alpha|-|\beta|} w_{\alpha,\beta}(x) \le \mathcal{W}_\delta^{\alpha,\beta}(x) \le 2^{|\alpha|+|\beta|} w_{\alpha,\beta}(x), \quad x \in \mathfrak{D}_{2\delta}. \tag{2.10}$$

Proof. For $x \in \mathfrak{D}_{2\delta}$ and $u \in [x - \delta\varphi(x)/2, x + \delta\varphi(x)/2]$, we have

$$(1-u)/2 \le (1 - x + \delta\varphi(x)/2)/2 \le 1 - x \le 2(1 - x - \delta\varphi(x)/2) \le 2(1-u)$$

and

$$(1+u)/2 \le (1 + x + \delta\varphi(x)/2)/2 \le 1 + x \le 2(1 + x - \delta\varphi(x)/2) \le 2(1+u).$$

This immediately yields (2.8). Now,

$$\mathcal{W}_\delta^{\alpha,\beta}(x) = w_{\alpha,0}(x + \delta\varphi(x)/2) w_{0,\beta}(x - \delta\varphi(x)/2) \le$$
$$\le 2^{|\alpha|} w_{\alpha,0}(x) 2^{|\beta|} w_{0,\beta}(x) = 2^{|\alpha|+|\beta|} w_{\alpha,\beta}(x)$$

and

$$w_{\alpha,\beta}(x) = w_{\alpha,0}(x) w_{0,\beta}(x) \le 2^{|\alpha|} w_{\alpha,0}(x + \delta\varphi(x)/2) 2^{|\beta|} w_{0,\beta}(x - \delta\varphi(x)/2) =$$
$$= 2^{|\alpha|+|\beta|} \mathcal{W}_\delta^{\alpha,\beta}(x).$$

Proposition 2.1 is proved.

Lemma 2.3. *If $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $r/2 + \alpha \ge 0$, $r/2 + \beta \ge 0$, $0 < p < \infty$, and $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$, then*

$$\lim_{t \to 0+} \omega_{k,r}^\varphi(f^{(r)}, t)_{\alpha,\beta,p} = 0.$$

Proof. Let $\epsilon > 0$. For convenience, denote $C_p := \max\{1, 2^{1/p-1}\}$. Since $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$, there is $\delta > 0$ such that

$$\left\| w_{\alpha,\beta} \varphi^r f^{(r)} \right\|_{L_p([-1,1] \setminus \mathfrak{D}_\delta)} < \frac{\epsilon}{2c_0 C_p},$$

where c_0 is the constant c from the statement of Lemma 2.2. Set

$$g^{(r)}(x) := \begin{cases} f^{(r)}(x), & \text{if} \quad x \in \mathfrak{D}_\delta, \\ 0, & \text{otherwise,} \end{cases}$$

and note that, since $g^{(r)} \in L_p[-1,1]$, there exists $t_0 > 0$ such that

$$\omega_k^\varphi(g^{(r)}, t)_p < \epsilon/(2^{|\alpha-\beta|+1} C_p), \quad 0 < t \le t_0.$$

Using Lemma 2.2 and the fact that, if $r/2+\alpha, r/2+\beta \geq 0$ and $x \in \mathfrak{D}_{kh}$, then $\mathcal{W}_{kh}^{r/2+\alpha,r/2+\beta}(x) \leq \leq 2^{|\alpha-\beta|}$, we have

$$\omega_{k,r}^{\varphi}(f^{(r)},t)_{\alpha,\beta,p} \leq C_p\omega_{k,r}^{\varphi}(g^{(r)},t)_{\alpha,\beta,p} + C_p\omega_{k,r}^{\varphi}(f^{(r)}-g^{(r)},t)_{\alpha,\beta,p} \leq$$

$$\leq 2^{|\alpha-\beta|}C_p\omega_k^{\varphi}(g^{(r)},t)_p + c_0C_p\left\|w_{\alpha,\beta}\varphi^r\left(f^{(r)}-g^{(r)}\right)\right\|_p <$$

$$< \epsilon/2 + c_0C_p\left\|w_{\alpha,\beta}\varphi^r f^{(r)}\right\|_{L_p([-1,1]\setminus\mathfrak{D}_\delta)} \leq \epsilon,$$

if $0 < t \leq t_0$.

Lemma 2.3 is proved.

We now turn our attention to the case $p=\infty$. It is clear that, in order for

$$\lim_{t\to 0+}\omega_{k,r}^{\varphi}(f^{(r)},t)_{\alpha,\beta,\infty} = 0$$

to hold we certainly need that $f \in C^r(-1,1)$, but this condition is not sufficient. If $f \in \mathbb{B}_\infty^r(w_{\alpha,\beta}) \cap \cap rC^r(-1,1)$ and $r/2+\alpha, r/2+\beta \geq 0$, then we can only conclude that $\omega_{k,r}^{\varphi}(f^{(r)},t)_{\alpha,\beta,\infty} < \infty$ for $t>0$. For example, if at least one of $r/2+\alpha$ and $r/2+\beta$ is not zero, and f is such that $f^{(r)}(x) := w_{\alpha,\beta}^{-1}(x)\varphi^{-r}(x)$, $r \in \mathbb{N}_0$, then $f \in \mathbb{B}_\infty^r(w_{\alpha,\beta}) \cap C^r(-1,1)$ and $\omega_{k,r}^{\varphi}(f^{(r)},t)_{\alpha,\beta,\infty} \geq 1$.

Lemma 2.4. *If* $k\in\mathbb{N}$, $r\in\mathbb{N}_0$, $r/2+\alpha\geq 0$, $r/2+\beta\geq 0$, *and* $f \in \mathbb{B}_\infty^r(w_{\alpha,\beta}) \cap C^r(-1,1)$, *then*

$$\lim_{t\to 0}\omega_{k,r}^{\varphi}(f^{(r)},t)_{\alpha,\beta,\infty} = 0 \tag{2.11}$$

if and only if

Case 1. $r/2+\alpha>0$ and $r/2+\beta>0$:

$$\lim_{x\to\pm 1} w_{\alpha,\beta}(x)\varphi^r(x)f^{(r)}(x) = 0. \tag{2.12}$$

Case 2. $r/2+\alpha>0$ and $r/2+\beta=0$:

$$\lim_{x\to 1} w_{\alpha,\beta}(x)\varphi^r(x)f^{(r)}(x) = 0, \qquad \text{and} \quad f^{(r)} \in C[-1,1). \tag{2.13}$$

Case 3. $r/2+\alpha=0$ and $r/2+\beta>0$:

$$\lim_{x\to -1} w_{\alpha,\beta}(x)\varphi^r(x)f^{(r)}(x) = 0, \qquad \text{and} \quad f^{(r)} \in C(-1,1]. \tag{2.14}$$

Case 4. $r/2+\alpha=0$ and $r/2+\beta=0$:

$$f^{(r)} \in C[-1,1]. \tag{2.15}$$

Note that since, for $f \in B_\infty^r(w_{\alpha,\beta})$, $f^{(r)}$ may not be defined at ± 1, when we write $f^{(r)} \in \in C[-1,1)$, for example, we mean that $f^{(r)}$ can be defined at -1 so that it becomes continuous there.

Proof. Since $\omega_{k,r}^{\varphi}(f^{(r)},t)_{\alpha,\beta,\infty} = \omega_{k,0}^{\varphi}(g,t)_{r/2+\alpha,r/2+\beta,\infty}$ with $g := f^{(r)}$, without loss of generality, we may assume that $r=0$ throughout this proof. Note also that Case 4 is trivial since

$\omega^{\varphi}_{k,0}(f,t)_{0,0,\infty} = \omega^k_{\varphi}(f,t)_{\infty}$, the regular DT modulus, tends to 0 as $t \to 0$ if and only if f is uniformly continuous (= continuous) on $[-1,1]$.

We now prove the lemma in Case 2, all other cases being similar.

Given $\varepsilon > 0$, assume that (2.13) holds, and let $\delta = \delta(\varepsilon) \in (0,1)$ be such that

$$w_{\alpha,\beta}(x)|f(x)| < 2^{-k}\varepsilon, \quad x \in [1-\delta, 1).$$

Denote

$$\omega(t) := \omega_k(f,t;[-1,1-\delta/3]),$$

the regular kth modulus of smoothness of f on the interval $[-1,1-\delta/3]$, and note that $\lim_{t\to 0}\omega(t) = 0$ because of the continuity of f on this interval. Thus, there exists $t_0 > 0$ such that $t_0 \le 2\delta/(3k)$ and $\omega(t_0) < \varepsilon/2^{\alpha}$, and we fix $0 < h \le t_0$.

For $x \in \mathfrak{D}_{kh}$, denote $J_x := [x - kh\varphi(x)/2, x + kh\varphi(x)/2] \subseteq [-1,1]$. If $x \le 1 - 2\delta/3$, then $J_x \subseteq [-1, 1-\delta/3]$. Hence,

$$|\mathcal{W}^{\alpha,\beta}_{kh}(x)\Delta^k_{h\varphi(x)}(f,x)| \le 2^{\alpha}|\Delta^k_{h\varphi(x)}(f,x)| < \varepsilon. \tag{2.16}$$

If, on the other hand, $x > 1 - 2\delta/3$, then $J_x \subseteq [1-\delta,1]$. Hence, for some $\theta \in J_x$,

$$|\mathcal{W}^{\alpha,\beta}_{kh}(x)\Delta^k_{h\varphi(x)}(f,x)| \le 2^k\mathcal{W}^{\alpha,\beta}_{kh}(x)|f(\theta)| \le 2^k w_{\alpha,\beta}(\theta)|f(\theta)| < \varepsilon. \tag{2.17}$$

Combining (2.16) and (2.17), we get (2.11).

Conversely, assume that $\alpha > 0$, $\beta = 0$ and (2.11) holds. Observing that $\lim_{t\to 0}\omega_k(f,t;[-1,0]) = 0$, we conclude that f is uniformly continuous on $[-1,0]$, i.e., $f \in C[-1,1)$. Also, given $\varepsilon > 0$, fix $0 < h < 1/(2k)$ such that $\omega^{\varphi}_{k,0}(f,h)_{\alpha,\beta,\infty} < \varepsilon$. Let $x \in (3/4,1)$, and let $\theta \in (1/2, x)$ be such that $\theta + kh\varphi(\theta)/2 = x$. Then

$$|f(x) - \Delta^k_{h\varphi(\theta)}(f,\theta)| \le (2^k - 1)\|f\|_{C[0,1-h^2/4]} =: A_h$$

which yields

$$|w_{\alpha,\beta}(x)f(x)| \le \frac{w_{\alpha,\beta}(x)}{\mathcal{W}^{\alpha,\beta}_{kh}(\theta)}|\mathcal{W}^{\alpha,\beta}_{kh}(\theta)\Delta^k_{h\varphi(\theta)}(f,\theta)| + w_{\alpha,\beta}(x)A_h \le$$

$$\le \omega^{\varphi}_{k,0}(f,h)_{\alpha,\beta,\infty} + w_{\alpha,\beta}(x)A_h.$$

Hence, $\limsup_{x\to 1}|w_{\alpha,\beta}(x)f(x)| \le \varepsilon$, and so $\lim_{x\to 1} w_{\alpha,\beta}(x)(x)f(x) = 0$.

Lemma 2.4 is proved.

3. Proof of the upper estimate in Theorem 1.1. We devote this section to proving that the moduli defined by (1.1) can be estimated from above by the appropriate K-functionals from Definition 1.4.

First, we need the following lemma.

Lemma 3.1. *Let $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $r/2+\alpha \ge 0$, $r/2+\beta \ge 0$, and $1 \le p \le \infty$. If $g \in \mathbb{B}^{r+k}_p(w_{\alpha,\beta})$, then*

$$\omega^{\varphi}_{k,r}(g^{(r)},t)_{\alpha,\beta,p} \le ct^k\left\|w_{\alpha,\beta}\varphi^{k+r}g^{(k+r)}\right\|_p. \tag{3.1}$$

Proof. We follow the lines of the proof of [10] (Lemma 4.1) and rely on the calculations there, modified to accommodate the additional weight $w_{\alpha,\beta}$.

We begin with the well known identity

$$\Delta_h^k(F,x) = \int_{-h/2}^{h/2} \dots \int_{-h/2}^{h/2} F^{(k)}(x+u_1+\dots+u_k)du_1\dots du_k \tag{3.2}$$

and write

$$\omega_{k,r}^{\varphi}(g^{(r)},t)_{\alpha,\beta,p} = \sup_{0<h\le t}\left\|\mathcal{W}_{kh}^{r/2+\alpha,r/2+\beta}\Delta_{h\varphi}^k(g^{(r)},\cdot)\right\|_{L_p(\mathfrak{D}_{kh})} =$$

$$= \sup_{0<h\le t}\left\|\mathcal{W}_{kh}^{r/2+\alpha,r/2+\beta}\int_{-h\varphi/2}^{h\varphi/2}\dots\int_{-h\varphi/2}^{h\varphi/2} g^{(k+r)}(\cdot+u_1+\dots+u_k)du_1\dots du_k\right\|_{L_p(\mathfrak{D}_{kh})} \le$$

$$\le \sup_{0<h\le t}\left\|\int_{-h\varphi/2}^{h\varphi/2}\dots\int_{-h\varphi/2}^{h\varphi/2}(w_{\alpha,\beta}\varphi^r|g^{(k+r)}|)(\cdot+u_1+\dots+u_k)du_1\dots du_k\right\|_{L_p(\mathfrak{D}_{kh})},$$

where, in the last inequality, we used the fact that $r/2+\alpha\ge 0$ and $r/2+\beta\ge 0$ implies

$$\mathcal{W}_{kh}^{r/2+\alpha,r/2+\beta}(x)\le w_{\alpha,\beta}(v)\varphi^r(v), \quad \text{if} \quad x-kh\varphi(x)/2\le v\le x+kh\varphi(x)/2.$$

By Hölder's inequality (with $1/p+1/q=1$), for each $x\in\mathfrak{D}_{kh}$ and $|u|\le(k-1)h\varphi(x)/2$, we have

$$\int_{-h\varphi(x)/2}^{h\varphi(x)/2}(w_{\alpha,\beta}\varphi^r|g^{(k+r)}|)(x+u+u_k)du_k = \int_{x+u-h\varphi(x)/2}^{x+u+h\varphi(x)/2}(w_{\alpha,\beta}\varphi^r|g^{(k+r)}|)(v)dv \le$$

$$\le \|w_{\alpha,\beta}\varphi^{k+r}g^{(k+r)}\|_{L_p(\mathcal{A}(x,u))}\left\|\varphi^{-k}\right\|_{L_q(\mathcal{A}(x,u))}\le$$

$$\le \mathcal{G}_p^{\alpha,\beta}(x;g,k,r)\left\|\varphi^{-k}\right\|_{L_q(\mathcal{A}(x,u))},$$

where

$$\mathcal{A}(x,u) := \left[x+u-\frac{h}{2}\varphi(x), x+u+\frac{h}{2}\varphi(x)\right]$$

and

$$\mathcal{G}_p^{\alpha,\beta}(x;g,k,r) := \left\|w_{\alpha,\beta}\varphi^{k+r}g^{(k+r)}\right\|_{L_p[x-kh\varphi(x)/2,x+kh\varphi(x)/2]}.$$

Thus, the proof is complete, once we show that

$$I(k,p)\le ch^k\left\|w_{\alpha,\beta}\varphi^{k+r}g^{(k+r)}\right\|_p, \tag{3.3}$$

where

$$I(k,p) := \left\| \mathcal{G}_p^{\alpha,\beta}(\cdot; g, k, r)\mathcal{F}_q(\cdot, k) \right\|_{L_p(\mathfrak{D}_{kh})},$$

$$\mathcal{F}_q(x,k) := \int\limits_{-h\varphi(x)/2}^{h\varphi(x)/2} \ldots \int\limits_{-h\varphi(x)/2}^{h\varphi(x)/2} \left\| \varphi^{-k} \right\|_{L_q(\mathcal{A}(x, u_1+\ldots+u_{k-1}))} du_1 \ldots du_{k-1}, \qquad \text{if} \quad k \geq 2,$$

and $\mathcal{F}_q(x,1) := \left\| \varphi^{-1} \right\|_{L_q(\mathcal{A}(x,0))}$.

To this end, we write

$$\|\cdot\|_{L_p(\mathfrak{D}_{kh})} \leq \|\cdot\|_{L_p(\mathfrak{D}_{2kh})} + \|\cdot\|_{L_p((\mathfrak{D}_{kh}\setminus\mathfrak{D}_{2kh})\cap[0,1])} + \|\cdot\|_{L_p((\mathfrak{D}_{kh}\setminus\mathfrak{D}_{2kh})\cap[-1,0])} =:$$

$$=: I_1(p) + I_2(p) + I_3(p).$$

In order to estimate $I_1(p)$, using (2.9), for $x \in \mathfrak{D}_{2kh}$, we have

$$\mathcal{F}_q(x,k) \leq 2^k (h\varphi(x))^{k-1} \varphi^{-k}(x) (h\varphi(x))^{1/q} = 2^k h^{k-1/p} \varphi^{-1/p}(x).$$

Exactly the same sequence of inequalities as in [10, p. 141, 142] with $\varphi^{k+r} g^{(k+r)}$ there replaced by $w_{\alpha,\beta}\varphi^{k+r} g^{(k+r)}$ yields the estimate

$$I_1(p) \leq ch^k \left\| w_{\alpha,\beta}\varphi^{k+r} g^{(k+r)} \right\|_p.$$

We now estimate $I_2(p)$, the estimate of $I_3(p)$ being analogous. Denoting

$$\mathcal{E}_{kh} := (\mathfrak{D}_{kh} \setminus \mathfrak{D}_{2kh}) \cap [0,1]$$

we note that, since $\mathcal{G}_p^{\alpha,\beta}(x; g, k, r) \leq \left\| w_{\alpha,\beta}\varphi^{k+r} g^{(k+r)} \right\|_p$, $x \in \mathfrak{D}_{kh}$, we are done if we show that

$$\left\| \mathcal{F}_q(\cdot, k) \right\|_{L_p(\mathcal{E}_{kh})} \leq ch^k. \tag{3.4}$$

It remains to observe that the estimates

$$\int\limits_{\mathcal{E}_{kh}} (\mathcal{F}_q(x,k))^p \, dx \leq ch^{kp} \qquad \text{and} \qquad \sup_{x\in\mathcal{E}_{kh}} \mathcal{F}_1(x,k) \leq ch^k$$

which are, respectively, inequalities (4.19) and (4.10) from [10], imply the validity of (3.4).

Lemma 3.1 is proved.

Lemma 3.2. *Let $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $r/2 + \alpha \geq 0$, $r/2 + \beta \geq 0$, and $1 \leq p \leq \infty$. If $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$, then*

$$\omega_{k,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,p} \leq cK_{k,r}^{\varphi}(f^{(r)}, t^k)_{\alpha,\beta,p}, \quad t > 0. \tag{3.5}$$

Proof. Take any $g \in \mathbb{B}_p^{r+k}(w_{\alpha,\beta})$. Then, by Lemma 2.1, $g \in \mathbb{B}_p^r(w_{\alpha,\beta})$, and using Lemmas 2.2 and 3.1 we have

$$\omega_{k,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,p} \leq \omega_{k,r}^{\varphi}(f^{(r)} - g^{(r)}, t)_{\alpha,\beta,p} + \omega_{k,r}^{\varphi}(g^{(r)}, t)_{\alpha,\beta,p} \leq$$

$$\leq c \left\| w_{\alpha,\beta}\varphi^r \left(f^{(r)} - g^{(r)} \right) \right\|_p + ct^k \left\| w_{\alpha,\beta}\varphi^{k+r} g^{(k+r)} \right\|_p,$$

which immediately yields (3.5).

Lemma 3.2 is proved.

4. Equivalence of the moduli and Realization functionals and proof of the lower estimate in Theorem 1.1. In this section, using some general results for special classes of doubling and A^* weights, we prove that, for all $0 < p \le \infty$, the $\omega_{k,r}^{\varphi}$ moduli are equivalent to certain Realization functionals. This, in turn, provides lower estimates of $\omega_{k,r}^{\varphi}$ by means of the appropriate K-functionals, thus proving the lower estimate in Theorem 1.1. This, of course, is meaningful only for $1 \le p \le \infty$, as we recall that, for $0 < p < 1$, the K-functionals may vanish while the moduli do not.

For general definitions of doubling weights, A^* weights, $\mathcal{W}(\mathcal{Z})$ and $\mathcal{W}^*(\mathcal{Z})$ see [5, 6]. We only mentioned that the Jacobi weights with nonnegative exponents belong to all of these classes (see [6] (Remark 3.3) and [5] (Example 2.7)). We now restate some definitions from [5, 6], adapting them to the weights $w_{\alpha,\beta}$ with $\alpha, \beta \ge 0$, and state corresponding theorems for these weights only.

Let $\mathcal{Z}_{A,h}^1 := [-1, -1 + Ah^2]$, $\mathcal{Z}_{A,h}^2 := [1 - Ah^2, 1]$ and $\mathcal{I}_{A,h} := [-1 + Ah^2, 1 - Ah^2]$.

The main part weighted modulus of smoothness and the averaged main part weighted modulus are defined, respectively, as

$$\Omega_{\varphi}^k(f, A, t)_{p,w} := \sup_{0<h\le t} \left\| w(\cdot)\Delta_{h\varphi(\cdot)}^k(f, \cdot; \mathcal{I}_{A,h}) \right\|_{L_p(\mathcal{I}_{A,h})}$$

and

$$\widetilde{\Omega}_{\varphi}^k(f, A, t)_{p,w} := \left(\frac{1}{t} \int_0^t \left\| w(\cdot)\Delta_{h\varphi(\cdot)}^k(f, \cdot; \mathcal{I}_{A,h}) \right\|_{L_p(\mathcal{I}_{A,h})}^p dh \right)^{1/p}.$$

The (complete) weighted modulus of smoothness and the (complete) averaged weighted modulus are defined as

$$\omega_{\varphi}^k(f, A, t)_{p,w} := \Omega_{\varphi}^k(f, A, t)_{p,w} + \sum_{j=1}^{2} E_k(f, \mathcal{Z}_{2A,t}^j)_{w,p}$$

and

$$\widetilde{\omega}_{\varphi}^k(f, A, t)_{p,w} := \widetilde{\Omega}_{\varphi}^k(f, A, t)_{p,w} + \sum_{j=1}^{2} E_k(f, \mathcal{Z}_{2A,t}^j)_{w,p},$$

respectively.

The following is an immediate corollary of [5] (Theorem 5.2) in the case $0 < p < \infty$ and [6] (Theorem 6.1) if $p = \infty$.

Theorem 4.1. *Let $k, \nu_0 \in \mathbb{N}$, $\nu_0 \ge k$, $0 < p \le \infty$, $\alpha \ge 0$, $\beta \ge 0$, $A > 0$, and $f \in L_p^{\alpha,\beta}$. Then, there exists $N \in \mathbb{N}$ depending on k, ν_0, p, α and β, such that for every $n \ge N$ and $\vartheta > 0$, there is a polynomial $P_n \in \mathbb{P}_n$ satisfying*

$$\|w_{\alpha,\beta}(f - P_n)\|_p \le c\widetilde{\omega}_{\varphi}^k(f, A, \vartheta/n)_{p,w_{\alpha,\beta}} \le c\omega_{\varphi}^k(f, A, \vartheta/n)_{p,w_{\alpha,\beta}}$$

and

$$n^{-\nu} \left\| w_{\alpha,\beta}\varphi^{\nu} P_n^{(\nu)} \right\|_p \le c\widetilde{\omega}_{\varphi}^k(f, A, \vartheta/n)_{p,w_{\alpha,\beta}} \le c\omega_{\varphi}^k(f, A, \vartheta/n)_{p,w_{\alpha,\beta}}, \quad k \le \nu \le \nu_0,$$

where constants c depend only on k, ν_0, p, A, α, β and ϑ.

The following theorem is proved in [11].

Theorem 4.2. *Let $k \in \mathbb{N}$, $\alpha \ge 0$, $\beta \ge 0$, $A > 0$, $0 < p \le \infty$, and $f \in L_p^{\alpha,\beta}$. Then, for any $0 < t \le \sqrt{2/A}$, we have*

$$E_k(f, \mathcal{Z}_{A,t})_{w_{\alpha,\beta},p} \le c\omega_{k,0}^{*\varphi}(f,t)_{\alpha,\beta,p} \le c\omega_{k,0}^{\varphi}(f,t)_{\alpha,\beta,p}, \tag{4.1}$$

where the interval $\mathcal{Z}_{A,t}$ is either $[1-At^2, 1]$ or $[-1, -1+At^2]$, and c depends only on k, p, α, β, and A.

In particular, if $A = 2$ and $t = 1$, then

$$E_k(f)_{w_{\alpha,\beta},p} \le c\omega_{k,0}^{*\varphi}(f,1)_{\alpha,\beta,p} \le c\omega_{k,0}^{\varphi}(f,1)_{\alpha,\beta,p}. \tag{4.2}$$

We now show that the moduli $\omega_\varphi^k(f,A,t)_{p,w_{\alpha,\beta}}$ and $\widetilde{\omega}_\varphi^k(f,A,t)_{p,w_{\alpha,\beta}}$ may be estimated from above by the moduli $\omega_{k,0}^{\varphi}(f,t)_{\alpha,\beta,p}$ and $\omega_{k,0}^{*\varphi}(f,t)_{\alpha,\beta,p}$, respectively.

Lemma 4.1. *Let $k \in \mathbb{N}$, $\alpha \ge 0$, $\beta \ge 0$, $A \ge 2k^2$ and $f \in L_p^{\alpha,\beta}$, $0 < p \le \infty$. Then, for $0 < t \le 1/\sqrt{A}$,*

$$\omega_\varphi^k(f,A,t)_{p,w_{\alpha,\beta}} \le c\omega_{k,0}^{\varphi}(f,t)_{\alpha,\beta,p}$$

and

$$\widetilde{\omega}_\varphi^k(f,A,t)_{p,w_{\alpha,\beta}} \le c\omega_{k,0}^{*\varphi}(f,t)_{\alpha,\beta,p},$$

where constants c depend only on k, p, α, β and A.

Proof. Recall that $\mathcal{I}_{A,h} = [-1+Ah^2, 1-Ah^2]$ and note that, if $A \ge 2k^2$, then

$$\mathcal{I}_{A,h} \subseteq \mathfrak{D}_{2kh} \subset \mathfrak{D}_{kh} \qquad \text{for all} \quad h > 0.$$

Since, by Proposition 2.1, $w_{\alpha,\beta}(x) \sim \mathcal{W}_{kh}^{\alpha,\beta}(x)$, $x \in \mathfrak{D}_{2kh}$, we have

$$\left\| w_{\alpha,\beta}(\cdot)\Delta_{h\varphi(\cdot)}^k(f,\cdot;\mathcal{I}_{A,h}) \right\|_{L_p(\mathcal{I}_{A,h})} \le c \left\| \mathcal{W}_{kh}^{\alpha,\beta}(\cdot)\Delta_{h\varphi(\cdot)}^k(f,\cdot) \right\|_{L_p(\mathfrak{D}_{kh})},$$

so that

$$\Omega_\varphi^k(f,A,t)_{p,w_{\alpha,\beta}} \le c\omega_{k,0}^{\varphi}(f,t)_{\alpha,\beta,p}$$

and

$$\widetilde{\Omega}_\varphi^k(f,A,t)_{p,w_{\alpha,\beta}} \le c\omega_{k,0}^{*\varphi}(f,t)_{\alpha,\beta,p}.$$

Now, Theorem 4.2 yields that, for $0 < t \le 1/\sqrt{A}$,

$$\max\left\{E_k(f,[1-2At^2,1])_{w_{\alpha,\beta},p}, E_k(f,[-1,-1+2At^2])_{w_{\alpha,\beta},p}\right\} \le$$

$$\le c\omega_{k,0}^{*\varphi}(f,t)_{\alpha,\beta,p} \le c\omega_{k,0}^{\varphi}(f,t)_{\alpha,\beta,p}.$$

Lemma 4.1 is proved.

The following is an immediate corollary of Theorem 4.1 and Lemma 4.1.

***Corollary* 4.1.** *Let $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $r/2+\alpha \ge 0$, $r/2+\beta \ge 0$, and $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$, $0 < p \le \infty$. Then, there exists $N \in \mathbb{N}$ depending on k, r, p, α and β, such that for every $n \ge N$ and $0 < \vartheta \le 1$, there is a polynomial $P_n \in \mathbb{P}_n$ satisfying*

$$\left\| w_{\alpha,\beta}\varphi^r(f^{(r)} - P_n^{(r)}) \right\|_p \le c\omega_{k,r}^{*\varphi}(f^{(r)}, \vartheta/n)_{\alpha,\beta,p} \le c\omega_{k,r}^{\varphi}(f^{(r)}, \vartheta/n)_{\alpha,\beta,p}$$

and

$$n^{-k}\left\| w_{\alpha,\beta}\varphi^{k+r}P_n^{(k+r)} \right\|_p \le c\omega_{k,r}^{*\varphi}(f^{(r)}, \vartheta/n)_{\alpha,\beta,p} \le c\omega_{k,r}^{\varphi}(f^{(r)}, \vartheta/n)_{\alpha,\beta,p},$$

where constants c depend only on k, r, p, α, β and ϑ.

Suppose now that $0 < t \le 2/k$, and $n \in \mathbb{N}$ is such that $n \ge N$ and $c_1/t \le n \le c_2/t$. Then, denoting $\mu := \max\{1, c_2\}$, Corollary 4.1 with $\vartheta = \min\{1, c_1\}$ implies that

$$K_{k,r}^{\varphi}(f^{(r)}, t^k)_{\alpha,\beta,p} \le \mu^k K_{k,r}^{\varphi}(f^{(r)}, (t/\mu)^k)_{\alpha,\beta,p} \le \mu^k R_{k,r}^{\varphi}(f^{(r)}, n^{-k})_{\alpha,\beta,p} \le$$
$$\le c\omega_{k,r}^{*\varphi}(f^{(r)}, \vartheta/n)_{\alpha,\beta,p} \le c\omega_{k,r}^{*\varphi}(f^{(r)}, t)_{\alpha,\beta,p} \le c\omega_{k,r}^{\varphi}(f^{(r)}, t)_{\alpha,\beta,p}. \tag{4.3}$$

Note that (4.3) is valid for all $0 < p \le \infty$. However, we remind the reader that, for $0 < p < 1$, the K-functional may become identically equal to zero.

Together with Lemma 3.2, the sequence of estimates (4.3) immediately yields Theorem 1.1.

We now show that the estimates in Lemma 4.1 may be reversed in some sense, i.e., there exists $0 < \theta \le 1$ such that moduli $\omega_{k,0}^{\varphi}(f, \theta t)_{\alpha,\beta,p}$ and $\omega_{k,0}^{*\varphi}(f, \theta t)_{\alpha,\beta,p}$ may be estimated from above, respectively, by $\omega_{\varphi}^{k}(f, A, t)_{p,w_{\alpha,\beta}}$ and $\widetilde{\omega}_{\varphi}^{k}(f, A, t)_{p,w_{\alpha,\beta}}$.

Lemma 4.2. *Let $k \in \mathbb{N}$, $\alpha \ge 0$, $\beta \ge 0$, $A > 0$ and $f \in L_p^{\alpha,\beta}$, $0 < p \le \infty$. Then, there exists $0 < \theta \le 1$ depending only on k and A, such that for all $0 < t \le \sqrt{1/A}$,*

$$\omega_{k,0}^{\varphi}(f, \theta t)_{\alpha,\beta,p} \le c\omega_{\varphi}^{k}(f, A, t)_{p,w_{\alpha,\beta}} \tag{4.4}$$

and

$$\omega_{k,0}^{*\varphi}(f, \theta t)_{\alpha,\beta,p} \le c\widetilde{\omega}_{\varphi}^{k}(f, A, t)_{p,w_{\alpha,\beta}}, \tag{4.5}$$

where constants c depend only on k, p, α, β and A.

Proof. Let $B := \max\{A^2, 4k^2\}$, $\theta := \min\left\{1, \sqrt{A/(kB)}\right\}$, $0 < t \le \sqrt{1/A}$ and $0 < h \le \theta t$. Note that $h \le \sqrt{1/B}$ and, if $x \in \mathcal{I}_{B,h}$, then $x \pm kh\varphi(x)/2 \in \mathcal{I}_{A,h}$. Also, $\mathcal{I}_{B,h} \subset \mathfrak{D}_{2kh}$, and so Proposition 2.1 implies that $w_{\alpha,\beta}(x) \sim \mathcal{W}_{kh}^{\alpha,\beta}(x)$, for all $x \in \mathcal{I}_{B,h}$. Hence,

$$\left\| \mathcal{W}_{kh}^{\alpha,\beta}(\cdot)\Delta_{h\varphi(\cdot)}^{k}(f, \cdot) \right\|_{L_p(\mathcal{I}_{B,h})} \le c \left\| w_{\alpha,\beta}(\cdot)\Delta_{h\varphi(\cdot)}^{k}(f, \cdot; \mathcal{I}_{A,h}) \right\|_{L_p(\mathcal{I}_{A,h})}. \tag{4.6}$$

Now, let $S_1 := [0,1] \cap (\mathfrak{D}_{kh} \setminus \mathcal{I}_{B,h})$. Then, denoting $x_0 = 1 - Bh^2$, we have

$$\widetilde{S}_1 := \bigcup_{x \in S_1} [x - kh\varphi(x)/2, x + kh\varphi(x)/2] = [x_0 - kh\varphi(x_0)/2, 1] \subset [1 - 2At^2, 1].$$

It now follows by Remark 2.3 that, for a polynomial of best weighted approximation $p_k \in \mathbb{P}_k$ to f on $[1 - 2At^2, 1]$,

$$\left\| \mathcal{W}_{kh}^{\alpha,\beta}(\cdot)\Delta_{h\varphi(\cdot)}^{k}(f, \cdot) \right\|_{L_p(S_1)} \le c \left\| w_{\alpha,\beta}(f - p_k) \right\|_{L_p(\widetilde{S}_1)} \le$$
$$\le cE_k(f, [1 - 2At^2, 1])_{w_{\alpha,\beta},p}, \tag{4.7}$$

where we used the fact that any kth difference of p_k is identically zero.

Similarly, for $S_2 := [-1, 0] \cap (\mathfrak{D}_{kh} \setminus \mathcal{I}_{B,h})$ and

$$\widetilde{S}_2 := \bigcup_{x \in S_2} [x - kh\varphi(x)/2, x + kh\varphi(x)/2] \subset [-1, -1 + 2At^2],$$

we have

$$\left\| \mathcal{W}_{kh}^{\alpha,\beta}(\cdot)\Delta_{h\varphi(\cdot)}^{k}(f,\cdot) \right\|_{L_p(S_2)} \le cE_k(f,[-1,-1+2At^2])_{w_{\alpha,\beta},p}. \tag{4.8}$$

Therefore, noting that $\mathfrak{D}_{kh} = \mathcal{I}_{B,h} \cup S_1 \cup S_2$ and combining (4.6) through (4.8), we have, for all $0 < h \le \theta t$,

$$\left\| \mathcal{W}_{kh}^{\alpha,\beta}(\cdot)\Delta_{h\varphi(\cdot)}^{k}(f,\cdot) \right\|_{L_p(\mathfrak{D}_{kh})} \le c \left\| w_{\alpha,\beta}(\cdot)\Delta_{h\varphi(\cdot)}^{k}(f,\cdot;\mathcal{I}_{A,h}) \right\|_{L_p(\mathcal{I}_{A,h})} + c\sum_{j=1}^{2} E_k(f,\mathcal{Z}_{2A,t}^{j})_{w,p}.$$

Estimates (4.4) and (4.5) now follow, respectively, by taking supremum and by integrating with respect to h over $(0,\theta t]$, and using the fact that $\theta \le 1$.

Lemma 4.2 is proved.

Using Lemmas 4.1 and 4.2 we immediately get Theorem 1.2 as a corollary of the following result that follows from [5] (Corollary 11.2).

Theorem 4.3. *Let $k \in \mathbb{N}$, $0 < p < 1$, $A > 0$, $\alpha \ge 0$, $\beta \ge 0$, and $f \in L_p^{\alpha,\beta}$. Then there exist $N \in \mathbb{N}$ depending on k, p, α and β, and $\vartheta > 0$ depending on k, p, A, α and β, such that, for any $\vartheta_1 \in (0,\vartheta]$, $n \ge N$, $\vartheta_1/n \le t \le \vartheta/n$, we have*

$$R_{k,0}^{\varphi}(f,n^{-k})_{\alpha,\beta,p} \sim \widetilde{\omega}_{\varphi}^{k}(f,A,t)_{p,w_{\alpha,\beta}} \sim \omega_{\varphi}^{k}(f,A,t)_{p,w_{\alpha,\beta}}.$$

5. Weighted DT moduli and alternative proof of the lower estimate via K-functionals. In this section, we provide an alternative proof, in the case $1 \le p \le \infty$, of the lower estimate of the moduli $\omega_{k,r}^{\varphi}(f^{(r)},t)_{\alpha,\beta,p}$ and $\omega_{k,r}^{*\varphi}(f^{(r)},t)_{\alpha,\beta,p}$ by appropriate K-functionals, using certain weighted DT moduli.

We denote the kth forward and the kth backward differences by $\overrightarrow{\Delta}_h^k(f,x) := \Delta_h^k(f,x+kh/2)$ and $\overleftarrow{\Delta}_h^k(f,x) := \Delta_h^k(f,x-kh/2)$, respectively.

Adapting the weighted DT moduli which were defined in [1] ((8.2.10)) for a weight w on $D := [-1,1]$, we set for $f \in L_{w,p}$,

$$\begin{aligned} \omega_{\varphi}^{k}(f,t)_{w,p} := \sup_{0<h\le t} \left\| w(\cdot)\Delta_{h\varphi}^{k}(f,\cdot) \right\|_{L_p[-1+t^*,1-t^*]} + \\ + \sup_{0<h\le t^*} \left\| w(\cdot)\overrightarrow{\Delta}_h^k(f,\cdot) \right\|_{L_p[-1,-1+12t^*]} + \\ + \sup_{0<h\le t^*} \left\| w(\cdot)\overleftarrow{\Delta}_h^k(f,\cdot) \right\|_{L_p[1-12t^*,1]}, \end{aligned} \tag{5.1}$$

where $t^* := 2k^2t^2$. The first term on the right in the above equation is called the main-part modulus and denoted by $\Omega_{\varphi}^{k}(f,t)_{w,p}$. Obviously, we have $\Omega_{\varphi}^{k}(f,t)_{w,p} \le \omega_{\varphi}^{k}(f,t)_{w,p}$.

Next, the weighted K-functional was defined in [1, p. 55] ((6.1.1)) as

$$K_{k,\varphi}(f,t^k)_{w,p} := \inf_{g \in \mathbb{B}_p^k(w)} \{\|w(f-g)\|_p + t^k\|w\varphi^k g^{(k)}\|_p\},$$

and we note that

$$K_{k,\varphi}(f,t^k)_{w_{\alpha,\beta},p} = K_{k,0}^{\varphi}(f,t^k)_{\alpha,\beta,p}.$$

It was shown in [1] (Theorem 6.1.1) that, given an appropriate weight w (all Jacobi weights with *nonnegative* exponents are included), the weighted K-functional is equivalent to the weighted DT modulus of f. Namely, by [1] (Theorem 6.1.1), for $1 \le p \le \infty$,

$$M^{-1}\omega_\varphi^k(f,t)_{w,p} \le K_{k,\varphi}(f,t^k)_{w,p} \le M\omega_\varphi^k(f,t)_{w,p}, \quad 0 < t \le t_0,$$

where t_0 is some sufficiently small constant. Hence, in particular, if $\alpha, \beta \ge 0$, then

$$\omega_\varphi^k(f,t)_{w_{\alpha,\beta},p} \sim K_{k,\varphi}(f,t^k)_{w_{\alpha,\beta},p}, \quad 0 < t \le t_0. \tag{5.2}$$

Note that, if $\alpha < 0$ or $\beta < 0$, then there are functions f in $L_p^{\alpha,\beta}$ for which $\omega_\varphi^k(f,\delta)_{w_{\alpha,\beta},p} = \infty$. Indeed, the following example was given in [4] (see also [1, p. 56] (Remark 6.1.2)) and, in fact, it was the starting point for our counterexample in Theorem 2.1. Suppose that $1 \le p < \infty$ and that $\delta > 0$ is fixed. If $f(x) := (x+1-\varepsilon)^{-\beta-1/p}\chi_{[-1+\varepsilon,-1+2\varepsilon]}(x)$ with $\beta < 0$ and $0 < \varepsilon < t^*$, then $\|w_{\alpha,\beta}f\|_p \le c(\alpha,\beta,p)$ (and so $f \in L_p^{\alpha,\beta}$), $\|w_{\alpha,\beta}(\cdot)f(\cdot+\varepsilon)\|_{L_p[-1,-1+12t^*]} = \infty$, and $\|w_{\alpha,\beta}(\cdot)f(\cdot+i\varepsilon)\|_p = 0$, $2 \le i \le k$, and therefore

$$\sup_{0<h\le t^*}\left\|w_{\alpha,\beta}(\cdot)\overrightarrow{\Delta}_h^k(f,\cdot)\right\|_{L_p[-1,-1+12t^*]} \ge \left\|w_{\alpha,\beta}(\cdot)\overrightarrow{\Delta}_\varepsilon^k(f,\cdot)\right\|_{L_p[-1,-1+12t^*]} =$$

$$= \|w_{\alpha,\beta}(\cdot)\left[f(\cdot) - kf(\cdot+\varepsilon)\right]\|_{L_p[-1,-1+12t^*]} = \infty.$$

Also, if $f \in L_p^{\alpha,\beta}$ then (choosing $g \equiv 0$) we have $K_{k,\varphi}(f,t^k)_{w_{\alpha,\beta},p} \le \|w_{\alpha,\beta}f\|_p < \infty$. Hence, (5.2) is not valid if $\alpha < 0$ or $\beta < 0$ (see also Theorem 2.1 with $r = 0$).

An equivalent averaged weighted DT modulus

$$\omega_\varphi^{*k}(f,t)_{w,p} := \left(\frac{1}{t}\int_0^t \int_{-1+t^*}^{1-t^*} |w(x)\Delta_{\tau\varphi(x)}^k(f,x)|^p\,dx\,d\tau\right)^{1/p} +$$

$$+\left(\frac{1}{t^*}\int_0^{t^*}\int_{-1}^{-1+At^*} |w(x)\overrightarrow{\Delta}_u^k(f,x)|^p\,dx\,du\right)^{1/p} +$$

$$+\left(\frac{1}{t^*}\int_0^{t^*}\int_{1-At^*}^{1} |w(x)\overleftarrow{\Delta}_u^k(f,x)|^p\,dx\,du\right)^{1/p}, \tag{5.3}$$

where $1 \le p < \infty$, $t^* := 2k^2t^2$, and A is some sufficiently large absolute constant, was defined in [1] ((6.1.9)). For $p = \infty$, set $\omega_\varphi^{*k}(f,t)_{w,\infty} := \omega_\varphi^k(f,t)_{w,\infty}$. It was shown in [1, p. 57] that, for an appropriate weight w (again, all Jacobi weights with nonnegative exponents are included), $1 \le p \le \infty$ and sufficiently small $t_0 > 0$,

$$K_{k,\varphi}(f,t^k)_{w,p} \le c\omega_\varphi^{*k}(f,t)_{w,p}, \quad 0 < t \le t_0. \tag{5.4}$$

We now provide an alternative proof of the inverse estimate to (3.5) independent of the results in Section 4. First, we need the following lemma.

Lemma 5.1. *Let* $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $r/2+\alpha \ge 0$, $r/2+\beta \ge 0$, $1 \le p < \infty$, *and* $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$. *Then*

$$\omega_\varphi^{*k}(f^{(r)},t)_{w_{\alpha,\beta}\varphi^r,p} \le c(k,r,\alpha,\beta)\omega_{k,r}^{*\varphi}(f^{(r)},c(k)t)_{\alpha,\beta,p}, \quad 0<t\le c(k).$$

Proof. The proof of this lemma is very similar to that of Lemma 6.1 in [10], but we still provide all details here for completeness. The three terms in the definition (5.3) are to be estimated separately, but the second and third are similar, so we will estimate the first two. Since $\omega_\varphi^{*k}(f^{(r)},t)_{w_{\alpha,\beta}\varphi^r,p} = \omega_\varphi^{*k}(g,t)_{w_{r/2+\alpha,r/2+\beta},p}$ and $\omega_{k,r}^{*\varphi}(f^{(r)},t)_{\alpha,\beta,p} = \omega_{k,0}^{*\varphi}(g,t)_{r/2+\alpha,r/2+\beta,p}$ with $g := f^{(r)}$, without loss of generality, we may assume that $r=0$ throughout this proof.

Note that $t^* = 2k^2t^2$ implies that $[-1+t^*, 1-t^*] \subset \mathfrak{D}_{2kt} \subset \mathfrak{D}_{2k\tau}$, $0 \le \tau \le t$, so that by (2.10) we have

$$\frac{1}{t}\int_0^t \int_{-1+t^*}^{1-t^*} |w_{\alpha,\beta}(x)\Delta_{\tau\varphi(x)}^k(f,x)|^p\,dx\,d\tau \le$$

$$\le \frac{2^{(\alpha+\beta)p}}{t}\int_0^t \int_{\mathfrak{D}_{2k\tau}} |\mathcal{W}_{k\tau}^{\alpha,\beta}(x)\Delta_{\tau\varphi(x)}^k(f,x)|^p\,dx\,d\tau \le$$

$$\le 2^{(\alpha+\beta)p}\omega_{k,0}^{*\varphi}(f,t)_{\alpha,\beta,p}^p.$$

In order to estimate the second term we follow the proof of [10] (Lemma 6.1) and assume that $t \le (2k\sqrt{A+k/2})^{-1}$. Then

$$\frac{1}{t^*}\int_0^{t^*}\int_{-1}^{-1+At^*} |w_{\alpha,\beta}(x)\overrightarrow{\Delta}_u^k(f,x)|^p\,dx\,du =$$

$$= \frac{1}{t^*}\int_0^{t^*}\int_{-1}^{-1+At^*} |w_{\alpha,\beta}(x)\Delta_u^k(f,x+ku/2)|^p\,dx\,du \le$$

$$\le \frac{1}{t^*}\int_0^{t^*}\int_{-1+ku/2}^{-1+(A+k/2)t^*} |w_{\alpha,\beta}(y-ku/2)\Delta_u^k(f,y)|^p\,dy\,du \le$$

$$\le \frac{1}{t^*}\int_{-1}^{-1+(A+k/2)t^*}\int_0^{2(y+1)/k} |w_{\alpha,\beta}(y-ku/2)\Delta_u^k(f,y)|^p\,du\,dy =$$

$$= \frac{1}{t^*}\int_{-1}^{-1+(A+k/2)t^*}\int_0^{2(y+1)/(k\varphi(y))} \varphi(y)|w_{\alpha,\beta}(y-kh\varphi(y)/2)\Delta_{h\varphi(y)}^k(f,y)|^p\,dh\,dy \le$$

$$\le c\frac{1}{t^*}\int_{-1}^{-1+(A+k/2)t^*}\int_0^{2(y+1)/(k\varphi(y))} \varphi(y)|\mathcal{W}_{kh}^{\alpha,\beta}(y)\Delta_{h\varphi(y)}^k(f,y)|^p\,dh\,dy \le$$

$$\le c\frac{1}{\sqrt{t^*}} \int\limits_{-1}^{-1+(A+k/2)t^*} \int\limits_{0}^{2(y+1)/(k\varphi(y))} |\mathcal{W}_{kh}^{\alpha,\beta}(y)\Delta_{h\varphi(y)}^k(f,y)|^p \, dh\, dy \le$$

$$\le c\frac{1}{\sqrt{t^*}} \int\limits_{0}^{c\sqrt{t^*}} \int\limits_{\mathfrak{D}_{kh}\cap[-1,-1+(A+k/2)t^*]} |\mathcal{W}_{kh}^{\alpha,\beta}(y)\Delta_{h\varphi(y)}^k(f,y)|^p \, dy\, dh \le$$

$$\le c\omega_{k,0}^{*\varphi}(f, c(k)t)_{\alpha,\beta,p}^p,$$

where for the third inequality we used the fact that, for $y \le -1/2$ and $0 \le h \le 2(y+1)/(k\varphi(y))$,

$$1 - y + kh\varphi(y)/2 \le 2\,(1 - y - kh\varphi(y)/2)\,,$$

and so

$$w_{\alpha,\beta}(y - kh\varphi(y)/2) \le 2^\alpha \mathcal{W}_{kh}^{\alpha,\beta}(y).$$

Lemma 5.1 is proved.

A similar proof yields (see [10], Lemma 6.2) an analogous result in the case $p = \infty$.

Lemma 5.2. *Let $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $r/2 + \alpha \ge 0$, $r/2 + \beta \ge 0$, and $f \in \mathbb{B}_\infty^r(w_{\alpha,\beta})$. Then*

$$\omega_\varphi^k(f^{(r)}, t)_{w_{\alpha,\beta}\varphi^r,\infty} \le c(k, r, \alpha, \beta)\omega_{k,r}^\varphi\big(f^{(r)}, c(k)t\big)_{\alpha,\beta,\infty}, \quad 0 < t \le c(k).$$

We are now ready to prove the inverse of the estimate (3.5).

Lemma 5.3. *Let $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $r/2+\alpha \ge 0$, $r/2+\beta \ge 0$, and $1 \le p \le \infty$. If $f \in \mathbb{B}_p^r(w_{\alpha,\beta})$, then*

$$K_{k,r}^\varphi(f^{(r)}, t^k)_{\alpha,\beta,p} \le c\omega_{k,r}^{*\varphi}(f^{(r)}, t)_{\alpha,\beta,p} \le c\omega_{k,r}^\varphi(f^{(r)}, t)_{\alpha,\beta,p}, \quad 0 < t \le 2/k. \tag{5.5}$$

Proof. Combining (5.4) with the weight $w = w_{\alpha,\beta}\varphi^r$ with Lemmas 5.1 and 5.2, we obtain, for $1 \le p \le \infty$,

$$K_{k,r}^\varphi(f^{(r)}, t^k)_{\alpha,\beta,p} = K_{k,\varphi}(f^{(r)}, t^k)_{w_{\alpha,\beta}\varphi^r,p} \le$$

$$\le c\omega_\varphi^{*k}(f^{(r)}, t)_{w_{\alpha,\beta}\varphi^r,p} \le c\omega_{k,r}^{*\varphi}(f^{(r)}, c(k)t)_{\alpha,\beta,p}\,, \quad 0 < t \le c\,.$$

Hence, we have

$$K_{k,r}^\varphi(f^{(r)}, t^k)_{\alpha,\beta,p} \le c\omega_{k,r}^{*\varphi}(f^{(r)}, c_1 t)_{\alpha,\beta,p}, \quad 0 < t \le c_2, \tag{5.6}$$

where c_1 and c_2 are some positive constants that may depend only on k.

Suppose now that $0 < t \le 2/k$. Then, denoting $\mu := \max\{1, c_1, 2/(kc_2)\}$ and using (5.6) we obtain

$$K_{k,r}^\varphi(f^{(r)}, t^k)_{\alpha,\beta,p} \le \mu^k K_{k,r}^\varphi(f^{(r)}, (t/\mu)^k)_{\alpha,\beta,p} \le$$

$$\le c\omega_{k,r}^{*\varphi}(f^{(r)}, c_1 t/\mu)_{\alpha,\beta,p} \le c\omega_{k,r}^{*\varphi}(f^{(r)}, t)_{\alpha,\beta,p}\,,$$

which is the first inequality in (5.5). Finally, the second inequality in (5.5) follows from (1.4).

Lemma 5.3 is proved.

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