KOLMOGOROV AND LINEAR WIDTHS OF WEIGHTED SOBOLEV–TYPE CLASSES ON A FINITE INTERVAL

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Abstract. Let I be a finite interval, $r \in \mathbb{N}$ and $\rho(t) = \operatorname{dist}\{t, \partial I\}, t \in I$. Denote by $W^r_{p,\alpha}$, $0 < \alpha < \infty$, the class of functions x on I with the seminorm $\|x^{(r)}\rho^\alpha\|_{L_p} \le 1$. We obtain two-sided estimates of the Kolmogorov widths $d_n\left(W^r_{p,\alpha}\right)_{L_q}$ and of the linear widths $d_n\left(W^r_{p,\alpha}\right)^{lin}_{L_q}$.

§1. Introduction and statement of the main results

Let X be a real linear space of vectors x with norm $\|x\|_X$, $W \subset X, W \neq \emptyset$, and L^n, a subspace in X of dimension $\dim L^n \le n$, $n \ge 0$. Let $M^n = M^n(x^0) := x^0 + L^n$ be a shift of the subspace L^n by an arbitrary vector $x^0 \in X$. Let

$$E(x, M^n)_X := \inf_{y \in M^n} \|x - y\|_X,$$

denote the best approximation of the vector $x \in X$ by M^n, and let

$$E(W, M^n)_X := \sup_{x \in W} E(x, M^n)_X, \tag{1.1}$$

denote the distance between the sets W and M^n.

The Kolmogorov n-width of W is defined by

$$d_n(W)_X := \inf_{M^n} E(W, M^n)_X, \quad n \ge 0. \tag{1.2}$$

We also let $\Lambda(X, L^n)$ be the set of all linear maps $\Lambda : X \to L^n$. Then

$$E(W, L^n)^{lin}_X := \inf_{\Lambda \in \Lambda(X, L^n)} \sup_{x \in W} \|x - \Lambda x\|_X$$

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denotes the best linear approximation of the set W by L^n. The linear n-width of W is defined by

$$(1.3) \qquad d_n(W)_X^{lin} := \inf_{L^n \subset X} E(W, L^n)_X^{lin}, \quad n \geq 0.$$

Let I be a finite interval in $\mathbb{R}$, and let $r \in \mathbb{N}$ and $0 \leq \alpha < \infty$. For $1 \leq p \leq \infty$, and $\rho(t) := \operatorname{dist}\{t, \partial I\}$, $t \in I$, we denote

$$W_{p,\alpha}^r := W_{p,\alpha}^r(I) := \{x : I \to \mathbb{R} \mid x^{(r-1)} \in AC_{loc}(I), \|x^{(r)}\rho^\alpha\|_{L_p(I)} \leq 1\}.$$

If $\alpha = 0$, then we write $W_p^r := W_p^r(I) := W_{p,0}^r(I)$. We also write L_q for $L_q(I)$.

The following two theorems are well known.

Theorem A. *Let $r \in \mathbb{N}$, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$ be so that $r - \frac{1}{p} + \frac{1}{q} > 0$. If $(r,p) \neq (1,1)$, and if $(r,p) = (1,1)$ and $1 \leq q \leq 2$, then*

$$d_n(W_p^r)_{L_q} \asymp n^{-r+(\max\{\frac{1}{p},\frac{1}{2}\}-\max\{\frac{1}{q},\frac{1}{2}\})_+}, \quad n \geq r,$$

where $(u)_+ := \max\{u, 0\}$ and $a_n \asymp b_n$ means that there exist two constants $0 < C_1 < C_2$, such that $C_1 a_n \leq b_n \leq C_2 a_n$, $\forall n$. If on the other hand, $(r,p) = (1,1)$ and $2 < q < \infty$, then

$$c_1 n^{-\frac{1}{2}} \leq d_n(W_1^1)_{L_q} \leq c_2 n^{-\frac{1}{2}} \big(\log(n+1)\big)^{\frac{3}{2}}, \quad n \geq 1,$$

where $c_1 > 0$ and c_2 do not depend on n.

The exact order of the Kolmogorov widths of the classes W_1^1 in L_q, $2 < q < \infty$, are unknown.

Theorem B. *Let $r \in \mathbb{N}$, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$ be so that $r - \frac{1}{p} + \frac{1}{q} > 0$. If $(r,p) \neq (1,1)$, or if $(r,p) = (1,1)$ and $1 \leq q \leq 2$, and if $(r,q) = (1,\infty)$ and $2 \leq p \leq \infty$, then*

$$d_n(W_p^r)_{L_q}^{lin} \asymp n^{-r+(\frac{1}{p}-\frac{1}{q})_+ - \min\{(\frac{1}{p}-\frac{1}{2})_+,(\frac{1}{2}-\frac{1}{q})_+\}}, \quad n \geq r.$$

If on the other hand, $(r,p) = (1,1)$ and $2 < q < \infty$, then

$$c_1 n^{-\frac{1}{2}} \leq d_n(W_1^1)_{L_q}^{lin} \leq c_2 n^{-\frac{1}{2}} \big(\log(n+1)\big)^{\frac{3}{2}}, \quad n \geq 1,$$

and if $(r,q)=(1,\infty)$ *and* $1<p<2$*, then*

$$c_1 n^{-\frac{1}{2}} \le d_n(W_p^1)_{L_\infty}^{lin} \le c_2 n^{-\frac{1}{2}}\big(\log(n+1)\big)^{\frac{3}{2}}, \quad n\ge 1,$$

where $c_1>0$ *and* c_2 *do not depend on* n*.*

The exact orders of the linear widths of the classes W_1^1 in L_q, $2<q<\infty$, and of W_p^1, $1<p<2$, in L_∞ are not known.

We refer the interested reader to the list of references for earlier results concerning the Kolmogorov and linear n- widths of Sobolev classes of periodic and of nonperiodic functions, as well as of finite dimensional sets.

The main result that we prove in Sections 2 and 3, is the following extension of Theorem A.

Theorem 1. *Let* $r\in\mathbb{N}$, $1\le p,q\le\infty$ *and* $0<\alpha<\infty$*, be so that* $r-\alpha-\frac{1}{p}+\frac{1}{q}>0$*. If* $(r,p)\ne(1,1)$*, and if* $(r,p)=(1,1)$ *and* $1\le q\le 2$*, then*

$$d_n(W_{p,\alpha}^r)_{L_q} \asymp n^{-r+(\max\{\frac{1}{p},\frac{1}{2}\}-\max\{\frac{1}{q},\frac{1}{2}\})_+}, \quad n\ge r. \tag{1.4}$$

If on the other hand, $(r,p)=(1,1)$ *and* $2<q<\infty$*, then*

$$c_1 n^{-\frac{1}{2}} \le d_n(W_{1,\alpha}^1)_{L_q} \le c_2 n^{-\frac{1}{2}}\big(\log(n+1)\big)^{\frac{3}{2}}, \quad n\ge 1, \tag{1.5}$$

where $c_1>0$ *and* c_2 *do not depend on* n*.*

And the main result of Section 4 is the following extension of Theorem B.

Theorem 2. *Let* $r\in\mathbb{N}$, $1\le p,q\le\infty$ *and* $0<\alpha<\infty$*, be so that* $r-\alpha-\frac{1}{p}+\frac{1}{q}>0$*. If* $(r,p)\ne(1,1)$*, or if* $(r,p)=(1,1)$ *and* $1\le q\le 2$*, and if* $(r,q)=(1,\infty)$ *and* $2\le p\le\infty$*, then*

$$d_n(W_{p,\alpha}^r)_{L_q}^{lin} \asymp n^{-r+(\frac{1}{p}-\frac{1}{q})_+-\min\{(\frac{1}{p}-\frac{1}{2})_+,(\frac{1}{2}-\frac{1}{q})_+\}}, \quad n\ge r. \tag{1.6}$$

If on the other hand, $(r,p)=(1,1)$ *and* $2<q<\infty$*, then*

$$c_1 n^{-\frac{1}{2}} \le d_n(W_{1,\alpha}^1)_{L_q}^{lin} \le c_2 n^{-\frac{1}{2}}\big(\log(n+1)\big)^{\frac{3}{2}}, \quad n\ge 1, \tag{1.7}$$

and if $(r,q)=(1,\infty)$ *and* $1<p<2$*, then*

$$c_1 n^{-\frac{1}{2}} \le d_n(W_{p,\alpha}^1)_{L_\infty}^{lin} \le c_2 n^{-\frac{1}{2}}\big(\log(n+1)\big)^{\frac{3}{2}}, \quad n\ge 1, \tag{1.8}$$

where $c_1>0$ *and* c_2 *do not depend on* n*.*

§2. Kolmogorow widths of the classes $W^r_{p,\alpha}$ in L_q–basic construction

In order to prove the upper bounds in (1.4) we first need the following well known inequalities. First, recall that the Taylor polynomial of order $r-1 \geq 0$,

$$\pi(x;t,t_0) := \sum_{s=0}^{r-1} \frac{1}{s!} x^{(s)}(t_0)(t-t_0)^s,$$

of a function $x : I \to \mathbb{R}$, such that $x^{(r-1)} \in AC_{loc}(I)$, where $t_0 \in I$, satisfies

$$|x^{(s)}(t) - \pi^{(s)}(t,t_0)| \leq \frac{1}{(r-s-1)!} \left| \int_{t_0}^{t} |x^{(r)}(\tau)||t-\tau|^{r-s-1}\, d\tau \right|, \qquad t \in I, \quad s = 0, \ldots, r-1. \tag{2.1}$$

Now for $n \in \mathbb{N}$ and $1 \leq p \leq \infty$, let l^n_p denote, as usual, the spaces of vectors $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with the norms

$$\|x\|_{l^n_p} := \begin{cases} \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \max_{1 \leq i \leq n} |x_i|, & p = \infty, \end{cases}$$

and let B^n_p be its unit ball. We recall the following two lemmas (see [1] and [5]),

Lemma G1. *Let $m, n \in \mathbb{N}$ be so that $m < n$, and set*

$$\Phi(m,n,p,q) := \begin{cases} \max\left\{n^{\frac{1}{q}-\frac{1}{p}}, \left(1-\frac{m}{n}\right)^{\left(\frac{1}{p}-\frac{1}{q}\right)/\left(\frac{2}{p}-1\right)}\right\}, \\ \quad 1 \leq p < q \leq 2, \\ \max\left\{n^{\frac{1}{q}-\frac{1}{p}}, \min\left\{1, n^{\frac{1}{q}} m^{-\frac{1}{2}}\right\}\left(1-\frac{m}{n}\right)^{\frac{1}{2}}\right\}, \\ \quad 1 \leq p < 2 \leq q < \infty, \\ \left(\min\left\{1, n^{\frac{1}{q}} m^{-\frac{1}{2}}\right\}\right)^{\left(\frac{1}{p}-\frac{1}{q}\right)/\left(\frac{1}{2}-\frac{1}{q}\right)}, \\ \quad 2 \leq p < q < \infty. \end{cases} \tag{2.2}$$

Then for any $1 \leq p < q < \infty$, we have

$$d_m\left(B^n_p\right)_{l^n_q} \asymp \Phi(m,n,p,q), \tag{2.3}$$

where the constants in these two-sided estimates are independent of m and n.

And

Lemma K. *Let $m, n \in \mathbb{N}$ be so that $m < n$. Then*

$$d_m\left(B_2^n\right)_{l_\infty^n} \le cm^{-\frac{1}{2}}\left(1+\log\frac{n}{m}\right)^{\frac{3}{2}}, \tag{2.4}$$

where $c > 0$ is an absolute constant.

We are ready for the proof of Theorem 1.

Proof of Theorem 1–part I. We achieve upper bounds for the approximation of the classes $W_{p,\alpha}^r$ in the L_q metric, using continuous piecewise polynomials. To show this take the generic interval $I = (-1, 1)$, and recall that

$$\rho(t) = \operatorname{dist}(t, \{-1, 1\}) = \min\{|1+t|, |1-t|\}, \quad t \in I.$$

Fix $r \in \mathbb{N}$, $0 \le \alpha < \infty$, $1 \le p \le \infty$ and $1 \le q \le \infty$ so that $r - \alpha - \frac{1}{p} + \frac{1}{q} > 0$, and write

$$\beta := \beta(r, \alpha, p, q) := \left(r - \tfrac{1}{p} + \tfrac{1}{q}\right)\left(r - \alpha - \tfrac{1}{p} + \tfrac{1}{q}\right)^{-1}. \tag{2.5}$$

Given $n \in \mathbb{N}$, let

$$t_{ni} := t_{ni}(r, \alpha, p, q) := \begin{cases} 1 - \left(\frac{n-i}{n}\right)^\beta, & i = 0, 1, \ldots, n, \\ -1 + \left(\frac{n+i}{n}\right)^\beta, & i = -n, \ldots, -1, \end{cases} \tag{2.6}$$

be a partition of I. Denote by

$$I_{ni} := I_{ni}(r, \alpha, p, q) := \begin{cases} [t_{n,i-1}, t_{ni}], & i = 1, \ldots, n, \\ [t_{ni}, t_{n,i+1}], & i = -n, \ldots - 1, \end{cases}$$

the intervals of the partition, and let

$$\bar{t}_{ni} := \begin{cases} t_{2n,2i-1}, & i = 1, \ldots, n, \\ t_{2n,2i+1}, & i = -n, \ldots, -1. \end{cases}$$

On each interval I_{ni}, define two complementary splines φ_{*ni} and φ_{ni}^* as follows. For $i = 1, \ldots, n$, set

$$\varphi_{*ni}''(t) := \begin{cases} -2(\bar{t}_{ni} - t_{n,i-1})^{-1}(t_{ni} - t_{n,i-1})^{-1}, & t_{n,i-1} < t < \bar{t}_{ni}, \\ 2(t_{ni} - \bar{t}_{ni})^{-1}(t_{ni} - t_{n,i-1})^{-1}, & \bar{t}_{ni} < t < t_{ni}, \end{cases}$$

and

$$\varphi_{ni}^{*\,\prime\prime} := -\varphi_{*ni}''.$$

Now let

$$\varphi'_{*ni}(t) := \int_{t_{n,i-1}}^{t} \varphi''_{*ni}(\tau)\, d\tau, \quad t \in I_{ni},$$

and

$$\varphi^{*}_{ni}{}'(t) := \int_{t_{ni}}^{t} \varphi^{*}_{ni}{}''(\tau)\, d\tau, \quad t \in I_{ni}.$$

Finally, let

$$\varphi_{*ni}(t) := \int_{t_{ni}}^{t} \varphi'_{*ni}(\tau)\, d\tau, \quad t \in I_{ni},$$

and

$$\varphi^{*}_{ni}(t) := \int_{t_{n,i-1}}^{t} \varphi^{*}_{ni}{}'(\tau)\, d\tau, \quad t \in I_{ni}.$$

Now for $i = -n, \ldots, -1$ we set

$$\varphi_{*ni}(t) := \varphi_{*n,-i}(-t), \quad t \in I_{ni},$$

and

$$\varphi^{*}_{ni}(t) := \varphi^{*}_{n,-i}(-t), \quad t \in I_{ni}.$$

All the above functions are quadratic polynomials on the respective intervals and it is readily to seen that

$$\varphi_{*ni}(t_{n,i-1}) = \varphi^{*}_{ni}(t_{ni}) = 1, \quad \varphi_{*ni}(t_{ni}) = \varphi^{*}_{ni}(t_{n,i-1}) = 0, \quad i = 1, \ldots, n,$$

$$\varphi_{*ni}(t_{n,i+1}) = \varphi^{*}_{ni}(t_{ni}) = 1, \quad \varphi_{*ni}(t_{ni}) = \varphi^{*}_{ni}(t_{n,i+1}) = 0, \quad i = -n, \ldots, -1.$$

Furthermore for all $-n \le i \le n$,

$$0 \le \varphi_{*ni}(t) \le 1, \quad 0 \le \varphi^{*}_{ni}(t) \le 1, \quad \text{and} \quad \varphi_{*ni}(t) + \varphi^{*}_{ni}(t) \equiv 1, \quad t \in I_{ni}. \tag{2.7}$$

Thus in particular,

$$\|\varphi_{*ni}\|_{L_\infty(I_{ni})} = \|\varphi^{*}_{ni}\|_{L_\infty(I_{ni})} = 1, \quad i = \pm 1, \ldots, \pm n. \tag{2.8}$$

For $x \in W^r_{p,\alpha}(I)$, and $1 \le i \le n$, let $\pi_{*,r-1}(x;i;t)$ and $\pi^{*}_{r-1}(x;i;t)$, denote the Taylor polynomials of degree $r-1$ of x, expanded respectively, about to the left-hand and the right-hand endpoints of the interval I_{ni}, that is,

$$\pi_{*,r-1}(x;i;t) := \sum_{s=0}^{r-1} \frac{1}{s!} x^{(s)}(t_{n,i-1})(t - t_{n,i-1})^s, i = 1, \ldots, n,$$

$$\pi^{*}_{r-1}(x;i;t) := \sum_{s=0}^{r-1} \frac{1}{s!} x^{(s)}(t_{ni})(t - t_{ni})^s, \quad i = 1, \ldots, n-1.$$

Symmetrically, for $-n \le i \le -1$, let $\pi_{*,r-1}(x;i;t)$, $i=-n,\dots,-1$, and let $\pi^*_{r-1}(x;i;t)$, $i=-n+1,\dots,-1$ denote the Taylor polynomials of degree $r-1$ of x, expanded respectively, about to the right-hand and the left-hand endpoints of the interval I_{ni}.

We are ready to define the approximating splines, namely, let

$$\sigma_{r,n}(x;t) := \begin{cases} \pi_{*,r-1}(x;i;t)\varphi_{*ni}(t) + \pi^*_{r-1}(x;i;t)\varphi^*_{ni}(t), \quad t \in I_{ni}, \\ \quad i = \pm 1, \dots, \pm(n-1), \\ \pi_{*,r-1}(x;\pm n;t), \quad t \in I_{n,\pm n}. \end{cases}$$

Evidently, $\sigma_{r,n}(x;\cdot) \in C^1(I)$, and it is a polynomial of degree $\le r+1$ on each interval of the partition (in fact on the two end intervals it is a polynomial of degree $\le r-1$). We are going to prove that condition $r-\alpha-\frac{1}{p}+\frac{1}{q}>0$ implies

$$\sup_{x\in W^r_{p,\alpha}(I)} \|x(\cdot)-\sigma_{r,n}(x;\cdot)\|_{L_q(I)} \le \begin{cases} cn^{-r+\frac{1}{p}-\frac{1}{q}}, & 1\le p\le q\le\infty, \\ cn^{-r}, & 1\le q\le p\le\infty. \end{cases} \tag{2.9}$$

where $c=c(r,\alpha,p,q)$ is independent of $n\ge 1$.

Indeed by (2.7) and (2.8) we have for each $1\le i\le n-1$,

$$\begin{aligned} &\|x(\cdot)-\sigma_{r,n}(x;\cdot)\|_{L_q(I_{ni})} \\ &= \|x(\cdot)-\pi_{*,r-1}(x;i;\cdot)\varphi_{*ni}(\cdot)+\pi^*_{r-1}(x;i;\cdot)\varphi^*_{ni}(\cdot)\|_{L_q(I_{ni})} \\ &= \|\big(x(\cdot)-\pi_{*,r-1}(x;i;\cdot)\big)\varphi_{*ni}(\cdot)+\big(x(\cdot)-\pi^*_{r-1}(x;i;\cdot)\big)\varphi^*_{ni}(\cdot)\|_{L_q(I_{ni})} \\ &\le \|x(\cdot)-\pi_{*,r-1}(x;i;\cdot)\|_{L_q(I_{ni})}+\|x(\cdot)-\pi^*_{r-1}(x;i;\cdot)\|_{L_q(I_{ni})}. \end{aligned} \tag{2.10}$$

By (2.1) and Hölder's inequality we get for $t\in I_{ni}$, $1\le i\le n-1$,

$$\begin{aligned} &|x(t)-\pi_{*,r-1}(x;i;t)| \\ &\le \frac{1}{(r-1)!}\int_{t_{n,i-1}}^t \big|x^{(r)}(\tau)\big|(t-\tau)^{r-1}d\tau \\ &= \frac{1}{(r-1)!}\int_{t_{n,i-1}}^t \big|x^{(r)}(\tau)\rho^\alpha(\tau)\big|\rho^{-\alpha}(\tau)(t-\tau)^{r-1}d\tau \\ &\le \frac{1}{(r-1)!}\int_{t_{n,i-1}}^t \big|x^{(r)}(\tau)\rho^\alpha(\tau)\big|(1-t_{ni})^{-\alpha}(t-\tau)^{r-1}d\tau \\ &\le \frac{1}{(r-1)!}(1-t_{ni})^{-\alpha}(t_{ni}-t_{n,i-1})^{r-\frac{1}{p}}\big\|x^{(r)}\rho^\alpha\big\|_{L_p(I_{ni})}. \end{aligned} \tag{2.11}$$

Similarly

$$|x(t)-\pi^*_{r-1}(x;i;t)| \le \frac{1}{(r-1)!}\int_t^{t_{ni}} \left|x^{(r)}(\tau)\right|(\tau-t)^{r-1}d\tau$$
$$\le \frac{1}{(r-1)!}(1-t_{ni})^{-\alpha}(t_{ni}-t_{n,i-1})^{r-\frac{1}{p}}\left\|x^{(r)}\rho^{\alpha}\right\|_{L_p(I_{ni})}.$$

Hence by (2.10), for $i=1,\ldots,n-1$

$$\|x(\cdot)-\sigma_{r,n}(x;\cdot)\|_{L_q(I_{ni})} \le \frac{2}{(r-1)!}(1-t_{ni})^{-\alpha}(t_{ni}-t_{n,i-1})^{r-\frac{1}{p}+\frac{1}{q}}\left\|x^{(r)}\rho^{\alpha}\right\|_{L_p(I_{ni})}. \tag{2.12}$$

Finally for $i=n$, we need a variant of (2.11), namely, for $t\in I_{nn}$,

$$|x(t)-\pi_{*,r-1}(x;n;t)|$$
$$\le \frac{1}{(r-1)!}\int_{t_{n,n-1}}^{t}\left|x^{(r)}(\tau)\right|(t-\tau)^{r-1}d\tau$$
$$= \frac{1}{(r-1)!}\int_{t_{n,n-1}}^{t}\left|x^{(r)}(\tau)\rho^{\alpha}(\tau)\right|\rho^{-\alpha}(\tau)(t-\tau)^{r-1}d\tau$$
$$\le \frac{1}{(r-1)!}\int_{t_{n,n-1}}^{t}\left|x^{(r)}(\tau)\rho^{\alpha}(\tau)\right|(t-\tau)^{r-\alpha-1}d\tau$$
$$\le \frac{1}{(r-1)!}\left\|x^{(r)}\rho^{\alpha}\right\|_{L_p(I_{nn})}\begin{cases}\max\left\{\left(1-t_{n,n-1}\right),(1-t)\right\}^{r-\alpha-1}, & p=1,\\ \left(\int_{t_{n,n-1}}^{t}(1-\tau)^{(r-\alpha-1)p'}d\tau\right)^{\frac{1}{p'}}, & 1<p\le\infty,\end{cases}$$

where $\frac{1}{p}+\frac{1}{p'}=1$. Now, if $1<p\le\infty$ and $(r-\alpha-1)p'\ne -1$, then

$$\left(\int_{t_{n,n-1}}^{t}(1-\tau)^{(r-\alpha-1)p'}d\tau\right)^{\frac{1}{p'}}$$
$$\le \left|(r-\alpha-1)p'+1\right|^{-\frac{1}{p'}}\max\left\{\left(1-t_{n,n-1}\right),(1-t)\right\}^{r-\alpha-\frac{1}{p}},$$

and if $1<p\le\infty$ and $(r-\alpha-1)p'=-1$, or equivalently $r-\alpha-\frac{1}{p}=0$, then we fix an $0<\epsilon<r-\alpha-\frac{1}{p}+\frac{1}{q}$,

and we have

$$\left(\int_{t_{n,n-1}}^{t}(1-\tau)^{(r-\alpha-1)p'}d\tau\right)^{\frac{1}{p'}} = \left(\int_{t_{n,n-1}}^{t}(1-\tau)^{(r-\alpha-1-\epsilon+\epsilon)p'}d\tau\right)^{\frac{1}{p'}}$$
$$\le \left(1-t_{n,n-1}\right)^{\epsilon}\left(\int_{t_{n,n-1}}^{t}(1-\tau)^{(r-\alpha-1-\epsilon)p'}d\tau\right)^{\frac{1}{p'}}$$
$$\le \left(\epsilon p'\right)^{-\frac{1}{p'}}\left(1-t_{n,n-1}\right)^{\epsilon}(1-t)^{-\epsilon}.$$

By the definition of $\sigma_{r,n}$ we conclude that

$$
(2.13)\qquad
\begin{aligned}
&\|x(\cdot)-\sigma_{r,n}(x;\cdot)\|_{L_q(I_{nn})} \le c\big\|x^{(r)}\rho^\alpha\big\|_{L_p(I_{nn})} \\
&\times \begin{cases}
\max\Big\{ \big(1-t_{n,n-1}\big)^{r-\alpha-\frac{1}{p}+\frac{1}{q}}, \Big(\int_{t_{n,n-1}}^1 (1-t)^{(r-\alpha-\frac{1}{p})q}dt\Big)^{\frac{1}{q}} \Big\}, \\
\quad \text{if} \quad p=1 \quad \text{or} \quad r-\alpha-\frac{1}{p}\neq 0, \\
\big(1-t_{n,n-1}\big)^{\epsilon}\Big(\int_{t_{n,n-1}}^1 (1-t)^{-\epsilon q}dt\Big)^{\frac{1}{q}}, \quad \text{otherwise},
\end{cases} \\
&\le c\big(1-t_{n,n-1}\big)^{r-\alpha-\frac{1}{p}+\frac{1}{q}}\big\|x^{(r)}\rho^\alpha\big\|_{L_p(I_{nn})},
\end{aligned}
$$

where we used the fact that $(r-\alpha-\frac{1}{p})q>-1$. Similarly, we have for $-n+1\le i\le -1$,

$$
(2.14)\qquad
\begin{aligned}
&\|x(\cdot)-\sigma_{r,n}(x;\cdot)\|_{L_q(I_{ni})} \\
&\le \frac{2}{(r-1)!}(1+t_{ni})^{-\alpha}(t_{n,i+1}-t_{n,i})^{r-\frac{1}{p}+\frac{1}{q}}\big\|x^{(r)}\rho^\alpha\big\|_{L_p(I_{ni})},
\end{aligned}
$$

and

$$
(2.15)\qquad \|x(\cdot)-\sigma_{r,n}(x;\cdot)\|_{L_q(I_{n,-n})} \le c\big(1+t_{n,-n+1}\big)^{r-\alpha-\frac{1}{p}+\frac{1}{q}}\big\|x^{(r)}\rho^\alpha\big\|_{L_p(I_{n,-n})},
$$

where $c=c(r,\alpha,p,q)$.

For later application we note that if $1\le p<q\le\infty$, then clearly $r-\alpha-\frac{1}{p}+\frac{1}{p}>r-\alpha-\frac{1}{p}+\frac{1}{q}>0$, and we may replace q by p in (2.12) through (2.15). Thus for all $i=1,\dots,n-1$ we obtain

(2.16)
$$
\|x(\cdot)-\sigma_{r,n}(x;\cdot)\|_{L_p(I_{n,\pm i})} \le \frac{2}{(r-1)!}\big|1\mp t_{n,\pm i}\big|^{-\alpha}\big|t_{n,\pm i}-t_{n,\pm(i-1)}\big|^{r}\big\|x^{(r)}\rho^\alpha\big\|_{L_p(I_{n,\pm i})},
$$

and

$$
(2.17)\qquad \|x(\cdot)-\sigma_{r,n}(x;\cdot)\|_{L_p(I_{n,\pm n})} \le c\big|1\mp t_{n,\pm(n-1)}\big|^{r-\alpha}\big\|x^{(r)}\rho^\alpha\big\|_{L_p(I_{n,\pm n})}.
$$

where $c=c(r,\alpha,p)$.

In order to conclude our estimates we thus need to estimate the distances $|t_{n,\pm i}-t_{n,\pm(i-1)}|$ and $|1\mp t_{n,\pm i}|$. To this end, it is readily seen that since $\beta\ge 1$,

$$
(2.18)\quad cn^{-\beta}(n-|i|+1)^{\beta-1}\le |t_{n,\pm i}-t_{n,\mp(i-1)}|\le Cn^{-\beta}(n-|i|+1)^{\beta-1}, \quad i=1,\dots,n,
$$

and

$$cn^{-\beta}(n-|i|+1)^{\beta} \le |1 \mp t_{n,\pm i}| \le Cn^{-\beta}(n-|i|+1)^{\beta}, \quad i=0,1,\dots,n-1, \tag{2.19}$$

where $c=c(\beta), C=C(\beta)$.

It now follows that

$$\begin{aligned}
&(1-t_{ni})^{-\alpha}(t_{ni}-t_{n,i-1})^{r-\frac{1}{p}+\frac{1}{q}} \\
&\le cn^{\alpha\beta}(n-i+1)^{-\alpha\beta}n^{-\beta(r-\frac{1}{p}+\frac{1}{q})}(n-i+1)^{(\beta-1)(r-\frac{1}{p}+\frac{1}{q})} \\
&= cn^{-\beta(r-\alpha-\frac{1}{p}+\frac{1}{q})}(n-i+1)^{\beta(r-\alpha-\frac{1}{p}+\frac{1}{q})}(n-i+1)^{-r+\frac{1}{p}-\frac{1}{q}} \\
&= cn^{-r+\frac{1}{p}-\frac{1}{q}}, \quad i=1,\dots,n-1,
\end{aligned} \tag{2.20}$$

where in the last equality we made use of the definition of β (see (2.5)). Similarly,

$$(1+t_{ni})^{-\alpha}(t_{n,i+1}-t_{ni})^{r-\frac{1}{p}+\frac{1}{q}} \le cn^{-r+\frac{1}{p}-\frac{1}{q}}, \quad i=-n+1,\dots,-1. \tag{2.21}$$

Also

$$\begin{aligned}
(1-t_{n,n-1})^{r-\alpha-\frac{1}{p}+\frac{1}{q}} &\le cn^{-r+\frac{1}{p}-\frac{1}{q}}, \\
(1+t_{n,-n+1})^{r-\alpha-\frac{1}{p}+\frac{1}{q}} &\le cn^{-r+\frac{1}{p}-\frac{1}{q}}.
\end{aligned} \tag{2.22}$$

Combining (2.12) through (2.15) with the estimates (2.20) through (2.22), we obtain

$$\|x(\cdot)-\sigma_{r,n}(x;\cdot)\|_{L_q(I_{ni})} \le c\big\|x^{(r)}\rho^{\alpha}\big\|_{L_p(I_{ni})}n^{-r+\frac{1}{p}-\frac{1}{q}}, \quad i=\pm1,\dots,\pm n, \tag{2.23}$$

where $c=c(r,\alpha,p,q)$. Applying the well known inequality

$$\left(\sum_{j=1}^{m}|a_j|^q\right)^{\frac{1}{q}} \le \left(\sum_{j=1}^{m}|a_j|^p\right)^{\frac{1}{p}}, \quad 1\le p\le q\le\infty,$$

we obtain

$$\begin{aligned}
&\|x(\cdot)-\sigma_{r,n}(x;\cdot)\|_{L_q(I)} \\
&= \left(\sum_{i=-n}^{-1}\|x(\cdot)-\sigma_{r,n}(x;\cdot)\|^q_{L_q(I_{ni})} + \sum_{i=1}^{n}\|x(\cdot)-\sigma_{r,n}(x;\cdot)\|^q_{L_q(I_{ni})}\right)^{\frac{1}{q}} \\
&\le cn^{-r+\frac{1}{p}-\frac{1}{q}}\left(\sum_{i=-n}^{-1}\|x^{(r)}\rho^{\alpha}\|^q_{L_p(I_{ni})} + \sum_{i=1}^{n}\|x^{(r)}\rho^{\alpha}\|^q_{L_p(I_{ni})}\right)^{\frac{1}{q}} \\
&\le cn^{-r+\frac{1}{p}-\frac{1}{q}}\left(\sum_{i=-n}^{-1}\|x^{(r)}\rho^{\alpha}\|^p_{L_p(I_{ni})} + \sum_{i=1}^{n}\|x^{(r)}\rho^{\alpha}\|^p_{L_p(I_{ni})}\right)^{\frac{1}{p}} \\
&= cn^{-r+\frac{1}{p}-\frac{1}{q}}\|x^{(r)}\rho^{\alpha}\|_{L_p(I)} \le cn^{-r+\frac{1}{p}-\frac{1}{q}}.
\end{aligned}$$

This completes the proof of (2.9) for $1 \le p \le q \le \infty$. We now turn to the case $1 \le q \le p \le \infty$. Here we apply the well known inequality

$$\left(\sum_{j=1}^{m} |a_j|^q\right)^{\frac{1}{q}} \le m^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{j=1}^{m} |a_j|^p\right)^{\frac{1}{p}}, \quad 1 \le q \le p \le \infty,$$

to get by virtue of (2.23),

$$\begin{aligned}
&\|x(\cdot) - \sigma_{r,n}(x;\cdot)\|_{L_q(I)} \\
&= \left(\sum_{i=-n}^{-1} \|x(\cdot) - \sigma_{r,n}(x;\cdot)\|^q_{L_q(I_{ni})} + \sum_{i=1}^{n} \|x(\cdot) - \sigma_{r,n}(x;\cdot)\|^q_{L_q(I_{ni})}\right)^{\frac{1}{q}} \\
&\le cn^{-r+\frac{1}{p}-\frac{1}{q}} \left(\sum_{i=-n}^{-1} \left\|x^{(r)}\rho^\alpha\right\|^q_{L_p(I_{ni})} + \sum_{i=1}^{n} \left\|x^{(r)}\rho^\alpha\right\|^q_{L_p(I_{ni})}\right)^{\frac{1}{q}} \\
&\le cn^{-r} \left(\sum_{i=-n}^{-1} \left\|x^{(r)}\rho^\alpha\right\|^p_{L_p(I_{ni})} + \sum_{i=1}^{n} \left\|x^{(r)}\rho^\alpha\right\|^p_{L_p(I_{ni})}\right)^{\frac{1}{p}} \\
&= cn^{-r} \left\|x^{(r)}\rho^\alpha\right\|_{L_p(I)} \le cn^{-r}.
\end{aligned}$$

This completes the proof of (2.9) for $1 \le q \le p \le \infty$.

For $n > 1$, let $\Sigma^1_{r,n} := \Sigma^1_{r,n}(I)$ be the space of piecewise polynomials $\zeta \in C^1(I)$, that are polynomials of degree $\le r+1$ in each interval $[t_{n,i-1}, \bar{t}_{ni}]$ and $[\bar{t}_{ni}, t_{ni}]$, $1 \le i \le n-1$, and in each interval $[t_{n,i+1}, \bar{t}_{ni}]$ and $[\bar{t}_{ni}, t_{ni}]$, $-n+1 \le i \le -1$, and which are polynomials of degree $\le r-1$ in $I_{n,\pm n}$. Then clearly, $\dim \Sigma^1_{r,n} = 4r(n-1) + 2r - 2$. For $n = 1$ we do even better by taking for $\Sigma^1_{r,1} := \Sigma^1_{r,1}(I) := P_{r-1}$, the space of polynomials of degree $\le r-1$, that is, of dimension r. By our construction, for each $x \in W^r_{p,\alpha}$, $\sigma_{r,n}(x;\cdot) \in \Sigma^1_{r,n}$, whence

$$E\left(W^r_{p,\alpha}, \Sigma^1_{r,n}\right)_{L_q} \le \begin{cases} cn^{-r+\frac{1}{p}-\frac{1}{q}}, & 1 \le p \le q \le \infty, \\ cn^{-r}, & 1 \le q \le p \le \infty, \end{cases} \tag{2.24}$$

where $c = c(r, \alpha, p, q)$. Therefore for $N := 4nr - 2r - 2$, if $n > 1$, and $N = r$, if $n = 1$, we get

$$d_N\left(W^r_{p,\alpha}\right)_{L_q} \le \begin{cases} cN^{-r+\frac{1}{p}-\frac{1}{q}}, & 1 \le p \le q \le \infty, \\ cN^{-r}, & 1 \le q \le p \le \infty, \end{cases}$$

where $c = c(r, \alpha, p, q)$. It is a standard technique now to obtain,

$$
(2.25) \qquad d_n\left(W^r_{p,\alpha}\right)_{L_q} \le \begin{cases} cn^{-r+\frac{1}{p}-\frac{1}{q}}, & 1 \le p \le q \le \infty, \quad n \ge r, \\ cn^{-r}, & 1 \le q \le p \le \infty, \quad n \ge r, \end{cases}
$$

where $c = c(r, \alpha, p, q)$. This concludes the first part of the proof. $\square$

§3. Kolmogorow widths of the classes $W^r_{p,\alpha}$ in L_q–refinement

The upper bound (2.25) in some cases already yields the upper estimate required in Theorem 1, but we have to improve it in the cases $1 \le p \le 2 < q \le \infty$ and $2 \le p < q \le \infty$ by using a well known discretization technique. In this section we will show that except for the case $r = p = 1$, for $n \ge r$, we have

$$
(3.1) \qquad d_n\left(W^r_{p,\alpha}\right)_{L_q} \le \begin{cases} cn^{-r+\frac{1}{p}-\frac{1}{2}}, & 1 \le p \le 2 < q \le \infty, \\ cn^{-r}, & 2 \le p < q \le \infty, \end{cases}
$$

where $c = c(r, \alpha, p, q)$. If $r = p = 1$ and $2 < q < \infty$, then we can only prove

$$
d_n\left(W^1_{1,\alpha}\right)_{L_q} \le cn^{-\frac{1}{2}}\left(\log(n+1)\right)^{\frac{3}{2}}.
$$

Proof of Theorem 1–part II. For $n > 1$ let $\Sigma^0_{r,n} := \Sigma^0_{r,n}(I)$ be the space of continuous piecewise polynomials $\zeta \in C(I)$, that are polynomials of degree $\le r+1$ in each interval $[t_{n,i-1}, \bar{t}_{ni}]$ and $[\bar{t}_{ni}, t_{ni}]$, $1 \le i \le n$, and in each interval $[\bar{t}_{ni}, t_{n,i+1}]$ and $[t_{ni}, \bar{t}_{ni}]$, $-n \le i \le -1$. Then clearly, $\dim \Sigma^0_{r,n} = 4(r+1)n + 1$. For $n = 1$ we take $\Sigma^0_{r,1} := \Sigma^1_{r,1}$. We also have for all $n \ge 1$,

$$
\Sigma^1_{r,n} \subseteq \Sigma^0_{r,n}, \quad \Sigma^1_{r,n} \subseteq \Sigma^1_{r,2n}, \quad \Sigma^0_{r,n} \subseteq \Sigma^0_{r,2n}.
$$

We first exclude the case $r = p = 1$ and $2 < q < \infty$, and we prove that in all other cases, if $r - \alpha - \frac{1}{p} + \frac{1}{q} > 0$, then there exist integers $\lambda = \lambda(r, p, q) > 1$, $a = a(r, p, q) > 0$ and subspaces $\Sigma_{r,a2^n} \subseteq \Sigma^0_{r,2^{\lambda n}}$, of dimensions $r \le \dim \Sigma_{r,a2^n} \le a2^n \le 2^{\lambda n}$, $n \ge 1$, such that

$$
(3.2) \qquad E\left(W^r_{p,\alpha}, \Sigma_{r,a2^n}\right)_{L_q} \le \begin{cases} c2^{-\left(r-\frac{1}{p}+\frac{1}{2}\right)n}, & 1 \le p \le 2 < q \le \infty, \\ c2^{-rn}, & 2 \le p < q \le \infty, \end{cases}
$$

where $c = c(r, \alpha, p, q)$. Set $n(r) := 2(r+1)n$, $n > 1$, and subdivide the intervals, refining the partition by letting

$$
(3.3) \qquad t_{n,i,k} := \begin{cases} t_{n(r),(r+1)(2i-1)+k}, & i = 1, \dots, n, \quad k = 0, \pm 1, \dots, \pm(r+1), \\ t_{n(r),(r+1)(2i+1)+k}, & i = -n, \dots, -1, \quad k = 0, \pm 1, \dots, \pm(r+1), \end{cases}
$$

where the points $t_{n(r),2(r+1)i\pm k}$ are defined by (2.6) for $n=n(r)$. Observe that

$$t_{n,i,0}=\bar{t}_{ni}, \quad i=\pm 1,\cdots,\pm n,$$

and

$$t_{n,i,-r-1}=t_{n,i-1}, \quad t_{n,i,r+1}=t_{ni}, \quad i=1,\ldots,n,$$
$$t_{n,i,r+1}=t_{n,i+1}, \quad t_{n,i,-r-1}=t_{ni}, \quad i=-n,\ldots,-1.$$

Now define a one-to-one correspondence between the spaces $\Sigma^0_{r,n}$ and $\mathbb{R}^{2n(r)+1}$, by the invertible discretization operator

$$A_{r,\beta,q,n}:\Sigma^0_{r,n}\ni\zeta\to y=\big(y_{-n(r)},\ldots,y_{-1},y_0,y_1,\ldots,y_{n(r)}\big)\in\mathbb{R}^{2n(r)+1},$$

where

$$y_j=n(r)^{-\frac{\beta}{q}}(n(r)-|j|+1)^{\frac{\beta-1}{q}}\zeta(t_{n(r),j}), \quad j=0,\pm 1,\ldots,\pm n(r).$$

The inverse operator is

$$A^{-1}_{r,\beta,q,n}:\mathbb{R}^{2n(r)+1}\ni y=\big(y_{-n(r)},\ldots,y_{-1},y_0,y_1,\ldots,y_{n(r)}\big)\to\zeta\in\Sigma^0_{r,n},$$

where ζ is uniquely defined by the interpolation equations

$$\zeta(t_{n(r),j})=n(r)^{\frac{\beta}{q}}(n(r)-|j|+1)^{-\frac{\beta-1}{q}}y_j, j=0,\pm 1,\ldots,\pm n(r).$$

We will show that the norms $\|A_{r,\beta,q,n}\zeta\|_{l_q^{2n(r)+1}}$ and $\|\zeta\|_{L_q}$ are equivalent, the equivalence constants depending only on p, q, r and α. Indeed, since ζ is a polynomial of degree $\le r+1$ on each interval $[t_{n,i-1},\bar{t}_{ni}]$ and $[\bar{t}_{ni},t_{ni}]$ etc., then there is a constant $c=c(q,r)>0$ such that

$$\text{(3.4)} \qquad c^{-1}|I_{ni}|^{\frac{1}{q}}\|\zeta\|_{L_\infty(I_{ni})}\le\|\zeta\|_{L_q(I_{ni})}\le|I_{ni}|^{\frac{1}{q}}\|\zeta\|_{L_\infty(I_{ni})}, \quad i=\pm 1,\ldots,\pm n,$$

where the right-hand inequality is simply Hölder's inequality. Also we have the representation

$$\zeta(t)=\begin{cases}\sum_{k=-r-1}^{0}\zeta(t_{n,i,k})\prod_{j=-r-1,j\neq k}^{0}(t-t_{n,i,j})(t_{n,i,k}-t_{n,i,j})^{-1}, & t\in[t_{n,i-1},\bar{t}_{ni}],\\ \sum_{k=r+1}^{0}\zeta(t_{n,i,k})\prod_{j=0,j\neq k}^{r+1}(t-t_{n,i,j})(t_{n,i,k}-t_{n,i,j})^{-1}, & t\in[\bar{t}_{ni},t_{ni}],\\ \text{for} \quad i=1,\ldots,n, & \\ \sum_{k=0}^{r+1}\zeta(t_{n,i,k})\prod_{j=0,j\neq k}^{r+1}(t-t_{n,i,j})(t_{n,i,k}-t_{n,i,j})^{-1}, & t\in[\bar{t}_{ni},t_{n,i+1}],\\ \sum_{k=-r-1}^{0}\zeta(t_{n,i,k})\prod_{j=-r-1,j\neq k}^{0}(t-t_{n,i,j})(t_{n,i,k}-t_{n,i,j})^{-1}, & t\in[t_{ni},\bar{t}_{ni}],\\ \text{for} \quad i=-n,\ldots,-1. & \end{cases}$$

Note that altogether we needed exactly $2n(r)+1$ values of ζ to represent it.

Hence we have to estimate ζ in two prototypes of intervals and we do that in one of each. From the above representation we see that for $1 \le i \le n$,

$$
\begin{aligned}
&\max_{-r-1\le k\le 0} |\zeta(t_{n,i,k})| \le \|\zeta\|_{L_\infty([t_{n,i-1},\bar{t}_{ni}])} \\
&\le (r+2)(\bar{t}_{ni}-t_{n,i-1})^{r+1}\big(\min_{-r\le k\le 0}|t_{n,i,k-1}-t_{n,i,k}|\big)^{-r-1}\max_{-r-1\le k\le 0}|\zeta(t_{n,i,k})|,
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
&\max_{0\le k\le r+1} |\zeta(t_{n,i,k})| \le \|\zeta\|_{L_\infty([\bar{t}_{ni},t_{ni}])} \\
&\le (r+2)(t_{ni}-\bar{t}_{ni})^{r+1}\big(\min_{0\le k\le r}|t_{n,i,k+1}-t_{n,i,k}|\big)^{-r-1}\max_{0\le k\le r+1}|\zeta(t_{n,i,k})|.
\end{aligned}
\tag{3.6}
$$

Now it follows readily by (2.18) and (2.19) that

$$
\begin{aligned}
|I_{ni}| &\asymp n(r)^{-\beta}(n(r)-|i|+1)^{\beta-1} \\
&\asymp n(r)^{-\beta}(n(r)-|(r+1)(2i\mp 1)+k|+1)^{\beta-1}, \\
&i=\pm 1,\dots,\pm n, \qquad k=0,\pm 1,\dots,\pm(r+1).
\end{aligned}
\tag{3.7}
$$

Hence by (3.4) through (3.7), we get for $i=\pm 1,\dots,\pm n$,

$$
\begin{aligned}
\|\zeta\|^q_{L_q(I_{ni})} &\le |I_{ni}|\|\zeta\|^q_{L_\infty(I_{ni})} \\
&\le c|I_{ni}|\max_{-r-1\le k\le r+1}|\zeta(t_{n,i,k})|^q \\
&\le c|I_{ni}|\sum_{k=-r-1}^{r+1}|\zeta(t_{n,i,k})|^q \\
&\le c\sum_{k=-r-1}^{r+1} n(r)^{-\beta}(n(r)-|(r+1)(2i\mp 1)+k|+1)^{\beta-1}|\zeta(t_{n,i,k})|^q,
\end{aligned}
$$

where $c=c(r,\alpha,p,q)$. Combining the inequalities for all the intervals yields,

$$
\begin{aligned}
\|\zeta\|_{L_q} &\le c\left(\sum_{j=-n(r)}^{n(r)} n(r)^{-\beta}(n(r)-|j|+1)^{\beta-1}|\zeta(t_{n(r),j})|^q\right)^{\frac{1}{q}} \\
&= c\|A_{r,\beta,q,n}\zeta\|_{l_q^{2n(r)+1}},
\end{aligned}
\tag{3.8}
$$

where $c=c(r,\alpha,p,q)$.

Conversely, let $y \in \mathbb{R}^{2n(r)+1}$ and write $\zeta := A^{-1}_{r,\beta,q,n} y$. Then

$$\begin{aligned}
&\|y\|^q_{l_q^{2n(r)+1}} \\
&= \sum_{j=-n(r)}^{n(r)} n(r)^{-\beta}(n(r) - |j| + 1)^{\beta-1} |\zeta(t_{n(r),j})|^q \\
&\leq \sum_{i=-n}^{-1} \sum_{k=-r-1}^{r+1} n(r)^{-\beta}(n(r) - |(r+1)(2i+1) + k| + 1)^{\beta-1} |\zeta(t_{n,i,k})|^q \\
&+ \sum_{i=1}^{n} \sum_{k=-r-1}^{r+1} n(r)^{-\beta}(n(r) - |(r+1)(2i-1) + k| + 1)^{\beta-1} |\zeta(t_{n,i,k})|^q.
\end{aligned}$$

Thus, in order to have a clear picture of the computations, we will take a typical sum, for instance, just the points in the interval $I_{ni}, 1 \leq i \leq n$. It follows by (3.4) through (3.7), that

$$\begin{aligned}
&\sum_{k=-r-1}^{r+1} n(r)^{-\beta}(n(r) - (r+1)(2i-1) - k + 1)^{\beta-1} |\zeta(t_{n,i,k})|^q \\
&\leq c|I_{ni}| \|\zeta\|^q_{L_\infty(I_{ni})} \leq c\|\zeta\|^q_{L_q(I_{ni})}.
\end{aligned}$$

where $c = c(r, \alpha, p, q)$. Hence we conclude that

$$\|A_{r,\beta,q,n}\zeta\|_{l_q^{2n(r)+1}} \leq c\|\zeta\|_{L_q},$$

where $c = c(r, \alpha, p, q)$. This together with (3.8) yields

$$\|\zeta\|_{L_q} \asymp \|A_{r,\beta,q,n}\zeta\|_{l_q^{2n(r)+1}}, \quad \zeta \in \Sigma^0_{r,n}. \tag{3.9}$$

This completes the proof of the boundedness of the operator $A_{r,\beta,q,n}$ and its inverse.

Now, fix some integer $N \in \mathbb{N}$. Let $m_0 := r$ and let the integers $m_\nu \leq 4(r+1)2^\nu + 1$, $\nu = 1, 2, \ldots, N$ be prescribed, and let $L^{m_\nu}, \nu = 1, 2, \ldots, N$ be subspaces of $\mathbb{R}^{4(r+1)2^\nu+1}$ of dimension $\dim L^{m_\nu} = m_\nu$. Set

$$\Sigma^{m_0} := \Sigma^0_{r,1}, \quad \Sigma^{m_\nu} := A^{-1}_{r,\beta,q,2^\nu} L^{m_\nu}, \quad \nu = 1, 2, \ldots, N.$$

Then clearly $\Sigma^{m_\nu} \subset \Sigma^0_{r,2^\nu}$ and $\dim \Sigma^{m_\nu} = m_\nu$, $\nu = 0, 1, 2, \ldots, N$. Denote

$$\Sigma^{m_0,\ldots,m_N} := \operatorname{span}\big(\cup_{\nu=0}^N \Sigma^{m_\nu}\big).$$

Thus $\Sigma^{m_0,\dots,m_N} \subset \Sigma^0_{r,2^N}$ and $\dim \Sigma^{m_0,\dots,m_N} \le m_0 + \cdots + m_N$. Each $x \in W^r_{p,\alpha}$ can be written as

$$x(t) = x(t) - \sigma_{r,2^N}(x;t) + \sigma_{r,1}(x;t) + \sum_{\nu=1}^{N} \big(\sigma_{r,2^\nu}(x;t) - \sigma_{r,2^{\nu-1}}(x;t)\big), \quad t \in I,$$

where $\sigma_{r,2^\nu}(x;\cdot) \in \Sigma^1_{r,2^\nu}$. Therefore,

$$\begin{aligned} E\big(x, \Sigma^{m_0,\dots,m_N}\big)_{L_q} &\le E\big(x(\cdot) - \sigma_{r,2^N}(x;\cdot), \Sigma^{m_0,\dots,m_N}\big)_{L_q} + E\big(\sigma_{r,1}(x;\cdot), \Sigma^{m_0,\dots,m_N}\big)_{L_q} \\ &+ \sum_{\nu=1}^{N} E\big(\sigma_{r,2^\nu}(x;\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot), \Sigma^{m_0,\dots,m_N}\big)_{L_q} \\ &\le \big\|x(\cdot) - \sigma_{r,2^N}(x;\cdot)\big\|_{L_q} + E\big(\sigma_{r,1}(x;\cdot), \Sigma^{m_0}\big)_{L_q} \\ &+ \sum_{\nu=1}^{N} E\big(\sigma_{r,2^\nu}(x;\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot), \Sigma^{m_\nu}\big)_{L_q} \\ &= \big\|x(\cdot) - \sigma_{r,2^N}(x;\cdot)\big\|_{L_q} \\ &+ \sum_{\nu=1}^{N} E\big(\sigma_{r,2^\nu}(x;\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot), \Sigma^{m_\nu}\big)_{L_q}, \end{aligned}$$

where we used the fact that $\sigma_{r,1} \in \Sigma^{m_0}$. Observe that if $\sigma \in \Sigma^{m_\nu}$, then $\big(\big(\sigma_{r,2^\nu}(x;\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot)\big) - \sigma\big) \in \Sigma^0_{r,2^\nu}(I)$. Hence by (3.9) if $\sigma \in \Sigma^{m_\nu}$, then

$$\begin{aligned} &\big\|\big(\sigma_{r,2^\nu}(x;\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot)\big) - \sigma\big\|_{L_q} \\ &\le c\big\|A_{r,\beta,q,2^\nu}\big(\big(\sigma_{r,2^\nu}(x;\cdot)\big) - \sigma_{r,2^{\nu-1}}(x;\cdot)\big) - A_{r,\beta,q,2^\nu}\sigma\big)\big\|_{l_q^{4(r+1)2^\nu+1}}, \end{aligned}$$

where $c = c(r,\alpha,p,q)$. Therefore,

$$\begin{aligned} &E\big(\big(\sigma_{r,2^\nu}(x;\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot)\big), \Sigma^{m_\nu}\big)_{L_q} \\ &\le c\big(A_{r,\beta,q,2^\nu}\big(\sigma_{r,2^\nu}(x;\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot)\big), A_{r,\beta,q,2^\nu}\Sigma^{m_\nu}\big)_{l_q^{4(r+1)2^\nu+1}} \\ &= cE\big(A_{r,\beta,q,2^\nu}\big(\sigma_{r,2^\nu}(x;\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot)\big), L^{m_\nu}\big)_{l_q^{4(r+1)2^\nu+1}}. \end{aligned}$$

Finally, we conclude that

$$\begin{aligned} E\big(x, \Sigma^{m_0,\dots,m_N}\big)_{L_q} &\le \big\|x(\cdot) - \sigma_{r,2^N}(x;\cdot)\big\|_{L_q} \\ &+ c\sum_{\nu=1}^{N} E\big(A_{r,\beta,q,2^\nu}\big(\sigma_{r,2^\nu}(x;\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot)\big), L^{m_\nu}\big)_{l_q^{4(r+1)2^\nu+1}}. \end{aligned} \tag{3.10}$$

We will show that whenever $1 \leq p < q \leq \infty$ and $r - \alpha - \frac{1}{p} + \frac{1}{q} > 0$, then for all $x \in W^r_{p,\alpha}$, the splines $\sigma_{r,2^\nu}(x;\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot)$ are mapped by $A_{r,\beta,q,2^\nu}$ into the ball $c2^{-(r-\frac{1}{p}+\frac{1}{q})\nu} B_p^{4(r+1)2^\nu+1}$, where $c = c(r,\alpha,p,q)$. Indeed,

$$\left\|\sigma_{r,2^\nu}(x;\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot)\right\|_{L_p(I_{2^\nu,i})} \leq \left\|x(\cdot) - \sigma_{r,2^\nu}(x;\cdot)\right\|_{L_p(I_{2^\nu,i})} + \left\|x(\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot)\right\|_{L_p(I_{2^\nu,i})}, \quad i = \pm 1, \ldots, \pm 2^\nu,$$

while the second term is estimated by

$$\left\|x(\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot)\right\|_{L_p(I_{2^\nu,2i-1})} \leq \left\|x(\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot)\right\|_{L_p(I_{2^{\nu-1},i})}, \quad i = 1, \ldots, 2^{\nu-1},$$
$$\left\|x(\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot)\right\|_{L_p(I_{2^\nu,2i+1})} \leq \left\|x(\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot)\right\|_{L_p(I_{2^{\nu-1},i})}, \quad i = -2^{\nu-1}, \ldots, -1$$

and

$$\left\|x(\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot)\right\|_{L_p(I_{2^\nu,2i})} \leq \left\|x(\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot)\right\|_{L_p(I_{2^{\nu-1},i})}, \quad i = \pm 1, \ldots, \pm 2^{\nu-1},$$

where we used the fact that $I_{2^\nu,2i\mp 1} \subset I_{2^{\nu-1},i}$ and $I_{2^\nu,2i} \subset I_{2^{\nu-1},i}$, $\quad i = \pm 1, \ldots, \pm 2^{\nu-1}$. Thus, with $\mu =$ either ν or $\nu - 1$, it now follows by virtue of (2.16) through (2.19) that

$$\begin{aligned}
&\|x(\cdot) - \sigma_{r,2^\mu}(x;\cdot)\|_{L_p(I_{2^\mu,\pm i})} \\
&\leq \frac{2}{(r-1)!}\left|1 \mp t_{2^\mu,\pm i}\right|^{-\alpha}\left|t_{2^\mu,i} - t_{2^\mu,\pm(i-1)}\right|^r \left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{2^\mu,\pm i})} \\
&\leq c2^{\beta\alpha\mu}\left(2^\mu - |i| + 1\right)^{-\beta\alpha} 2^{-\beta r\mu}\left(2^\mu - |i| + 1\right)^{(\beta-1)r}\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{2^\mu,\pm i})} \\
&= 2^{-\beta(r-\alpha)\mu}\left(2^\mu - |i| + 1\right)^{\beta(r-\alpha)-r}\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{2^\mu,\pm i})} \\
&\qquad i = \pm 1, \ldots, \pm(2^\mu - 1),
\end{aligned}$$

and

$$\begin{aligned}
\|x(\cdot) - \sigma_{r,2^\mu}(x;\cdot)\|_{L_p(I_{2^\mu,\pm 2^\mu})} &\leq c\left|1 \mp t_{2^\mu,\pm(2^\mu-1)}\right|^{r-\alpha}\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{2^\mu,\pm 2^\mu})} \\
&\leq c2^{-\beta(r-\alpha)\nu}\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{2^\mu,\pm 2^\mu})},
\end{aligned}$$

where we recall that β is defined in (2.5) and that $c = c(r,\alpha,p,q)$. Hence, for all $i = 1, \ldots, 2^{\nu-1}$, we have

$$\begin{aligned}
&\left\|\sigma_{r,2^\nu}(x;\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot)\right\|_{L_p(I_{2^\nu,\pm 2i})} \\
&\leq c2^{-\beta(r-\alpha)\nu}\left(2^\nu - |2i| + 1\right)^{\beta(r-\alpha)-r}\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{2^{\nu-1},\pm i})},
\end{aligned}$$

and

$$\begin{aligned}&\left\|\sigma_{r,2^\nu}(x;\cdot)-\sigma_{r,2^{\nu-1}}(x;\cdot)\right\|_{L_p(I_{2^\nu,\pm 2i\mp 1})}\\&\le c2^{-\beta(r-\alpha)\nu}\left(2^\nu-|2i-1|+1\right)^{\beta(r-\alpha)-r}\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{2^{\nu-1},\pm i})}.\end{aligned}$$

Now, similar to (3.4),

$$\begin{aligned}&\left\|\sigma_{r,2^\nu}(x;\cdot)-\sigma_{r,2^{\nu-1}}(x;\cdot)\right\|_{L_\infty(I_{2^\nu,\pm 2i})}\\&\le c\left|I_{2^\nu,\pm 2i}\right|^{-\frac{1}{p}}\left\|\sigma_{r,2^\nu}(x;\cdot)-\sigma_{r,2^{\nu-1}}(x;\cdot)\right\|_{L_p(I_{2^\nu,\pm 2i})}\\&\le c2^{-\beta(r-\alpha-\frac{1}{p})\nu}\left(2^\nu-2i+1\right)^{\beta(r-\alpha-\frac{1}{p})-r+\frac{1}{p}}\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{2^{\nu-1},\pm i})},\end{aligned}$$

and

$$\begin{aligned}&\left\|\sigma_{r,2^\nu}(x;\cdot)-\sigma_{r,2^{\nu-1}}(x;\cdot)\right\|_{L_\infty(I_{2^\nu,\pm 2i\mp 1})}\\&\le c\left|I_{2^\nu,\pm 2i\mp 1}\right|^{-\frac{1}{p}}\left\|\sigma_{r,2^\nu}(x;\cdot)-\sigma_{r,2^{\nu-1}}(x;\cdot)\right\|_{L_p(I_{2^\nu,\pm 2i\mp 1})}\\&\le c2^{-\beta(r-\alpha-\frac{1}{p})\nu}\left(2^\nu-2i+2\right)^{\beta(r-\alpha-\frac{1}{p})-r+\frac{1}{p}}\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{2^{\nu-1},\pm i})}.\end{aligned}$$

Therefore, it follows that for all $i=1,\dots,2^{\nu-1}$,

$$\begin{aligned}&2^{-\frac{\beta}{q}\nu}\left(2^\nu-2i+1\right)^{\frac{\beta-1}{q}\nu}\left|\sigma_{r,2^\nu}(x;t_{2^\nu,\pm 2i,k})-\sigma_{r,2^{\nu-1}}(x;t_{2^\nu,\pm 2i,k})\right|\\&\le c2^{-\beta(r-\alpha-\frac{1}{p}+\frac{1}{q})\nu}\left(2^\nu-2i+1\right)^{\beta(r-\alpha-\frac{1}{p}+\frac{1}{q})-r+\frac{1}{p}-\frac{1}{q}}\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{2^{\nu-1},\pm i})}\\&=c2^{-(r-\frac{1}{p}+\frac{1}{q})\nu}\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{2^{\nu-1},\pm i})},\\&\quad k=0,\pm1,\dots,\pm(r+1),\end{aligned}\tag{3.11}$$

and

$$\begin{aligned}&2^{-\frac{\beta}{q}\nu}\left(2^\nu-2i+2\right)^{\frac{\beta-1}{q}\nu}\left|\sigma_{r,2^\nu}(x;t_{2^\nu,\pm 2i\mp 1,k})-\sigma_{r,2^{\nu-1}}(x;t_{2^\nu,\pm 2i\mp 1,k})\right|\\&\le c2^{-\beta(r-\alpha-\frac{1}{p}+\frac{1}{q})\nu}\left(2^\nu-2i+2\right)^{\beta(r-\alpha-\frac{1}{p}+\frac{1}{q})-r+\frac{1}{p}-\frac{1}{q}}\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{2^{\nu-1},\pm i})}\\&=c2^{-(r-\frac{1}{p}+\frac{1}{q})\nu}\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{2^{\nu-1},\pm i})},\\&\quad k=0,\pm1,\dots,\pm(r+1),\end{aligned}\tag{3.12}$$

where the points $t_{2^\nu,i,k}$, $k=0,\pm1,\dots,\pm(r+1)$, $i=\pm1,\dots,\pm 2^\nu$ were defined in (3.3), and where we used the definition of β in (2.5) to simplify the final estimates in (3.11) and (3.12). Recall that

$$A_{r,\beta,q,2^\nu}\left(\sigma_{r,2^\nu}(x;\cdot)-\sigma_{r,2^{\nu-1}}(x;\cdot)\right)=:\left(z_{-2(r+1)2^\nu},\dots,z_0,\dots,z_{2(r+1)2^\nu}\right)=:z,$$

where

$$z_j = \big(2(r+1)2^\nu\big)^{-\frac{\beta}{q}}\big(2(r+1)2^\nu - |j| + 1\big)^{\frac{\beta-1}{q}}$$
$$\times\big(\sigma_{r,2^\nu}(x; t_{2(r+1)2^\nu,j}) - \sigma_{r,2^{\nu-1}}(x; t_{2(r+1)2^\nu,j})\big),$$
$$j = 0, \pm 1, \ldots, \pm 2(r+1)2^\nu.$$

Then in view of

$$2^\nu - |i| + 1 \asymp 2(r+1)2^\nu - \big|(r+1)(2i \mp 1) + k\big| + 1,$$
$$k = 0, \pm 1, \ldots, \pm(r+1), \quad i = \pm 1, \ldots, \pm 2^\nu, \quad \nu = 1, 2, \ldots,$$

we obtain by (3.11) and (3.12),

$$\big\|A_{r,\beta,q,2^\nu}\big(\sigma_{r,2^\nu}(x;\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot)\big)\big\|_{l_p^{4(r+1)2^\nu+1}} = \|z\|_{l_p^{4(r+1)2^\nu+1}}$$
$$\leq c2^{-(r-\frac{1}{p}+\frac{1}{q})\nu}\big\|x^{(r)}\rho^\alpha\big\|_{L_p(I)} \leq c2^{-(r-\frac{1}{p}+\frac{1}{q})\nu},$$

where $c = c(r, \alpha, p, q)$. This shows that for every $x \in W^r_{p,\alpha}$, $A_{r,\beta,q,2^\nu}$ maps $\big(\sigma_{r,2^\nu}(x;\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot)\big)$ into the ball $c2^{-(r-\frac{1}{p}+\frac{1}{q})\nu}B_p^{4(r+1)2^\nu+1}$.

Note that by (2.9),

$$\big\|x(\cdot) - \sigma_{r,2^N}(x;\cdot)\big\|_{L_q} \leq c2^{-(r-\frac{1}{p}+\frac{1}{q})N}, \quad x \in W^r_{p,\alpha}.$$

Thus (3.10) yields

$$E\big(W^r_{p,\alpha}, \Sigma^{m_0,\ldots,m_N}\big)_{L_q} \leq c2^{-(r-\frac{1}{p}+\frac{1}{q})N}$$
$$+ c\sum_{\nu=1}^N 2^{-(r-\frac{1}{p}+\frac{1}{q})\nu}E\big(B_p^{4(r+1)2^\nu+1}, L^{m_\nu}\big)_{l_q^{4(r+1)2^\nu+1}}, \tag{3.13}$$

where $c = c(r, \alpha, p, q)$. Let L^{m_ν} be so that

$$E\big(B_p^{4(r+1)2^\nu+1}, L^{m_\nu}\big)_{l_q^{4(r+1)2^\nu+1}} \leq 2d_{m_\nu}\big(B_p^{4(r+1)2^\nu+1}\big)_{l_q^{4(r+1)2^\nu+1}}, \quad \nu = 1, \ldots, N.$$

Then by (3.13),

$$E\big(W^r_{p,\alpha}, \Sigma^{m_0,\ldots,m_N}\big)_{L_q} \leq c2^{-(r-\frac{1}{p}+\frac{1}{q})N}$$
$$+ c\sum_{\nu=1}^N 2^{-(r-\frac{1}{p}+\frac{1}{q})\nu}d_{m_\nu}\big(B_p^{4(r+1)2^\nu+1}\big)_{l_q^{4(r+1)2^\nu+1}}, \tag{3.14}$$

where $c = c(r, \alpha, p, q)$.

Excluding the case $r = p = 1$, let $1 \le p \le 2 < q \le \infty$. We fix a positive integer λ so big that

$$(3.15) \qquad r - \frac{1}{p} - \frac{1}{2}\frac{1}{\lambda - 1} > 0, \quad \text{and} \quad \lambda \ge \left(r - \frac{1}{p} + \frac{1}{2}\right)\left(r - \frac{1}{p} + \frac{1}{q}\right)^{-1},$$

and put $N := \lambda n$, $n \in \mathbb{N}$. Set

$$(3.16) \qquad \begin{aligned} m_\nu &:= 4(r+1)2^\nu + 1, \quad \nu = 1, \dots, n-1, \\ m_\nu &:= 4(r+1)\left\lceil 2^{\frac{N-\nu}{\lambda-1}} \right\rceil, \quad \nu = n, \dots, N, \end{aligned}$$

where $\lceil u \rceil$ denotes the integer ceiling of u. Evidently,

$$\dim \Sigma^{m_0, \dots, m_N} = m_0 + m_1 + \cdots + m_N \le a2^n,$$

where $a = a(r, p, q) \in \mathbb{N}$. Moreover

$$d_{m_\nu}\left(B_p^{4(r+1)2^\nu+1}\right)_{l_q^{4(r+1)2^\nu+1}} = 0, \quad \nu = 1, \dots, n-1.$$

Hence, by (3.14) and (3.15) we have,

$$(3.17) \qquad \begin{aligned} &E\left(W^r_{p,\alpha}, \Sigma^{m_0,\dots,m_N}\right)_{L_q} \le c2^{-(r-\frac{1}{p}+\frac{1}{2})n} \\ &+ c\sum_{\nu=n}^{N} 2^{-(r-\frac{1}{p}+\frac{1}{q})\nu} d_{m_\nu}\left(B_p^{4(r+1)2^\nu+1}\right)_{l_q^{4(r+1)2^\nu+1}}, \end{aligned}$$

where $c = c(r, \alpha, p, q)$. Note that we have used the second inequality in (3.15) to replace the exponent in the first term on the right in (3.14) by the one in the same term in (3.17). If $q < \infty$, then we apply Lemma G1 to obtain,

$$d_m\left(B_p^n\right)_{l_q^n} \le cn^{\frac{1}{q}} m^{-\frac{1}{2}}, \quad m < n,$$

where $c = c(p, q)$. Thus,

$$\begin{aligned} \sum_{\nu=n}^{N} 2^{-(r-\frac{1}{p}+\frac{1}{q})\nu} d_{m_\nu}\left(B_p^{4(r+1)2^\nu+1}\right)_{l_q^{4(r+1)2^\nu+1}} &\le c\sum_{\nu=n}^{N} 2^{-(r-\frac{1}{p})\nu - \frac{1}{2}\frac{N-\nu}{\lambda-1}} \\ &= c2^{-\frac{1}{2}\frac{\lambda n}{\lambda-1}} \sum_{\nu=n}^{\lambda n} 2^{-(r-\frac{1}{p}-\frac{1}{2}\frac{1}{\lambda-1})\nu} \\ &\le c2^{-\frac{1}{2}\frac{\lambda n}{\lambda-1}} 2^{-(r-\frac{1}{p}-\frac{1}{2}\frac{1}{\lambda-1})n} \\ &= c2^{-(r-\frac{1}{p}+\frac{1}{2})n}, \end{aligned}$$

thence by (3.17) we conclude that

$$E\big(W^r_{p,\alpha}, \Sigma^{m_0,\dots,m_N}\big)_{L_q} \le c2^{-(r-\frac{1}{p}+\frac{1}{2})n}, \tag{3.18}$$

where $c = c(r, \alpha, p, q)$.

If $q = \infty$, then we apply Lemma K, noting that $B^n_p \subseteq B^n_2$, $1 \le p \le 2$. Since

$$\begin{aligned}
&\sum_{\nu=n}^{N} 2^{-(r-\frac{1}{p})\nu} d_{m_\nu}\left(B_2^{4(r+1)2^\nu+1}\right)_{l_\infty^{4(r+1)2^\nu+1}} \\
&\le c\sum_{\nu=n}^{N} 2^{-(r-\frac{1}{p})\nu} m_\nu^{-\frac{1}{2}}\left(1+\log\tfrac{4(r+1)2^\nu+1}{m_\nu}\right)^{\frac{3}{2}} \\
&\le c\sum_{\nu=n}^{N} 2^{-(r-\frac{1}{p})\nu} 2^{-\frac{1}{2}\frac{N-\nu}{\lambda-1}}\left(1+\log 2^{\nu-\frac{N-\nu}{\lambda-1}}\right)^{\frac{3}{2}} \\
&= c2^{-\frac{1}{2}\frac{\lambda}{\lambda-1}n}\sum_{\nu=n}^{\lambda n} 2^{-(r-\frac{1}{p}-\frac{1}{2}\frac{1}{\lambda-1})\nu}\left(1+\log 2^{\frac{\lambda}{\lambda-1}(\nu-n)}\right)^{\frac{3}{2}} \\
&\le c2^{-(r-\frac{1}{p}+\frac{1}{2})n}\sum_{\nu=n}^{\lambda n} 2^{-(r-\frac{1}{p}-\frac{1}{2}\frac{1}{\lambda-1})(\nu-n)}(\nu-n+1)^{\frac{3}{2}} \\
&\le c2^{-(r-\frac{1}{p}+\frac{1}{2})n}\int_1^\infty t^{\frac{3}{2}} 2^{-(r-\frac{1}{p}-\frac{1}{2}\frac{1}{\lambda-1})t}\,dt \\
&\le c2^{-(r-\frac{1}{p}+\frac{1}{2})n},
\end{aligned}$$

where $c = c(r, p)$. Hence by (3.17)

$$E\big(W^r_{p,\alpha}, \Sigma^{m_0,\dots,m_N}\big)_{L_\infty} \le c2^{-(r-\frac{1}{p}+\frac{1}{2})n}, \tag{3.19}$$

where $c = c(r, \alpha, p)$.

In case $2 \le p < q \le \infty$, we fix a positive integer λ so big that

$$\lambda \ge \max\left\{3, r\left(r-\frac{1}{p}+\frac{1}{q}\right)^{-1}\right\}. \tag{3.20}$$

Then in particular,

$$r-\frac{1}{p}-\frac{1}{2}\frac{1}{\lambda-1} \ge r-\frac{1}{2}-\frac{1}{2}\frac{1}{\lambda-1} \ge r-\frac{3}{4} > 0, \tag{3.21}$$

that in turn implies

$$\eta(r,p,q) \tag{3.22}$$
$$:= r - \frac{1}{p} + \frac{1}{q} - \frac{1}{q}\left(\frac{1}{p}-\frac{1}{q}\right)\left(\frac{1}{2}-\frac{1}{q}\right)^{-1} - \frac{1}{2}\frac{1}{\lambda-1}\left(\frac{1}{p}-\frac{1}{q}\right)\left(\frac{1}{2}-\frac{1}{q}\right)^{-1} > 0.$$

Again, put $N := \lambda n$, $n \in \mathbb{N}$, and set m_ν according to (3.16). Again, we clearly have $m_0 + m_1 + \cdots + m_N \leq a2^n$, where $a = a(r,p,q) \in \mathbb{N}$. By (3.13) and (3.20) we have

$$(3.23)\qquad \begin{aligned} &E\big(W^r_{p,\alpha}, \Sigma^{m_0,\ldots,m_N}\big)_{L_q} \leq c2^{-rn} \\ &+ c\sum_{\nu=n}^{N} 2^{-(r-\frac{1}{p}+\frac{1}{q})\nu} d_{m_\nu}\left(B_p^{4(r+1)2^\nu+1}\right)_{l_q^{4(r+1)2^\nu+1}}, \end{aligned}$$

We again apply Lemma G1 to obtain

$$d_m\left(B_p^n\right)_{l_q^n} \leq cn^{\frac{1}{q}\left(\frac{1}{p}-\frac{1}{q}\right)/\left(\frac{1}{2}-\frac{1}{q}\right)} m^{-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)/\left(\frac{1}{2}-\frac{1}{q}\right)}, \quad m < n,$$

where $c = c(p,q)$. Thus by virtue of (3.22),

$$\begin{aligned} &\sum_{\nu=n}^{N} 2^{-(r-\frac{1}{p}+\frac{1}{q})\nu} d_{m_\nu}\left(B_p^{4(r+1)2^\nu+1}\right)_{l_q^{4(r+1)2^\nu+1}} \\ &\leq c\sum_{\nu=n}^{N} 2^{-\left(r-\frac{1}{p}+\frac{1}{q}\right)\nu} 2^{\frac{1}{q}\left(\frac{1}{p}-\frac{1}{q}\right)/\left(\frac{1}{2}-\frac{1}{q}\right)\nu} 2^{-\frac{1}{2}\frac{N-\nu}{\lambda-1}\left(\frac{1}{p}-\frac{1}{q}\right)/\left(\frac{1}{2}-\frac{1}{q}\right)} \\ &= c2^{-\frac{1}{2}\frac{\lambda n}{\lambda-1}\left(\frac{1}{p}-\frac{1}{q}\right)/\left(\frac{1}{2}-\frac{1}{q}\right)} \sum_{\nu=n}^{\lambda n} 2^{-\eta(r,p,q)\nu} \\ &\leq c2^{-\frac{1}{2}\frac{\lambda}{\lambda-1}\left(\frac{1}{p}-\frac{1}{q}\right)/\left(\frac{1}{2}-\frac{1}{q}\right)n} 2^{-\eta(r,p,q)n} = c2^{-rn}, \end{aligned}$$

where $c = c(r,p,q)$. Thence by (3.23) we have

$$E\big(W^r_{p,\alpha}, \Sigma^{m_0,\ldots,m_N}\big)_{L_q} \leq c2^{-rn}, \tag{3.24}$$

where $c = c(r,\alpha,p,q)$.

Finally in case $2 \leq p < q = \infty$, we apply Lemma K, noting that

$$\|x\|_{l_2^n} \leq n^{\frac{1}{2}-\frac{1}{p}}\|x\|_{l_p^n}, \quad x \in \mathbb{R}^n.$$

Then $B_p^{4(r+1)2^\nu+1} \subseteq \big(4(r+1)2^\nu+1\big)^{\frac{1}{2}-\frac{1}{p}} B_2^{4(r+1)2^\nu+1}$ so that

$$d_{m_\nu}\big(B_p^{4(r+1)2^\nu+1}\big)_{l_\infty^{4(r+1)2^\nu+1}} \le \big(4(r+1)2^\nu+1\big)^{\frac{1}{2}-\frac{1}{p}} d_{m_\nu}\big(B_2^{4(r+1)2^\nu+1}\big)_{l_\infty^{4(r+1)2^\nu+1}}.$$

It follows by (3.23) that

$$E\big(W_{p,\alpha}^r, \Sigma^{m_0,\dots,m_N}\big)_{L_\infty} \le c2^{-rn} + c\sum_{\nu=n}^{N} 2^{-(r-\frac{1}{2})\nu} d_{m_\nu}\big(B_2^{4(r+1)2^\nu+1}\big)_{l_\infty^{4(r+1)2^\nu+1}},$$

where $c = c(r,\alpha,p)$. Applying Lemma K and (3.21) to obtain

$$\begin{aligned}
&\sum_{\nu=n}^{N} 2^{-(r-\frac{1}{2})\nu} d_{m_\nu}\big(B_2^{4(r+1)2^\nu+1}\big)_{l_\infty^{4(r+1)2^\nu+1}} \\
&\le c\sum_{\nu=n}^{N} 2^{-(r-\frac{1}{2})\nu} m_\nu^{-\frac{1}{2}} \left(1+\log\tfrac{4(r+1)2^\nu+1}{m_\nu}\right)^{\frac{3}{2}} \\
&\le c\sum_{\nu=n}^{\lambda n} 2^{-(r-\frac{1}{2})\nu} 2^{-\frac{1}{2}\frac{\lambda n-\nu}{\lambda-1}} \left(1+\log 2^{\nu-\frac{\lambda n-\nu}{\lambda-1}}\right)^{\frac{3}{2}} \\
&= c2^{-\frac{1}{2}\frac{\lambda}{\lambda-1}n} \sum_{\nu=n}^{\lambda n} 2^{-(r-\frac{1}{2}\frac{\lambda}{\lambda-1})\nu} \left(1+\log 2^{\frac{\lambda}{\lambda-1}(\nu-n)}\right)^{\frac{3}{2}} \\
&\le c2^{-rn} \sum_{\nu=n}^{\lambda n} 2^{-(r-\frac{1}{2}\frac{\lambda}{\lambda-1})(\nu-n)} (\nu-n+1)^{\frac{3}{2}} \\
&\le c2^{-rn} \int_1^\infty t^{\frac{3}{2}} 2^{-(r-\frac{1}{2}\frac{\lambda}{\lambda-1})t}\,dt \le c2^{-rn},
\end{aligned}$$

we conclude that

$$E\big(W_{p,\alpha}^r, \Sigma^{m_0,\dots,m_N}\big)_{L_\infty} \le c2^{-rn}, \tag{3.25}$$

where $c = c(r,\alpha,p)$.

If we put $\Sigma_{r,a2^n} := \Sigma^{m_0,\dots,m_N}$, then (3.2) follows readily from (3.18), (3.19), (3.24) and (3.25), and it in turn implies

$$d_{a2^n}\big(W_{p,\alpha}^r(I)\big)_{L_q(I)} \le \begin{cases} c2^{-\left(r-\frac{1}{p}+\frac{1}{2}\right)n}, & 1 \le p \le 2 < q \le \infty, \\ c2^{-rn}, & 2 \le p < q \le \infty, \end{cases}$$

where $c = c(r,\alpha,p,q)$. It is now a standard technique to obtain (3.1) and the upper bounds in (1.4) are proven.

The one remaining case is $r = p = 1$ and $2 < q < \infty$ (note that $q = \infty$ is excluded by the assumption that $r - \alpha - \frac{1}{p} + \frac{1}{q} > 0$). We put $N := \lceil \frac{q}{2} n \rceil$, $n \in \mathbb{N}$, and set

$$
\begin{aligned}
m_\nu &:= 2^{\nu+3} + 1, \quad \nu = 1, \dots, n-1, \\
m_\nu &:= \lceil n^{-1} 2^n \rceil, \quad \nu = n, \dots, N,
\end{aligned} \tag{3.26}
$$

Then $m_0 + m_1 + \cdots + m_N \le a 2^n$, where $a = a(q) \in \mathbb{N}$. From (3.14) we have

$$
\begin{aligned}
E\big(W^1_{1,\alpha}, \Sigma^{m_0,\dots,m_N}\big)_{L_q} &\le c 2^{-\frac{N}{q}} + c \sum_{\nu=n}^{N} 2^{-\frac{\nu}{q}} d_{m_\nu} \big(B_1^{2^{\nu+3}+1}\big)_{l_q^{2^{\nu+3}+1}} \\
&\le c 2^{-\frac{n}{2}} + c \sum_{\nu=n}^{N} 2^{-\frac{\nu}{q}} d_{m_\nu} \big(B_1^{2^{\nu+3}+1}\big)_{l_q^{2^{\nu+3}+1}},
\end{aligned}
$$

where $c = c(\alpha, q)$. We apply Lemma G1 once more and get

$$
\sum_{\nu=n}^{N} 2^{-\frac{\nu}{q}} d_{m_\nu} \big(B_1^{2^{\nu+3}+1}\big)_{l_q^{2^{\nu+3}+1}} \le c \sum_{\nu=n}^{N} 2^{-\frac{\nu}{q}} 2^{\frac{\nu}{q}} m_\nu^{-\frac{1}{2}} = c(N - n + 1) n^{\frac{1}{2}} 2^{-\frac{n}{2}} \le c n^{\frac{3}{2}} 2^{-\frac{n}{2}},
$$

which in turn implies

$$
E\big(W^1_{1,\alpha}, \Sigma^{m_0,\dots,m_N}\big)_{L_q} \le c n^{\frac{3}{2}} 2^{-\frac{n}{2}},
$$

where $c = c(\alpha, q)$. Hence

$$
d_{a2^n} \big(W^1_{1,\alpha}\big)_{L_q(I)} \le c n^{\frac{3}{2}} 2^{-\frac{n}{2}}, \quad 2 < q < \infty.
$$

Again it is a standard technique to obtain the upper bound in (1.5).

The lower bounds in (1.4) and (1.5) readily follow from Theorem A and the fact that $W^r_p(I) \subseteq W^r_{p,\alpha}(I)$. This completes the proof of Theorem 1. $\square$

§4. Linear widths of the classes $W^r_{p,\alpha}$ in L_q

In order to prove the upper bounds in Theorem 2 we recall the following lemma (see [1]).

Lemma G2. *Let $n, m \in \mathbb{N}$ be so that $m < n$, and let $1 \le p < q \le \infty$, excluding the case $p = 1$ and $q = \infty$. Then*

$$
d_m \big(B_p^n\big)^{lin}_{l_q^n} \asymp \Psi(m, n, p, q) := \begin{cases} \Phi(m, n, p, q), & 1 \le p < q \le p', \\ \Phi(m, n, q', p'), & \max\{p, p'\} < q \le \infty, \end{cases} \tag{4.1}
$$

where $\Phi(m, n, p, q)$ is defined by (2.2), and $\frac{1}{p} + \frac{1}{p'} = 1$. The constants in these two-sided estimates do not depend on m and n.

The following is a corollary of [11, Lemma 3].

Lemma M1. *For all $n, m \in \mathbb{N}$ so that $m < n$ we have*

$$d_{2m+1}\left(B_1^{2n+1}\right)_{l_\infty^{2n+1}}^{lin} \leq \begin{cases} cn^{-\frac{1}{2}}, & n < m^2, \\ cn^{\frac{1}{2}}m^{-\frac{3}{2}}, & n \geq m^2, \end{cases}$$

where $c > 0$ is an absolute constant.

Also, as a corollary of [11, Lemma 5], we get

Lemma M2. *For all $n, m \in \mathbb{N}$ so that $m < n$ we have*

$$d_m\left(B_1^n\right)_{l_\infty^n}^{lin} \leq cm^{-\frac{1}{2}}\left(\log n\right)^{\frac{1}{2}},$$

where $c > 0$ is an absolute constant.

Proof of Theorem 2. Evidently, the mapping $x \mapsto \sigma_{r,n}(x;\cdot)$, defined in the proof of Theorem 1, is a linear mapping defined on $W_{p,\alpha}^r$. Therefore, it follows by (2.25) that

$$E\left(W_{p,\alpha}^r, \Sigma_{r,n}^1\right)_{L_q}^{lin} \leq \begin{cases} cN^{-r+\frac{1}{p}-\frac{1}{q}}, & 1 \leq p \leq q \leq \infty, \\ cN^{-r}, & 1 \leq q \leq p \leq \infty, \end{cases}$$

where $c = c(r, \alpha, p, q)$. Hence, for $N := 4nr - 2r - 2$, if $n > 1$, and $N = r$, if $n = 1$, we obtain

$$d_N\left(W_{p,\alpha}^r\right)_{L_q}^{lin} \leq \begin{cases} cN^{-r+\frac{1}{p}-\frac{1}{q}}, & 1 \leq p \leq q \leq \infty, \\ cN^{-r}, & 1 \leq q \leq p \leq \infty, \end{cases}$$

where $c = c(r, \alpha, p, q)$, which in turn applying standard technique, implies

$$d_n\left(W_{p,\alpha}^r\right)_{L_q}^{lin} \leq \begin{cases} cn^{-r+\frac{1}{p}-\frac{1}{q}}, & 1 \leq p \leq q \leq \infty, \quad n \geq r, \\ cn^{-r}, & 1 \leq q \leq p \leq \infty, \quad n \geq r, \end{cases} \tag{4.2}$$

where $c = c(r, \alpha, p, q)$. The upper bound (4.2) in some cases is the upper estimate we had to show in (1.6), but we have to improve it in the case $1 \leq p < 2 < q \leq \infty$. Indeed, in this case excluding the cases where $r = p = 1$ and $2 < q < \infty$, and $(r, q) = (1, \infty)$ and $1 < p < 2$, we will show that for $n \geq r$,

$$d_n\left(W_{p,\alpha}^r\right)_{L_q}^{lin} \leq \begin{cases} cn^{-r+\frac{1}{p}-\frac{1}{2}}, & 1 \leq p < 2 < q \leq p', \\ cn^{-r+\frac{1}{2}-\frac{1}{q}}, & 1 \leq p < 2 < p' < q \leq \infty, \end{cases} \tag{4.3}$$

where $c = c(r, \alpha, p, q)$.

Observing that the discretization operators $A_{r,\beta,q,2^\nu}$ are linear, we obtain by virtue of (3.10),

$$\begin{aligned} E\big(W^r_{p,\alpha}, \Sigma^{m_0,\dots,m_N}\big)^{lin}_{L_q} &\le c2^{-(r-\frac{1}{p}+\frac{1}{q})N} \\ &+ c\sum_{\nu=1}^{N} 2^{-(r-\frac{1}{p}+\frac{1}{q})\nu} E\big(B_p^{4(r+1)2^\nu+1}, L^{m_\nu}\big)^{lin}_{l_q^{4(r+1)2^\nu+1}}, \end{aligned}$$

where as before $m_0 := r$, $c = c(r,\alpha,p,q)$. We now take subspaces L^{m_ν} so that

$$E\big(B_p^{4(r+1)2^\nu+1}, L^{m_\nu}\big)^{lin}_{l_q^{4(r+1)2^\nu+1}} \le 2d_{m_\nu}\big(B_p^{4(r+1)2^\nu+1}\big)^{lin}_{l_q^{4(r+1)2^\nu+1}}, \quad \nu = 1,\dots,N,$$

then

$$\begin{aligned} E\big(W^r_{p,\alpha}, \Sigma^{m_0,\dots,m_N}\big)^{lin}_{L_q} &\le c2^{-(r-\frac{1}{p}+\frac{1}{q})N} \\ &+ c\sum_{\nu=1}^{N} 2^{-(r-\frac{1}{p}+\frac{1}{q})\nu} d_{m_\nu}\big(B_p^{4(r+1)2^\nu+1}\big)^{lin}_{l_q^{4(r+1)2^\nu+1}}, \end{aligned} \tag{4.4}$$

where $c = c(r,\alpha,p,q)$.

We first consider the case $1 \le p \le 2 < q \le p'$. Recall that we have excluded the case $r = p = 1$, so that we may define the integers λ, N and $\{m_\nu\}$ by (3.15) and (3.16). We note that if $1 \le p \le 2 < q \le p'$ and $q < \infty$, then it follows by Lemma G1 and Lemma G2 that

$$d_m\big(B_p^n\big)^{lin}_{l_q^n} \asymp d_m\big(B_p^n\big)_{l_q^n}.$$

Hence, just as in the proof of (3.18), it follows that (4.4) implies

$$E\big(W^r_{p,\alpha}, \Sigma^{m_0,\dots,m_N}\big)^{lin}_{L_q} \le c2^{-(r-\frac{1}{p}+\frac{1}{2})n}, \tag{4.5}$$

where $c = c(r,\alpha,p,q)$.

For the case $p = 1$, $q = \infty$ and $r > 1$, we use Lemma M1 and Lemma M2. To this end, set

$$M := \left\lceil \frac{2\lambda}{\lambda+1} n \right\rceil,$$

and note that

$$2^\nu \le c\left\lceil 2^{\frac{N-\nu}{\lambda-1}} \right\rceil^2, \quad \nu = n,\dots,M-1,$$

where $c > 0$ is an absolute constant. Thus by (4.4) we have

$$
\begin{aligned}
E\big(W^r_{1,\alpha}, \Sigma^{m_0,\dots,m_N}\big)^{lin}_{L_\infty} \le\ & c2^{-(r-1)N} \\
& + c\sum_{\nu=n}^{M-1} 2^{-(r-1)\nu} d_{m_\nu}\big(B_1^{4(r+1)2^\nu+1}\big)^{lin}_{l_\infty^{4(r+1)2^\nu+1}} \\
& + c\sum_{\nu=M}^{N} 2^{-(r-1)\nu} d_{m_\nu}\big(B_1^{4(r+1)2^\nu+1}\big)^{lin}_{l_\infty^{4(r+1)2^\nu+1}},
\end{aligned}
\tag{4.6}
$$

where $c = c(r,\alpha)$. Lemma M1 yields

$$
\begin{aligned}
\sum_{\nu=n}^{M-1} 2^{-(r-1)\nu} d_{m_\nu}\big(B_1^{4(r+1)2^\nu+1}\big)^{lin}_{l_\infty^{4(r+1)2^\nu+1}} &\le c\sum_{\nu=n}^{M-1} 2^{-(r-1)\nu}2^{-\frac{1}{2}\frac{N-\nu}{\lambda-1}} \\
&= c2^{-\frac{1}{2}\frac{\lambda}{\lambda-1}n}\sum_{\nu=n}^{M-1} 2^{-\left(r-1-\frac{1}{2}\frac{1}{\lambda-1}\right)\nu} \\
&\le c2^{-\frac{1}{2}\frac{\lambda}{\lambda-1}n}2^{-\left(r-1-\frac{1}{2}\frac{1}{\lambda-1}\right)n} \\
&= c2^{-(r-\frac{1}{2})n}.
\end{aligned}
$$

And by virtue of Lemma M2 we have,

$$
\begin{aligned}
&\sum_{\nu=M}^{N} 2^{-(r-1)\nu} d_{m_\nu}\big(B_1^{4(r+1)2^\nu+1}\big)^{lin}_{l_\infty^{4(r+1)2^\nu+1}} \\
&\le c2^{-\frac{1}{2}\frac{\lambda}{\lambda-1}n} c\sum_{\nu=M}^{N} 2^{-(r-1)\nu}2^{\frac{1}{2}\frac{1}{\lambda-1}\nu}\big(\log\big(4(r+1)2^\nu+1\big)\big)^{\frac{1}{2}} \\
&\le c2^{-\frac{1}{2}\frac{\lambda}{\lambda-1}n}\sum_{\nu=M}^{N} 2^{-\left(r-1-\frac{1}{2}\frac{1}{\lambda-1}\right)\nu}\nu^{\frac{1}{2}} \\
&= c2^{-(r-\frac{1}{2})n}\sum_{\nu=M}^{N} 2^{-\left(r-1-\frac{1}{2}\frac{1}{\lambda-1}\right)(\nu-n)}\nu^{\frac{1}{2}} \\
&\le c2^{-(r-\frac{1}{2})n}\sum_{\nu=M}^{N} 2^{-\left(r-1-\frac{1}{2}\frac{1}{\lambda-1}\right)(\nu-n)}(\nu-n)^{\frac{1}{2}} \\
&\le c2^{-(r-\frac{1}{2})n}\int_0^\infty t^{\frac{1}{2}}2^{-\left(r-1-\frac{1}{2}\frac{1}{\lambda-1}\right)t}\,dt \le c2^{-(r-\frac{1}{2})n},
\end{aligned}
$$

where we have used the fact that

$$
\nu \le \tfrac{\lambda+1}{\lambda-1}(\nu-n), \quad \nu = M,\dots,N.
$$

Incorporating the last two inequalities into (4.6) yields,

$$E\big(W^r_{1,\alpha}, \Sigma^{m_0,\dots,m_N}\big)^{lin}_{L_\infty} \le c2^{-(r-\frac{1}{2})n}, \tag{4.7}$$

where $c = c(r,\alpha)$.

Next we consider the case $1 \le p < 2 < p' \le q \le \infty$. Since we exclude the case $(r,q) = (1,\infty)$, we may fix a positive integer λ so that

$$r - 1 + \frac{1}{q} - \frac{1}{2}\frac{1}{\lambda - 1} > 0, \quad \lambda \ge \left(r + \frac{1}{q} - \frac{1}{2}\right)\left(r - \frac{1}{p} + \frac{1}{q}\right)^{-1}.$$

We put $N := \lambda n$, $n \in \mathbb{N}$ and define m_ν by (3.16). Then by (4.4) we have,

$$\begin{aligned} E\big(W^r_{p,\alpha}, \Sigma^{m_0,\dots,m_N}\big)^{lin}_{L_q} &\le c2^{-(r+\frac{1}{q}-\frac{1}{2})n} \\ &+ c\sum_{\nu=1}^{N} 2^{-(r-\frac{1}{p}+\frac{1}{q})\nu} d_{m_\nu}\big(B_p^{4(r+1)2^\nu+1}\big)^{lin}_{l_q^{4(r+1)2^\nu+1}}, \end{aligned} \tag{4.8}$$

Now, it follows from Lemma G2 that

$$d_m\big(B^n_p\big)^{lin}_{l^n_q} \le cn^{1-\frac{1}{p}}m^{-\frac{1}{2}},$$

where $c = c(p,q)$. Hence,

$$\begin{aligned} \sum_{\nu=n}^{N} 2^{-(r-\frac{1}{p}+\frac{1}{q})\nu} d_{m_\nu}\big(B_p^{4(r+1)2^\nu+1}\big)^{lin}_{l_q^{4(r+1)2^\nu+1}} &\le c\sum_{\nu=n}^{N} 2^{-(r-\frac{1}{p}+\frac{1}{q})\nu}2^{(1-\frac{1}{p})\nu}2^{-\frac{1}{2}\frac{N-\nu}{\lambda-1}} \\ &= c2^{-\frac{1}{2}\frac{\lambda}{\lambda-1}n}\sum_{\nu=n}^{\lambda n} 2^{-\left(r-1+\frac{1}{q}-\frac{1}{2}\frac{1}{\lambda-1}\right)\nu} \\ &\le c2^{-(r+\frac{1}{q}-\frac{1}{2})n}, \end{aligned}$$

which together with (4.8) implies

$$E\big(W^r_{p,\alpha}, \Sigma^{m_0,\dots,m_N}\big)^{lin}_{L_q} \le c2^{-(r+\frac{1}{q}-\frac{1}{2})n} \tag{4.9}$$

where $c = c(r,\alpha,p,q)$. We obtain (4.3) by standard methods from (4.5), (4.7) and (4.9).

We finally deal with the two outstanding cases. First, let $r = p = 1$ and $2 < q < \infty$. We set $N := \lceil \frac{q}{2} n \rceil$ and define m_ν by (3.26). Then clearly, $m_0 + m_1 + \cdots + m_N \leq a2^n$, where $a = a(q) \in \mathbb{N}$. From (4.4) we have

$$
\begin{aligned}
E\big(W^1_{1,\alpha}, \Sigma^{m_0,\ldots,m_N}\big)^{lin}_{L_q} &\leq c2^{-\frac{N}{q}} + c\sum_{\nu=n}^{N} 2^{-\frac{\nu}{q}} d_{m_\nu}\big(B_1^{2^{\nu+3}+1}\big)^{lin}_{l_q^{2^{\nu+3}+1}} \\
&\leq c2^{-\frac{n}{2}} + c\sum_{\nu=n}^{N} 2^{-\frac{\nu}{q}} d_{m_\nu}\big(B_1^{2^{\nu+3}+1}\big)^{lin}_{l_q^{2^{\nu+3}+1}},
\end{aligned}
\tag{4.10}
$$

where $c = c(\alpha, q)$. Now, Lemma G2 yields

$$
\begin{aligned}
\sum_{\nu=n}^{N} 2^{-\frac{\nu}{q}} d_{m_\nu}\big(B_1^{2^{\nu+2}+1}\big)^{lin}_{l_q^{2^{\nu+2}+1}} &\leq c\sum_{\nu=n}^{N} 2^{-\frac{\nu}{q}} 2^{\frac{\nu}{q}} m_\nu^{-\frac{1}{2}} \\
&= c(N-n+1)n^{\frac{1}{2}}2^{-\frac{n}{2}} \\
&\leq cn^{\frac{3}{2}}2^{-\frac{n}{2}},
\end{aligned}
$$

which combined with (4.10) gives

$$
E\big(W^1_{1,\alpha}, \Sigma^{m_0,\ldots,m_N}\big)^{lin}_{L_q} \leq cn^{\frac{3}{2}}2^{-\frac{n}{2}},
$$

whence

$$
d_{a2^n}\big(W^1_{1,\alpha}\big)^{lin}_{L_q} \leq cn^{\frac{3}{2}}2^{-\frac{n}{2}}, \quad 2 < q < \infty.
$$

Again standard technique yields (1.7).

Secondly, if $r = 1$, $q = \infty$ and $1 < p < 2$, then we set $N := \lceil \frac{p'}{2} n \rceil$, $n \in \mathbb{N}$ and define m_ν by (3.26). Again we clearly have $m_0 + m_1 + \cdots + m_N \leq a2^n$, where this time, $a = a(p) \in \mathbb{N}$. From (4.4) we have

$$
\begin{aligned}
E\big(W^1_{1,\alpha}, \Sigma^{m_0,\ldots,m_N}\big)^{lin}_{L_\infty} &\leq c2^{-(1-\frac{1}{p})N} + c\sum_{\nu=n}^{N} 2^{-(1-\frac{1}{p})\nu} d_{m_\nu}\big(B_p^{2^{\nu+3}+1}\big)^{lin}_{l_\infty^{2^{\nu+3}+1}} \\
&\leq c2^{-\frac{n}{2}} + c\sum_{\nu=n}^{N} 2^{-\frac{1}{p'}\nu} d_{m_\nu}\big(B_p^{2^{\nu+3}+1}\big)^{lin}_{l_\infty^{2^{\nu+3}+1}}.
\end{aligned}
\tag{4.11}
$$

Again, Lemma G2 implies that

$$
\begin{aligned}
\sum_{\nu=n}^{N} 2^{-\frac{\nu}{p'}} d_{m_\nu}\big(B_p^{2^{\nu+2}+1}\big)^{lin}_{l_\infty^{2^{\nu+2}+1}} &\leq c\sum_{\nu=n}^{N} 2^{-\frac{\nu}{p'}} 2^{\frac{\nu}{p'}} m_\nu^{-\frac{1}{2}} \\
&= c(N-n+1)n^{\frac{1}{2}}2^{-\frac{n}{2}} \leq cn^{\frac{3}{2}}2^{-\frac{n}{2}},
\end{aligned}
$$

which combined with (4.11) yields,

$$E\big(W^1_{p,\alpha}, \Sigma^{m_0,\dots,m_N}\big)^{lin}_{L_\infty} \leq cn^{\frac{3}{2}}2^{-\frac{n}{2}},$$

and in turn,

$$d_{a2^n}\big(W^1_{p,\alpha}\big)^{lin}_{L_\infty} \leq cn^{\frac{3}{2}}2^{-\frac{n}{2}}, \quad 1 < p < 2.$$

Again by standard technique, this concludes the proof of (1.8), and thus that of the upper bounds. The lower bounds follow from Theorem B, as we observe that $W^r_p \subseteq W^r_{p,\alpha}$. This completes our proof. □

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