More on comonotone polynomial approximation

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Abstract. The main achievement of this article is that we show what was to us a surprising conclusion, namely, twice continuously differentiable functions in (0,1) (with some regular behavior at the endpoints) which change monotonicity at least once in the interval are approximable better by comonotone polynomials, than such functions which are merely monotone. We obtain Jackson type estimates for the comonotone polynomial approximation of such functions which are impossible to achieve for monotone approximation.

1. Introduction and main results

Denote by $\mathbf{Y_s}$, $s \geq 1$, the collection of all sets $Y := Y_s : 0 < y_s < \cdots < y_1 < 1$, and let $\Delta^{(1)}(Y)$, be the collection of all functions $f \in \mathbf{C}[0,1]$ which change monotonicity at the points of Y, and are nondecreasing in $[y_1, 1]$. Set $\Pi(x) := \prod_{i=1}^{s}(x - y_i)$. Then for $f \in \mathbf{C}^1(0,1)$, $f \in \Delta^{(1)}(Y)$ iff

$$f'(x)\Pi(x) \geq 0, \quad x \in (0,1).$$

The degree of comonotone approximation by polynomials p_{n-1} of degree not exceeding $n-1$ will be denoted by

$$E_n^{(1)}(f, Y) := \inf_{p_{n-1} \in \Delta^{(1)}(Y)} \|f - p_{n-1}\|.$$

Here and below $\|\cdot\|$ denotes the uniform norm over $[0,1]$.

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Let $\varphi(x) := \sqrt{x(1-x)}$ and set

$$\varphi_\delta(x) := \sqrt{(1-x-\tfrac{\delta}{2}\varphi(x))(x-\tfrac{\delta}{2}\varphi(x))}, \qquad x \pm \tfrac{\delta}{2}\varphi(x) \in [0,1].$$

Let $\mathbf{C}^r_\varphi$, $r \in \mathbf{N}$, be the space of those functions $f \in \mathbf{C}^r(0,1)$ for which

$$\lim_{x \to 1/2 \pm 1/2} \varphi^r(x) f^{(r)}(x) = 0,$$

and for completion, put $\mathbf{C}^0_\varphi := \mathbf{C}[0,1]$. For $f \in \mathbf{C}^r_\varphi$ and $k \geq 1$, denote

$$\omega^\varphi_{k,r}(f^{(r)}, t) := \sup_{0 \leq h \leq t} \sup_x |\varphi^r_{kh}(x) \Delta^k_{h\varphi(x)} f^{(r)}(x)|, \quad t \geq 0,$$

where

$$\Delta^k_{h\varphi(x)} g(x) := \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} g(x - (k-j)h\varphi(x)/2),$$

and the inner supremum is taken over all x so that

$$x \pm \frac{k}{2} h\varphi(x) \in (0,1).$$

Note that for $f \in \mathbf{C}[0,1]$,

$$\omega^\varphi_{k,0}(f,t) := \omega^\varphi_k(f,t),$$

where the latter is the Ditzian–Totik [1] modulus of smoothness. In a recent paper [5] we have investigated the validity of the statements: for every $f \in \Delta^{(1)}(Y) \cap \mathbf{C}^r_\varphi$,

$$E^{(1)}_n(f,Y) \leq \frac{C}{n^r} \omega^\varphi_{k,r}(f^{(r)}, \frac{1}{n}), \quad n \geq N, \tag{1}$$

where C and N are constants, independent of n and f. Among other results we have shown in [5] that

Theorem LS1. *If $f \in \mathbf{C}^r_\varphi \cap \Delta^{(1)}(Y)$, with $r > 2$, then*

$$E^{(1)}_n(f,Y) \leq \frac{C(k,r,Y)}{n^r} \omega^\varphi_{k,r}(f^{(r)}, \frac{1}{n}), \quad n \geq k + r,$$

and

$$E^{(1)}_n(f,Y) \leq \frac{C(k,r,s)}{n^r} \omega^\varphi_{k,r}(f^{(r)}, \frac{1}{n}), \quad n \geq N(k,r,Y).$$

We are now able to answer what happens in most cases where $r \leq 2$. In the sequel $c = c(s)$, denotes different constants which are either absolute constants or depend only on s, while $C(Y)$ and $N(Y)$ denote constants which depend on Y. We first have

Theorem 1. *If $f \in \mathbf{C}^1_\varphi \cap \Delta^{(1)}(Y)$, then*

$$E_n^{(1)}(f,Y) \le c\frac{1}{n}\omega_{2,1}^{\varphi}(f',\frac{1}{n}), \quad n \ge N(Y), \tag{2}$$

and

$$E_n^{(1)}(f,Y) \le C(Y)\frac{1}{n}\omega_{2,1}^{\varphi}(f',\frac{1}{n}), \quad n \ge 3. \tag{2'}$$

An immediate consequence of Theorem 1 is,

Corollary 1. *If $f \in \mathbf{C}^2_\varphi \cap \Delta^{(1)}(Y)$, then*

$$E_n^{(1)}(f,Y) \le c\frac{1}{n^2}\omega_{1,2}^{\varphi}(f'',\frac{1}{n}), \quad n \ge N(Y), \tag{3}$$

and

$$E_n^{(1)}(f,Y) \le C(Y)\frac{1}{n^2}\omega_{1,2}^{\varphi}(f'',\frac{1}{n}), \quad n \ge 3. \tag{3'}$$

Remark 1. It should be noted that in the cases $r = 1$, $k = 2$, and $r = 2$, $k = 1$, the estimates are better than in the purely monotone approximation case in that we have no estimates of the form (3) and (3′), and (2) and (2′), respectively (see [3]).

Another consequence of Theorem 1 is,

Corollary 2. *If $f \in \Delta^{(1)}(Y)$, and*

$$E_n(f) \le \frac{1}{n^2}, \quad \forall n \ge 3,$$

then

$$E_n^{(1)}(f,Y) \le C(Y)\frac{1}{n^2}, \quad \forall n \ge 3,$$

and

$$E_n^{(1)}(f,Y) \le c\frac{1}{n^2}, \quad \forall n \ge N(Y).$$

On the other hand we are going to show that,

Theorem 2. *For each $Y \in \mathbf{Y}_s$ there is a function $f \in \mathbf{C}^2[0,1] \cap \Delta^{(1)}(Y)$, such that*

$$\limsup_{n\to\infty} \frac{n^2 E_n^{(1)}(f,Y)}{\omega_{3,2}^{\varphi}(f'',\frac{1}{n})} = \infty.$$

By virtue of properties of the moduli of smoothness, we immediately get,

Corollary 3. *Let $0 \le r \le 2$, and $k \ge 5 - r$. Then for each $Y \in \mathbf{Y}_s$ there is a function $f \in \mathbf{C}^r_\varphi \cap \Delta^{(1)}(Y)$, such that*

$$\limsup_{n\to\infty} \frac{n^r E_n^{(1)}(f,Y)}{\omega_{k,r}^{\varphi}(f^{(r)},\frac{1}{n})} = \infty. \tag{4}$$

Remark 2. Corollary 3 is partially known. The cases $r=1$, $k\ge 4$ follow by [2], and the cases $r=0$, $k\ge 5$ follow by [7]. Actually, [7] yields that (4) is true for $r=0$, $k\ge 3$.

Remark 3. The proof of Theorem 2 implies also the validity of the following result for purely monotone approximation. *Let $0\le r\le 2$ and $k\ge 5-r$. Then there is a function $f\in \mathbf{C}^r_\varphi\cap\Delta^{(1)}$, such that*

$$\lim_{n\to\infty}\sup\frac{n^r E_n^{(1)}(f)}{\omega^\varphi_{k,r}(f^{(r)},\frac{1}{n})}=\infty. \tag{5}$$

Here $\Delta^{(1)}:=\Delta^{(1)}(\emptyset)$ is the set of monotone functions, and $E_n^{(1)}(f)$ is the degree of monotone approximation.

Again the cases $r=0$, $k\ge 5$ are known (see [7] where it is proved that (5) is actually true for $r=0$, $k\ge 4$).

Remark 4. Combining all the known results, we can now describe them in truth tables for the inequalities (1).

In the following tables the sign "+" in entry (r,k) means that (1) is valid for this pair of numbers (and the proper s for which the table is given) with constants C and N which depend only on k, r and s; the sign "$\oplus$" means that (1) holds with some of the constants depending also on Y; and "$-$" means that (1) does not hold at all.

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
4	$+$	$+$	$+$	$+$	$+$	$\cdots$
3	$+$	$+$	$+$	$+$	$+$	$\cdots$
2	$-$	$-$	$-$	$-$	$-$	$\cdots$
1	$+$	$-$	$-$	$-$	$-$	$\cdots$
0	$+$	$+$	$-$	$-$	$-$	$\cdots$
	1	2	3	4	5	k

Fig. 1: $s=0$ (the monotone case)

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
$2s+4$	$+$	$+$	$+$	$+$	$+$	$\cdots$
$2s+3$	$+$	$+$	$+$	$+$	$+$	$\cdots$
$2s+2$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
3	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
2	$\oplus$	?	$-$	$-$	$-$	$\cdots$
1	$\oplus$	$\oplus$	?	$-$	$-$	$\cdots$
0	$+$	$\oplus$	$-$	$-$	$-$	$\cdots$
	1	2	3	4	5	k

Fig. 2: $s\ge 1$

Proofs

Let $I=[0,1]$ and put $x_j:=\frac{1}{2}(1-\cos j\pi/n)$, $j=0,\dots,n$. In order to prove Theorem 1 we first denote by $\mathbf{S}_n$ the set of continuous piecewise quadratics with knots at the x_j's. Then we recall the following lemma from [5].

Lemma LS2. *If $s_n\in\mathbf{S}_n\cap\Delta^{(1)}(Y)$, then*

$$E^{(1)}_{cn}(s_n,Y)\le c\omega^\varphi_{3,0}(s_n,\frac{1}{n}).$$

Now we put $I_1 := [0, x_1]$ and $I_n := [x_{n-1}, 1]$, and let

$$l_1(x) := f'(x_1) + (x - x_1)[x_1, 2x_1; f'],$$

be the linear function which interpolates f' at x_1 and $2x_1$; and symmetrically we set

$$l_n(x) := f'(x_{n-1}) + (x - x_{n-1})[x_{n-1} - (1 - x_{n-1}), x_{n-1}; f'].$$

Here and below $[z_0, z_1; g]$ stands for the first divided difference of a function g at the knots z_0 and z_1; and $[z_0, z_1, z_2; g]$ denotes the second divided difference at the knots z_0, z_1 and z_2.

The following lemma is proved along the same lines as [4, Lemma 2].

Lemma LS3. *If $f \in \mathbf{C}^1_\varphi \cap \Delta^{(1)}(Y)$, then there is a continuous piecewise quadratic $\tilde{s}_n \in \mathbf{S}_n$, such that*

$$\|f - \tilde{s}_n\|_I \le \frac{c}{n}\omega^\varphi_{2,1}(f', \frac{1}{n}), \tag{6}$$

$$\Pi(x)\tilde{s}'_n(x) \ge 0, \quad x \in [x_1, x_{n-1}],$$
$$\tilde{s}'_n(x) = l_1(x), \quad x \in I_1,$$

and

$$\tilde{s}'_n(x) = l_n(x), \quad x \in I_n.$$

Finally, recall that for $f \in \mathbf{C}^1_\varphi$, we have

$$\omega^\varphi_{3,0}(f, t) \le ct\omega^\varphi_{2,1}(f', t),$$

and

$$\omega^\varphi_{2,1}(f', \lambda t) \le c\lambda^2\omega^\varphi_{2,1}(f', t), \quad \lambda > 1.$$

We are ready to prove Theorem 1.

Proof of Theorem 1. In order to prove Theorem 1 we first take n sufficiently large so that f is monotone in I_1 and in I_n. Then in view of Lemma LS3 at most what we have to do is correct the behavior of $\tilde{s}_n$ on I_1 and I_n while keeping it close to the original function. To this end it suffices to construct a spline s_n satisfying,

$$\begin{aligned} s_n(x) &= \tilde{s}_n(x), \quad x \in [x_1, x_{n-1}], \\ \Pi(0)s'_n(0) &\ge 0, \\ \Pi(1)s'_n(1) &\ge 0, \end{aligned}$$

and

$$\|s'_n - \tilde{s}'_n\|_{I_1 \cup I_n} \le cn\omega^\varphi_{2,1}(f', \frac{1}{n}). \tag{7}$$

Indeed, since $|I_1|, |I_n| \le c/n^2$, then (7) implies that

$$\|s_n - \tilde{s}_n\|_{I_1\cup I_n} \le \frac{c}{n}\omega^{\varphi}_{2,1}(f', \frac{1}{n}),$$

which combined with (6) implies (2). We will construct s_n on I_1, the other part being similar. If $\Pi(0)l_1(0) \ge 0$, then there is no need for correction and we put

$$s_n'(x) := \tilde{s}_n'(x) = l_1(x), x \in I_1,$$

Thus, we may assume that $\Pi(0)l_1(0) < 0$, furthermore, without loss of generality we may assume that $\Pi(0) > 0$, so that $l_1(0) < 0$. This in turn implies that

$$f'(2x_1) - f'(x_1) > 0.$$

Hence, if we put

$$s_n'(x) := f'(x_1), \quad x \in I_1,$$

then

$$\|s_n' - \tilde{s}_n'\|_{I_1} = \|f'(x_1) - l_1\|_{I_1} = f'(2x_1) - f'(x_1).$$

Now, in order to prove (7), we need to estimate $f'(2x_1) - f'(x_1)$. Put j^* to be so that $2^{j^*}x_1 \le y_s < 2^{j^*+1}x_1$, and note that by taking n sufficiently large we may assume that $j^* \ge 3$. For $j \ne j*$, put $t_j = 2^j x_1$, and set $t_{j^*} := y_s$. Then, for all $j = 0, ..., j^* - 2$,

$$\begin{aligned}
|[t_j, t_{j+1}; f'] - [t_{j+1}, t_{j+2}; f']| &= (t_{j+2} - t_j)|[t_j, t_{j+1}, t_{j+2}, f']| \\
&\le \frac{c}{t_j\sqrt{t_j}}\omega^{\varphi}_{2,1}(f', \sqrt{t_j}) \le \frac{c2^{j/2}}{t_j\sqrt{x_1}}\omega^{\varphi}_{2,1}(f', \sqrt{x_1}) \\
&\le \frac{cn^3}{2^{j/2}}\omega^{\varphi}_{2,1}(f', \frac{1}{n}),
\end{aligned}$$

where for the first inequality we used Newton's formula

$$(t_{j+2} - t_j)(t_{j+2} - t_{j+1})[t_j, t_{j+1}, t_{j+2}, f'] = f'(t_{j+2}) - L_1(f', t_{j+2}; t_j, t_{j+1}),$$

(here $L_1(f', t_{j+2}; t_j, t_{j+1})$ stands for the linear function which interpolates f' at t_j and t_{j+1} evaluated at t_{j+2}) and an easy extension of [5, (2.24)] (note that we need to extend [5, (2.24)] to cover also nonequidistant points since we use it with $t_{j+1} - t_j \ne t_{j+2} - t_{j+1}$). Then for the third inequality we apply the fact that $x_1 \sim \frac{1}{n^2}$. Hence,

$$\begin{aligned}
\frac{f'(2x_1) - f'(x_1)}{x_1} &\le [x_1, 2x_1, f'] - \frac{f'(t_{j^*-1})}{t_{j^*-1} - y_s} \\
&= [t_0, t_1, f'] - [t_{j^*-1}, t_{j^*}; f'] \\
&\le \sum_{j=0}^{j^*-2}(t_{j+2} - t_j)|[t_j, t_{j+1}, t_{j+2}, f']| \\
&\le cn^3\omega^{\varphi}_{2,1}(f', \frac{1}{n})\sum_{j=0}^{\infty} 2^{-j/2} \le cn^3\omega^{\varphi}_{2,1}(f', \frac{1}{n}),
\end{aligned}$$

where we applied the fact that f is nondecreasing at t_{j^*-1}. Recalling again that $x_1 \sim \frac{1}{n^2}$ we obtain (7) and conclude the proof of (2), as well as the proof of (2′) for $n \geq N(Y)$. In order to prove (2′) for $n < N(Y)$,when $s > 1$, one can approximate f by the constant polynomial $p_{n-1}(x) := f(0)$. When $s = 1$, then one can take $p'_{n-1}(x) := L_1(f', x; y_1, a)$, where $|y_1 - a| = 1/4$. $\square$

For the proof of Theorem 2, we begin with $b \in (0,1)$, and we let $S_b(x)$, be a function satisfying,

$$S_b \in \mathbf{C}^4(I); \quad S'_b(x) \geq 0, x \in I; \quad S_b(x) = 1, x \geq 2b; \quad S_b(x) = 0, x \leq b;$$

and

$$||S'_b|| \leq \frac{c}{b},$$

for some absolute constant c. We put

$$g_b(x) := \Pi(x) \int_b^x \frac{x-u}{u^2} du, \quad x \in (0,1],$$

and

$$\tilde{g}_b(x) := \begin{cases} S_b(x) g_b(x), & x \in (0,1], \\ 0 & x = 0; \end{cases}$$

and set

$$G_b(x) := \int_{2b}^x g_b(u) du, \quad x \in (0,1],$$

and

$$\tilde{G}_b(x) := \int_{2b}^x \tilde{g}_b(u) du, \quad x \in [0,1].$$

It is readily seen that

$$\tilde{G}_b \in \mathbf{C}^5(I) \cap \Delta^{(1)}(Y),$$

and

$$||\tilde{G}_b|| \leq \frac{c}{b}, \quad ||\tilde{g}_b|| \leq \frac{c}{b}, \quad ||\tilde{g}'_b|| \leq \frac{c}{b}. \tag{8}$$

First we prove

Lemma 1. We have

$$\omega^{\varphi}_{3,2}(\tilde{G}''_b, t) \leq c(1 + \frac{t^3}{b}).$$

Proof. Since $\tilde{G}''_b(x) - G''_b(x) = 0$, if $x \geq 2b$, and for $x \in (0, 2b)$, we have

$$|\tilde{G}''_b(x) - G''_b(x)| \leq |g'_b(x)| + |g_b(x) S'_b(x)| \leq \frac{c}{x}.$$

Hence

$$\begin{aligned}\omega^{\varphi}_{3,2}(\tilde{G}''_b, t) &\le \omega^{\varphi}_{3,2}(G''_b, t) + \omega^{\varphi}_{3,2}(\tilde{G}''_b - G''_b, t)\\ &\le \omega^{\varphi}_{3,2}(G''_b, t) + 2^3\|\varphi^2(G''_b - \tilde{G}''_b)\|\\ &\le \omega^{\varphi}_{3,2}(G''_b, t) + c.\end{aligned}$$

Rewrite g_b as

$$g_b(x) := \Pi(x)\int_b^1 \frac{x-u}{u^2}du - \Pi(x)\int_x^1 \frac{x-u}{u^2}du =: g_{b,1}(x) - g_{b,2}(x).$$

Then $|g'_{b,2}(x)| \le \frac{c}{x}$, which in turn implies,

$$\omega^{\varphi}_{3,2}(g'_{b,2}, t) \le 8\|\varphi^2 g'_{b,2}\| \le c.$$

Finally, since $\|g^{(4)}_{b,1}\| \le \frac{c}{b}$, then

$$\omega^{\varphi}_{3,2}(g'_{b,1}, t) \le t^3\|g^{(4)}_{b,1}\| \le \frac{ct^3}{b}. \quad \square$$

Denote by P^*_n the set of polynomials p_n of degree $\le n$, such that

$$\Pi(0)p'_n(0) \ge 0.$$

Then we show

Lemma 2. For each $b \in (0, \frac{1}{n^2})$, any $d \in (0,2)$ and every polynomial $p_n \in P^*_n$, we have

$$\|\tilde{G}_b - p_n\|_{[0,d]} \ge \frac{cd}{n^2}\log\frac{1}{n^2 b} - \frac{1}{n^2}.$$

Proof. Put

$$l(x) := \Pi(x)\int_b^{1/n^2} \frac{x-u}{u^2}du, \quad g^*_b(x) := \Pi(x)\int_{1/n^2}^x \frac{x-u}{u^2}du,$$

and

$$L(x) := \int_{2b}^x l(u)du, \quad G^*_b(x) := \int_{2b}^x g^*_b(u)du.$$

Hence,

$$G_b(x) = L(x) + G^*_b(x).$$

It follows that for $p_n \in P^*_n$, we have

$$\Pi(0)p'_n(0) - \Pi(0)L'(0) \ge -\Pi(0)l(0) = \Pi^2(0)\log\frac{1}{n^2 b}. \tag{9}$$

Now, if $x \geq 2b$, then by integration we get,

$$|\tilde{G}_b(x) - L(x)| = |G_b(x) - L(x)| = |G_b^*(x)| \leq cn^2x^2 + \frac{c}{n^2},$$

and at the same time,

$$\begin{aligned}
||\tilde{G}_b - L||_{[0,2b]} &\leq ||\tilde{G}_b||_{[b,2b]} + ||L||_{[0,2b]} \\
&\leq cb||g_b||_{[b,2b]} + cb||l||_{[b,2b]} \\
&\leq cb + cb\log\frac{1}{bn^2} \leq \frac{c}{n^2}.
\end{aligned}$$

Hence, for $x \in [0, d]$,

$$\begin{aligned}
|p_n(x) - L(x)| &\leq ||p_n - \tilde{G}_b||_{[0,d]} + |\tilde{G}_b(x) - L(x)| \\
&\leq ||p_n - \tilde{G}_b||_{[0,d]} + \frac{c}{n^2} + cn^2x^2 \\
&\leq c(||p_n - \tilde{G}_b||_{[o,d]} + \frac{1}{n^2})(1 + n^4x^2) \\
&\leq cn^4(||p_n - \tilde{G}_b||_{[o,d]} + \frac{1}{n^2})(1/n^2 + x)^2.
\end{aligned}$$

Whence, we may apply the following Dzjadyk type inequality (see e.g. Lemma 14.1 and (14.9) in [6]),

$$\begin{aligned}
|p_n'(0) - L'(0)| &\leq \frac{cn^4}{d}(||p_n - \tilde{G}_b||_{[o,d]} + \frac{1}{n^2})(1/n^2)^{2-1} \\
&= \frac{cn^2}{d}(||p_n - \tilde{G}_b||_{[o,d]} + \frac{1}{n^2}),
\end{aligned}$$

which combined with (9) yields,

$$||p_n - \tilde{G}_b||_{[o,d]} \geq cd\frac{|\Pi(0)|}{n^2}\log\frac{1}{n^2b} - \frac{1}{n^2}. \quad \square$$

We are now ready to prove Theorem 2 by constructing the counterexample.

Proof of Theorem 2. Let

$$b_n := \frac{1}{n^2\log n}, \quad d_n := \frac{1}{\sqrt{\log\log n}}, n > 2,$$

and set

$$f_n(x) := cb_n\tilde{G}_{b_n}(x),$$

where c is taken so small that (11) and (12) below are fulfilled. We summarize the properties of f_n as follow from (8) and Lemma 1. Namely, for every $n > 2$,

$$f_n \in \mathbf{C}^5(I) \cap \Delta^{(1)}(Y),$$

$$f_n'(x) = 0, \quad x \in [0, b_n], \tag{10}$$

$$||f_n|| < 1, \quad ||f_n'|| < 1, \quad ||f_n''|| < 1, \tag{11}$$

and

$$\omega_{3,2}^{\varphi}(f_n'', \frac{1}{n}) \leq b_n. \tag{12}$$

Lemma 2 yields the existence of n_1 such that for all $n \geq n_1$ and $p_n \in P_n^*$ we have,

$$\begin{aligned} ||f_n - p_n||_{[0,d_n]} &= cb_n||\tilde{G}_{b_n} - q_n||_{[0,d_n]} \\ &\geq c_* \sqrt{\log\log n}\, b_n \frac{1}{n^2}, \end{aligned}$$

for some suitable constant $c_* > 0$, where we note that $q_n := \frac{1}{b_n} p_n \in P_n^*$.

Finally we put $D_0 := 1$ and

$$D_\sigma := \frac{b_{n_1}}{n_1^2} \cdots \frac{b_{n_\sigma}}{n_\sigma^2},$$

where n_σ is defined by induction as follows. Suppose that $n_1, \ldots, n_{\sigma-1}$, $\sigma > 1$, have been selected. We write

$$F_{\sigma-1}(x) := \sum_{j=1}^{\sigma-1} D_{j-1} f_{n_j}(x)$$

and take $n_\sigma > n_{\sigma-1}$ so big that

$$d_{n_\sigma} = \frac{1}{\sqrt{\log\log n_\sigma}} < b_{n_{\sigma-1}}, \tag{13}$$

and

$$\frac{||F_{\sigma-1}^{(5)}||}{n_\sigma^3} < D_{\sigma-1} b_{n_\sigma}.$$

Now let

$$\Phi_\sigma(x) := \sum_{j=\sigma}^{\infty} D_{j-1} f_{n_j}(x),$$

where the uniform convergence of the series as well as its twice term-by-term differentiation, is justified by (11). In fact by virtue of (11),

$$||\Phi_\sigma|| < 2D_{\sigma-1}, \quad ||\Phi_\sigma''|| < 2D_{\sigma-1}.$$

Finally setting

$$f(x) := \Phi_1(x),$$

we have

$$\begin{aligned}
\omega^{\varphi}_{3,2}(f'', \frac{1}{n_\sigma}) &\le \omega^{\varphi}_{3,2}(F''_{\sigma-1}, \frac{1}{n_\sigma}) + \omega^{\varphi}_{3,2}(D_{\sigma-1} f''_{n_\sigma}, \frac{1}{n_\sigma}) + \omega^{\varphi}_{3,2}(\Phi''_{\sigma+1}, \frac{1}{n_\sigma}) \\
&\le \frac{1}{n_\sigma^3} \|F^{(5)}_{\sigma-1}\| + D_{\sigma-1} b_{n_\sigma} + 2^4 D_\sigma \\
&< 18 D_{\sigma-1} b_{n_\sigma}.
\end{aligned}$$

On the other hand, by (10) and (13),

$$f(x) =: A_\sigma + \Phi_\sigma(x) = A_\sigma + D_{\sigma-1} f_{n_\sigma}(x) + \Phi_{\sigma+1}(x), \quad x \in [0, d_{n_\sigma}],$$

for some constant A_σ. Hence for each $p_{n_\sigma} \in P^*_{n_\sigma}$,

$$\begin{aligned}
\|f - p_{n_\sigma}\| &\ge \|f - p_{n_\sigma}\|_{[0.d_{n_\sigma}]} = \|\Phi_\sigma - q_{n_\sigma}\|_{[0.d_{n_\sigma}]} \\
&\ge D_{\sigma-1} \|f_{n_\sigma} - \frac{1}{D_{\sigma-1}} q_{n_\sigma}\|_{[0.d_{n_\sigma}]} - \|\Phi_{\sigma+1}\| \\
&\ge c_* \sqrt{\log\log n_\sigma} D_\sigma - 2D_\sigma = D_\sigma (c_* \sqrt{\log\log n_\sigma} - 2),
\end{aligned}$$

where we used $q_{n_\sigma} := p_{n_\sigma} - A_\sigma \in P^*_{n_\sigma}$. Hence, for all $\sigma \ge 1$,

$$\begin{aligned}
\frac{\|f - p_{n_\sigma}\|}{\frac{1}{n_\sigma^2} \omega^{\varphi}_{3,2}(f'', \frac{1}{n_\sigma})} &\ge \frac{D_\sigma (c_* \sqrt{\log\log n_\sigma} - 2)}{\frac{1}{n_\sigma^2} 18 D_{\sigma-1} b_{n_\sigma}} \\
&= \frac{c_*}{18} \sqrt{\log\log n_\sigma} - \frac{1}{9}.
\end{aligned}$$

This concludes the proof of Theorem 2. $\square$

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