NEARLY COMONOTONE APPROXIMATION II

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Dedicated to Ferenc Móricz on the occasion of his 60th birthday

Abstract. When we approximate a continuous function f which changes its monotonicity finitely many, say s times, in $[-1,1]$, we wish sometimes that the approximating polynomials follow these changes in monotonicity. However, it is well known that this requirement restricts very much the degree of approximation that the polynomials can achieve, namely, only the rate of $\omega_2(f,1/n)$ and even this not with a constant (dependent only on s), rather with a constant which depends on the location of the interior extrema. Recently, we proved that relaxing the comonotonicity requirement in intervals of length proportional to $1/n$ about the interior extrema of the function and in intervals of length $1/n^2$ near the endpoints, what we called nearly comonotone approximation, allows the polynomials to achieve a pointwise approximation rate of ω_3 (moreover, with a constant which depends only on s). We show here that even when we relax the requirement of monotonicity of the polynomials on sets of measures approaching 0 (no matter how slowly or how fast), ω_4 is not reachable. We conclude the paper with results on the sizes of the deleted sets which allow the ω_3 degree of approximation in the norm; and when f is differentiable, allow estimates involving the kth modulus of smoothness of f'.

§1. Introduction

Let $f \in \mathbb{C}[-1,1]$ change monotonicity finitely many, say $s \ge 1$ times, on $I := [-1,1]$. Specifically, let $\mathbb{Y}_s$ be the set of all collections $Y := \{y_i\}_{i=1}^s$ of points, $-1 < y_s < \dots < y_1 < 1$. In the proof it will be convenient to put $y_{s+1} := -1$ and $y_0 := 1$. For $Y \in \mathbb{Y}_s$ we set

$$\Pi(x) = \Pi(x,Y) := \prod_{i=1}^{s}(x-y_i),$$

and denote by $\Delta^{(1)}(Y)$ the collection of functions $f \in \mathbb{C}[-1,1]$, which change monotonicity at the points y_i, $1 \le i \le s$, and which are nondecreasing in $(y_1,1)$, that is, f is nondecreasing in the intervals (y_{2j+1},y_{2j}) and it is nonincreasing in (y_{2j},y_{2j-1}). A polynomial $p_n \in \mathcal{P}_n$, the space of polynomials of degree not exceeding n, is said to be comonotone with f, on a set $E \subset I$, if and only if

$$p_n'(x)\Pi(x,Y) \ge 0, \quad x \in E.$$

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The degree of approximation of $f \in \Delta^{(1)}(Y)$, by comonotone polynomials, is measured in the sup-norm $\|\cdot\| := \|\cdot\|_I$. Since we need the sup-norm on other intervals we will in all cases of an interval $J \neq I$, indicate that by writing $\|\cdot\|_J$. For $f \in \Delta^{(1)}(Y)$, Shvedov [Sv] proved that there exist polynomials which are comonotone with f on I, and such that

$$\|f - P_n\| \le c(Y)\omega_2(f, 1/n), \quad n \ge 1, \tag{1.1}$$

where $\omega_k(f,\cdot)$ denotes the modulus of smoothness of order k, of f. The constant $c(Y)$ depends on the location of the points Y and cannot be replaced by a constant which depends only on the number of interior extrema. (If one substitute $\omega := \omega_1$ for ω_2 in (1.1), then such an estimate does hold with a constant c which depends only on the number of interior extrema (see e.g. Newman [N])). On the other hand, one cannot replace ω_2 in (1.1) by ω_k with any $k \ge 3$, not even with a constant c which depends on f and only for sufficiently large n, (see Wu and Zhou [WZ]). It is quite natural to ask whether one can strengthen (1.1) in the sense of being able to replace ω_2 by moduli of smoothness of higher order, if one is willing to allow P_n not to be comonotone with f on a rather "small" subset of I. This indeed is the case if we allow the polynomials not to be comonotone with f in intervals of length proportional to $1/n$ about each interior extrema and in intervals of length $1/n^2$ near the end points, what we have called nearly comonotone approximation. We recently proved [LS1], that in such a case it is possible to achieve the estimates

$$\|f - P_n\| \le c(s)\omega_3(f, 1/n), \tag{1.2}$$

(see also Theorem 2 below). However, this improvement cannot be extended to ω_4, and thus not to ω_k for any $k > 3$. In [DLS], we have shown a similar phenomenon in monotone approximation and in order to prove our assertion here we will use similar notation and technique. In general it seems more difficult to force the approximating polynomials to be comonotone with the function when the latter changes monotonicity than when it is nondecreasing throughout the interval, thus the degree of approximation suffers. Thus, at first it is not clear why should one present a proof of Theorem 1 below, however, we should mention that we recently found out that when f is twice differentiable, then one can guarantee better degree of comonotone approximation if f changes monotonicity at least once, than in the monotone case (see [LS3]).

For our first result we need some notation. Given $Y \in \mathbb{Y}_s$, an $\epsilon > 0$ and a function $f \in \Delta^{(1)}(Y)$, we denote

$$E_n^{(1)}(f;\epsilon;Y) := \inf_{P_n} \|f - P_n\|,$$

where the infimum is taken over all polynomials $P_n \in \mathcal{P}_n$ satisfying

$$\operatorname{meas}(\{x \,:\, P_n'(x)\Pi(x,Y) \ge 0\} \cap I) \ge 2 - \epsilon.$$

We will write $E_n^{(1)}(f;Y) := E_n^{(1)}(f;0;Y)$.

In Section 2 we prove

Theorem 1. *Given $Y \in \mathbb{Y}_s$. For each sequence $\bar{\epsilon} = \{\epsilon_n\}_{n=1}^{\infty}$, of nonnegative numbers tending to 0, there exists a function $f := f_{\bar{\epsilon}} \in \Delta^{(1)}(Y)$ such that*

$$\limsup_{n\to\infty} \frac{E_n^{(1)}(f;\epsilon_n;Y)}{\omega_4(f,1/n)} = \infty. \tag{1.3}$$

Note that we allow the relaxing of monotonicity (on small sets) anywhere in I, not necessarily near the interior extrema or the endpoints, and ω_4 cannot be had.

In [LS1], we proved (1.2) provided the comonotonicity requirements of the polynomials P_n, are relaxed in intervals about the extrema of lengths proportional to $1/n$, provided the proportion is not too small. This was due to our proof in [LS1] of the stronger pointwise estimates. Here we are interested in norm estimates and we show that in contrast to pointwise approximation, the proportion constant can be taken arbitrarily small if we are willing to pay in C.

In order to state our next result we need some more notation. For each $h > 0$ and $x \in I$ we write $\varphi(x) := \sqrt{1-x^2}$ and $\rho_h(x) := h^2 + h\varphi(x)$, and we let

$$O(h,Y) := I \cap \cup_{i=1}^{s}(y_i - \rho_h(y_i), y_i + \rho_h(y_i)),$$
$$O^*(h,Y) := O(h,Y) \cup [-1,-1+h^2] \cup [1-h^2,1].$$

Finally we use $\omega_k^{\varphi}(g,t)$ to denote the kth Ditzian–Totik [DT] modulus of smoothness. Then we have,

Theorem 2. *For each $A > 0$, there is a constant $C = C(s,A)$, for which if $Y \in \mathbb{Y}_s$ and $f \in \Delta^{(1)}(Y)$, then for every $n \ge 2$ a polynomial $P_n \in \mathcal{P}_n$ which is comonotone with f on $I \setminus O^*(A/n,Y)$ exists, such that*

$$\|f - P_n\| \le C\omega_3^{\varphi}(f,1/n) \le C\omega_3(f,1/n). \tag{1.4}$$

If we assume that f is differentiable, then we can further obtain,

Theorem 3. *For each $k \ge 1$ and $A > 0$, there is a constant $C = C(k,s,A)$, for which if $Y \in \mathbb{Y}_s$ and $f \in \Delta^{(1)}(Y) \cap \mathbb{C}^1[-1,1]$, then for every $n \ge k$, a polynomial $P_n \in \mathcal{P}_n$, which is comonotone with f on $I \setminus O(A/n,Y)$ exists, such that*

$$\|f - P_n\| \le C\frac{1}{n}\omega_k^{\varphi}(f',1/n) \le C\frac{1}{n}\omega_k(f',1/n). \tag{1.5}$$

If we are willing to allow P_n not to be comonotone with f also in neighborhoods of the endpoints ± 1, then we may strengthen (1.5). To this end, for $f \in \mathbb{C}^1(-1,1)$, so that $\lim_{x\to\pm 1}\varphi(x)f'(x) = 0$, we denote

$$\omega_{k,1}^{\varphi}(f',t) := \sup_{0\le h\le t}\sup_{x}(1 - x - \frac{kh}{2}\varphi(x))^{1/2}(1 + x - \frac{kh}{2}\varphi(x))^{1/2}|\Delta_{h\varphi(x)}^{k} f'(x)|, \quad t \ge 0,$$

where the inner supremum is taken over all x so that $x \pm (kh/2)\varphi(x) \in (-1,1)$, and

$$\Delta_{h\varphi(x)}^{k} g(x) := \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} g(x - (k-j)\frac{h}{2}\varphi(x)).$$

Then we have

Theorem 4. *For each $k \ge 1$ and $A > 0$, there is a constant $C = C(k,s,A)$, for which if $Y \in \mathbb{Y}_s$ and $f \in \Delta^{(1)}(Y) \cap \mathbb{C}^1(-1,1)$ and $\lim_{x\to\pm1}\varphi(x)f'(x) = 0$, then for every $n \ge k$, a polynomial $P_n \in \mathcal{P}_n$ which is comonotone with f on $I \setminus O^*(A/n, Y)$ exists, such that*

$$\|f - P_n\| \le C\frac{1}{n}\omega^{\varphi}_{k,1}(f', 1/n). \tag{1.6}$$

Remark. Although we have assumed that $s \ge 1$, we remark that Theorems 2–4 are valid also in the monotone case, i.e., $s = 0$ and $O = \emptyset$. Furthermore, Theorems 2 and 4 are new also in that case (the deleted neighborhoods obviously are just about the endpoints of the interval), while Theorem 3 is well known.

In Section 3 we prove some auxiliary lemmas, and then we prove Theorems 2–4 in Section 4.

§2. Construction of a counterexample

Given $Y \in \mathbb{Y}_s$, let $[y_{i^*+1}, y_{i^*}]$ be an interval of the largest length, and let I_0 be its middle one third. Then it is readily seen that $|[y_{i^*+1}, y_{i^*}]| \ge 1/(s+1)$, $\|\Pi\|_{[-2,2]} < 3^s$ and

$$\min_{x\in I_0}|\Pi(x)| \ge \left(\frac{1}{3(s+1)}\right)^s =: a. \tag{2.1}$$

For $\nu > 10$, let

$$t_\nu(x) := \cos\nu\arccos\frac{x}{2}, \quad x \in [-2,2],$$

denote the Chebyshev polynomial of degree ν, for the interval $[-2,2]$, and let $z_j := 2\cos\frac{j\pi}{\nu}$, $j = 0,\dots,\nu$, be its extrema. Given $0 < b < \frac{1}{2}$, we take two points on both sides of z_j, $j = 1,\dots,\nu-1$, namely, we set $z_{j,l} := 2\cos(\frac{(j+b)\pi}{\nu})$ and $z_{j,r} := 2\cos(\frac{(j-b)\pi}{\nu})$. Then

$$|t_\nu(z_{j,l})| = |t_\nu(z_{j,r})| = \cos\pi b,$$

and

$$z_{j,r} - z_{j,l} = 4\sin\frac{j\pi}{\nu}\sin\frac{\pi b}{\nu} < 4\pi\frac{b}{\nu}. \tag{2.2}$$

We also need the truncated Chebyshev polynomial, namely,

$$t^*_\nu(x) := t^*_{\nu,b}(x) := \begin{cases} -\cos\pi b, & t_\nu(x) < -\cos\pi b \\ t_\nu(x), & \textit{otherwise}, \end{cases}$$

and we put

$$t_{\nu,b} := a^{-1}(t_\nu + \cos\pi b)\Pi, \quad \text{and} \quad \tilde{t}_{\nu,b} := a^{-1}(t^*_{\nu,b} + \cos\pi b)\Pi.$$

It is readily seen that

$$
\begin{aligned}
(2.3) \qquad \|\tilde{t}_{\nu,b}\| &\le \|t_{\nu,b}\| \le \|t_{\nu,b}\|_{[-2,2]} \\
&\le 2a^{-1}\|\Pi\|_{[-2,2]} < 2a^{-1}3^s =: M^*.
\end{aligned}
$$

Now, define a function $g = g_{\nu,b}$, on $[-1,1]$, by defining it on each interval $[z_{2j+1}, z_{2j-1}] \cap [-1,1]$. To this end, first we have at most $2s+2$ intervals $[z_{2j+1}, z_{2j-1}]$ such that $Y \cap (z_{2j+2}, z_{2j-2}) \ne \emptyset$, or $1 \in (z_{2j+1}, z_{2j-1}]$, or $-1 \in [z_{2j+1}, z_{2j-1})$, which we call of type I. For these we put

$$g(x) := \tilde{t}_{\nu,b}(x), \quad x \in [z_{2j+1}, z_{2j-1}].$$

All other intervals $[z_{2j+1}, z_{2j-1}] \subset [-1,1]$, are of type II and satisfy $Y \cap (z_{2j+2}, z_{2j-2}) = \emptyset$. For these intervals our definition requires some preliminaries. First we note that for each $1 \le i \le s$ and $x \in [z_{2j+1}, z_{2j-1}]$, we have

$$(2.4) \qquad \frac{1}{4} \le \frac{x - y_i}{z_{2j} - y_i} \le 4,$$

which in turn implies that

$$(2.5) \qquad \frac{1}{4^s} \le \frac{\Pi(x)}{\Pi(z_{2j})} \le 4^s.$$

Indeed, without loss of generality assume that $y_i \le z_{2j+2}$. Then if $x > z_{2j}$, then

$$
\begin{aligned}
1 \le \frac{x - y_i}{z_{2j} - y_i} &\le \frac{z_{2j-1} - y_i}{z_{2j} - y_i} \\
&\le 1 + \frac{z_{2j-1} - z_{2j}}{z_{2j} - z_{2j+1}} \le 4,
\end{aligned}
$$

while if $x < z_{2j}$, then

$$
\begin{aligned}
1 \ge \frac{x - y_i}{z_{2j} - y_i} &\ge \frac{z_{2j+1} - y_i}{z_{2j} - y_i} \\
&\ge 1 - \frac{z_{2j} - z_{2j+1}}{z_{2j} - z_{2j+2}} \ge \frac{1}{4}.
\end{aligned}
$$

Thus (2.4) is proved. Now, by virtue of (2.5), it follows that there is a constant $B < 1/2$, depending only on s, such that if $b < B$, then

$$(2.6) \qquad \Pi(z_{2j}) \int_{z_{2j+1}}^{z_{2j-1}} t_{\nu,b}(x) > 0.$$

For each $0 \le M \le M^*$, where M^* is defined in (2.3), and $x \in [z_{2j+1}, z_{2j-1}]$, we set

$$g_{j,M}(x) := \begin{cases} M\operatorname{sgn}\Pi(z_{2j}), & \text{if} \quad |\tilde{t}_{\nu,b}(x)| > M, \\ \tilde{t}_{\nu,b}(x), & \text{otherwise.} \end{cases}$$

We now define the function g in intervals $[z_{2j+1}, z_{2j-1}]$, of type II where $\Pi(z_{2j}) > 0$. The case where $\Pi(z_{2j}) < 0$ requires obvious modifications and is left to the reader. Now,

$$\int_{z_{2j+1}}^{z_{2j-1}} (t_{\nu,b}(x) - g_{j,M^*}(x))\,dx = \int_{z_{2j+1}}^{z_{2j-1}} (t_{\nu,b}(x) - \tilde{t}_{\nu,b}(x))\,dx < 0,$$

and by (2.6),

$$\int_{z_{2j+1}}^{z_{2j-1}} (t_{\nu,b}(x) - g_{j,0}(x))\,dx = \int_{z_{2j+1}}^{z_{2j-1}} t_{\nu,b}(x) > 0.$$

Hence a constant $0 < M_j < M^*$ exists, such that

$$\int_{z_{2j+1}}^{z_{2j-1}} t_{\nu,b}(x)\,dx = \int_{z_{2j+1}}^{z_{2j-1}} g_{j,M_j}(x)\,dx. \tag{2.7}$$

Thus we define

$$g(x) := g_{j,M_j}(x), \quad x \in [z_{2j+1}, z_{2j-1}].$$

This completes the definition of $g_{\nu,b}$, and in view of the fact that $g(z_{2j\pm1}) = 0$, for all $z_{2j\pm1} \in [-1,1]$, it follows that it is continuous in $[-1,1]$. We also note that

$$g(x)\Pi(x) \geq 0, \quad x \in [-1,1]. \tag{2.8}$$

Next we observe that for any interval $[z_{2j+1}, z_{2j-1}]$, of either type I or type II, we have by virtue of (2.2) and (2.7),

$$\begin{aligned}
&\int_{I\cap[z_{2j+1},z_{2j-1}]} |t_{\nu,b}(x) - g(x)|\,dx \\
&\leq 2\Big(\int_{z_{2j+1}}^{z_{2j+1,r}} + \int_{z_{2j-1,l}}^{z_{2j-1}}\Big)|t_{\nu,b}(x)|\,dx \\
&\leq 2((z_{2j+1,r} - z_{2j+1}) + (z_{2j-1} - z_{2j-1,l}))(1-\cos\pi b)\|\Pi\| \\
&\leq 16\pi\frac{b}{\nu}5b^2 3^s := C_0\frac{b^3}{\nu},
\end{aligned} \tag{2.9}$$

where this defines C_0, and where we made use of

$$1 - \cos\pi b = 2\sin^2\frac{\pi b}{2} < 5b^2.$$

Here and in the sequel we denote by C_0, C_1, and so on, different constants which may depend only on s.

Given $n \geq 1$ and $0 < b < B$, we let $\nu := [b^{\frac{3}{4}}n] + 11$, and put

$$T_{\nu,b}(x) := \int_0^x t_{\nu,b}(u)\,du \quad \text{and} \quad f_{n,b}(x) := \int_0^x g_{\nu,b}(u)\,du, \quad x \in I.$$

From (2.8) we conclude that $f_{n,b}$ is in $\Delta^{(1)}(Y)$. Also, it readily follows by (2.7) and (2.9) that

$$\|f_{n,b} - T_{\nu,b}\| \le C_0(2s+4)\frac{b^3}{\nu} =: C_1\frac{b^3}{\nu} \le C_1\frac{b^{\frac{9}{4}}}{n}, \tag{2.10}$$

where we have counted the $2s+2$ intervals of type I, and possibly two intervals of type II, namely, those containing 0 and x, respectively.

We now proceed as in [DLS] with minor changes due to the fact that we have to work inside the interval I_0, the middle third of $[y_{i^*+1}, y_{i^*}]$, defined above (2.1). Note that [DLS, (2.4)] is exactly our inequality (2.10); inequality [DLS, (2.6)] holds for odd j and $x \in I_0 \cap [\tilde{z}_{j.l}, \tilde{z}_{j,r}]$; and [DLS, Lemma 1] holds for any $J \subset I_0$, instead of $J \subset I$. Also, in view of the somewhat different definition of $T_{\nu,b}$, we need the analogous estimate of $\|T^{(4)}_{\nu,b}\|$, and to this end we observe that since $I \subset [-2,2]$, it follows by the Bernstein inequality that

$$\begin{aligned} \|T^{(4)}_{\nu,b}\| &= \|t^{(3)}_{\nu,b}\| \\ &\le \|t_{\nu,b}\|_{[-2,2]}\frac{C_2(\nu+s)^3}{(1-(\frac{1}{2})^2)^{3/2}} \\ &\le M^*\frac{C_2(1+s)^3\nu^3}{(1-(\frac{1}{2})^2)^{3/2}} =: C_3\nu^3. \end{aligned} \tag{2.11}$$

We now define $f_{\overline{\epsilon}} := f$, for a given sequence $\overline{\epsilon} = \{\epsilon_n\}$, along similar lines to [DLS]. Let $b_n := (\max\{\epsilon_n^2, \frac{1}{n}\})^{2/5}$, and set $d_0 := 1$, and

$$d_j := \frac{b_{n_j}^{9/4}}{n_j}d_{j-1} = \prod_{\nu=1}^{j}\frac{b_{n_\nu}^{9/4}}{n_\nu}, \quad j \ge 1,$$

where the sequence $\{n_\nu\}$ is defined by induction as follows. First, we choose n_1 so large that $b_{n_1} < \min\{B, |I_0|^8\}$, so that it satisfies (2.13) below and $J_0 := I_0$. Suppose that $\{n_1, \dots, n_{\sigma-1}\}$ and $J_{\sigma-2} \subseteq J_{\sigma-3} \subseteq \cdots \subseteq J_0$, $\sigma \ge 2$, have been defined. Then put

$$F_{\sigma-1} := \sum_{j=1}^{\sigma-1} d_{j-1}f_{n_j,b_{n_j}},$$

and let $J_{\sigma-1}$ be an interval such that $J_{\sigma-1} \subseteq J_{\sigma-2}$ and

$$F'_{\sigma-1}(x) = 0, \quad x \in J_{\sigma-1}. \tag{2.12}$$

(The induction process will guarantee the existence of such intervals.) Let $N_{1,\sigma}$ be such that

$$\min\{B, |J_{\sigma-1}|^8\} \ge b_n, \quad n \ge N_{1,\sigma}, \tag{2.13}$$

and let

$$N_{2,\sigma} := \left(\frac{\|F^{(2)}_{\sigma-1}\|}{d_{\sigma-1}} \right)^{10}.$$

Finally, we take

$$n_\sigma > \max\{n_{\sigma-1}, N_{1,\sigma}, N_{2,\sigma}\}$$

so big that the function $f'_{n_\sigma,b_{n_\sigma}}$ oscillates a few times inside the interval $J_{\sigma-1}$ and since it vanishes on some interval in each oscillation, that is, inside $J_{\sigma-1}$, there exists an interval $J_\sigma \subset J_{\sigma-1}$ as required in (2.12).

Now denote

$$\Phi_\sigma := \sum_{j=\sigma}^{\infty} d_{j-1} f_{n_j,b_{n_j}},$$

where the convergence of the series is justified by the definition of the d_j's and the fact that $\|f_{n,b_n}\| \le M^*$, for all n. Indeed,

$$\begin{aligned} \|\Phi_\sigma\| &\le M^* d_{\sigma-1}\left(1 + \frac{b_{n_\sigma}^{9/4}}{n_\sigma} + \frac{b_{n_\sigma}^{9/4} b_{n_{\sigma+1}}^{9/4}}{n_\sigma n_{\sigma+1}} + \dots\right) \\ &\le M^* d_{\sigma-1} \sum_{j=0}^{\infty} 2^{-j} = 2M^* d_{\sigma-1}. \end{aligned} \tag{2.14}$$

Also, since $\|f'_{n,b_n}\| \le M^*$, for all n, then the series is differentiable term by term, which in turn implies that $\Phi_\sigma \in \Delta^{(1)}(Y)$, for all $\sigma \ge 1$. So we define

$$f := f_{\overline{\epsilon}} := \sum_{j=1}^{\infty} d_{j-1} f_{n_j,b_{n_j}},$$

then $f \in \Delta^{(1)}(Y)$ and in particular f is monotone in I_0. Without loss of generality we may assume that f is nondecreasing in I_0. Now it follows by (2.11) and (2.14) (see [DLS, Lemma 2]),

Lemma 1. *For each $\sigma \ge 1$ we have*

$$\omega_4(f, 1/n_\sigma) \le C_4 d_\sigma.$$

The last lemma that we need is

Lemma 2. *There is an absolute constant C_5 such that whenever a measurable $E \subset I$ satisfies*

$$|E| \le \epsilon_{n_\sigma},$$

and P_{n_σ} is a polynomial satisfying

$$P'_{n_\sigma}(x) \ge 0, \quad x \in I_0 \setminus E,$$

then

$$\|f-P_{n_\sigma}\| \ge (b_{n_\sigma}^{-1/8}-C_5)d_\sigma.$$

Proof. Since f is nondecreasing on I_0, our proof follows using [DLS, Lemma 1], exactly as in [DLS, Lemma 3]. □

The proof of (1.3) now follows from Lemmas 1 and 2 since

$$\limsup_{n\to\infty}\frac{E_n^{(1)}(f;\epsilon_n;Y)}{\omega_4(f,1/n)} \ge \limsup_{\sigma\to\infty}\frac{E_{n_\sigma}^{(1)}(f;\epsilon_{n_\sigma};Y)}{\omega_4(f,1/n_\sigma)}$$
$$\ge \limsup_{\sigma\to\infty}\frac{1}{C_4}(b_{n_\sigma}^{-1/8}-C_5)=\infty.$$

This concludes our proof. □

§3. Some auxiliary results

In this section we replace I by a general fixed interval $[a,b]$ and we will use the notation $\omega_k(f;t;[a,b])$ to be the usual kth modulus of smoothness of f restricted to the $[a,b]$. Also, we assume that Y is a collection of s points in (a,b). For $m\ge 1$, let

$$I_j := \left[a+\tfrac{(j-1)(b-a)}{m}, a+\tfrac{j(b-a)}{m}\right], \quad j=1,\dots,m,$$

and set

$$\hat O_i := \hat O_{i,m}(Y) := \left[a+\tfrac{(j-2)(b-a)}{m}, a+\tfrac{(j+1)(b-a)}{m}\right]\cap[a,b],$$
$$\text{if} \quad y_i\in\left[a+\tfrac{(j-1)(b-a)}{m}, a+\tfrac{j(b-a)}{m}\right), \quad 1\le i\le s,$$

and

$$\hat O := \cup_{i=1}^s \hat O_i, \quad \text{and} \quad \hat O^* := \hat O\cup I_1\cup I_m.$$

For $f\in\mathbb{C}[a,b]$, denote by

$$L(x;f;t_1,\dots,t_k)$$

the Lagrange polynomial of degree $\le k-1$, interpolating f at the points $t_1,\dots,t_k\in[a,b]$. Recall that

$$\|f-L(\cdot;f;t_1,\dots,t_k)\| \le c(k,m)\omega_k(f;b-a;[a,b]), \tag{3.1}$$

provided that $|t_\nu-t_\mu|\ge\frac{b-a}{km}$, $1\le\nu,\mu\le k$. In particular (3.1) implies that if $\tau_\mu-\tau_\nu\ge\frac{b-a}{m}$, then

$$\|f\|_{[a,b]} \le c(k,m)\|f\|_{[\tau_\nu,\tau_\mu]}+c(k,m)\omega_k(f;b-a;[a,b]). \tag{3.2}$$

Here and for the rest of Section 3, constants $c(k,m)$ may depend only on k and m, while constants $c(m)$ may depend only on m. Constants may differ in different occurrences, even if they appear in the same line.

Denote by $\Delta^{(1)}(Y)[a,b]$ the collection of functions $f\in\mathbb{C}[a,b]$, which change monotonicity at the points y_i, $1\le i\le s$, and which are nondecreasing in (y_1,b). We first have

Lemma 3. *If $f \in \Delta^{(1)}(Y)[a,b]$, then a quadratic polynomial P exists, such that*

$$\|f - P\|_{[a,b]} \le c(m)\omega_3(f; b-a; [a,b]), \tag{3.3}$$

and P is comonotone with f on $I \setminus \hat{O}^*$.

Proof. Let $J_\nu = [a_\nu, b_\nu]$ denote the closed interval components of $\hat{O}^*$, enumerated so that $a = a_0 < b_0 < a_1 < \cdots < a_l < b_l = b$, and $l \le s+1$. We have four cases to consider.
(i). f is monotone on $[a,b] \setminus \hat{O}^*$. Then, either we have that one of the polynomials

$$P_1(x) := L(x; f; b_0 - (b-a)/m, b_0, a_l), \quad P_2(x) := L(x; f; b_0, a_l, a_l + (b-a)/m),$$

is monotone on $[b_0, a_l]$, obviously having the same monotonicity as f outside $\hat{O}^*$, and by (3.1) satisfying Lemma 3; or the linear polynomial $P_3(x)$, which interpolates f at the points b_0 and a_l, necessarily is enclosed in between P_1 and P_2 on $[a,b]$. Since both P_1 and P_2 satisfy (3.1), we conclude that P_3 satisfies Lemma 3.
Otherwise,
(ii). Suppose that there exists $\nu \ne 0, l$, such that f is nonincreasing on $[a, a_\nu] \setminus \hat{O}^*$ and is nondecreasing on $[b_\nu, b] \setminus \hat{O}^*$. Since the changes in monotonicity of f occur at distances of at least $\frac{b-a}{m}$ from both a_ν and b_ν, it is readily seen that either $f(a_\nu) \le f(b_\nu)$, and one can find a point $t_{\nu_1} \in [a_\nu + \frac{b-a}{m}, b_\nu]$, so that $f(t_{\nu_1}) = f(a_\nu)$; or $f(a_\nu) > f(b_\nu)$, and then there is a point $t_{\nu_2} \in [a_\nu, b_\nu - \frac{b-a}{m}]$, so that $f(t_{\nu_2}) = f(b_\nu)$. Since $f(b_0) \ge f(a_\nu)$ and $f(a_l) \ge f(b_\nu)$, hence (respectively), either the polynomial

$$P(x) := L(x; f; b_0, a_\nu, t_{\nu_1}),$$

or the polynomial

$$P(x) := L(x; f; t_{\nu_2}, b_\nu, a_l),$$

satisfy Lemma 3.
(iii). If there exists $\nu \ne 0, l$, such that f is nondecreasing on $[a, a_\nu] \setminus \hat{O}^*$ and is nonincreasing on $[b_\nu, b] \setminus \hat{O}^*$, then obviously we have a case similar to (ii), and the proof is along the same lines.
Finally,
(iv). In general none of the above holds. Then we observe that there exist four points $a \le \tau_0 < \tau_1 < \tau_2 < \tau_3 \le b$, such that $\tau_j - \tau_{j-1} \ge \frac{b-a}{m}, \quad j = 1, \dots, 3$, for which either

$$f(\tau_0) \le f(\tau_1), \quad f(\tau_1) \ge f(\tau_2), \quad \text{and} \quad f(\tau_2) \le f(x), \quad x \in [\tau_2, \tau_3];$$

or

$$f(\tau_0) \ge f(\tau_1), \quad f(\tau_1) \le f(\tau_2), \quad \text{and} \quad f(\tau_2) \ge f(x), \quad x \in [\tau_2, \tau_3].$$

Now for $g(x) := f(x) - f(\tau_2)$, let $L(x) := L(x; g; \tau_0, \tau_1, \tau_2)$. Then

$$g(x)L(x) \le 0 \quad x \in [\tau_2, \tau_3].$$

Therefore

$$|g(x)| \le |g(x) - L(x)|, \quad x \in [\tau_2, \tau_3].$$

Hence by virtu of (3.1),

$$\|g\|_{[\tau_2,\tau_3]} \le c(m)\omega_3(f; b-a; [a,b]).$$

This in turn yields by (3.2),

$$\|g\|_{[a,b]} \le c(m)\|g\|_{[\tau_2,\tau_3]} + c(m)\omega_3(f; b-a; [a,b]) \le c(m)\omega_3(f; b-a; [a,b]).$$

Thus polynomial $P(x) \equiv f(\tau_2)$ satisfies Lemma 3. This completes the proof of Lemma 3. □

Next we need a lemma on copositive approximation which is interesting in itself. We say that $f \in \Delta^{(0)}(Y)[a,b]$, if $f \in \mathbb{C}[a,b]$ and $f(x)\Pi(x,Y) \ge 0$ on $[a,b]$, and that a polynomial p is copositive with f on a set $E \subset [a,b]$, if $p(x)\Pi(x,Y) \ge 0$ on E. Then we prove

Lemma 4. *If $f \in \Delta^{(0)}(Y)[a,b]$, then for each $k \ge 1$ a polynomial p of degree $k-1$ exists, such that*

$$\|f - p\|_{[a,b]} \le c(k,m)\omega_k(f; b-a; [a,b]),$$

and p is copositive with f on $[a,b] \setminus \hat{O}$.

Proof. We denote by $J_\nu = [a_\nu, b_\nu]$ the closed interval components of the set $\hat{O}$, enumerated so that $a \le a_1 < \cdots < a_l < b_l \le b$, and $l \le s$.

First assume that $l \ge k$. Then we can choose arbitrary points $t_1 \in Y \cap J_1$, $t_2 \in Y \cap J_2$,..., $t_k \in Y \cap J_k$. Since $f \in \Delta^{(0)}(Y)[a,b]$, then we have $f(t_j) = 0$, $j = 1, \dots, k$, and evidently $t_j - t_{j-1} \ge \frac{b-a}{m}$. Hence by (3.1)

$$\|f\| = \|f - L(\cdot; f; t_1, \dots, t_k)\| \le c(k,m)\omega_k(f; b-a; [a,b]).$$

Thus, $p \equiv 0$ satisfies Lemma 4.

In all other cases we have $k > l$. Let $J_{j_1}, \dots, J_{j_r}$, $0 \le r \le l < k$, be those intervals which contain an odd number of the points y_i. (Note that there may be none, in which case $r = 0$.) Denote by $t_1, \dots, t_r$ the middle points of these intervals and put

$$R(x) := \prod_{\nu=1}^{r} (x - t_\nu).$$

(If $r = 0$, then $R \equiv 1$.)

Evidently, R is copositive with f on $[a,b] \setminus \hat{O}$ and we have,

$$|R(x)| \ge c_1, \quad x \in [a,b] \setminus \hat{O},$$

where $c_1 = c_1(k,m)$. If p^* is the polynomial of degree $k-1$ of best approximation to f on $[a,b]$, and $E := \|f - p^*\|_{[a,b]}$, then it is readily seen that

$$p(x) := p^*(x) + \frac{E}{c_1} R(x),$$

satisfies the requirements of Lemma 4. This completes the proof. □

We apply Lemma 4 in

Lemma 5. *If $f \in \Delta^{(1)}(Y)[a,b] \cap \mathbb{C}^1[a,b]$, then for each $k \geq 1$ a polynomial P of degree k exists, such that*

$$\|f - P\|_{[a,b]} \leq c(k,m)(b-a)\omega_k(f'; b-a; [a,b]). \tag{3.4}$$

and P is comonotone with f on $[a,b] \setminus \hat{O}$.

Proof. Since $f' \in \Delta^{(0)}(Y)[a,b]$, let p be the polynomial of degree $k-1$ which is copositive with it, which is guaranteed by Lemma 4. Then the polynomial

$$P(x) := f(a) + \int_a^x p(u)du,$$

satisfies the requirements of Lemma 5. □

§4. Proofs of Theorems 2–4

It is more convenient to prove Theorems 2–4, when we replace the neighborhoods $(y_i - \rho_h(y_i), y_i + \rho_h(y_i))$, by equivalent neighborhoods which are described by the Chebyshev partition. Since there is going to be no confusion, we will basically use for the new neighborhoods the same notation. This switch also allows us to use results and notations from [LS2]. Namely, let $x_{j,n} := \cos j\pi/n$, $j = 0, \ldots, n$, and put $x_{-1,n} := 1$ and $x_{n+1,n} := -1$. Finally, let $I_{j,n} = [x_{j,n}, x_{j-1,n}]$, $j = 1, \ldots, n$, be the intervals of the Chebyshev partition. Given Y, let

$$O_i(Y,n) := [x_{j+1,n}, x_{j-2,n}] \quad \text{if} \quad y_i \in [x_{j,n}, x_{j-1,n}),$$

and set

$$O(Y,n) := \cup_{i=1}^s O_i(Y,n),$$
$$O^*(Y,n) := O(Y,n) \cup I_{1,n} \cup I_{n,n}.$$

Recall that in [LS2] we have denoted by $\tilde{O}_\nu$ the closed interval components of $O := O(Y,n)$, and we let $\Sigma_{k,O}$, denote the set of continuous piecewise polynomials of degree $k-1$, on the nth Chebyshev partition, which are polynomials on each $\tilde{O}_\nu$. Here we need the analogous notation for $O^* := O^*(Y,n)$, namely, we let $\tilde{O}^*_\nu$ be the closed interval components of O^*, and we set Σ_{k,O^*}, to be the subset of $\Sigma_{k,O}$, containing those piecewise polynomials which are polynomials on each $\tilde{O}^*_\nu$. In the proofs below we need notation for the endpoints of $\tilde{O}^*_\nu$, namely we let $[a_\nu, b_\nu] := \tilde{O}^*_\nu$. Further, we use in addition to O and O^*, also the sets $O(Y,nm)$ and $O^*(Y,nm)$ for different Y's and for various m's. We assume throughout this section that $m \geq 2$, to be properly chosen in each case.

In the course of the proofs in this section we make use of the following proposition (see [LS2, Proposition 3]).

Proposition A. *If* $S \in \Sigma_{k,O(Y,n)} \cap \Delta^{(1)}(Y)$*, then*

$$E^{(1)}_{C_1 n}(S,Y) \le C_2 \omega^{\varphi}_m(S, \frac{1}{n}),$$

where $C_1 = C_1(k,s)$ *and* $C_2 = C_2(k,s)$*.*

We first prove

Proof of Theorem 2. We construct a piecewise polynomial

$$S \in \Sigma_{3,O^*(Y,n)} \tag{4.1}$$

which is comonotone with f on $I \setminus O^*(Y,nm)$, and sufficiently close to it. To this end, we note that by [LS1, Lemma 1], for $I_{j,n} \not\subset O^*(Y,n)$, there is a quadratic polynomial $p_j = p_{j,n}$, which interpolates f at both endpoints of $I_{j,n}$, and is comonotone with f on $I_{j,n}$, for which

$$\|f - p_j\|_{I_{j,n}} \le C\omega_3(f; |I_{j,n}|; [x_{j+1,n}, x_{j-2,n}]).$$

Since $\omega_3(f; |I_{j,n}|; [x_{j+1,n}, x_{j-2,n}]) \le C\omega^{\varphi}_3(f, 1/n)$, in turn we have

$$\|f - p_j\|_{I_{j,n}} \le C_3 \omega^{\varphi}_3\Big(f, \frac{1}{n}\Big). \tag{4.2}$$

Here and for the rest of the proof, constants C are either absolute constants, or depend only on s, while constants c may depend on s and m, they may differ in different occurrences, even if they appear in the same line.

Each $\tilde{O}^*_\nu$ consists of at most $(3s+1)$ intervals $I_{j,n}$, for each of which we have $|I_{j,n}| \le |\tilde{O}^*_\nu| \le C|I_{j,n}|$. It also consists of at most $(3s+1)m$ intervals $I_{j,nm}$, and if we put $\tilde{O}^{**}_\nu$ for $\tilde{O}^*_\nu$ augmented on both sides (or one side, if $\pm 1 \in \tilde{O}^*_\nu$), by the adjacent intervals $I_{j,nm}$, then $|\tilde{O}^*_\nu| \sim |\tilde{O}^{**}_\nu| \sim sm|I_{j,nm}|$, (where the notation $\mathcal{A} \sim \mathcal{B}$ ($\mathcal{A}$ and $\mathcal{B}$ depend on n) means as usual, that there are absolute constants $0 < \alpha < \beta$, so that $\alpha < \mathcal{A}/\mathcal{B} < \beta$). Hence, as in Lemma 3, we construct a quadratic polynomial q_ν, which is comonotone with f on $\tilde{O}^*_\nu \setminus O^*(Y,nm)$, and such that

$$\|f - q_\nu\|_{\tilde{O}^*_\nu} \le c\omega_3(f; |\tilde{O}^{**}_\nu|; \tilde{O}^{**}_\nu).$$

Also, it follows that $\omega_3(f; |\tilde{O}^{**}_\nu|; \tilde{O}^{**}_\nu) \le C\omega^{\varphi}_3(f, 1/n)$, which in turn implies

$$\|f - q_\nu\|_{\tilde{O}^*_\nu} \le c_1 \omega^{\varphi}_3\Big(f, \frac{1}{n}\Big). \tag{4.3}$$

Now let

$$\tilde{S}(x) := \begin{cases} q_\nu(x), & \text{if} \quad x \in \tilde{O}^*_\nu, \\ p_j(x), & \text{if} \quad x \in I_{j,n} \setminus O^*, \end{cases}$$

then we have a piecewise polynomial with possible discontinuities at the endpoints a_ν and b_ν which are in $(-1,1)$. We create out of it a continuous function, in the following way. Denote

$$\alpha_\nu := q_\nu(a_\nu) - f(a_\nu) \quad \text{and} \quad \beta_\nu := f(b_\nu) - q_\nu(b_\nu),$$

and let

$$\alpha(x) := \sum_{a_\nu > x} \alpha_\nu + \sum_{b_\nu \ge x} \beta_\nu, \quad -1 \le x \le 1.$$

Since the number of $\tilde{O}_\nu^*$'s does not exceed $s+2$, it readily follows by (4.3) that

$$|\alpha(x)| \le 2(s+2)c_1\omega_3^\varphi\Big(f, \frac{1}{n}\Big), \quad x \in I. \tag{4.4}$$

Thus we put $S := \tilde{S} + \alpha$, and we have a continuous piecewise polynomial, which is clearly comonotone with f on $I \setminus O^*(Y, nm)$, satisfying (4.1), and by virtue of (4.2) through (4.4),

$$\begin{aligned} \|f - S\| &\le (\max\{C_3, c_1\} + 2(s+2)c_1)\omega_3^\varphi\Big(f, \frac{1}{n}\Big) \\ &=: c_2\omega_3^\varphi\Big(f, \frac{1}{n}\Big). \end{aligned} \tag{4.5}$$

Since S is a quadratic polynomial on each $\tilde{O}_\nu^*$, then it changes monotonicity at most once there. Hence, there is $Y_* \subset O^*(Y, nm)$ containing $s_* \le s+2$ points, such that either $S \in \Delta^{(1)}(Y_*)$, or $-S \in \Delta^{(1)}(Y_*)$. Since $m \ge 2$, it follows that $O(Y_*, 2n) \subset O^*(Y, n)$, whence

$$S \in \Sigma_{3, O(Y_*, 2n)}. \tag{4.6}$$

We apply Proposition A, and conclude that there is a polynomial p of degree $\le C_4 n$, which is comonotone with S everywhere on I, and such that

$$\|S - p\| \le C_5\omega_3^\varphi\Big(S, \frac{1}{2n}\Big). \tag{4.7}$$

It is readily seen that p is comonotone with f on $I \setminus O^*(Y, nm)$, and combining (4.5) and (4.7) we have

$$\begin{aligned} \|f - p\| &\le \|f - S\| + \|S - p\| \le \|f - S\| + C_5\omega_3^\varphi(S, \tfrac{1}{2n}) \\ &\le \|f - S\| + C_5\omega_3^\varphi(S - f, \tfrac{1}{2n}) + C_5\omega_3^\varphi(f, \tfrac{1}{2n}) \\ &\le (1 + 2^3C_5)\|f - S\| + C_5\omega_3^\varphi(f, \tfrac{1}{2n}) \\ &\le (1 + 2^3C_5)c_2\omega_3^\varphi(f, \tfrac{1}{n}) + C_5\omega_3^\varphi(f, \tfrac{1}{2n}) \\ &\le c_3\omega_3^\varphi(f, \tfrac{1}{n}). \end{aligned} \tag{4.8}$$

Given $n \ge C_4$ and $m \ge 2$, we apply the above to $n_1 := [n/C_4]$, and $m_1 \ge mC_4$ so big that nm divides n_1m_1. Then we observe that $O^*(Y, n_1m_1) \subset O^*(Y, nm)$, and we may conclude that there is a polynomial P_n of degree $\le C_4 n_1 \le n$, which is comonotone with f on $I \setminus O^*(Y, nm) \subset I \setminus O^*(Y, n_1 m_1)$, such that (1.4) is satisfied. Thus, a proper choice of $m = m(A)$ yields Theorem 2 for $n \ge C_4$. For $2 \le n \le C_4$, Theorem 2 follows from Lemma 3. This completes the proof. $\square$

Next we prove,

Proof of Theorem 3. The proof follows along the lines of the proof of Theorem 2. Let $k \ge 1$ be fixed. As in the proof of Theorem 2, we construct a piecewise polynomial

$$S \in \Sigma_{k+1, O(Y,n)} \tag{4.9}$$

which is comonotone with f on $I \setminus O(Y, nm)$, and sufficiently close to it. To this end, we note that by [LS1, Lemma 2], for $I_{j,n} \not\subset O(Y,n)$, there is a polynomial $p_j = p_{j,n}$, of degree $\le k$, which interpolates f at both endpoints of $I_{j,n}$, and is comonotone with f on $I_{j,n}$, for which

$$\|f - p_j\|_{I_{j,n}} \le C|I_{j,n}|\omega_k(f'; |I_{j,n}|; I_{j,n}). \tag{4.10}$$

Since $\omega_k(f'; |I_{j,n}|; I_{j,n}) \le C\omega_k^\varphi(f', 1/n)$, in turn we have

$$\|f - p_j\|_{I_{j,n}} \le \frac{C_6}{n}\omega_k^\varphi\Big(f', \frac{1}{n}\Big). \tag{4.11}$$

Here and for the rest of the proof, constants C are either absolute constants, or may depend only on k and s, while constants c may depend on k, s and m, they may differ in different occurrences, even if they appear in the same line.

For each $\tilde{O}_\nu$, we obtain by Lemma 5, just as in the proof of Theorem 2, a polynomial q_ν of degree $\le k$, which is comonotone with f on $\tilde{O}_\nu \setminus O(Y, nm)$, and such that

$$\begin{aligned} \|f - q_\nu\|_{\tilde{O}_\nu} &\le c|\tilde{O}_\nu|\omega_k(f'; |\tilde{O}_\nu|; \tilde{O}_\nu) \\ &\le \frac{c}{n}\omega_k^\varphi\Big(f', \frac{1}{n}\Big). \end{aligned} \tag{4.12}$$

Again the piecewise polynomial

$$\tilde{S}(x) := \begin{cases} q_\nu(x), & \text{if } \ x \in \tilde{O}_\nu, \\ p_j(x), & \text{if } \ x \in I_{j,n} \setminus O, \end{cases}$$

perhaps has discontinuities at the endpoints of $\tilde{O}_\nu$, and we alter it to obtain a continuous piecewise S which is comonotone with f on $I \setminus O(Y, nm)$, satisfies (4.9), and by virtue of (4.11) and (4.12),

$$\|f - S\| \le \frac{c_4}{n}\omega_k^\varphi\Big(f', \frac{1}{n}\Big). \tag{4.13}$$

Since S is a polynomial of degree at most k on each $\tilde{O}_\nu$, then it changes monotonicity at most $k-1$ there. Hence, there is $Y_* \subset O(Y, nm)$ containing $s_* \le (k-1)s$ points, such that either $S \in \Delta^{(1)}(Y_*)$, or $-S \in \Delta^{(1)}(Y_*)$. Since $m \ge 2$, it follows that $O(Y_*, 2n) \subset O(Y, n)$, whence

$$S \in \Sigma_{k+1, O(Y_*, 2n)}.$$

Again, we apply Proposition A, and conclude that there is a polynomial p of degree $\le C_7 n$, which is comonotone with S everywhere on I, and such that

$$\|S - p\| \le C_8 \omega_{k+1}^{\varphi}\Big(S, \frac{1}{2n}\Big). \tag{4.14}$$

It is readily seen that p is comonotone with f on $I \setminus O(Y, nm)$, and combining (4.13) and (4.14) we have

$$\begin{aligned}
\|f - p\| &\le \|f - S\| + \|S - p\| \le \|f - S\| + C_8 \omega_{k+1}^{\varphi}(S, \tfrac{1}{2n}) \\
&\le \|f - S\| + C_8 \omega_{k+1}^{\varphi}(S - f, \tfrac{1}{2n}) + C_8 \omega_{k+1}^{\varphi}(f, \tfrac{1}{2n}) \\
&\le (1 + 2^{k+1} C_8) \|f - S\| + C_8 \omega_{k+1}^{\varphi}(f, \tfrac{1}{2n}) \\
&\le (1 + 2^{k+1} C_8) \frac{c_4}{n} \omega_k^{\varphi}(f', \tfrac{1}{n}) + C_8 \omega_{k+1}^{\varphi}(f, \tfrac{1}{2n}) \\
&\le \frac{c_5}{n} \omega_k^{\varphi}(f', \tfrac{1}{n}).
\end{aligned}$$

Given $n \ge C_7$ and $m \ge 2$, we continue as in the proof of Theorem 2, applying the above to $n_1 := [n/C_7]$, and $m_1 \ge m C_7$ so big that nm divides $n_1 m_1$. We obtain a polynomial P_n of degree $\le n$, which is comonotone with f on $I \setminus O(Y, nm)$, such that (1.5) is satisfied. Thus, a proper choice of $m = m(A)$ yields Theorem 3 for $n \ge C_7$, and for $k \le n \le C_7$, Theorem 3 follows from Lemma 5. This completes the proof. $\square$

Finally we prove,

Proof of Theorem 4. In this proof we work with the lesser known moduli of smoothness $\omega_{k,1}^{\varphi}(f', t)$. We refer the reader to our paper [LS2] for details. (One can also find some discussion in [DT].) The proof follows along the lines of the proof of Theorem 3, actually we only have to modify the latter near ± 1. Let $k \ge 1$ be fixed. As before, we construct a piecewise polynomial

$$S \in \Sigma_{k+1, O^*(Y, n)} \tag{4.15}$$

which is comonotone with f on $I \setminus O^*(Y, nm)$, and sufficiently close to it. As a generic case we are going to assume that

$$O^*(Y, n) \setminus I_{1,n} = O(Y, n). \tag{4.16}$$

This will allow us to demonstrate how we deal with an end interval which is in $O(Y, n)$ and with another which is not there. For all j, such that $I_{j,n} \not\subset O^*(Y, n)$, we take $p_j = p_{j,n}$ to be the same polynomial which was obtained in the proof of Theorem 3. Since, for $j \ne 1, n$,

$$|I_{j,n}| \omega_k(f'; |I_{j,n}|; I_{j,n}) \le \frac{C}{n} \omega_{k,1}^{\varphi}(f', 1/n), \tag{4.17}$$

then (4.10) implies

$$\|f-p_j\|_{I_{j,n}} \le \frac{C}{n}\omega^{\varphi}_{k,1}\Big(f',\frac{1}{n}\Big).$$

By (4.16) $[-1,b_0]=\tilde{O}_0=\tilde{O}_0^*$. Hence, for all $\nu\neq 0$, we take q_ν to be the same polynomial we have obtained in the proof of Theorem 3. Then (4.16) and (4.17) imply

$$\|f-q_\nu\|_{\tilde{O}_\nu} \le \frac{c}{n}\omega^{\varphi}_{k,1}\Big(f',\frac{1}{n}\Big).$$

Thus we are left with the construction of p_1 and q_0. We begin with p_1 . To this end, we observe that f is nondecreasing on $I^m_{1,n}:=[x_{1,n},x_{1,nm}]$ (even on $I_{1,n}$). Therefore by [LS1, Lemma 2], there is a polynomial $p_1=p_{1,n}$, of degree $\le k$, which interpolates f at the endpoints of $I^m_{1,n}$, and which is comonotone there with it (that is, on $I_{1,n}\setminus O^*(Y,nm)$), for which

$$\|f-p_j\|_{I^m_{1,n}} \le C|I^m_{1,n}|\omega_k(f';|I^m_{1,n}|;I^m_{1,n}).$$

Since similar to (4.17) we have

$$|I^m_{1,n}|\omega_k(f';|I^m_{1,n}|;I^m_{1,n}) \le \frac{c}{n}\omega^{\varphi}_{k,1}\Big(f',\frac{1}{n}\Big),$$

then by (3.2)

$$\begin{aligned}\|f-p_1\|_{I_{1,n}} &\le \frac{c}{n}\omega^{\varphi}_{k,1}\Big(f',\frac{1}{n}\Big)+C\omega_{k+1}\Big(f;|I_{1,n}|;I_{1,n}\Big)\\ &\le \frac{c}{n}\omega_{k,1}\Big(f',\frac{1}{n}\Big)+C\omega^{\varphi}_{k+1}\Big(f;\frac{1}{n}\Big)\\ &\le \frac{c}{n}\omega_{k,1}\Big(f',\frac{1}{n}\Big),\end{aligned}\tag{4.18}$$

where we used the inequality

$$\omega^{\varphi}_{k+1}\Big(f;\frac{1}{n}\Big) \le \frac{C}{n}\omega_{k,1}\Big(f',\frac{1}{n}\Big).$$

In order to construct q_0, we set $\tilde{O}^m_0:=[x_{nm-1,nm},b_0]$, so that as in Lemma 5 we obtain a polynomial q_0 of degree $\le k$, which is comonotone with f on $\tilde{O}^m_0\setminus O(Y,nm)$ (that is, on $\tilde{O}_0\setminus O^*(Y,nm)$) and such that

$$\|f-q_0\|_{\tilde{O}^m_0} \le c|\tilde{O}^m_0|\omega_k(f';|\tilde{O}^m_0|;\tilde{O}^m_0).$$

As in (4.18) we have

$$\begin{aligned}\|f-q_0\|_{\tilde{O}_0} &\le \frac{c}{n}\omega^{\varphi}_{k,1}\Big(f',\frac{1}{n}\Big)+C\omega^{\varphi}_{k+1}\Big(f;\frac{1}{n}\Big)\\ &\le \frac{c}{n}\omega^{\varphi}_{k,1}\Big(f',\frac{1}{n}\Big).\end{aligned}$$

Thus we have arrived at a point where we proceed as in the proofs of Theorems 2 and 3, and establish the existence of a polynomial p of degree $\le C_9 n$, comonotone with f on $I\setminus O^*(Y,nm)$ and such that

$$\|f-p\| \le \frac{c_4}{n}\omega_{k,1}\Big(f',\frac{1}{n}\Big).$$

We conclude the proof of Theorem 4 as in Theorem 3. $\square$

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