KOLMOGOROV AND LINEAR WIDTHS OF WEIGHTED SOBOLEV-TYPE CLASSES ON A FINITE INTERVAL II

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Abstract. Let I be a finite interval, $r \in \mathbb{N}$ and $\rho(t) = \operatorname{dist}\{t, \partial I\}$, $t \in I$. Denote by $\Delta_+^s W_{p,\alpha}^r$, $0 \le \alpha < \infty$, the class of functions x on I with the seminorm $\|x^{(r)}\rho^\alpha\|_{L_p} \le 1$ for which $\Delta_\tau^s x$, $\tau > 0$, is nonnegative on I. We obtain two-sided estimates of the Kolmogorov widths $d_n\left(\Delta_+^s W_{p,\alpha}^r\right)_{L_q}$ and of the linear widths $d_n\left(\Delta_+^s W_{p,\alpha}^r\right)_{L_q}^{lin}$, $s = 0, 1, \dots, r+1$.

§1. Introduction and statement of the main results

Let X be a real linear space of vectors x with norm $\|x\|_X$, $W \subset X, W \ne \emptyset$, and L^n, a subspace in X of dimension $\dim L^n \le n$, $n \ge 0$. Let $M^n = M^n(x^0) := x^0 + L^n$ be a shift of the subspace L^n by an arbitrary vector $x^0 \in X$. Let

$$E(x, M^n)_X := \inf_{y \in M^n} \|x - y\|_X,$$

denote the best approximation of the vector $x \in X$ by M^n, and let

$$E(W, M^n)_X := \sup_{x \in W} E(x, M^n)_X, \tag{1.1}$$

denote the distance between the sets W and M^n.

The Kolmogorov n-width of W is defined by

$$d_n(W)_X := \inf_{M^n} E(W, M^n)_X, \quad n \ge 0. \tag{1.2}$$

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We also let $\Lambda(X, L^n)$ be the set of all linear maps $\Lambda : X \to L^n$. Then

$$E(W, L^n)_X^{lin} := \inf_{\Lambda \in \Lambda(X,L^n)} \sup_{x \in W} \|x - \Lambda x\|_X$$

denotes the best linear approximation of the set W by L^n. The linear n-width of W is defined by

$$d_n(W)_X^{lin} := \inf_{L^n \subset X} E(W, L^n)_X^{lin}, \quad n \geq 0. \tag{1.3}$$

Let I be a finite interval in $\mathbb{R}$, and let $r \in \mathbb{N}$ and $0 \leq \alpha < \infty$. For $1 \leq p \leq \infty$, and $\rho(t) := \operatorname{dist}\{t, \partial I\}$, $t \in I$, we denote

$$W_{p,\alpha}^r := W_{p,\alpha}^r(I) := \{x : I \to \mathbb{R} \mid x^{(r-1)} \in AC_{loc}(I), \|x^{(r)} \rho^\alpha\|_{L_p(I)} \leq 1\}.$$

If $\alpha = 0$, then we write $W_p^r := W_p^r(I) := W_{p,0}^r(I)$. We also write L_q for $L_q(I)$. Let

$$\Delta_\tau^s x(t) := \sum_{k=0}^{s} (-1)^{s-k} \binom{s}{k} x(t + k\tau), \quad \{t, t + s\tau\} \subset I, \quad s = 0, 1, \ldots,$$

be the s-th difference of the function x, with step $\tau > 0$, and denote by $\Delta_+^s W_{p,\alpha}^r = \Delta_+^s W_{p,\alpha}^r(I)$, $s = 0, 1, \ldots$ the subclasses of functions $x \in W_{p,\alpha}^r$ for which $\Delta_\tau^s x(t) \geq 0$, for all $\tau > 0$ such that $[t, t + s\tau] \subseteq I$. If $\alpha = 0$, then we write $W_p^r := W_{p,0}^r(I)$ and $\Delta_+^s W_p^r := \Delta_+^s W_{p,0}^r(I)$. We also write L_q for $L_q(I)$. Throughout this paper we take the generic finite interval $I = [-1, 1]$.

The behavior of the Kolmogorov and linear widths in the classical case $\alpha = 0$, i.e., for the Sobolev classes $W_{p,0}^r = W_p^r$, has been thoroughly investigated. We refer the interested reader to the list of references for earlier results. We have recently proved [8],

Theorem KL1. *Let I be a finite interval and let $r \in \mathbb{N}$, $1 \leq p, q \leq \infty$ and $0 < \alpha < \infty$, be so that $r - \alpha - \frac{1}{p} + \frac{1}{q} > 0$. If $(r, p) \neq (1, 1)$, and if $(r, p) = (1, 1)$ and $1 \leq q \leq 2$, then*

$$d_n(W_{p,\alpha}^r)_{L_q} \asymp n^{-r + (\max\{\frac{1}{p}, \frac{1}{2}\} - \max\{\frac{1}{q}, \frac{1}{2}\})_+}, \quad n \geq r,$$

where $(u)_+ := \max\{u, 0\}$ and $a_n \asymp b_n$ means that there exist constants $0 < C_1 < C_2$, such that $C_1 a_n \leq b_n \leq C_2 a_n$, $\forall n$. If on the other hand, $(r, p) = (1, 1)$ and $2 < q < \infty$, then

$$c_1 n^{-\frac{1}{2}} \leq d_n(W_{1,\alpha}^1)_{L_q} \leq c_2 n^{-\frac{1}{2}} \big(\log(n+1)\big)^{\frac{3}{2}}, \quad n \geq 1,$$

where $c_1 > 0$ *and* c_2 *do not depend on* n.

It turns out that the Kolmogorov widths of the smaller classes $\Delta_+^s W_{p,\alpha}^r$, $0 \le s \le r$, are, in general, of the same order of magnitude as those of the classes $W_{p,\alpha}^r$. However, they are significantly smaller for the class $\Delta_+^{r+1} W_{p,\alpha}^r$. Thus we first have

Theorem 1. *Let* $r \in \mathbb{N}$, $1 \le p, q \le \infty$ *and* $0 \le \alpha < \infty$, *be so that* $r - \alpha - \frac{1}{p} + \frac{1}{q} > 0$. *If* $(r,p) \ne (1,1)$ *and if* $(r,p) = (1,1)$ *and* $1 \le q \le 2$, *then for each* $s = 0, 1, \dots, r$,

$$d_n(\Delta_+^s W_{p,\alpha}^r)_{L_q} \asymp n^{-r+(\max\{\frac{1}{p},\frac{1}{2}\}-\max\{\frac{1}{q},\frac{1}{2}\})_+}. \quad n \ge r. \tag{1.4}$$

If on the other hand, $(r,p) = (1,1)$ *and* $2 < q < \infty$, *then for* $s = 0, 1$,

$$c_1 n^{-\frac{1}{2}} \le d_n(\Delta_+^s W_{1,\alpha}^1)_{L_q} \le c_2 n^{-\frac{1}{2}} \big(\log(n+1)\big)^{\frac{3}{2}}, \quad n \ge 1, \tag{1.5}$$

where $c_1 > 0$ *and* c_2 *do not depend on* n.

But in case $s = r + 1$, we prove the following.

Theorem 2. *Let* $r \in \mathbb{N}$, $1 \le p, q \le \infty$ *and* $0 \le \alpha < \infty$, *be so that* $r - \alpha - \frac{1}{p} + \frac{1}{q} > 0$. *Then*

$$d_n(\Delta_+^{r+1} W_{p,\alpha}^r)_{L_q} \asymp n^{-r-\max\{\frac{1}{q},\frac{1}{2}\}}. \quad n > r. \tag{1.6}$$

For linear widths we have proved the following ([8]).

Theorem KL2. *Let* $r \in \mathbb{N}$, $1 \le p, q \le \infty$ *and* $0 < \alpha < \infty$ *be so that* $r - \alpha - \frac{1}{p} + \frac{1}{q} > 0$. *If* $(r,p) \ne (1,1)$, *or if* $(r,p) = (1,1)$ *and* $1 \le q \le 2$, *and if* $(r,q) = (1,\infty)$ *and* $2 \le p \le \infty$, *then*

$$d_n(W_{p,\alpha}^r)_{L_q}^{lin} \asymp n^{-r+(\frac{1}{p}-\frac{1}{q})_+-\min\{(\frac{1}{p}-\frac{1}{2})_+,(\frac{1}{2}-\frac{1}{q})_+\}}, \quad n \ge r.$$

If on the other hand, $(r,p) = (1,1)$ *and* $2 < q < \infty$, *then*

$$c_1 n^{-\frac{1}{2}} \le d_n(W_{1,\alpha}^1)_{L_q}^{lin} \le c_2 n^{-\frac{1}{2}} \big(\log(n+1)\big)^{\frac{3}{2}}, \quad n \ge 1,$$

and if $(r,q)=(1,\infty)$ *and* $1<p<2$*, then*

$$c_1 n^{-\frac{1}{2}} \le d_n(W^1_{p,\alpha})^{lin}_{L_\infty} \le c_2 n^{-\frac{1}{2}}\big(\log(n+1)\big)^{\frac{3}{2}}, \quad n \ge 1,$$

where $c_1>0$ *and* c_2 *do not depend on* n*.*

We have here the same phenomenon as for the Kolmogorov widths, namely, the linear widths of the smaller classes $\Delta^s_+ W^r_{p,\alpha}$, $0 \le s \le r$, are, in general, of the same order of magnitude as those of the classes $W^r_{p,\alpha}$. However, they are significantly smaller for the class $\Delta^{r+1}_+ W^r_{p,\alpha}$. Here we have

Theorem 3. *Let* $r\in\mathbb{N}$, $1\le p,q\le\infty$ *and* $0\le\alpha<\infty$*, be so that* $r-\alpha-\frac{1}{p}+\frac{1}{q}>0$*. If* $(r,p)\ne(1,1)$*, or if* $(r,p)=(1,1)$ *and* $1\le q\le 2$ *and if* $(r,q)=(1,\infty)$ *and* $2\le p\le\infty$*, then for each* $s=0,1,\dots,r$*,*

$$d_n(\Delta^s_+ W^r_{p,\alpha})^{lin}_{L_q} \asymp n^{-r+(\frac{1}{p}-\frac{1}{q})_+-\min\{(\frac{1}{p}-\frac{1}{2})_+,(\frac{1}{2}-\frac{1}{q})_+\}}, \quad n\ge r. \tag{1.7}$$

If on the other hand, $(r,p)=(1,1)$ *and* $2<q<\infty$ *then for* $s=0,1$*,*

$$c_1 n^{-\frac{1}{2}} \le d_n(\Delta^s_+ W^1_{1,\alpha})^{lin}_{L_q} \le c_2 n^{-\frac{1}{2}}\big(\log(n+1)\big)^{\frac{3}{2}}, \quad n \ge 1, \tag{1.8}$$

and if $(r,q)=(1,\infty)$ *and* $1<p<2$*, then for* $s=0,1$*,*

$$c_1 n^{-\frac{1}{2}} \le d_n(\Delta^s_+ W^1_{p,\alpha})^{lin}_{L_\infty} \le c_2 n^{-\frac{1}{2}}\big(\log(n+1)\big)^{\frac{3}{2}}, \quad n \ge 1, \tag{1.9}$$

where $c_1>0$ *and* c_2 *do not depend on* n*.*

And in case $s=r+1$, we show that

Theorem 4. *Let* $r\in\mathbb{N}$, $1\le p,q\le\infty$ *and* $0\le\alpha<\infty$*, be so that* $r-\alpha-\frac{1}{p}+\frac{1}{q}>0$*. Then*

$$d_n(\Delta^{r+1}_+ W^r_{p,\alpha})^{lin}_{L_q} \asymp n^{-r-\max\{\frac{1}{q},\frac{1}{2}\}}, \quad n>r. \tag{1.10}$$

Remark. Note that for each fixed q and all p such that $r-\alpha-\frac{1}{p}+\frac{1}{q}>0$, the Kolmogorov and linear widths of the classes $\Delta^{r+1}_+ W^r_{p,\alpha}$ are of the same order of magnitude.

§2. Kolmogorow widths of the classes $\Delta_+^s W_{p,\alpha}^r$ in L_q– auxiliary lemmas

For $n \in \mathbb{N}$ and $1 \le p \le \infty$, let l_p^n denote, as usual, the space of vectors $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with the norm

$$\|x\|_{l_p^n} := \begin{cases} \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \max_{i=1,\ldots,n} |x_i|, & p = \infty, \end{cases}$$

and let B_p^n be its unit ball. We recall the following lemmas.

Lemma KPS. (see [7], [13] and [16]) *Let $m, n \in \mathbb{N}$ be so that $m < n$. Then*

$$d_m(B_1^n)_{l_2^n} = \left(1 - \frac{m}{n}\right)^{\frac{1}{2}}.$$

Lemma G1. (see [1]) *Let $m, n \in \mathbb{N}$ be so that $m < n$, and set*

$$\Phi(m,n,p,q) := \begin{cases} \max\left\{n^{\frac{1}{q}-\frac{1}{p}}, \left(1 - \frac{m}{n}\right)^{\left(\frac{1}{p}-\frac{1}{q}\right)/\left(\frac{2}{p}-1\right)}\right\}, \\ \quad \text{if} \quad 1 \le p < q \le 2, \\ \max\left\{n^{\frac{1}{q}-\frac{1}{p}}, \min\left\{1, n^{\frac{1}{q}} m^{-\frac{1}{2}}\right\}\left(1 - \frac{m}{n}\right)^{\frac{1}{2}}\right\}, \\ \quad \text{if} \quad 1 \le p < 2 \le q \le \infty, \\ \left(\min\left\{1, n^{\frac{1}{q}} m^{-\frac{1}{2}}\right\}\right)^{\left(\frac{1}{p}-\frac{1}{q}\right)/\left(\frac{1}{2}-\frac{1}{q}\right)}, \\ \quad \text{if} \quad 2 \le p < q \le \infty. \end{cases} \tag{2.1}$$

Then for any $1 \le p < q < \infty$,

$$d_m\left(B_p^n\right)_{l_q^n} \asymp \Phi(m,n,p,q), \tag{2.2}$$

where the constants in these two-sided estimates are independent of m and n.

Lemma K. (see [5]) *Let $m, n \in \mathbb{N}$ be so that $m < n$. Then*

$$d_m\left(B_2^n\right)_{l_\infty^n} \le c m^{-\frac{1}{2}} \left(1 + \log \frac{n}{m}\right)^{\frac{3}{2}}, \tag{2.3}$$

where $c > 0$ is an absolute constant.

Finally,

Lemma PS. (see [14] and [17]) *Let $m, n \in \mathbb{N}$ be so that $m < n$. Then for any $1 \le q \le p \le \infty$*

$$d_m\big(B_p^n\big)_{l_q^n} = (n-m)^{\frac{1}{q}-\frac{1}{p}}. \tag{2.4}$$

We need some more lemmas and begin with some notation. Let $Y^n := \big\{y^{(i)}\big\}_{i=1}^n$ be a system of vectors in X, and let $1 \le p \le \infty$. The set

$$S_p^+\big(Y^n\big) := \Big\{y \mid y := \sum_{i=1}^n a_i y^{(i)},\ a = (a_1, \dots, a_n) \in \mathbb{R}^n,\ a_i \ge 0,\ i = 1, \dots, n,\ \|a\|_{l_p^n} \le 1\Big\},$$

is called a positive p-sector over the system Y^n in X, and we denote $-S_p^+\big(Y^n\big) := \big\{y \mid -y \in S_p^+\big(Y^n\big)\big\}$. The following lemma is a consequence of Lemmas KPS and PS.

Lemma 1. *Let $m, n \in \mathbb{N}$ be so that $m < n$, and let $1 \le p, q \le \infty$. If $E^n := \big\{e^{(i)}\big\}_{i=1}^n$ denotes the standard system of the vectors $e^{(1)} = (1, 0, \dots, 0)$, ... , $e^{(n)} = (0, \dots, 0, 1)$ in $\mathbb{R}^n$, then*

$$d_m\big(S_p^+\big(E^n\big)\big)_{l_q^n} \ge \max\left\{\frac{1}{2} n^{-\frac{1}{p}}(n-m)^{\frac{1}{q}}, \left(1 - \frac{m+1}{n}\right)^{\frac{1}{2}}\right\} n^{-\left(\frac{1}{2}-\frac{1}{q}\right)_+}. \tag{2.5}$$

Proof. First $n^{-\frac{1}{p}} S_\infty^+\big(E^n\big) \subseteq S_p^+\big(E^n\big)$, where we observe that $S_\infty^+\big(E^n\big)$ is just the L_∞ ball of radius $\frac{1}{2}$ about $x^0 = \big(\frac{1}{2}, \dots, \frac{1}{2}\big)$. Thus, by virtue of Lemma PS it follows that for every $1 \le p, q \le \infty$,

$$\begin{aligned} d_m\big(S_p^+\big(E^n\big)\big)_{l_q^n} &\ge n^{-\frac{1}{p}} d_m\big(S_\infty^+\big(E^n\big)\big)_{l_q^n} \\ &= \frac{1}{2} n^{-\frac{1}{p}}(n-m)^{\frac{1}{q}}. \end{aligned} \tag{2.6}$$

At the same time,

$$d_m\big(S_1^+\big(E^n\big)\big)_{l_2^n} \ge d_{m+1}\big(B_1^n\big)_{l_2^n}.$$

For B_1^n is the convex hull of $S_1^+\big(E^n\big) \cup -S_1^+\big(E^n\big)$, and we approximate it at least as well by the linear span of the m-dimensional linear manifold, which is in general of dimension $m+1$. Hence by Lemma KPS we have

$$d_m\big(S_1^+\big(E^n\big)\big)_{l_2^n} \ge \left(1 - \frac{m+1}{n}\right)^{\frac{1}{2}},$$

and since $S_1^+(E^n) \subseteq S_p^+(E^n)$, $1 \le p \le \infty$, we conclude that

$$d_m\left(S_p^+(E^n)\right)_{l_2^n} \ge \left(1 - \frac{m+1}{n}\right)^{\frac{1}{2}}.$$

If $1 \le q \le 2$, then $\|x\|_{l_q^n} \ge \|x\|_{l_2^n}$ so that

$$d_m\left(S_p^+(E^n)\right)_{l_q^n} \ge \left(1 - \frac{m+1}{n}\right)^{\frac{1}{2}}, \quad 1 \le p \le \infty, \tag{2.7}$$

and if $2 \le q \le \infty$, then $\|x\|_{l_q^n} \ge n^{\frac{1}{q}-\frac{1}{2}}\|x\|_{l_2^n}$ whence

$$d_m\left(S_p^+(E^n)\right)_{l_q^n} \ge n^{\frac{1}{q}-\frac{1}{2}}\left(1 - \frac{m+1}{n}\right)^{\frac{1}{2}}, \quad 1 \le p \le \infty. \tag{2.8}$$

Combining (2.6) through (2.8), completes the proof of Lemma 1. $\square$

Next we have

Lemma 2. *Let $n \in \mathbb{N}$ and $1 \le p, q \le \infty$. Given $a_i > 0$, and $b_i > 0$, $i = 1, \ldots, n$, let $\tau := (\tau_1, \ldots, \tau_n)$ belong to the set*

$$T_p := \begin{cases} \left\{\tau : \tau_i \ge 0, 1 \le i \le n,\ \left(\sum_{i=1}^n b_i^p \left(\sum_{j=1}^i \tau_i\right)^p\right)^{\frac{1}{p}} \le 1\right\}, & 1 \le p < \infty, \\ \left\{\tau : \tau_i \ge 0, 1 \le i \le n,\ \max_{1 \le i \le n} b_i \tau_i \le 1\right\}, & p = \infty. \end{cases} \tag{2.9}$$

Let

$$f_q(\tau) := \begin{cases} \left(\sum_{i=1}^n a_i^q \tau_i^q\right)^{\frac{1}{q}}, & 1 \le q < \infty, \\ \max_{1 \le i \le n} a_i \tau_i, & q = \infty. \end{cases} \tag{2.10}$$

Then setting $a_{n+1} := 0$ and $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\max_{\tau \in T_p} f_q(\tau) \le \begin{cases} \left(\sum_{i=1}^n \left(|a_i - a_{i+1}|\, b_i^{-1}\right)^{p'}\right)^{\frac{1}{p'}}, & 1 < p \le \infty, \\ \max_{1 \le i \le n} |a_i - a_{i+1}| b_i^{-1}, & p = 1. \end{cases} \tag{2.11}$$

Note that the estimates in (2.11) are independent of q.

Proof. Since

$$\left(\sum_{i=1}^n a_i^q \tau_i^q\right)^{\frac{1}{q}} \le \sum_{i=1}^n a_i \tau_i, \quad 1 \le q \le \infty,$$

we may assume that $q=1$. Thus substituting

$$\theta_i := b_i\big(\sum_{k=1}^{i}\tau_k\big),\quad i=1,\dots,n,\quad \text{i.e.,}\quad \tau_1=b_1^{-1}\theta_1,\quad \tau_i=b_i^{-1}\theta_i-b_{i-1}^{-1}\theta_{i-1},\quad i=2,\dots,n,$$

then, applying the Abelian transformation, (2.10) takes the form

$$g(\theta):=f_1(\tau)=\sum_{i=1}^{n}(a_i-a_{i+1})\,b_i^{-1}\theta_i,$$

where $a_{n+1}:=0$, and the set T_p of (2.9) becomes

$$\Theta_p := T_p = \begin{cases} \{0\le b_1^{-1}\theta_1\le\cdots\le b_n^{-1}\theta_n,\ (\sum_{i=1}^{n}\theta_i^p)^{\frac{1}{p}}\le 1\}, & 1\le p<\infty,\\ \{0\le b_1^{-1}\theta_1\le\cdots\le b_n^{-1}\theta_n,\ \max_{1\le i\le n}\theta_i\le 1\}, & p=\infty.\end{cases}$$

Clearly $\Theta_p\subseteq B_p^n$, thus we will instead estimate the linear functional $g(\theta)$, over the bigger set. But this is just the norm $\|g\|_{p'}$, that is, (2.11). $\square$

Finally, the following lemma is straightforward.

Lemma 3. *Let $r\in\mathbb{N}$, $1\le p,q\le\infty$ and $0\le\alpha<\infty$, be so that $r-\alpha-\frac{1}{p}+\frac{1}{q}>0$. Then $x\in\Delta_+^{r+1}W_{p,\alpha}^r$ if and only if $x^{(r-1)}\in AC_{loc}$ and it is convex in I, $\|x^{(r)}\rho^\alpha\|_{L_p}\le 1$ and $x^{(r)}$ is equal a.e. on I to a nondecreasing function.*

§3. Kolmogorov widths of the classes $\Delta_+^s W_{p,\alpha}^r$, $0\le s\le r$

We begin with a construction that will be used in proving the lower bounds in Theorems 1 and 2. To this end, let $m,N\in\mathbb{N}$ and set $\delta_{m,N}:=\frac{2}{(m+1)N}$. Now let

$$t_{m,N,i}:=-1+i\delta_{m,N},\quad i=0,1,\dots,(m+1)N,$$

and

$$I_{m,N,i}:=[t_{m,N,(m+1)(i-1)},t_{m,N,(m+1)(i-1)+1}],\quad i=1,\dots,N.$$

For each $i=1,\dots,N$, set

$$\psi_{m,p,N,i}(t):=\delta_{m,N}^{-\frac{1}{p}}\frac{1}{m!}\left(\big(t-t_{m,N,(m+1)(i-1)}\big)_+^m-\big(t-t_{m,N,(m+1)(i-1)+1}\big)_+^m\right), \tag{3.1}$$

so that

$$
\begin{aligned}
&\psi^{(m-1)}_{m,p,N,i}(t) \\
(3.2)\qquad &= \begin{cases} 0, & t < t_{m,N,(m+1)(i-1)}, \\ \delta_{m,N}^{-\frac{1}{p}}\big(t - t_{m,N,(m+1)(i-1)}\big), & t_{m,N,(m+1)(i-1)} < t < t_{m,N,(m+1)(i-1)+1}, \\ \delta_{m,N}^{1-\frac{1}{p}}, & t > t_{m,N,(m+1)(i-1)+1}, \end{cases}
\end{aligned}
$$

and

$$
(3.3)\qquad \psi^{(m)}_{m,p,N,i}(t) = \begin{cases} 0, & t < t_{m,N,(m+1)(i-1)} \quad \text{and} \quad t > t_{m,N,(m+1)(i-1)+1}, \\ \delta_{m,N}^{-\frac{1}{p}}, & t_{m,N,(m+1)(i-1)} < t < t_{m,N,(m+1)(i-1)+1}. \end{cases}
$$

It is clear that $\psi_{m,p,N,i}(\cdot) \in \Delta^s_+ W^m_p$, $s = 0, 1, \ldots, m$. Evidently, for all $i, j = 1, \ldots, N$,

$$
\int_{I_{m,N,j}} \Delta^m_{\delta_{m,N}} \psi_{m,p,N,i}(t)dt = \int_{I_{m,N,j}} \int_0^{\delta_{m,N}} \cdots \int_0^{\delta_{m,N}} \psi^{(m)}_{m,p,N,i}(t+\tau_1 \cdots + \tau_m)\, d\tau_1 \cdots d\tau_m\, dt,
$$

so that it follows from (3.3) that

$$
(3.4)\qquad \int_{I_{m,N,j}} \int_0^{\delta_{m,N}} \cdots \int_0^{\delta_{m,N}} \psi^{(m)}_{m,p,N,i}(t + \tau_1 \cdots + \tau_m)\, d\tau_1 \cdots d\tau_m\, dt = 0, \quad j \neq i.
$$

On the other hand when $j = i$ we take $t \in I_{m,N,i}$ and for $\bar\tau := (\tau_1, \ldots, \tau_m)$, we denote

$$
S_{m,N,i}(t) := \left\{\bar\tau \mid \tau_k \geq 0,\ 1 \leq k \leq m,\ \tau_1 + \cdots + \tau_m \leq t_{m,N,(m+1)(i-1)+1} - t\right\}.
$$

Then

$$
S_{m,N,i}(t) = \big(t_{m,N,(m+1)(i-1)+1} - t\big)\, S_1^+(E^m),
$$

and $\psi^{(m)}_{m,p,N,i}(t + \tau_1 \cdots + \tau_m) = 0$ if $\bar\tau \notin S_{m,N,i}(t)$. Thus by (3.3) we obtain,

$$
\begin{aligned}
&\int_{I_{m,N,i}} \int_0^{\delta_{m,N}} \cdots \int_0^{\delta_{m,N}} \psi^{(m)}_{m,p,N,i}(t + \tau_1 \cdots + \tau_m)\, d\tau_1 \cdots d\tau_m\, dt \\
&= \delta_{m,N}^{-\frac{1}{p}} \int_{I_{m,N,i}} \int_{S_{m,N,i}(t)} d\bar\tau\, dt \\
(3.5)\qquad &= \delta_{m,N}^{-\frac{1}{p}} \frac{1}{m!} \int_{I_{m,N,i}} \big(t_{m,n,(m+1)(i-1)+1} - t\big)^m\, dt \\
&= \frac{1}{(m+1)!} \delta_{m,N}^{m+1-\frac{1}{p}}.
\end{aligned}
$$

Hence

$$2^{-m}\delta_{m,N}^{\frac{1}{q}-1}\int_{I_{m,N,i}}\Delta_{\delta_{m,N}}^{m}\psi_{m,p,N,i}(t)\,dt=\frac{2^{-m}}{(m+1)!}\left(\frac{m+1}{2}\right)^{-m+\frac{1}{p}-\frac{1}{q}}N^{-m+\frac{1}{p}-\frac{1}{q}}$$
$$=\frac{2^{\frac{1}{q}-\frac{1}{p}}}{(m+1)!}(m+1)^{-m+\frac{1}{p}-\frac{1}{q}}N^{-m+\frac{1}{p}-\frac{1}{q}}. \tag{3.6}$$

Define the discretization operators

$$L_q\ni x\to A_{m,N,q}x:=\big(y_1,\ldots,y_N\big)\in l_q^N,$$

by setting

$$y_j:=2^{-m}\delta_{m,N}^{\frac{1}{q}-1}\int_{I_{m,N,j}}\Delta_{\delta_{m,N}}^{m}x(t)\,dt,\quad j=1,\ldots,N, \tag{3.7}$$

where

$$I_{m,N,j}:=\big(t_{m,N,(m+1)(j-1)},t_{m,N,(m+1)(j-1)+1}\big),\quad j=1,\ldots,N,$$

so that $|I_{m,N,j}|=\delta_{m,N}$. Then for $x\in L_q$

$$\|x\|_{L_q}\geq\|A_{m,N,q}x\|_{l_q^N}. \tag{3.8}$$

Indeed, for $1\leq q<\infty$, it follows by Hölder's and Jensen's inequalities that

$$\left|\int_{I_{m,N,j}}\Delta_{\delta_{m,N}}^{m}x(t)\,dt\right|^q=\left|\int_{I_{m,N,j}}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}x(t+k\delta_{m,N})\,dt\right|^q$$
$$\leq\left(\sum_{k=0}^{m}\binom{m}{k}\int_{I_{m,N,j}}|x(t+k\delta_{m,N})|\,dt\right)^q$$
$$\leq 2^{mq}|I_{m,N,j}|^{q-1}\sum_{k=0}^{m}2^{-m}\binom{m}{k}\int_{I_{m,N,j}}|x(t+k\delta_{m,N})|^q\,dt$$
$$\leq 2^{mq}\delta_{m,N}^{1-q}\sum_{k=0}^{m}\int_{I_{m,N,j}}|x(t+k\delta_{m,N})|^q\,dt,$$

which in turn yields

$$
\begin{aligned}
\Big(\sum_{j=1}^{N}|y_j|^q\Big)^{1/q} &= \left(\sum_{j=1}^{N} 2^{-mq}\delta_{m,N}^{q-1}\left|\int_{I_{m,N,j}} \Delta^m_{\delta_{m,N}} x(t)\,dt\right|^q\right)^{\frac{1}{q}} \\
&\le \left(\sum_{j=1}^{N}\sum_{k=0}^{m}\int_{I_{m,N,j}} |x(t+k\delta_{m,N})|^q\,dt\right)^{\frac{1}{q}} \\
&= \left(\int_I |x(t)|^q\,dt\right)^{\frac{1}{q}} = \|x\|_{L_q},
\end{aligned}
$$

and (3.8) is proved for $1 \le q < \infty$. If $q = \infty$, then

$$
\begin{aligned}
\max_{1\le j\le N}|y_j| &= \max_{1\le j\le N} 2^{-m}\delta_{m,N}^{-1}\left|\int_{I_{m,N,j}} \Delta^m_{\delta_{m,N}} x(t)\,dt\right| \\
&\le \max_{1\le j\le N} 2^{-m}\delta_{m,N}^{-1}\sum_{k=0}^{m}\binom{m}{k}\int_{I_{m,N,j}} |x(t+k\delta_{m,N})|\,dt \\
&\le \max_{-1\le t\le 1}|x(t)| = \|x\|_{L_\infty},
\end{aligned}
$$

and (3.8) is proved for $q = \infty$.

We are ready to prove Theorem 1.

Proof of Theorem 1. Since $\Delta^s_+ W^r_{p,\alpha} \subseteq W^r_{p,\alpha}$, then obviously the upper bounds in (1.4) and (1.5) follow from Theorem KL1. Thus we have to prove only the lower bounds. Then again in view of the inclusion $\Delta^s_+ W^r_p \subseteq \Delta^s_+ W^r_{p,\alpha}$, $0 \le \alpha < \infty$, it suffices to prove that

$$
d_n\left(\Delta^s_+ W^r_p\right)_{L_q} \ge cn^{-r+\left(\max\{\frac{1}{p},\frac{1}{2}\}-\max\{\frac{1}{q},\frac{1}{2}\}\right)_+}, n \ge r. \tag{3.9}
$$

Set $\Psi^N_{r,p} := \{\psi_{r,p,N,i}(\cdot)\}_{i=1}^N$, where $\psi_{r,p,N,i}$ was defined in (3.1) with $m = r$, and denote by $S^+_p\left(\Psi^N_{r,p}\right)$ the positive p-sector over the system $\Psi^N_{r,p}$. Then it follows by (3.3) that $S^+_p\left(\Psi^N_{r,p}\right) \subseteq \Delta^s_+ W^r_p$ for all $s = 0, 1, \ldots, r$. Hence

$$
d_n\left(\Delta^s_+ W^r_p\right)_{L_q} \ge d_n\left(S^+_p\left(\Psi^N_{r,p}\right)\right)_{L_q}. \tag{3.10}
$$

Denote

$$
\psi^{(i)}_{r,p,q,N} := A_{r,N,q}\psi_{r,p,N,i}(\cdot), \quad i = 1, \ldots, N
$$

and let $\Psi^N_{r,p,q} := \left\{\psi^{(i)}_{r,p,q,N}\right\}_{i=1}^N$. Then (3.8) implies

$$d_n\left(S_p^+\left(\Psi^N_{r,p}\right)\right)_{L_q} \geq d_n\left(S_p^+\left(\Psi^N_{r,p,q}\right)\right)_{l_q^N}. \tag{3.11}$$

If we set

$$C_{r,p,q} := \frac{2^{\frac{1}{q}-\frac{1}{p}}}{(r+1)!}(r+1)^{-r+\frac{1}{p}-\frac{1}{q}},$$

then from (3.6) we get,

$$\psi^{(i)}_{r,p,q,N} = C_{r,p,q}N^{-r+\frac{1}{p}-\frac{1}{q}}e^{(i)}, \quad i = 1, \dots, N,$$

where we recall that $e^{(i)} = (0, \dots, 0, 1, 0, \dots, 0)$ (with the 1 in the ith coordinate), so that

$$S_p^+\left(\Psi^N_{r,p,q}\right) = C_{r,p,q}N^{-r+\frac{1}{p}-\frac{1}{q}}S_p^+\left(E^N\right).$$

In conclusion,

$$d_n\left(S_p^+\left(\Psi^N_{r,p,q}\right)\right)_{L_q} = C_{r,p,q}N^{-r+\frac{1}{p}-\frac{1}{q}}d_n\left(S_p^+\left(E^N\right)\right)_{l_q^N}, \tag{3.12}$$

which by virtue of (3.10) and (3.11) becomes

$$d_n\left(\Delta^s_+W^r_p\right)_{L_q} \geq C_{r,p,q}N^{-r+\frac{1}{p}-\frac{1}{q}}d_n\left(S_p^+\left(E^N\right)\right)_{l_q^N}, \quad 0 \leq s \leq r.$$

Finally, substituting $N = 4n$, we get by (2.5),

$$\begin{aligned} d_n\left(\Delta^s_+W^r_p\right)_{L_q} &\geq cn^{-r+\frac{1}{p}-\frac{1}{q}}d_n\left(S_p^+\left(E^{4n}\right)\right)_{l_q^{4n}} \\ &\geq cn^{-r+\frac{1}{p}-\frac{1}{q}}\max\left\{\frac{1}{2}(4n)^{-\frac{1}{p}}(4n-n)^{\frac{1}{q}}, \left(1-\frac{n+1}{4n}\right)^{\frac{1}{2}}(4n)^{-(\frac{1}{2}-\frac{1}{q})_+}\right\} \\ &\geq cn^{-r+\frac{1}{p}-\frac{1}{q}}\max\left\{n^{-\frac{1}{p}+\frac{1}{q}}, n^{-(\frac{1}{2}-\frac{1}{q})_+}\right\} \\ &= cn^{-r+(\max\{\frac{1}{p},\frac{1}{2}\}-\max\{\frac{1}{q},\frac{1}{2}\})_+}, \quad n \in \mathbb{N}, \end{aligned}$$

where $c = c(r, p, q)$. This completes the proof of (3.9) and concludes the proof of Theorem 1. $\square$

§4. Kolmogorov widths of the classes $\Delta_+^{r+1}W_{p,\alpha}^r$,

It is not surprising that we have smaller widths for these classes as the elements in $\Delta_+^{r+1}W_{p,\alpha}^r$ have a nondecreasing rth derivative thus their $r+1$st derivative exists a.e. and is locally in L_1 in I. Indeed, we are using this in both directions in the following proof.

Proof of Theorem 2. We begin with the shorter proof of the lower bounds in (1.6). Since $\Delta_+^{r+1}W_\infty^r \subseteq \Delta_+^{r+1}W_{p,\alpha}^r$, $0 \le \alpha < \infty$, it suffices to prove that

$$d_n\left(\Delta_+^{r+1}W_\infty^r\right)_{L_q} \ge cn^{-r-\max\{\frac{1}{q},\frac{1}{2}\}}, \quad n \ge r. \tag{4.1}$$

To this end recall the functions $\psi_{r+1,1,N,i}$ from (3.1) with $m=r+1$ and $p=1$. Then by (3.2), $\|\psi_{r+1,1,N,i}^{(r)}\|_{L_\infty} = 1$, so that $\psi_{r+1,1,N,i} \in \Delta_+^{r+1}W_\infty^r$. Hence, if we denote $\Psi_{r+1,1}^N := \{\psi_{r+1,1,N,i}(\cdot)\}_{i=1}^N$, and we let $S_1^+\left(\Psi_{r+1,1}^N\right)$ be the positive 1-sector over the system $\Psi_{r+1,1}^N$, then $S_1^+\left(\Psi_{r+1,1}^N\right) \subseteq \Delta_+^{r+1}W_\infty^r$, whence

$$d_n\left(\Delta_+^{r+1}W_\infty^r\right)_{L_q} \ge d_n\left(S_1^+\left(\Psi_{r+1,1}^N\right)\right)_{L_q}. \tag{4.2}$$

Now we apply the discretization operators $A_{r+1,N,q}$, which were defined by (3.7) with $m=r+1$, to obtain $\psi_{r+1,1,q,N}^{(i)} := A_{r+1,N,q}\psi_{r+1,1,N,i}(\cdot)$, $i=1,\dots,N$, and the system $\Psi_{r+1,1,q}^N := \left\{\psi_{r+1,1,N,q}^{(i)}\right\}_{i=1}^N$. Then like (3.10), it follows by (3.8) that

$$d_n\left(S_1^+\left(\Psi_{r+1,1}^N\right)\right)_{L_q} \ge d_n\left(S_1^+\left(\Psi_{r+1,1,q}^N\right)\right)_{l_q^N}.$$

Now, by virtue of (3.12) we have

$$d_n\left(S_1^+\left(\Psi_{r+1,1,q}^N\right)\right)_{L_q} = C_{r+1,1,q}N^{-(r+1)+1-\frac{1}{q}}d_n\left(S_1^+\left(E^N\right)\right)_{l_q^N}.$$

Therefore, taking $N=4n$ and combining with (4.2) we obtain

$$\begin{aligned}
d_n\left(\Delta_+^{r+1}W_\infty^r\right)_{L_q} &\ge cn^{-r-\frac{1}{q}}d_n\left(S_1^+\left(E^{4n}\right)\right)_{l_q^{4n}} \\
&\ge cn^{-r-\frac{1}{q}}\max\left\{\frac{1}{2}(4n)^{-1}(4n-n)^{\frac{1}{q}}, \left(1-\frac{n+1}{4n}\right)^{\frac{1}{2}}(4n)^{-(\frac{1}{2}-\frac{1}{q})_+}\right\} \\
&\ge cn^{-r-\max\left\{\frac{1}{q},\frac{1}{2}\right\}}, \quad n \in \mathbb{N},
\end{aligned}$$

where $c = c(r, q)$. This completes the proof of (4.1).

We now turn to the proof of the upper bounds. We apply an extension of ideas of V. M. Tikhomirov for obtaining the Kolmogorov widths of the classes $\Delta_+^2 W_p^1$ in L_q. We are indebted to Tikhomirov for improving our previous proof of this direction. Following Tikhomirov, we reduce our problem to that of the isoperimetric problem in $\mathbb{R}^n$. We fix

$$\beta > \left(r + \frac{1}{q}\right)\left(r - \alpha - \frac{1}{p} + \frac{1}{q}\right)^{-1}, \tag{4.3}$$

and let

$$t_{n,i} := t_{\beta,n,i} := \begin{cases} 1 - \left(\frac{n-i}{n}\right)^\beta, & i = 0, 1, \dots, n, \\ -1 + \left(\frac{n+i}{n}\right)^\beta, & i = -n, \dots, -1. \end{cases} \tag{4.4}$$

Set

$$I_{n,i} := I_{\beta,n,i} := \begin{cases} [t_{n,i-1}, t_{n,i}], & i = 1, \dots, n, \\ [t_{n,i}, t_{n,i+1}], & i = -n, \dots, -1. \end{cases} \tag{4.5}$$

Denote by $l_{r,n,i}(x;t)$, $t \in I$, $i = \pm 1, \dots, \pm(n-1)$, the Lagrange polynomials of degree r, which interpolate $x \in \Delta_+^{r+1} W_{p,\alpha}^r$ at the $r+1$ equidistant points $t_{n,i,k}$, $k = 0, 1 \dots, r$, partitioning $I_{n,i}$, that is,

$$l_{r,n,i}(x; t_{n,i,k}) = x(t_{n,i,k}), \quad k = 0, 1, \dots, r, \quad i = \pm 1, \dots, \pm(n-1),$$

and set

$$\sigma_{r,n}(x;t) := \begin{cases} l_{r,n,i}(x;t), & t \in I_{n,i}, \quad i = \pm 1, \dots, \pm(n-1), \\ l_{r,n,\pm(n-1)}(x;t), & t \in I_{n,\pm n}. \end{cases}$$

We first prove that

$$\sup_{x \in \Delta_+^{r+1} W_{p,\alpha}^r} \|x(\cdot) - \sigma_{r,n}(x;\cdot)\|_{L_q} \le c n^{-r-\frac{1}{q}}, \tag{4.6}$$

where $c = c(r, \alpha, \beta, p, q)$.

To this end, if $x \in \Delta_+^{r+1} W_{p,\alpha}^r$, then by Lemma 3, $x^{(r-1)}$ is convex and $x^{(r)}$ in nondecreasing, so that we may take an rth degree polynomial $\pi_r(x;\cdot)$ such that its $(r-1)$st derivative is tangent to $x^{(r-1)}$ at 0, namely,

$$\pi_r^{(r-1)}(x;0) = x^{(r-1)}(0), \quad \pi_r^{(r-1)}(x;t) \le x^{(r-1)}(t), \quad t \in I,$$

and the constant $\pi_r^{(r)}(x;t) = \pi_r^{(r)}(x;0)$, satisfies

$$x_-^{(r)}(0) \le \pi_r^{(r)}(x;t) \le x_+^{(r)}(0).$$

Thus, let

$$\breve{x}(t) := x(t) - \pi_r(x;t), \quad t \in I,$$

and we clearly have

$$\breve{x}(t) - \sigma_{r,n}(\breve{x};t) = x(t) - \sigma_{r,n}(x;t), \quad t \in I.$$

Since $x^{(r)}$ is nondecreasing, then it readily follows that

$$\left\|\breve{x}^{(r)}\rho^\alpha\right\|_{L_p} \le 3\left\|x^{(r)}\rho^\alpha\right\|_{L_p}, \quad x \in \Delta_+^{r+1}W_{p,\alpha}^r.$$

Indeed, if $\pi_r^{(r)}(x;0) \ge 0$, then we use the inequality $\pi_r^{(r)}(x;t) \le x_+^{(r)}(0) \le x^{(r)}(t)$ a.e. in $[0,1)$. Hence

$$\begin{aligned}
\left\|\breve{x}^{(r)}\rho^\alpha\right\|_{L_p} &\le \left\|x^{(r)}\rho^\alpha\right\|_{L_p} + \left\|\pi_r^{(r)}\rho^\alpha\right\|_{L_p} \\
\blacksquare = \blacksquare &\left\|x^{(r)}\rho^\alpha\right\|_{L_p} + 2\left\|\pi_r^{(r)}\rho^\alpha\right\|_{L_p[0,1]} \\
&\le \left\|x^{(r)}\rho^\alpha\right\|_{L_p} + 2\left\|x^{(r)}\rho^\alpha\right\|_{L_p[0,1]} \\
&\le 3\left\|x^{(r)}\rho^\alpha\right\|_{L_p}.
\end{aligned}$$

Otherwise $\pi_r^{(r)}(x;0) < 0$, so that $\pi_r^{(r)}(x;t) \ge x_-^{(r)}\blacksquare(0)\blacksquare \ge x^{(r)}(t)$ a.e. in $(-1,0]$, and the proof is similar.

It is well known that by Whitney's theorem we have

$$\max_{t\in I_{n,i}} |\breve{x}(t) - l_{r,n,i}(\breve{x};t)| \le c\omega_{r+1}\left(\breve{x}; \tfrac{|I_{n,i}|}{r+1}; I_{n,i}\right), \quad i = \pm 1, \ldots, \pm(n-1),$$

where $c = c(r)$ and $\omega_{r+1}(x;\delta;[a,b])$ is the usual $(r+1)$st modulus of smoothness of x with step δ, in the interval $[a,b]$. Thus for $x \in \Delta_+^{r+1}W_{p,\alpha}^r$, if $\tau > 0$ is so that $\{t, t+(r+1)\tau\} \in$

$I_{n,\pm i}$, $1 \le i \le n-1$, then

$$\begin{aligned}
\left|\Delta_\tau^{r+1}\breve{x}(t)\right| &= \left|\int_0^\tau \cdots \int_0^\tau \Delta_\tau^1 \breve{x}^{(r)}(t+\tau_1+\ldots\tau_r)d\tau_1\ldots d\tau_r\right| \\
&\le \int_0^\tau \cdots \int_0^\tau \left|\Delta_\tau^1 \breve{x}^{(r)}(t+\tau_1+\ldots\tau_r)\right| d\tau_1\ldots d\tau_r \\
&\le |I_{n,\pm i}|^r \operatorname{esssup}_{t_1,t_2\in I_{n,\pm i}}\left(\breve{x}^{(r)}(t_1)-\breve{x}^{(r)}(t_2)\right) =: |I_{n,\pm i}|^r \omega_{r,n,\pm i},
\end{aligned}$$

implying that

$$\max_{t\in I_{n,i}} |\breve{x}(t) - l_{r,n,i}(\breve{x};t)| \le c|I_{n,i}|^r \omega_{r,n,i}, \quad i=\pm 1,\ldots,\pm(n-1),$$

where $c = c(r)$. In view of the definition of $\sigma_{r,n}$ we conclude that for all $1 \le q \le \infty$,

$$\begin{aligned}
\|\breve{x}(\cdot) - \sigma_{r,n}(\breve{x};\cdot)\|_{L_q(I_{n,i})} &= \|\breve{x}(\cdot) - l_{r,n,i}(\breve{x};\cdot)\|_{L_q(I_{n,i})} \\
&\le c|I_{n,i}|^{r+\frac{1}{q}}\omega_{r,n,i}, \quad i=\pm 1,\ldots,\pm(n-1).
\end{aligned} \tag{4.7}$$

In order to estimate the norm of $\breve{x}(\cdot) - \sigma_{r,n}(\breve{x};\cdot)$ in $L_q(I_{n,\pm n})$, we denote by

$$\theta_{r-1}(\breve{x};t_{n,n-1};t) := \sum_{s=1}^{r-1} \breve{x}^{(s)}(t_{n,n-1})\frac{(t-t_{n,n-1})^s}{s!},$$

the Taylor polynomial of degree $r-1$ of $\breve{x}$. Then

$$\begin{aligned}
\breve{x}(t) - \sigma_{r,n}(\breve{x};t) &= \breve{x}(t) - l_{r,n,n-1}(\breve{x};t) \\
&= \breve{x}(t) - \theta_{r-1}(\breve{x};t_{n,n-1};t) - l_{r,n,n-1}\left(\breve{x}(\cdot)\blacksquare + \blacksquare\theta_{r-1}(\breve{x};t_{n,n-1};\cdot);t\right),
\end{aligned}$$

whence

$$\begin{aligned}
\|\breve{x}(\cdot) - \sigma_{r,n}(\breve{x};\cdot)\|_{L_q(I_{n,n})} &\le \|\breve{x}(\cdot) - \theta_{r-1}(\breve{x};t_{n,n-1};\cdot)\|_{L_q(I_{n,n})} \\
&\quad + \left\|l_{r,n,n-1}\left(\breve{x}(\cdot) - \theta_{r-1}(\breve{x};t_{n,n-1};\cdot);\cdot\right)\right\|_{L_q(I_{n,n})}.
\end{aligned}$$

It is readily seen by Minkowski's inequality (applied to the parameter q) and Hölder's inequality (applied to the parameter p) that

$$\|\breve{x}(\cdot) - \theta_{r-1}(\breve{x};t_{n,n-1};\cdot)\|_{L_q(I_{n,n})} \le c|I_{n,n}|^{r-\alpha-\frac{1}{p}+\frac{1}{q}}\|\breve{x}^{(r)}\rho^\alpha\|_{L_p(I_{n,n})},$$

where $c = c(r, \alpha, p, q)$. It should be noted that in the above we use the fact that $r - \alpha - \frac{1}{p} + \frac{1}{q} > 0$. Since

$$c_1|I_{n,n-1}| \leq |I_{n,n}| \leq c_2|I_{n,n-1}|,$$

where $c_1 = c_1(r, \alpha, \beta, p, q)$, $c_2 = c_2(r, \alpha, \beta, p, q)$, it follows that

$$\begin{aligned}\|l_{r,n,n-1}(\breve{x}(\cdot) - \theta_{r-1}(\breve{x}; t_{n,n-1}; \cdot))\|_{L_q(I_{n,n})} &\leq c|I_{n,n}|^{\frac{1}{q}} \|\breve{x}(\cdot) - \theta_{r-1}(\breve{x}; t_{n,n-1}; \cdot)\|_{L_\infty(I_{n,n-1})} \\ &\leq c|I_{n,n}|^{r-\alpha-\frac{1}{p}+\frac{1}{q}} \|\breve{x}^{(r)}\rho^\alpha\|_{L_p(I_{n,n-1})},\end{aligned}$$

where $c = c(r, \alpha, \beta, p, q)$. Recalling that $\|\breve{x}^{(r)}\rho^\alpha\|_p \leq 3$, we conclude that

$$\|\breve{x}(\cdot) - \sigma_{r,n}(\breve{x}; \cdot)\|_{L_q(I_{n,n})} \leq c|I_{n,n}|^{r-\alpha-\frac{1}{p}+\frac{1}{q}}, \tag{4.8}$$

where $c = c(r, \alpha, \beta, p, q)$. Similarly

$$\|\breve{x}(\cdot) - \sigma_{r,n}(\breve{x}; \cdot)\|_{L_q(I_{n,-n})} \leq c|I_{n,-n}|^{r-\alpha-\frac{1}{p}+\frac{1}{q}}, \tag{4.9}$$

where $c = c(r, \alpha, \beta, p, q)$, so combining with (4.7) and (4.8) yields,

$$\begin{aligned}&\|x(\cdot) - \sigma_{r,n}(x; \cdot)\|_{L_q[0,1]} \\ &\leq \begin{cases} c\left(\sum_{i=1}^{n-1} |I_{n,i}|^{rq+1}\omega_{r,n,i}^q\right)^{\frac{1}{q}} + c|I_{n,n}|^{r-\alpha-\frac{1}{p}+\frac{1}{q}}, & 1 \leq q < \infty, \\ c\max_{1\leq i<n} |I_{n,i}|^r\omega_{r,n,i} + c|I_{n,n}|^{r-\alpha-\frac{1}{p}}, & q = \infty. \end{cases}\end{aligned} \tag{4.10}$$

and similarly,

$$\begin{aligned}&\|x(\cdot) - \sigma_{r,n}(x; \cdot)\|_{L_q[-1,0]} \\ &\leq \begin{cases} c\left(\sum_{i=-n+1}^{-1} |I_{n,i}|^{rq+1}\omega_{r,n,i}^q\right)^{\frac{1}{q}} + c|I_{n,-n}|^{r-\alpha-\frac{1}{p}+\frac{1}{q}}, & 1 \leq q < \infty, \\ c\max_{-n<i\leq-1} |I_{n,i}|^r\omega_{r,n,i} + c|I_{n,-n}|^{r-\alpha-\frac{1}{p}}, & q = \infty. \end{cases}\end{aligned}$$

where $c = c(r, \alpha, \beta, p, q)$.

On the other hand, if $1 \leq p < \infty$, then

$$\begin{aligned}\left\|\breve{x}^{(r)}\rho^\alpha\right\|_{L_p[0,1]} &= \left(\sum_{i=1}^{n} \left\|\breve{x}^{(r)}\rho^\alpha\right\|_{L_p(I_{n,i})}^p\right)^{\frac{1}{p}} \\ &\geq \left(\sum_{i=1}^{n-1} \left\|\breve{x}^{(r)}\rho^\alpha\right\|_{L_p(I_{n,i+1})}^p\right)^{\frac{1}{p}}.\end{aligned}$$

Note that a.e. in $I_{n,i+1}$, $1 \le i \le n-1$,

$$\sum_{j=1}^{i} \omega_{r,n,j} \le \breve{x}^{(r)}(t),$$

so that for each $1 \le p \le \infty$,

$$\|\rho^\alpha\|_{L_p(I_{n,i+1})} \sum_{j=1}^{i} \omega_{r,n,j} \le \left\| \breve{x}^{(r)} \rho^\alpha \right\|_{L_p(I_{n,i+1})}.$$

Hence, for $p = \infty$,

$$\max_{1 \le i \le n-1} 3^{-1} \|\rho^\alpha\|_{L_\infty(I_{n,i+1})} \sum_{j=1}^{i} \omega_{r,n,j} \le 1, \tag{4.11}$$

and for $1 \le p < \infty$, we have

$$\left(\sum_{i=1}^{n-1} \|\rho^\alpha\|_{L_p(I_{n,i+1})}^p \left(\sum_{j=1}^{i} \omega_{r,n,j} \right)^p \right)^{\frac{1}{p}} \le \left\| \breve{x}^{(r)} \rho^\alpha \right\|_{L_p[0,1]} \le 3,$$

or

$$\left(\sum_{i=1}^{n-1} 3^{-p} \|\rho^\alpha\|_{L_p(I_{n,i+1})}^p \left(\sum_{j=1}^{i} \omega_{r,n,j} \right)^p \right)^{\frac{1}{p}} \le 1. \tag{4.12}$$

Similarly, for $p = \infty$,

$$\max_{1 \le i \le n-1} 3^{-1} \|\rho^\alpha\|_{L_\infty(I_{n,-i-1})} \sum_{j=1}^{i} \omega_{r,n,-j} \le 1, \tag{4.13}$$

and for $1 \le p < \infty$,

$$\left(\sum_{i=1}^{n-1} 3^{-p} \|\rho^\alpha\|_{L_p(I_{n,-i-1})}^p \left(\sum_{j=1}^{i} \omega_{r,n,-j} \right)^p \right)^{\frac{1}{p}} \le 1. \tag{4.14}$$

Write

$$a_i := c|I_{n,i}|^{r+\frac{1}{q}}, \quad i = 1, \dots, n-1,$$
$$b_i := \frac{1}{3} \|\rho^\alpha\|_{L_p(I_{n,i+1})}, \quad i = 1, \dots, n-1,$$

so that if we replace $\omega_{r,n,i}$ by τ_i, then we are in the setup of Lemma 2. Thus in view of (4.10), we may conclude by (2.11) that

$$\|x(\cdot)-\sigma_{r,n}(x;\cdot)\|_{L_q[0,1]} \le \begin{cases} \left(\sum_{i=1}^{n-1}\left(|a_i-a_{i+1}|\, b_i^{-1}\right)^{p'}\right)^{\frac{1}{p'}} + c|I_{n,n}|^{r-\alpha-\frac{1}{p}+\frac{1}{q}}, & 1<p\le\infty, \\ \max_{1\le i\le n-1}|a_i-a_{i+1}|b_i^{-1} + c|I_{n,n}|^{r-\alpha-1-\frac{1}{q}}, & p=1. \end{cases}$$

where $\frac{1}{p}+\frac{1}{p'}=1$ and where by definition $a_n := 0$. Now, by (4.3) through (4.5),

$$\begin{aligned} |a_i-a_{i+1}| &= \left||I_{n,i}|^{r+\frac{1}{q}} - |I_{n,i+1}|^{r+\frac{1}{q}}\right| \\ &= \left|(t_{n,i}-t_{n,i-1})^{r+\frac{1}{q}} - (t_{n,i+1}-t_{n,i})^{r+\frac{1}{q}}\right| \\ &= n^{-\beta(r+\frac{1}{q})}\left|\left((n-i+1)^\beta-(n-i)^\beta\right)^{r+\frac{1}{q}} - \left((n-i)^\beta-(n-i-1)^\beta\right)^{r+\frac{1}{q}}\right| \\ &\le cn^{-\beta(r+\frac{1}{q})}(n-i)^{(\beta-1)(r+\frac{1}{q})-1}, \quad i=1,\dots,n-2, \end{aligned} \tag{4.15}$$

and

$$|a_{n-1}-a_n| = |a_{n-1}| = |I_{n,n-1}|^{r+\frac{1}{q}}| \le cn^{-\beta(r+\frac{1}{q})}, \tag{4.16}$$

where $c=c(r,\alpha,\beta,p,q)$. Note that (4.16) is exactly (4.15) with $i=n-1$.

Now, if $p=\infty$, then

$$b_i = 3^{-1}\|\rho^\alpha\|_{L_\infty(I_{n,i+1})} = 3^{-1}\rho^\alpha(t_{n,i}) = 3^{-1}n^{-\beta\alpha}(n-i)^{\beta\alpha}, \quad i=1,\dots,n-1,$$

and if $1\le p<\infty$, then

$$\begin{aligned} \|\rho^\alpha\|_{L_p(I_{n,i+1})} &= \left(\int_{t_{n,i}}^{t_{n,i+1}}(1-t)^{\alpha p}dt\right)^{\frac{1}{p}} \\ &= (\alpha p+1)^{-\frac{1}{p}}\left((1-t_{n,i})^{\alpha p+1}-(1-t_{n,i+1})^{\alpha p+1}\right)^{\frac{1}{p}} \\ &= (\alpha p+1)^{-\frac{1}{p}}n^{-\beta(\alpha+\frac{1}{p})}\left((n-i)^{\beta(\alpha p+1)}-(n-i-1)^{\beta(\alpha p+1)}\right)^{\frac{1}{p}} \\ &\ge cn^{-\beta(\alpha+\frac{1}{p})}(n-i)^{\beta(\alpha+\frac{1}{p})-\frac{1}{p}}, \quad i=1,\dots,n-1, \end{aligned}$$

where $c=c(r,\alpha,\beta,p,q)$. Therefore for all $1\le p\le\infty$,

$$b_i \ge cn^{-\beta(\alpha+\frac{1}{p})}(n-i)^{\beta(\alpha+\frac{1}{p})-\frac{1}{p}}, \quad i=1,\dots,n-1,$$

where $c = c(r, \alpha, \beta, p, q)$.

Hence, if $p = 1$, then

$$
\begin{aligned}
&\max_{i=1,\dots,n-1} |a_i - a_{i+1}| b_i^{-1} \\
&\leq c \max_{i=1,\dots,n-1} n^{-\beta(r+\frac{1}{q})}(n-i)^{(\beta-1)(r+\frac{1}{q})-1} n^{\beta(\alpha+1)}(n-i)^{-\beta(\alpha+1)+1} \\
&= c \max_{i=1,\dots,n-1} n^{-\beta(r-\alpha-1+\frac{1}{q})}(n-i)^{\beta(r-\alpha-1+\frac{1}{q})-(r+\frac{1}{q})} \\
&\leq c n^{-\beta(r-\alpha-1+\frac{1}{q})} n^{\beta(r-\alpha-1+\frac{1}{q})-(r+\frac{1}{q})} \\
&= c n^{-r-\frac{1}{q}},
\end{aligned}
\tag{4.17}
$$

and if $1 < p \leq \infty$, then

$$
\begin{aligned}
&\left(\sum_{i=1}^{n-1} \left(|a_i - a_{i+1}|\, b_i^{-1} \right)^{p'} \right)^{\frac{1}{p'}} \\
&\leq c \left(\sum_{i=1}^{n-1} \left(n^{-\beta(r+\frac{1}{q})}(n-i)^{(\beta-1)(r+\frac{1}{q})-1} n^{\beta(\alpha+\frac{1}{p})}(n-i)^{-\beta(\alpha+\frac{1}{p})+\frac{1}{p}} \right)^{p'} \right)^{\frac{1}{p'}} \\
&= c n^{-\beta(r-\alpha-\frac{1}{p}+\frac{1}{q})} \left(\sum_{i=1}^{n-1} (n-i)^{(\beta(r-\alpha-\frac{1}{p}+\frac{1}{q})-(r+\frac{1}{q})-1+\frac{1}{p})p'} \right)^{\frac{1}{p'}} \\
&\leq c n^{-\beta(r-\alpha-\frac{1}{p}+\frac{1}{q})} n^{\beta(r-\alpha-\frac{1}{p}+\frac{1}{q})-(r+\frac{1}{q})-1+\frac{1}{p}+1-\frac{1}{p}} \\
&= n^{-r-\frac{1}{q}}.
\end{aligned}
\tag{4.18}
$$

Note that we used (4.3) in both (4.17) and (4.18). At the same time using (4.3) once more,

$$
|I_{n,n}|^{r-\alpha-\frac{1}{p}+\frac{1}{q}} = n^{-\beta(r-\alpha-\frac{1}{p}+\frac{1}{q})} \leq n^{-r-\frac{1}{q}},
$$

which combined with (4.17) and (4.18) turns (4.10) into

$$
\|x(\cdot) - \sigma_{r,n}(x;\cdot)\|_{L_q([0,1))} \leq c n^{-r-\frac{1}{q}},
$$

where $c = c(r, \alpha, \beta, p, q)$. Similarly we obtain

$$
\|x(\cdot) - \sigma_{r,n}(x;\cdot)\|_{L_q((-1,0])} \leq c n^{-r-\frac{1}{q}},
$$

and (4.6) follows. This completes the proof of the upper bound in (1.6) for $1 \le q \le 2$.

We have to improve the estimates for $2 < q \le \infty$ and we do it by applying discretization techniques. To this end, we first show that for each $x \in \Delta_+^{r+1} W_{p,\alpha}^r$,

$$\left\| (x(\cdot) - \sigma_{r,n}(x;\cdot))\, w_n^{-1+\frac{1}{q}} \right\|_{L_1} \le c n^{-r-\frac{1}{q}}, \tag{4.19}$$

where $c = c(r, \alpha, \beta, p, q)$ and

$$w_n(t) := n^{-1} \left(1 - |t| + n^{-\beta}\right)^{\frac{\beta-1}{\beta}}, \quad t \in I.$$

Observe that

$$c_1 |I_{n,i}| \le \min_{t \in I_{n,i}} w_n(t) \le \max_{t \in I_{n,i}} w_n(t) \le c_2 |I_{n,i}|, \quad i = \pm 1, \ldots, \pm n,$$

where $0 < c_1 = c_1(\beta) \le c_2 = c_2(\beta)$. For if we let $t \in I_{n,i}$, $1 \le i \le n$, then

$$\begin{aligned}
\min_{t \in I_{n,i}} w_n(t) &= n^{-1} \left(1 - t_{n,i} + n^{-\beta}\right)^{\frac{\beta-1}{\beta}} \\
&= n^{-1} \left(\left(\frac{n-i}{n}\right)^{\beta} + n^{-\beta} \right)^{\frac{\beta-1}{\beta}} \\
&= n^{-\beta} \left((n-i)^{\beta} + 1\right)^{\frac{\beta-1}{\beta}} \\
&\ge c_1 |I_{n,i}|,
\end{aligned}$$

and

$$\begin{aligned}
\max_{t \in I_{n,i}} w_n(t) &= n^{-1} \left(1 - t_{n,i-1} + n^{-\beta}\right)^{\frac{\beta-1}{\beta}} \\
&= n^{-1} \left(\left(\frac{n-i+1}{n}\right)^{\beta} + n^{-\beta} \right)^{\frac{\beta-1}{\beta}} \\
&= n^{-\beta} \left((n-i+1)^{\beta} + 1\right)^{\frac{\beta-1}{\beta}} \\
&\le c_2 |I_{n,i}|.
\end{aligned}$$

Hence by virtue of (4.7) and (4.8) (with $q=1$), we obtain

$$\left\| (\breve{x}(\cdot) - \sigma_{r,n}(\breve{x};\cdot))\, w_n^{-1+\frac{1}{q}} \right\|_{L_1(I_{n,i})} \le c|I_{n,i}|^{r+\frac{1}{q}} \omega_{r,n,i}.$$
$$i = \pm 1, \dots, \pm(n-1),$$

and

$$\left\| (\breve{x}(\cdot) - \sigma_{r,n}(\breve{x};\cdot))\, w_{\beta,n}^{-1+\frac{1}{q}} \right\|_{L_1(I_{n,\pm n})} \le c|I_{n,\pm n}|^{r-\alpha-\frac{1}{p}+\frac{1}{q}},$$

where $c = c(r,\alpha,\beta,p,q)$. Therefore

$$\left\| (x(\cdot) - \sigma_{r,n}(x;\cdot))\, w_n^{-1+\frac{1}{q}} \right\|_{L_1[0,1]} \le \sum_{i=1}^{n-1} c|I_{\beta,n,i}|^{r+\frac{1}{q}} \omega_{r,n,i}$$
$$+ c|I_{n,n}|^{r-\alpha-\frac{1}{p}+\frac{1}{q}},$$

In view of (4.11) and (4.12), we are again in the setup of Lemma 2 and we may conclude that

$$\left\| (x(\cdot) - \sigma_{r,n}(x;\cdot))\, w_n^{-1+\frac{1}{q}} \right\|_{L_1[0,1]} \le cn^{-r-\frac{1}{q}}.$$

Similar estimate is obtained for the interval $[-1,0]$, by virtue of (4.7), (4.9), (4.13) and (4.14). Thus (4.19) is established.

We proceed now to prove that

$$d_n\left(\Delta_+^{r+1} W_{p,\alpha}^r\right)_{L_q} \le cn^{-r-\frac{1}{2}}, \quad 2 < q \le \infty, \tag{4.20}$$

where $c = c(r,\alpha,\beta,p,q)$.

For $n > 1$ let $\Sigma_{r,n} := \Sigma_{r,n}(I)$ be the space of continuous piecewise polynomials $\zeta \in C(I)$, which are polynomials of degree $\le r$ in each interval $I_{n,i}$. Then clearly, $\dim \Sigma_{r,n} = 2rn+1$. For $n = 1$ we take $\Sigma_{r,1} := P_{r-1}$, the space of all polynomials of degree $< r$, i.e., $\dim \Sigma_{r,1} = r$. Evidently, for any $n \ge 1$, $\Sigma_{r,n} \subseteq \Sigma_{r,2n}$. We will prove the existence of integers $\lambda = \lambda(r,\alpha,p,q) > 1$, $a = a(r,\alpha,p,q) > 0$ and subspaces $\Sigma_{a2^n} \subseteq \Sigma_{r,2^{\lambda n}}$, of dimension $r \le \dim \Sigma_{a2^n} \le a2^n < 2^{\lambda n}$, $n > 1$, such that

$$E\big(\Delta_+^{r+1} W_{p,\alpha}^r, \Sigma_{a2^n}\big)_{L_q} \le c2^{-\left(r+\frac{1}{2}\right)n}, \quad 2 < q \le \infty. \tag{4.21}$$

where $c = c(r, \alpha, \beta, p, q)$.

For $n > 1$ we subdivide the intervals $I_{n,\pm i}$, $i = 1, \dots, n-1$, by

$$t_{n,\pm i,k} := t_{n,\pm(i-1)} + \tfrac{t_{n,\pm i} - t_{n,\pm(i-1)}}{r} k, \quad k = 0, 1, \dots, r,$$

and note that $t_{n,i,0} = t_{n,i\mp 1}$ and $t_{n,i,r} = t_{n,i}$, $i = \pm 1, \dots, \pm n$. Define a one-to-one correspondence between the spaces $\Sigma_{r,n}$ and $\mathbb{R}^{2nr+1}$, by the invertible discretization operator

$$A_{r,\beta,q,n} : \Sigma_{r,n} \ni \zeta \to y = \big(y_{-nr}, \dots, y_{-1}, y_0, y_1, \dots, y_{nr}\big) \in \mathbb{R}^{2nr+1},$$

where

$$\begin{aligned}
y_j &= (nr)^{-\frac{\beta}{q}} (nr - j + 1)^{\frac{\beta-1}{q}} \zeta(t_{n,i,k}), \\
&\qquad j = (i-1)r + k,\ k = 0, 1, \dots, r,\ i = 1, \dots, n, \\
y_j &= (nr)^{-\frac{\beta}{q}} (nr + j + 1)^{\frac{\beta-1}{q}} \zeta(t_{n,-i,k}), \\
&\qquad j = -(i-1)r - k,\ k = 0, 1, \dots, r,\ i = 1, \dots, n.
\end{aligned}$$

Note that altogether $j = 0, \pm 1, \dots, \pm nr$. The inverse operator is

$$A^{-1}_{r,\beta,q,n} : \mathbb{R}^{2nr+1} \ni y = \big(y_{-nr}, \dots, y_{-1}, y_0, y_1, \dots, y_{nr}\big) \to \zeta \in \Sigma_{r,n},$$

where ζ is uniquely defined by the interpolation equations

$$\begin{aligned}
\zeta(t_{n,i,k}) &= (nr)^{\frac{\beta}{q}} (nr - j + 1)^{-\frac{\beta-1}{q}} y_j, \quad j = (i-1)r + k,\ k = 0, 1, \dots, r,\ i = 1, \dots, n, \\
\zeta(t_{n,-i,k}) &= (nr)^{\frac{\beta}{q}} (nr + j + 1)^{-\frac{\beta-1}{q}} y_j, \quad j = -(i-1)r - k,\ k = 0, 1, \dots, r,\ i = 1, \dots, n.
\end{aligned}$$

Similarly to what was shown in [8, Theorem 1], it follows that

$$\|\zeta\|_{L_q} \asymp \|A_{r,\beta,q,n}\zeta\|_{l_q^{2nr+1}}, \quad \zeta \in \Sigma_{r,n}.$$

We proceed as in the above proof fixing an integer $N \in \mathbb{N}$ and prescribing integers $m_0 := r$ and $m_\nu \le 2r2^\nu + 1$, $\nu = 1, 2, \dots, N$. Let L^{m_ν}, $\nu = 1, 2, \dots, N$ be subspaces of $\mathbb{R}^{2r2^\nu+1}$ of dimension $\dim L^{m_\nu} = m_\nu$ and set

$$\Sigma^{m_0} := \Sigma_{r,1}, \quad \Sigma^{m_\nu} := A^{-1}_{r,\beta,q,2^\nu} L^{m_\nu}, \nu = 1, 2, \dots, N.$$

Then $\Sigma^{m_\nu} \subset \Sigma_{r,2^\nu}$ and $\dim \Sigma^{m_\nu} = m_\nu$, $\nu = 0, 1, 2, \ldots, N$. If we denote

$$\Sigma^{m_0,\ldots,m_N} := \operatorname{span}\bigl(\cup_{\nu=0}^N \Sigma^{m_\nu}\bigr),$$

then $\Sigma^{m_0,\ldots,m_N} \subset \Sigma_{r,2^N}$ and $\dim \Sigma^{m_0,\ldots,m_N} \le m_0 + \cdots + m_N$. Each $x \in \Delta_+^{r+1} W_{p,\alpha}^r$ may be expanded into

$$x(t) = x(t) - \sigma_{r,2^N}(x;t) + \sigma_{r,1}(x;t) + \sum_{\nu=1}^N \bigl(\sigma_{r,2^\nu}(x;t) - \sigma_{r,2^{\nu-1}}(x;t)\bigr), \quad t \in I,$$

where $\sigma_{r,2^\nu}(x;\cdot) \in \Sigma_{r,2^\nu}$. Therefore we conclude that,

$$\begin{aligned} E\bigl(x, \Sigma^{m_0,\ldots,m_N}\bigr)_{L_q} &\le \bigl\| x(\cdot) - \sigma_{r,2^N}(x;\cdot) \bigr\|_{L_q} \\ &\quad + c \sum_{\nu=1}^N E\Bigl(A_{r,\beta,q,2^\nu}\bigl(\sigma_{r,2^\nu}(x;\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot)\bigr), L^{m_\nu}\Bigr)_{l_q^{2r2^\nu+1}}. \end{aligned} \tag{4.22}$$

We will show that for all $x \in \Delta_+^{r+1} W_{p,\alpha}^r$ belong to the splines $\sigma_{r,2^\nu}(x;\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot)$ are mapped by $A_{r,\beta,q,2^\nu}$ into the ball $c 2^{-(r+\frac{1}{q})\nu} B_1^{2r2^\nu+1}$, where $c = c(r, \alpha, \beta, p, q)$. Indeed, it is readily seen that for each $k = 0, \ldots, r$,

$$(r2^\nu)^{-\beta} (r2^\nu - ((i-1)r + k) + 1)^{\beta-1} \le c\, |I_{2^\nu,\pm i}|, \quad i = 1, \ldots, 2^\nu,$$

where $c = c(r, \beta)$. Hence,

$$\begin{aligned} &\bigl\| A_{r,\beta,q,2^\nu}\bigl(\sigma_{r,2^\nu}(x;\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot)\bigr) \bigr\|_{l_1^{2r2^\nu+1}} \\ &= \sum_{i=1}^{2^\nu} \sum_{k=0}^{r} (r2^\nu)^{-\frac{\beta}{q}} (r2^\nu - ((i-1)r + k) + 1)^{\frac{\beta-1}{q}} \left| \sigma_{r,2^\nu}(x; t_{2^\nu,i,k}) - \sigma_{r,2^{\nu-1}}(x; t_{2^\nu,i,k}) \right| \\ &+ \sum_{i=-2^\nu}^{-1} \sum_{k=0}^{r} (r2^\nu)^{-\frac{\beta}{q}} (r2^\nu - |(i+1)r - k| + 1)^{\frac{\beta-1}{q}} \left| \sigma_{r,2^\nu}(x; t_{2^\nu,i,k}) - \sigma_{r,2^{\nu-1}}(x; t_{2^\nu,i,k}) \right| \\ &\le c \sum_{i=1}^{2^\nu} \sum_{k=0}^{r} |I_{2^\nu,i}|^{\frac{1}{q}} \left| \sigma_{r,2^\nu}(x; t_{2^\nu,i,k}) - \sigma_{r,2^{\nu-1}}(x; t_{2^\nu,i,k}) \right| \\ &+ c \sum_{i=-2^\nu}^{-1} \sum_{k=0}^{r} |I_{2^\nu,i}|^{\frac{1}{q}} \left| \sigma_{r,2^\nu}(x; t_{2^\nu,i,k}) - \sigma_{r,2^{\nu-1}}(x; t_{2^\nu,i,k}) \right|. \end{aligned} \tag{4.23}$$

Evidently,

$$\begin{aligned}
&\sum_{k=0}^{r} \left|\sigma_{r,2^\nu}(x;t_{2^\nu,i,k}) - \sigma_{r,2^{\nu-1}}(x;t_{2^\nu,i,k})\right| \\
&\le (r+1)\left\|\sigma_{r,2^\nu}(x;\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot)\right\|_{L_\infty(I_{2^\nu,i})} \\
&\le c\,|I_{2^\nu,i}|^{-1}\left\|\sigma_{r,2^\nu}(x;\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot)\right\|_{L_1(I_{2^\nu,i})},
\end{aligned}$$

where $c = c(r)$ and where we used the fact that $\sigma_{r,2^\nu}(x;\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot)$ is a polynomial of degree $\le r$ on $I_{2^\nu,i}$. Now, it is readily seen that

$$|I_{2^\nu,i}| \ge cw_{2^\nu}(t) \ge cw_{2^{\nu-1}}(t), \quad t \in I_{2^\nu,i},$$

whence,

$$\begin{aligned}
&|I_{2^\nu,i}|^{-1+\frac{1}{q}}\left\|\sigma_{r,2^\nu}(x;\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot)\right\|_{L_1(I_{2^\nu,i})} \\
&\le |I_{2^\nu,i}|^{-1+\frac{1}{q}}\left\|x(\cdot) - \sigma_{r,2^\nu}(x;\cdot)\right\|_{L_1(I_{2^\nu,i})} + |I_{2^\nu,i}|^{-1+\frac{1}{q}}\left\|x(\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot)\right\|_{L_1(I_{2^\nu,i})} \\
&\le c\left\|\left(x(\cdot) - \sigma_{r,2^\nu}(x;\cdot)\right)w_{2^\nu}^{-1+\frac{1}{q}}\right\|_{L_1(I_{2^\nu,i})} + c\left\|\left(x(\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot)\right)w_{2^{\nu-1}}^{-1+\frac{1}{q}}\right\|_{L_1(I_{2^\nu,i})}.
\end{aligned}$$

Therefore, (4.23) and (4.19) yield,

$$\begin{aligned}
&\left\|A_{r,\beta,q,2^\nu}\left(\sigma_{r,2^\nu}(x;\cdot) - \sigma_{r,2^{\nu-1}}(x;\cdot)\right)\right\|_{l_1^{2r2^\nu+1}} \\
&\le c\sum_{i=-2^\nu}^{2^\nu} |I_{2^\nu,i}|^{-1+\frac{1}{q}}\left\|\sigma_{\beta,r,2^\nu}(x;\cdot) - \sigma_{\beta,r,2^{\nu-1}}(x;\cdot)\right\|_{L_1(I_{\beta,2^\nu,i})} \\
&\le c\left\|\left((x)\cdot) - \sigma_{r,2^\nu}(x;\cdot)\right)w_{2^\nu}^{-1+\frac{1}{q}}\right\|_{L_1} \qquad (4.24)\\
&\quad + c\left\|\left(x(\cdot) - \sigma_{r,2^\nu}(x;\cdot)\right)w_{2^{\nu-1}}^{-1+\frac{1}{q}}\right\|_{L_1} \\
&\le c2^{-(r+\frac{1}{q})\nu},
\end{aligned}$$

where $c = c(r,\alpha,\beta,p,q)$. Hence, by virtue of (4.22),

$$\begin{aligned}
&E\left(\Delta_+^{r+1}W_{p,\alpha}^r, \Sigma^{m_0,\dots,m_N}\right)_{L_q} \\
&\le \sup_{x\in\Delta_+^{r+1}W_{p,\alpha}^r}\left\|x(\cdot) - \sigma_{r,2^N}(x;\cdot)\right\|_{L_q} \qquad (4.25)\\
&\quad + c\sum_{\nu=1}^{N} 2^{-(r+\frac{1}{q})\nu}E\left(B_1^{2r2^\nu+1}, L^{m_\nu}\right)_{l_q^{2r2^\nu+1}},
\end{aligned}$$

where $c = c(r,\alpha,\beta,p,q)$. If we set $m_\nu := 2r2^\nu + 1$, $\nu = 1,2,\dots,n-1$, that is, $L^{m_\nu} := \mathbb{R}^{2r2^\nu+1}$, then clearly

$$E\big(B^{2r2^\nu+1}, L^{m_\nu}\big)_{l_q^{2r2^\nu+1}} = 0, \quad \nu = 1,2,\dots,n-1.$$

Also by (4.6),

$$\big\|x(\cdot) - \sigma_{r,2^N}(x;\cdot)\big\|_{L_q} \le c2^{-(r+\frac{1}{q})N}, \quad x \in \Delta_+^{r+1}W_{p,\alpha}^r.$$

Thus, we take subspaces L^{m_ν} so that

$$E\big(B_1^{2r2^\nu+1}, L^{m_\nu}\big)_{l_q^{2r2^\nu+1}} \le 2d_{m_\nu}\big(B_1^{2r2^\nu+1}\big)_{l_q^{2r2^\nu+1}}, \quad \nu = 1,\dots,N,$$

and we obtain by (4.25),

$$\begin{aligned} E\big(\Delta_+^{r+1}W_{p,\alpha}^r, \Sigma^{m_0,\dots,m_N}\big)_{L_q} &\le c2^{-(r+\frac{1}{q})N} \\ &+ c\sum_{\nu=n}^{N} 2^{-(r+\frac{1}{q})\nu} d_{m_\nu}\big(B_1^{2r2^\nu+1}\big)_{l_q^{2r2^\nu+1}}, \end{aligned} \tag{4.26}$$

where $c = c(r,\alpha,\beta,p,q)$.

In order to obtain the upper bounds that we require we apply now a standard technique. Namely, we fix an positive integer λ so that

$$r - \frac{1}{2}\frac{1}{\lambda-1} > 0, \quad \text{and} \quad \lambda \ge \left(r+\frac{1}{2}\right)\left(r+\frac{1}{q}\right)^{-1},$$

and put $N := \lambda n$, $n \in \mathbb{N}$. Set

$$m_\nu := 2r\left\lceil 2^{\frac{N-\nu}{\lambda-1}}\right\rceil, \quad \nu = n,\dots,N,$$

where $\lceil u\rceil$ is the integrer ceiling of $u \in \mathbb{R}$. Evidently, $m_0 + m_1 + \cdots + m_N \le a2^n$, where $a = a(r,p,q) \in \mathbb{N}$. Hence, by (4.3) and (4.26) we have,

$$\begin{aligned} E\big(\Delta_+^{r+1}W_{p,\alpha}^r, \Sigma^{m_0,\dots,m_N}\big)_{L_q} &\le c2^{-(r+\frac{1}{2})n} \\ &+ c\sum_{\nu=n}^{n^*} 2^{-(r+\frac{1}{q})\nu} d_{m_\nu}\big(B_1^{2r2^\nu+1}\big)_{l_q^{2r2^\nu+1}}. \end{aligned} \tag{4.27}$$

If $2 < q < \infty$, then we use Lemma G1 to obtain

$$d_m \left(B_1^n\right)_{l_q^n} \leq c n^{\frac{1}{q}} m^{-\frac{1}{2}},$$

where $c = c(q)$. Therefore by virtue of (4.27),

$$\begin{aligned}
&E\big(\Delta_+^{r+1} W_{p,\alpha}^r, \Sigma^{m_0,\dots,m_N}\big)_{L_q} \\
&\leq c2^{-(r+\frac{1}{2})n} + \sum_{\nu=n}^{N} 2^{-(r+\frac{1}{q})\nu} d_{m_\nu}\big(B_1^{2r2^\nu+1}\big)_{l_q^{2r2^\nu+1}} \\
&\leq c\sum_{\nu=n}^{N} 2^{-r\nu-\frac{1}{2}\frac{N-\nu}{\lambda-1}} \qquad (4.28)\\
&= c2^{-\frac{1}{2}\frac{\lambda n}{\lambda-1}} \sum_{\nu=n}^{\lambda n} 2^{-(r-\frac{1}{2}\frac{1}{\lambda-1})\nu} \\
&\leq c2^{-\frac{1}{2}\frac{\lambda n}{\lambda-1}} 2^{-(r-\frac{1}{2}\frac{1}{\lambda-1})n} \\
&= c2^{-(r+\frac{1}{2})n},
\end{aligned}$$

where $c = c(r, \alpha, \beta, p, q)$.

For the case $q = \infty$ we observe that $B_1^n \subseteq B_2^n$, so that applying Lemma K and (4.27) we obtain

$$
\begin{aligned}
&E\big(\Delta_+^{r+1}W^r_{p,\alpha},\Sigma^{m_0,\dots,m_N}\big)_{L_\infty}\\
&\le c2^{-(r+\frac12)n}+c\sum_{\nu=n}^{N}2^{-r\nu}d_{m_\nu}\big(B_2^{2r2^\nu+1}\big)_{l_\infty^{2r2^\nu+1}}\\
&\le c2^{-(r+\frac12)n}+c\sum_{\nu=n}^{N}2^{-r\nu}d_{m_\nu}\big(B_2^{2r2^\nu+1}\big)_{l_\infty^{2r2^\nu+1}}\\
&\le c2^{-(r+\frac12)n}+c\sum_{\nu=n}^{N}2^{-r\nu}m_\nu^{-\frac12}\left(1+\log\tfrac{2r2^\nu+1}{m_\nu}\right)^{\frac32}\\
&\le c2^{-(r+\frac12)n}+c\sum_{\nu=n}^{N}2^{-r\nu}2^{-\frac12\frac{N-\nu}{\lambda-1}}\left(1+\log 2^{\nu-\frac{N-\nu}{\lambda-1}}\right)^{\frac32} &&(4.29)\\
&=c2^{-(r+\frac12)n}+c2^{-\frac12\frac{\lambda}{\lambda-1}n}\sum_{\nu=n}^{\lambda n}2^{-(r-\frac12\frac{1}{\lambda-1})\nu}\left(1+\log 2^{\frac{\lambda}{\lambda-1}(\nu-n)}\right)^{\frac32}\\
&\le c2^{-(r+\frac12)n}+c2^{-(r+\frac12)n}\sum_{\nu=n}^{\lambda n}2^{-(r-\frac12\frac{1}{\lambda-1})(\nu-n)}(\nu-n+1)^{\frac32}\\
&\le c2^{-(r+\frac12)n}+c2^{-(r+\frac12)n}\int_1^\infty t^{\frac32}2^{-(r-\frac12\frac{1}{\lambda-1})t}dt\\
&\le c2^{-(r+\frac12)n},
\end{aligned}
$$

where $c=c(r,\alpha,\beta,p)$. If we let $\Sigma_{a2^n}:=\Sigma^{m_0,\dots,m_N}$, then (4.21) follows from (4.28) and (4.29) and in turn implies

$$d_{a2^n}\big(\Delta_+^{r+1}W^r_{p,\alpha}\big)_{L_q}\le c2^{-\left(r+\frac12\right)n},\quad 2<q\le\infty,$$

where $c=c(r,\alpha,\beta,p,q)$. It is a standard technique now to obtain (4.20). This concludes the proof of the upper bounds in (1.6) and thus completes the proof of Theorem 2. □

§5. The linear widths of the classes $\Delta_+^sW^r_{p,\alpha}$ in L_q– auxiliary lemmas

We begin by recalling some lemmas.

Lemma G2. (see [1]) *Let $n,m\in\mathbb{N}$ be so that $m<n$, and let $1\le p<q\le\infty$, excluding the case $p=1$ and $q=\infty$. Then*

$$d_m\left(B_p^n\right)^{lin}_{l_q^n}\asymp\Psi(m,n,p,q):=\begin{cases}\Phi(m,n,p,q), & 1\le p<q\le p',\\ \Phi(m,n,q',p'), & \max\{p,p'\}<q\le\infty,\end{cases}\tag{5.1}$$

where $\Phi(m,n,p,q)$ is defined by (2.1), and $\frac{1}{p}+\frac{1}{p'}=1$. The constants in these two-sided estimates do not depend on m and n.

The following is the corollary of [12, Lemma 3].

Lemma M1. *For all $n,m\in\mathbb{N}$ so that $m<n$, we have*

$$d_{2m+1}\left(B_1^{2n+1}\right)_{l_\infty^{2n+1}}^{lin}\le\begin{cases}cn^{-\frac{1}{2}}, & n<m^2,\\ cn^{\frac{1}{2}}m^{-\frac{3}{2}}, & n\ge m^2,\end{cases}$$

where $c>0$ is an absolute constant.

Also, as a corollary of [12, Lemma 5], we get

Lemma M2. *For all $n,m\in\mathbb{N}$ so that $m<n$ we have*

$$d_m\left(B_1^n\right)_{l_\infty^n}^{lin}\le cm^{-\frac{1}{2}}\left(\log n\right)^{\frac{1}{2}},$$

where $c>0$ is an absolute constant.

We need one new result

Lemma 4. *Let $n,m\in\mathbb{N}$ be so that $m<n$ and $2\le p\le q\le\infty$ or $2\le\frac{p}{p-1}\le q\le\infty$. Then*

$$d_m\left(S_p^+\left(E^n\right)\right)_{l_q^n}^{lin}\ge 2^{-\frac{1}{2}}\left(1-\frac{m}{n}\right)^{\frac{1}{2}}n^{-(\frac{1}{p}-\frac{1}{2})_+}.$$

Proof. Let X be real linear normed space with X^* its dual, and let $L^m:=\operatorname{span}\{x_k\}_{k=1}^m$ be an arbitrary subspace in X. Then by definition

$$d_m(W)_X^{lin}=\inf_{\{x_k\}_{k=1}^m\subseteq X}\ \inf_{\{x_k^*\}_{k=1}^m\subseteq X^*}\ \sup_{x\in W}\left\|x-\sum_{k=1}^m<x_k^*,x>x_k\right\|_X.$$

Therefore

$$d_m\left(S_2^+\left(E^n\right)\right)_{l_\infty^n}^{lin}=\inf_{\{\zeta^{(k)}\}_{k=1}^m\subseteq l_\infty^n}\ \inf_{\{\xi^{(k)}\}_{k=1}^m\subseteq l_1^n}\ \sup_{x\in S_2^+\left(E^n\right)}\left\|x-\sum_{k=1}^m\left(\xi^{(k)},x\right)\zeta^{(k)}\right\|_{l_\infty^n},$$

where $(\cdot,\cdot)$ is usual scalar product. Since

$$\left\| x-\sum_{k=1}^{m}\big(\xi^{(k)},x\big)\zeta^{(k)}\right\|_{l_\infty^n}=\sup_{y\in B_1^n}\left|\Big(x-\sum_{k=1}^{m}\big(\xi^{(k)},x\big)\zeta^{(k)},y\Big)\right|,$$

it follows that

$$\begin{aligned}\sup_{x\in S_2^+\left(E^n\right)}\left\| x-\sum_{k=1}^{m}\big(\xi^{(k)},x\big)\zeta^{(k)}\right\|_{l_\infty^n}&=\sup_{x\in S_2^+\left(E^n\right)}\sup_{y\in B_1^n}\left|\Big(x-\sum_{k=1}^{m}\big(\xi^{(k)},x\big)\zeta^{(k)},y\Big)\right|\\&=\sup_{y\in B_1^n}\sup_{x\in S_2^+\left(E^n\right)}\left|\Big(x-\sum_{k=1}^{m}\big(\xi^{(k)},x\big)\zeta^{(k)},y\Big)\right|\\&=\sup_{y\in B_1^n}\sup_{x\in S_2^+\left(E^n\right)}\left|\Big(y-\sum_{k=1}^{m}\big(\zeta^{(k)},y\big)\xi^{(k)},x\Big)\right|,\end{aligned}$$

so that

$$\begin{aligned}d_m\big(S_2^+\left(E^n\right)\big)_{l_\infty^n}^{lin}&=\inf_{\{\zeta^{(k)}\}_{k=1}^m\subseteq l_\infty^n}\ \inf_{\{\xi^{(k)}\}_{k=1}^m\subseteq l_1^n}\ \sup_{y\in B_1^n}\ \sup_{x\in S_2^+\left(E^n\right)}\left|\Big(y-\sum_{k=1}^{m}\big(\zeta^{(k)},y\big)\xi^{(k)},x\Big)\right|\\&=\inf_{\{\xi^{(k)}\}_{k=1}^m\subseteq l_1^n}\ \inf_{\{\zeta^{(k)}\}_{k=1}^m\subseteq l_\infty^n}\ \sup_{y\in B_1^n}\ \sup_{x\in S_2^+\left(E^n\right)}\left|\Big(y-\sum_{k=1}^{m}\big(\zeta^{(k)},y\big)\xi^{(k)},x\Big)\right|.\end{aligned}$$

For any $z\in\mathbb{R}^n$ such that $\|z\|_{l_2^n}\neq 0$ we have

$$\begin{aligned}\sup_{x\in S_2^+\left(E^n\right)}|(z,x)|&=\|z\|_{l_2^n}\sup_{x\in S_2^+\left(E^n\right)}\left(\frac{z}{\|z\|_{l_2^n}},x\right)\\&\geq\|z\|_{l_2^n}\inf_{\|e\|_{l_2^n}=1}\ \sup_{x\in S_2^+\left(E^n\right)}|(e,x)|.\end{aligned}\tag{5.2}$$

We claim that

$$\inf_{\|e\|_{l_2^n}=1}\ \sup_{x\in S_2^+\left(E^n\right)}|(e,x)|\geq 2^{-\frac{1}{2}}.\tag{5.3}$$

Indeed, let $e=(e_1,\ldots,e_n)$ be an arbitrary vector such that $e_1^2+\cdots+e_n^2=1$. We divide coordinates of e into two sets, that of the negative coordinates and that of the nonnegative ones. Let $e_{i_\nu} < 0$, $1 \le \nu \le N$ and $e_{i_\nu} \ge 0$, $N < \nu \le n$. Then either $e_{i_1}^2 + \cdots + e_{i_N}^2 \ge \frac{1}{2}$, or $e_{i_{N+1}}^2 + \cdots + e_{i_n}^2 \ge \frac{1}{2}$, so without loss of generality, we may assume the former. Hence, setting $x_{i_\nu} := \left(e_{i_1}^2 + \cdots + e_{i_N}^2\right)^{-\frac{1}{2}} e_{i_\nu}$, $1 \le \nu \le N$ and $x_{i_\nu} := 0$, $N < \nu \le n$, we obtain $x = x(e) \in S_2^+\left(E^n\right)$ and $|(e,x)| \ge 2^{-\frac{1}{2}}$ and (5.3) is proven.

Now, by virtue of (5.2) and (5.3) we conclude that

$$\begin{aligned} d_m\left(S_2^+\left(E^n\right)\right)_{l_\infty^n}^{lin} &\ge 2^{-\frac{1}{2}} \inf_{\{\xi^{(k)}\}_{k=1}^m \subseteq l_1^n} \inf_{\{\zeta^{(k)}\}_{k=1}^m \subseteq l_\infty^n} \sup_{y \in B_1^n} \left\| y - \sum_{k=1}^m \left(\zeta^{(k)}, y\right) \xi^{(k)} \right\|_{l_2^n} \\ &= 2^{-\frac{1}{2}} \inf_{\{\xi^{(k)}\}_{k=1}^m \subseteq l_2^n} \inf_{\{\zeta^{(k)}\}_{k=1}^m \subseteq l_2^n} \sup_{y \in B_1^n} \left\| y - \sum_{k=1}^m \left(\zeta^{(k)}, y\right) \xi^{(k)} \right\|_{l_2^n} \\ &= 2^{-\frac{1}{2}} d_m\left(B_1^n\right)_{l_2^n}^{lin} \ge 2^{-\frac{1}{2}} d_m\left(B_1^n\right)_{l_2^n}. \end{aligned}$$

Hence by Lemma KPS,

$$d_m\left(S_2^+\left(E^n\right)\right)_{l_\infty^n}^{lin} \ge 2^{-\frac{1}{2}} \left(1 - \frac{m}{n}\right)^{\frac{1}{2}}. \tag{5.4}$$

If $2 \le p \le q \le \infty$, then $S_2^+\left(E^n\right) \subseteq S_p^+\left(E^n\right)$ and $\|x\|_{l_q^n} \ge \|x\|_{l_\infty^n}$. Then by (5.4) we have

$$d_m\left(S_p^+\left(E^n\right)\right)_{l_q^n}^{lin} \ge d_m\left(S_2^+\left(E^n\right)\right)_{l_\infty^n}^{lin} \ge 2^{-\frac{1}{2}} \left(1 - \frac{m}{n}\right)^{\frac{1}{2}}. \tag{5.5}$$

On the other hand, if $2 \le p' \le q \le \infty$, then $n^{\frac{1}{2}-\frac{1}{p}} S_2^+\left(E^n\right) \subseteq S_p^+\left(E^n\right)$ and $\|x\|_{l_q^n} \ge \|x\|_{l_\infty^n}$. Therefore, by (5.4) we obtain

$$d_m\left(S_p^+\left(E^n\right)\right)_{l_q^n}^{lin} \ge n^{\frac{1}{2}-\frac{1}{p}} d_m\left(S_2^+\left(E^n\right)\right)_{l_\infty^n}^{lin} \ge 2^{-\frac{1}{2}} \left(1 - \frac{m}{n}\right)^{\frac{1}{2}} n^{\frac{1}{2}-\frac{1}{p}}.$$

Combining this with (5.5) yields Lemma 4. $\square$

§6. The linear widths of the classes $\Delta_+^s W_{p,\alpha}^r$ in L_q

We are in a position to prove our assertions.

Proof of Theorem 3. The upper bounds in Theorem 3 follow from Theorem KL2, and the lower bounds for the cases $1 \le q \le p \le \infty$, $1 \le p \le q \le 2$ and $2 \le q \le \frac{p}{p-1} \le \infty$ are immediate consequences of Theorem 1 since $d_n\left(\Delta_+^s W_{p,\alpha}^r\right)_{L_q}^{lin} \ge d_n\left(\Delta_+^s W_{p,\alpha}^r\right)_{L_q}$.

Thus we concentrate on the two remaining cases, namely, $2 \le p \le q \le \infty$ and $2 \le p' \le q \le \infty$.

Since $\Delta_+^s W_p^r \subseteq \Delta_+^s W_{p,\alpha}^r$, $0 \le \alpha < \infty$, it suffices to prove that

$$d_n \left(\Delta_+^s W_p^r\right)_{L_q}^{lin} \ge c n^{-r+\frac{1}{p}-\frac{1}{q}}, \quad 2 \le p \le q \le \infty, \tag{6.1}$$

and

$$d_n \left(\Delta_+^s W_p^r\right)_{L_q}^{lin} \ge c n^{-r+\frac{1}{2}-\frac{1}{q}}, \quad 2 \le p' \le q \le \infty. \tag{6.2}$$

Just as in the proof of Theorem 1 we obtain,

$$d_n \left(\Delta_+^s W_p^r\right)_{L_q}^{lin} \ge C N^{-r+\frac{1}{p}-\frac{1}{q}} d_n \left(S_p^+ \left(E^N\right)\right)_{l_q^N}^{lin}, \quad 0 \le s \le r,$$

where $C = C(r,p,q)$. Thus, substituting $N = 2n$ we get

$$d_n \left(\Delta_+^s W_p^r\right)_{L_q}^{lin} \ge C (2n)^{-r+\frac{1}{p}-\frac{1}{q}} d_n \left(S_p^+ \left(E^{2n}\right)\right)_{l_q^{2n}}^{lin},$$

which by virtue of Lemma 4 turns into

$$\begin{aligned} d_n \left(\Delta_+^s W_p^r\right)_{L_q}^{lin} &\ge C (2n)^{-r+\frac{1}{p}-\frac{1}{q}} 2^{-\frac{1}{2}} \left(1 - \frac{n}{2n}\right)^{\frac{1}{2}} (2n)^{-\left(\frac{1}{p}-\frac{1}{2}\right)_+} \\ &\ge c n^{-r+\frac{1}{p}-\frac{1}{q}-\left(\frac{1}{p}-\frac{1}{2}\right)_+}, \end{aligned}$$

where $c = c(r,p,q)$ and (6.1) and (6.2) follow. This completes the proof of Theorem 3. $\square$

We proceed to the proof of Theorem 4.

Proof of Theorem 4. Again, the lower bounds follow from Theorem 2, since

$$d_n \left(\Delta_+^{r+1} W_{p,\alpha}^r\right)_{L_q}^{lin} \ge d_n \left(\Delta_+^{r+1} W_{p,\alpha}^r\right)_{L_q}.$$

Hence we only have to prove the upper bounds. We observe that all the operators which we used in the proof of Theorem 2 were linear. Therefore, just as in that proof we can obtain,

$$d_n(\Delta_+^{r+1} W_{p,\alpha}^r)_{L_q}^{lin} \le c n^{-r-\frac{1}{q}}, \quad 1 \le q \le 2, \tag{6.3}$$

and for $2 < q \le \infty$,

$$
\begin{aligned}
E\big(\Delta_+^{r+1}W^r_{p,\alpha}, \Sigma^{m_0,\dots,m_N}\big)^{lin}_{L_q} &\le c2^{-(r+\frac{1}{q})N} \\
&+ c\sum_{\nu=n}^{N} 2^{-(r+\frac{1}{q})\nu} d_{m_\nu}\big(B_1^{2r2^\nu+1}\big)^{lin}_{l_q^{2r2^\nu+1}}.
\end{aligned}
\tag{6.4}
$$

If $2 < q \le \infty$, then we prescribe the integers λ, N and m_ν as in the proof of Theorem 2. For $(p,q) \ne (1,\infty)$, we apply Lemma G2 to (6.4) and obtain

$$
E\big(\Delta_+^{r+1}W^r_{p,\alpha}, \Sigma^{m_0,\dots,m_N}\big)^{lin}_{L_q} \le c2^{-(r+\frac{1}{2})n}, \quad 2 < q < \infty,
\tag{6.5}
$$

where $c = c(r,\alpha,p,q)$.

In case $(p,q) = (1,\infty)$ we take

$$
M := \left\lceil \frac{2\lambda}{\lambda+1} n \right\rceil,
$$

and note that

$$
2^\nu \le c\left[2^{\frac{N-\nu}{\lambda-1}}\right]^2, \quad \nu = n,\dots,M-1,
$$

where $c > 0$ is an absolute constant. We rewrite (6.4) in the form

$$
\begin{aligned}
E\big(\Delta_+^{r+1}W^r_{1,\alpha}, \Sigma^{m_0,\dots,m_N}\big)^{lin}_{L_\infty} &\le c2^{-rN} \\
&+ c\sum_{\nu=n}^{M-1} 2^{-r\nu} d_{m_\nu}\big(B_1^{2r2^\nu+1}\big)^{lin}_{l_\infty^{2r2^\nu+1}} \\
&+ c\sum_{\nu=M}^{N} 2^{-r\nu} d_{m_\nu}\big(B_1^{2r2^\nu+1}\big)^{lin}_{l_\infty^{2r2^\nu+1}},
\end{aligned}
\tag{6.6}
$$

where $c = c(r,\alpha)$. Lemma M1 yields

$$
\begin{aligned}
\sum_{\nu=n}^{M-1} 2^{-r\nu} d_{m_\nu}\big(B_1^{2r2^\nu+1}\big)^{lin}_{l_\infty^{2r2^\nu+1}} &\le c\sum_{\nu=n}^{M-1} 2^{-r\nu}2^{-\frac{1}{2}\frac{N-\nu}{\lambda-1}} \\
&= c2^{-\frac{1}{2}\frac{\lambda}{\lambda-1}n} \sum_{\nu=n}^{M-1} 2^{-\left(r-\frac{1}{2}\frac{1}{\lambda-1}\right)\nu} \\
&\le c2^{-\frac{1}{2}\frac{\lambda}{\lambda-1}n} 2^{-\left(r-\frac{1}{2}\frac{1}{\lambda-1}\right)n} \\
&= c2^{-(r+\frac{1}{2})n}.
\end{aligned}
$$

And by virtue of Lemma M2 we have,

$$
\begin{aligned}
&\sum_{\nu=M}^{N} 2^{-r\nu} d_{m_\nu}\big(B_1^{2r2^\nu+1}\big)^{lin}_{l_\infty^{2r2^\nu+1}} \\
&\le c2^{-\frac{1}{2}\frac{\lambda}{\lambda-1}n} c \sum_{\nu=M}^{N} 2^{-r\nu} 2^{-\left(r-\frac{1}{2}\frac{1}{\lambda-1}\right)\nu} \big(\log(2r2^\nu+1)\big)^{\frac{1}{2}} \\
&\le c2^{-\frac{1}{2}\frac{\lambda}{\lambda-1}n} \sum_{\nu=M}^{N} 2^{-\left(r-\frac{1}{2}\frac{1}{\lambda-1}\right)\nu} \nu^{\frac{1}{2}} \\
&= c2^{-(r+\frac{1}{2})n} \sum_{\nu=M}^{N} 2^{-\left(r-\frac{1}{2}\frac{1}{\lambda-1}\right)(\nu-n)} \nu^{\frac{1}{2}} \\
&\le c2^{-(r+\frac{1}{2})n} \sum_{\nu=M}^{N} 2^{-\left(r-\frac{1}{2}\frac{1}{\lambda-1}\right)(\nu-n)} (\nu-n)^{\frac{1}{2}} \\
&\le c2^{-(r+\frac{1}{2})n} \int_0^\infty t^{\frac{1}{2}} 2^{-\left(r-\frac{1}{2}\frac{1}{\lambda-1}\right)t}\, dt \le c2^{-(r+\frac{1}{2})n},
\end{aligned}
$$

where we have used the fact that

$$\nu \le \tfrac{\lambda+1}{\lambda-1}(\nu-n), \quad \nu = M, \dots, N.$$

Substituting the last two inequalitites in (6.6), we obtain

$$E\big(\Delta_+^{r+1} W_{p,\alpha}^r, \Sigma^{m_0,\dots,m_N}\big)^{lin}_{L_\infty} \le c2^{-(r+\frac{1}{2})n},$$

which together with (6.3) and (6.5) implies the upper bounds in (1.10) and concludes the proof. □

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