Degree of Simultaneous Coconvex Polynomial Approximation

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Dedicated to Professor Paul Butzer on the occasion of his 70th birthday

Abstract: Let $f \in C^1[-1,1]$ change its convexity finitely many times in the interval, say s times, at $Y_s : -1 < y_s < \cdots < y_1 < 1$. We estimate the degree of simultaneous approximation of f and its derivative by polynomials of degree n, which change convexity exactly at the points Y_s, and their derivatives. We show that provided n is sufficiently large, depending on the location of the points Y_s, the rate of approximation can be estimated by $C(s)/n$ times the second Ditzian–Totik modulus of smoothness of f'. This should be compared to a recent paper by the authors together with I. A. Shevchuk where f is merely assumed to be continuous and estimates of coconvex approximation are given by means of the third Ditzian–Totik modulus of smoothness. However, no simultaneous approximation is given there.

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1 Introduction and main results

Let $f \in C^1[-1,1]$ change its convexity finitely many times, say s, at the points $Y_s : -1 < y_s < \cdots < y_1 < 1$ in $[-1,1]$. For later reference set $y_0 := 1$ and $y_{s+1} := -1$. We wish to approximate f by means of polynomials which are coconvex with f, that is, which change convexity exactly at the points Y_s. Estimates on the degree of simultaneous coconvex approximation were first obtained by Kopotun [2] for a twice continuously differentiable function. We are going to improve those results in that we do not assume the existence everywhere and continuity of the second derivative. We are going to make use of some special polynomials related to the function f which were constructed in that article [2]. Recently, the authors together with Shevchuk [5] have removed completely the assumption on the existence of derivatives but then of course one does not obtain simultaneous approximation. In order to state our main result we recall the definition of the mth order Ditzian–Totik moduli of smoothness $\omega_m^\varphi(f,t)$. For $f \in C[-1,1]$, we set

$$\omega_m^\varphi(f,t) = \sup_{0<h\le t} \left\|\Delta^m_{h\varphi(\cdot)} f(\cdot)\right\|,$$

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where $\varphi(x) := \sqrt{1-x^2}$, and

$$\Delta_\eta^m f(x) := \begin{cases} \sum_{i=0}^m \binom{m}{i}(-1)^{m-i} f(x - \frac{m}{2}\eta + i\eta), & \text{if } x \pm \frac{m}{2}\eta \in [-1,1], \\ 0, & \text{otherwise.} \end{cases}$$

is the symmetric mth difference.

Our main result is the following.

Theorem 1 *Let $f \in C^1[-1,1]$ have s changes of convexity at $Y_s : -1 < y_1 < \cdots < y_s < 1$, and denote $d(Y_s) := \min\{1+y_1, y_2-y_1, \ldots, y_s - y_{s-1}, 1-y_s\}$. Then there exists a constant $A = A(s)$ which depends only on the number of convexity changes–s, such that for $n > \frac{A(s)}{d(Y_s)}$, there is a polynomial p_n of degree not exceeding n, which is coconvex with f and satisfies*

$$\|f - p_n\| \le C(s) n^{-1} \omega_2^\varphi(f', n^{-1}), \tag{1}$$

and

$$\|f' - p_n'\| \le C(s) \omega_2^\varphi(f', n^{-1}). \tag{2}$$

The constant $C(s)$ is independent of f and n and of the location of the convexity changes.

Estimate (1) follows from the following result of the authors together with Shevchuk [5]. Note however that the following says nothing about simultaneous approximation.

Theorem A *Let $f \in C[-1,1]$ have s changes of convexity at $Y_s : -1 < y_1 < \cdots < y_s < 1$, and denote $d(Y_s) := \min\{1+y_1, y_2-y_1, \ldots, y_s - y_{s-1}, 1-y_s\}$. Then there exists a constant $A = A(s)$ which depends only on the number of convexity changes–s, such that for $n > \frac{A(s)}{d(Y_s)}$, there is a polynomial p_n of degree not exceeding n, which is coconvex with f and satisfies*

$$\|f - p_n\| \le C(s) \omega_3^\varphi(f, n^{-1}),$$

which in turn implies (1) for $f \in C^1[-1,1]$. The constant $C(s)$ is independent of f and n and of the location of the convexity changes.

As an immediate consequence of Theorem 1, we get an estimate for comonotone approximation, which is also the special case for the sup-norm, of a recent result by the authors [4]. Namely,

Corollary 2 *If $f \in C[-1,1]$ changes its monotonicity at the points Y_s, then for $n > \frac{A(s)}{d(s)}$, there is a polynomial p_n of degree not exceeding n which changes monotonicity exactly at the points Y_s and satisfies*

$$\|f - p_n\| \le C(s) \omega_2^\varphi(f, n^{-1}). \tag{3}$$

The proof of Corollary 2 follows immediately from (2) and the observation that the indefinite integral of f changes convexity at the points Y_s. We would also like to emphasize that all of the estimates (1), (2) and (3) are sharp in the sense that ω_2^φ cannot be replaced by ω_3^φ in any of them (see Zhou [6]).

2 Proof of Theorem 1

Without loss of generality we may assume that f is convex in $[-1, y_1]$, and by subtracting a linear function we may assume that $f(y_1) = f'(y_1) = 0$. Then by virtue of the continuity of the f', it follows that the flipped function

$$\hat{f}(x) := \begin{cases} -f(x) & \text{if } x \in [-1, y_1], \\ f(x) & \text{otherwise}, \end{cases} \tag{4}$$

is in $C^1[-1,1]$, $\hat{f}(y_1) = \hat{f}'(y_1) = 0$, it is concave in $[-1, y_2]$, and changes convexity at $Y_s \setminus \{y_1\} =: Y'_{s-1}$. For the sake of simplicity in notation in the sequel we rename $\alpha := y_1$. We are going to use an induction assumption on the number of convexity changes. For $s = 0$ Theorem 1 becomes a theorem on simultaneous convex approximation which is a simple consequence of Theorem 2 of [3]. Suppose now that Theorem 1 is true for a function f which has $s \geq 1$ convexity changes. Since $\hat{f}$ has fewer convexity changes $(s-1)$ we can assume for $n > \frac{A(s-1)}{d(Y'_{s-1})}$, the existence of an nth degree polynomial q_n which is coconvex with $\hat{f}$, and which satisfies the analogues of (1) and (2). Namely,

$$\|\hat{f} - q_n\| \leq C(s-1) n^{-1} \omega_2^{\varphi}(\hat{f}', n^{-1}), \tag{5}$$

and

$$\|\hat{f}' - q'_n\| \leq C(s-1) \omega_2^{\varphi}(\hat{f}', n^{-1}). \tag{6}$$

Note that since $f(\alpha) = 0$, we may assume that $q_n(\alpha) = 0$ doubling the constant in (5). The idea of flipping part of f, was originally introduced by Beatson and Leviatan [1] for their proof of the case of comonotone approximation. It is crucial to our proof, and is the main reason why we need the assumption that f is continuously differentiable and cannot do with just continuity of f.

Now let $x_j := x_{j,n} := \cos \frac{j\pi}{n}$, $j = 0, \dots, n$, be the Chebyshev partition of $[-1,1]$, and denote $h_j := h_{j,n} := x_{j-1} - x_j$ and

$$\psi_j(x) := \psi_{j,n}(x) := \frac{h_j}{|x - x_j| + h_j}.$$

It is well known that $h_{j\pm 1} < 3h_j$ and that for $x \in [x_j, x_{j-1}]$ $\Delta_n(x) \leq h_j < 5\Delta_n(x)$, where as always, $\Delta_n(x) := \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}$. We assume that $\alpha \in [x_{j_0}, x_{j_0-1})$. Then, if $n > N_\alpha := \max\{\frac{50}{y_2 - \alpha}, \frac{50}{1+\alpha}\}$, we are assured that $x_{j_0+3} \geq -1$ and that $x_{j_0-4} \leq y_2$. Also it follows that

$$2\varphi(\alpha) > n\Delta_n(\alpha) > \varphi(\alpha) > n^{-1}, \tag{7}$$

Thus, fix $n > \max\{\frac{A(s-1)}{d(Y'_{s-1})}, N_\alpha\}$, which readily leads to the definition of $A(s)$.

Kopotun [2] has constructed for α and q_n for each n like above, two polynomials V_n and W_n of degrees at most $20n(s+1) =: 4n\mu$, with the properties that for all $x \in [-1,1]$,

$$V_n(x) \mathrm{sgn}\,(x - \alpha) \geq 0,$$

$$V_n'(x)q_n''(x)(q_n'(x)-q_n'(\alpha))\mathrm{sgn}\,(x-\alpha)\ge 0\,,$$

$$|V_n(x)-\mathrm{sgn}\,(x-\alpha)|\le C(s)\psi_{j_0}^{\mu}, \tag{8}$$

$$|W_n(x)-\mathrm{sgn}\,(x-\alpha)|\le C(s)\psi_{j_0}^{\mu}; \tag{9}$$

and finally,

$$W_n'(x)\mathrm{sgn}\,q_n'(\alpha)\ge 0,\quad x\in[y_{2j},y_{2j+1}],\quad j=0,\dots,[s/2],$$

and

$$W_n'(x)\mathrm{sgn}\,q_n'(\alpha)\le 0,\quad x\in[y_{2j+1},y_{2j+2}],\quad j=0,\dots,[s/2].$$

We are ready to define the polynomial

$$p_n(x):=\int_\alpha^x\Big[\Big(q_n'(u)-q_n'(\alpha)\Big)V_n(u)+q_n'(\alpha)W_n(u)\Big]\,du\,,$$

of degree at most $5n\mu$, which is coconvex with f (see [2]). Hence we conclude the induction step by proving (1) and (2). To this end we need the estimates

$$|\hat f'(x)|=|f'(x)|\le C\psi_{j_0}^{-1}\omega_2^{\varphi}(f',n^{-1})\,,\qquad x\in[-1,1]\,, \tag{10}$$

and

$$\omega_2^{\varphi}(\hat f',n^{-1})\le C\omega_2^{\varphi}(f',n^{-1})\ . \tag{11}$$

We delay the proof of (10) and (11) to the end of this section as we continue with the flow of the proof.

In order to prove (2) we observe that,

$$\begin{aligned}|f'(x)-p_n'(x)| &= \Big|\Big(\hat f'(x)-q_n'(x)\Big)V_n(x)+\hat f'(x)\Big(\mathrm{sgn}\,(x-\alpha)-V_n(x)\Big)\\ &\qquad +q_n'(\alpha)\Big(V_n(x)-W_n(x)\Big)\Big|\\ &\le C(s)\omega_2^{\varphi}(f',n^{-1})\,(1+\psi_{j_0}^{\mu}+\psi_{j_0}^{\mu-1})\ ,\end{aligned}$$

where we estimated the first term by inequalities (6), (8) and (11), the second term by (8) and (10); and the third by inequalities (6), (8), (9) and (11), bearing in mind that $\hat f'(\alpha)=0$. This proves (2).

Recalling that $q_n(\alpha)=0$, we note that $\int_\alpha^x q_n'(u)\mathrm{sgn}\,(u-\alpha)\,du=q_n(x)\mathrm{sgn}\,(x-\alpha)$. Hence,

$$\begin{aligned}|f(x)-p_n(x)| &= \Big|\Big(\hat f(x)-q_n(x)\Big)\mathrm{sgn}\,(x-\alpha)+\int_\alpha^x\Big(q_n'(u)-\hat f'(u)\Big)\Big(\mathrm{sgn}\,(u-\alpha)-V_n(u)\Big)\,du\\ &\qquad +\int_\alpha^x\hat f'(u)\Big(\mathrm{sgn}\,(u-\alpha)-V_n(u)\Big)\,du+q_n'(\alpha)\int_\alpha^x\Big(V_n(u)-W_n(u)\Big)\,du\Big|\\ &\le C(s)n^{-1}\omega_2^{\varphi}(f',n^{-1})\ \left(1+n\left|\int_\alpha^x\psi_{j_0}^{\mu}(u)\,du\right|+n\left|\int_\alpha^x\psi_{j_0}^{\mu-1}(u)\,du\right|\right)\\ &\le C(s)n^{-1}\omega_2^{\varphi}(f',n^{-1})\ ,\end{aligned}$$

where we estimated the first term by (5) and (11), the second term by (6), (8) and (11), the third term by (8) and (10); and finally the forth term by (6), (8), (9) and (11), observing that $\hat{f}'(\alpha) = 0$.

We conclude the proof by proving (10) and (11). First let $x \in J_0 := [x_{j_0+1}, x_{j_0-2}]$. The concavity of $\hat{f}$ in $[-1, y_2]$, together with $\hat{f}'(\alpha) = 0$, implies that $\hat{f}' \geq 0$ in $[-1, \alpha]$ and $\hat{f}' \leq 0$ in $[\alpha, y_2]$. Hence,

$$\begin{aligned} |\hat{f}'(x)| = |f'(x)| &\leq |f'(x) - 2f'(\alpha) + f'(2\alpha - x)| \\ &= |\Delta^2_{h\varphi(\alpha)} f'(\alpha)| \leq C\omega_2^\varphi(f', n^{-1}) \end{aligned} \tag{12}$$

where $h\varphi(\alpha) = |x - a| \leq C\Delta_n(\alpha)$, so that by virtue of (7), $h \leq \frac{C\Delta_n(\alpha)}{\varphi(\alpha)} \leq \frac{2C}{n}$.

Now, if $x \notin J_0$, then by virtue of (7),

$$|\Delta^2_{h\varphi(x)} \hat{f}'(x)| = |\Delta^2_{h\varphi(x)} f'(x)| \leq \omega_2^\varphi(f', n^{-1}) \quad \text{for} \quad 0 < h \leq n^{-1} , \tag{13}$$

and if $x \in J_0$, then (13) follows by the same argument as (12). Thus (11) is proved.

Denote by $L(f')$ the linear function interpolating f' at α and x_{j_o-2}. Then by Whitney's Theorem

$$\max_{x \in J_0} |f'(x) - L(f')(x)| \leq C\omega_2(f', |J_0|; J_0) \leq C\omega_2^\varphi(f', n^{-1}) , \tag{14}$$

where $|J_0|$ denotes the length of J_0, thus $|J_0| \sim h_{j_0}$. Since $\tilde{L}$, the linear polynomial of best approximation of f' in $[-1, 1]$, satisfies

$$\|f' - \tilde{L}\| \leq C\omega_2^\varphi(f', n^{-1}) , \tag{15}$$

it follows by (14) that

$$|L(f')(x) - \tilde{L}(x)| \leq C\frac{|x - \alpha| + h_{j_0}}{h_{j_0}} \omega_2^\varphi(f', n^{-1}) ,$$

where we used the fact that we know how much a linear function grows outside an interval. This together with (15), in turn implies

$$|f'(x) - L(f')(x)| \leq C\frac{|x - \alpha| + h_{j_0}}{h_{j_0}} \omega_2^\varphi(f', n^{-1}) .$$

At the same time it follows from (12) that for all $x \in [-1, 1]$,

$$|L(f')(x)| = |f'(x_{j_0-2})| \left| \frac{x - \alpha}{x_{j_0-2} - \alpha} \right| \leq C\frac{|x - \alpha| + h_{j_0}}{h_{j_0}} \omega_2^\varphi(f', n^{-1}) .$$

Hence (10) follows. This completes our proof.

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