SHAPE PRESERVING APPROXIMATION BY POLYNOMIALS

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Abstract. We are going to survey recent developments and achievements in shape preserving approximation by polynomials. We wish to approximate a function f defined on a finite interval, say $[-1,1]$, while preserving certain intrinsic "shape" properties. To be specific we demand that the approximation process preserve properties of f, like its sign in all or part of the interval, its monotonicity, convexity, etc. We will refer to these properties as the shape of the function.

§1. Introduction

We are going to discuss the degree of constrained approximation of a function f in either the uniform norm or in the $\mathbb{L}_p[-1,1]$, norm $0<p<\infty$, and we will use the notation $\mathbb{L}_\infty[-1,1]$ for $\mathbb{C}[-1,1]$, whenever we state a result which is valid both for $\mathbb{C}[-1,1]$ as well as for $\mathbb{L}_p[-1,1]$, for a proper range of p's. The degree of approximation will be measured by the appropriate (quasi-)norm which we denote by $\|\cdot\|_p$. The approximation will be carried out by polynomials $p_n\in\Pi_n$, the space of polynomials of degree not exceeding n, which have the same shape in which we are interested, as f, namely, have the same sign as f does in various parts of $[-1,1]$, or change their monotonicity or convexity exactly where f does in $[-1,1]$. Most of the proofs of the statements in this survey and especially those of the affirmative results, are technically involved and will be omitted. All we are going to say about the technique of proof is that we usually first approximate f well by splines or just continuous piecewise polynomials with the same shape as f, and then we replace the polynomial pieces by polynomials of the same shape. Thus, while this survey deals only with polynomial approximation, there are similar affirmative results for continuous

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piecewise polynomials and in many cases for splines. We will sometime indicate a proof or construct a counterexample which we consider illustrative while not too involved.

Interest in the subject began in the 1960's with work on monotone approximation by Shisha and by Lorentz and Zeller. It gained momentum in the 1970's and early 1980's with the work on monotone approximation of DeVore, and the work on comonotone approximation of Shvedov, of Newman and of Beatson and Leviatan. The last fifteen years have seen extensive research and many new results, the most advanced of which are being summarized here. We are not going to give an elaborate historical account and we direct the interested reader to an earlier survey by the author [22] and to the references therein.

The theory we are going to develop is much richer when dealing with the uniform norm and much less is known for the L_p-(quasi-)norm, when $0 < p < \infty$. We are not going to state specifically too many open problems, however, the reader will only have to compare the results for the former and for the latter norms for the questions to be apparent. Also comparison between the results in the various sections will show where work is still to be done.

To be specific, let $s \ge 0$ and let $\mathbb{Y}_s$ be the set of all collections $Y_s := \{y_i\}_{i=1}^s$ of points, so that $y_{s+1} := -1 < y_s < ... < y_1 < 1 =: y_0$, where for $s = 0$, $Y_0 = \emptyset$. For $Y_s \in \mathbb{Y}_s$ we set

$$\Pi(x, Y_s) := \prod_{i=1}^{s}(x - y_i) \ ,$$

where the empty product =1.

We let $\Delta^0(Y_s)$, be the set of functions f which change their sign (in the weak sense) exactly at the points $y_i \in Y_s$, and which are nonnegative in $(y_1, 1)$ (if $s = 0$, then this means that $f \ge 0$ in $[-1, 1]$, and we will write $f \in \Delta^0$, suppressing the $Y_0 = \emptyset$). Note that our assumption is equivalent to

$$f(x)\Pi(x, Y_s) \ge 0, \quad -1 \le x \le 1.$$

For $f \in \Delta^0(Y_s) \cap \mathbb{L}_p[-1, 1]$, we denote by

$$E_n^{(0)}(f, Y_s)_p := \inf_{p_n \in \Pi_n \cap \Delta^0(Y_s)} \|f - p_n\|_p,$$

the degree of copositive approximation of f by algebraic polynomials. If $Y_0 = \emptyset$, then we write $E_n^{(0)}(f)_p := E_n^{(0)}(f,\emptyset)_p$, which is usually referred to as the degree of positive approximation.

Also, we let $\Delta^1(Y_s)$, be the set of functions f which change monotonicity at the points $y_i \in Y_s$, and which are nondecreasing in $(y_1, 1)$, that is, f is nondecreasing in the intervals (y_{2j+1}, y_{2j}) and it is nonincreasing in (y_{2j}, y_{2j-1}). In particular, if $s = 0$, then f is nondecreasing in $[-1,1]$, and we will write $f \in \Delta^1$. Moreover, if f is differentiable in $(-1,1)$, then

$$f \in \Delta^1(Y_s), \quad \text{iff} \quad f'(x)\Pi(x, Y_s) \geq 0, \quad -1 < x < 1.$$

Now for $f \in \Delta^1(Y_s) \cap \mathbb{L}_p[-1,1]$, we denote by

$$E_n^{(1)}(f, Y_s)_p := \inf_{p_n \in \Pi_n \cap \Delta^1(Y_s)} \|f - p_n\|_p,$$

the degree of comonotone polynomial approximation. Again if $Y_0 = \emptyset$, then we write $E_n^{(1)}(f)_p := E_n^{(1)}(f,\emptyset)_p$, which is usually referred to as the degree of monotone approximation.

Finally, we let $\Delta^2(Y_s)$, be the set of functions f which change convexity at the points $y_i \in Y_s$, and which are convex in $(y_1, 1)$, (again Δ^2 if $Y_0 = \emptyset$), and for $f \in \Delta^2(Y_s) \cap \mathbb{L}_p[-1,1]$, we denote by

$$E_n^{(2)}(f, Y_s)_p := \inf_{p_n \in \Pi_n \cap \Delta^2(Y_s)} \|f - p_n\|_p,$$

the degree of coconvex approximation. Once again if $Y_0 = \emptyset$, then we write $E_n^{(2)}(f)_p := E_n^{(2)}(f,\emptyset)_p$, which is usually referred to as the degree of convex approximation.

Remark. While it is obvious that f may belong to $\Delta^0(Y_{s_0}^0) \cap \Delta^1(Y_{s_1}^1)$, say, where $Y_{s_0}^0 \neq Y_{s_1}^1$ and $s_0 \neq s_1$, it should be emphasized that f may belong $\Delta^\nu(Y_s)$, where $0 \leq \nu \leq 2$ is fixed, for many different sets Y_s and different s's, since we assumed weak changes in the sign, the monotonicity or the convexity. Thus we find it useful for such a function to introduce the best degree of constrained approximation, namely, for the appropriate $0 \leq \nu \leq 2$ and a fixed s, we denote

$$e_n^{(\nu,s)}(f)_p := \inf E_n^{(\nu)}(f, Y_s)_p, \tag{1.1}$$

where the infimum is taken over all admissible sets Y_s of s points in which f changes its sign, monotonicity or convexity according to whether $\nu = 0, 1$ or 2, respectively. In this survey we make use of this notation only in negative results in comonotone approximation.

For comparison purposes we need the degree of unconstrained approximation, so for $f \in \mathbb{L}_p[-1,1]$, let us write

$$E_n(f)_p := \inf_{p_n \in \Pi_n} \|f - p_n\|_p.$$

Suppose $f \in \mathbb{C}[-1,1]$, $f \geq 0$. Then for $n \geq 0$, $P_n \in \Pi_n$ exists such that,

$$\|f - P_n\|_\infty = E_n(f)_\infty.$$

Thus

$$P_n(x) - f(x) \geq -E_n(f)_\infty,$$

so that

$$Q_n(x) := P_n(x) + E_n(f)_\infty \geq f(x) \geq 0.$$

Hence Q_n is nonnegative and we have

$$\|f - Q_n\|_\infty \leq 2E_n(f)_\infty,$$

which yields

$$E_n^{(0)}(f)_\infty \leq 2E_n(f)_\infty, \quad n \geq 0. \tag{1.2}$$

Thus there is nothing to investigate in this case. However, the situation is completely different when asking for either pointwise estimates for the approximation of nonnegative functions by nonnegative polynomials, or for L_p estimates of positive polynomial approximation. We will discuss recent results on these subjects and in copositive polynomial approximation in Section 2.

Now suppose $f \in \mathbb{C}^1[-1,1]$ is monotone nondecreasing. Then of course $f' \geq 0$. By the above, for $n \geq 1$, a nonnegative $q_{n-1} \in \Pi_{n-1}$ exists such that,

$$\|f' - q_{n-1}\|_\infty \leq 2E_{n-1}(f')_\infty.$$

Put $Q_n(x) := \int_0^x q_{n-1} + f(0)$. Then Q_n is nondecreasing and

$$\|f - Q_n\|_\infty \le 2E_{n-1}(f')_\infty.$$

Hence,

$$E_n^{(1)}(f)_\infty \le 2E_{n-1}(f')_\infty. \quad n \ge 1. \tag{1.3}$$

Similarly, if $f \in \mathbb{C}^2[-1,1]$ is convex, i.e., $f'' \ge 0$, then we get

$$E_n^{(2)}(f)_\infty \le E_{n-2}(f'')_\infty. \quad n \ge 2. \tag{1.4}$$

Recall that if $f \in W_p^1[-1,1]$, $1 \le p \le \infty$, the Sobolev space of locally absolutely continuous functions on $[-1,1]$, such that $f' \in \mathbb{L}_p[-1,1]$, then in unconstrained approximation by polynomials we have

$$E_n(f)_p \le \frac{C}{n} E_{n-1}(f')_p, \quad n \ge 1, \tag{1.5}$$

where $C = C(p)$ is an absolute constant. It should be emphasized that (1.5) is not valid for $0 < p < 1$. Evidently, (1.5) in turn implies for $f \in W_p^r[-1,1]$, $1 \le p \le \infty$,

$$E_n(f)_p \le \frac{C}{n^r} E_{n-r}(f^{(r)})_p, \quad n \ge r,$$

where $C = C(p)$ is an absolute constant.

Thus, in (1.3) we have lost an order of n and in (1.4) we have a loss of order of n^2. We will try to retrieve some of these losses in the estimates we present in this paper. These will be Jackson type estimates which are analogous to those in unconstrained approximation, namely, on the right-hand side of (1.3) and (1.4), we will have various moduli of smoothness of different kinds which we are going to define below. However, we will also show that constrained approximation restricts the validity of these estimates. At this stage we just point out that Shevchuk separately and with the author [33], [23] have proved,

Theorem 1.1. *There exists a constant $C > 0$ such that for any $n \ge 1$, a function $f = f_n \in \mathbb{C}^1[-1,1] \cap \Delta^1$ exists, such that*

$$E_n^{(1)}(f)_\infty \ge CE_{n-1}(f')_\infty > 0.$$

And for any $n \geq 2$, a function $f \in \mathbb{C}^2[-1,1] \cap \Delta^2$ exists, such that

$$E_n^{(2)}(f)_\infty \geq C E_{n-2}(f'')_\infty > 0.$$

Hence, it is clear that (1.3) and (1.4) by themselves, cannot be improved.

In fact if $0 < p < \infty$, the situation is even more pronounced. Indeed, Kopotun [18] proved that

Theorem 1.2. *Let $0 < p < \infty$ and $\nu = 1, 2$. Then for each $n \geq \nu$, and every constant $A > 0$, there exists an $f = f_{p,n,A} \in \mathbb{C}^\infty[-1,1] \cap \Delta^\nu$ for which*

$$E_n^{(\nu)}(f)_p \geq A E_{n-\nu}(f^{(\nu)})_p.$$

We will discuss monotone and comonotone approximation in Section 3 and convex and coconvex approximation in Section 4.

In order to use consistent notation we will use $r \geq 0$ for the number of derivatives that the function possesses, and $k \geq 0$ for the order of the moduli of smoothness. In addition to the ordinary moduli of smoothness $\omega_k(f,t)_p$, $0 < p \leq \infty$, (where $\omega_0(f,t)_p := \|f\|_p$), defined for $f \in \mathbb{L}_p[-1,1]$, we recall the Ditzian–Totik moduli of smoothness which are defined for such an f, as follows. With $\varphi(x) := \sqrt{1-x^2}$, we let

$$\Delta_{h\varphi}^k f(x) := \begin{cases} \sum_{i=0}^k (-1)^i \binom{k}{i} f(x + (i - \frac{k}{2})h\varphi(x)), & x \pm \frac{k}{2}h\varphi(x) \in [-1,1] \\ 0, & \text{otherwise}, \end{cases}$$

and we set $\Delta_{h\varphi}^0 f := f$. Then we denote

$$\omega_k^\varphi(f,t)_p := \sup_{0<h\leq t} \|\Delta_{h\varphi}^k f\|_p, \qquad \omega^\varphi(f,t)_p := \omega_1^\varphi(f,t)_p.$$

When $p = \infty$ we will also use a modification of these moduli where we will take into account not only the position of x in the interval when setting $\Delta_{h\varphi}^k f$, but also how far the endpoints of the interval $[x - \frac{k}{2}h\varphi(x), x + \frac{k}{2}h\varphi(x)]$ are from the endpoints of $[-1,1]$. To this end we set

$$\varphi_\delta(x) := \sqrt{(1 - x - \tfrac{\delta}{2}\varphi(x))(1 + x - \tfrac{\delta}{2}\varphi(x))}, \qquad x \pm \tfrac{\delta}{2}\varphi(x) \in [-1,1],$$

and we restrict $f \in \mathbb{C}[-1,1]$, to the space $\mathbb{C}_\varphi^r$, of functions $f \in \mathbb{C}^r(-1,1)$, such that $\lim_{x\to\pm 1} \varphi^r(x) f^{(r)}(x) = 0$. We denote

$$\omega_{k,r}^\varphi(f^{(r)},t) := \sup_{0\le h\le t} \sup_x |\varphi_{kh}^r(x)\Delta_{h\varphi(x)}^k f^{(r)}(x)|, \quad t \ge 0,$$

where the inner supremum is taken over all x so that

$$x \pm \frac{k}{2} h\varphi(x) \in (-1,1).$$

Note that for $k = 0$, we have

$$\omega_{0,r}^\varphi(f^{(r)},t) = \|\varphi^r f^{(r)}\|_\infty, \tag{1.6}$$

and that for $r = 0$,

$$\omega_{k,0}^\varphi(f^{(0)},t) := \omega_k^\varphi(f,t).$$

The above restriction guarantees that for $k \ge 1$, $\omega_{k,r}^\varphi(f^{(r)},t) \to 0$, as $t \to 0$. Also, it can be shown that if $f \in \mathbb{C}_\varphi^r$ and $0 \le m < r$, then

$$\omega_{k+r-m,m}^\varphi(f^{(m)},t) \le C(k,r)t^{r-m}\omega_{k,r}^\varphi(f^{(r)},t), \quad t \ge 0,$$

and conversely if $f \in \mathbb{C}[-1,1]$, $m < \alpha < k$ and $\omega_k^\varphi(f,t) \le t^\alpha$, then $f \in \mathbb{C}_\varphi^m$ and

$$\omega_{k-m,m}^\varphi(f^{(m)},t) \le C(\alpha,k)t^{\alpha-m}, \quad t \ge 0.$$

Finally, if $f \in \mathbb{C}_\varphi^m$ and $\omega_{r-m,m}^\varphi(f^{(m)},t) \le t^{r-m}$, then

$$\|\varphi^r f^{(r)}\|_\infty \le C(r).$$

We will denote the collection of functions satisfying the last inequality by $\mathbb{B}^r$, and the converse is also valid, namely, if $f \in \mathbb{B}^r$ and $0 \le m < r$, then $f \in \mathbb{C}_\varphi^m$, and

$$\omega_{r-m,m}^\varphi(f^{(m)},t) \le C(r)t^{r-m}\|\varphi^r f^{(r)}\|_\infty, \quad t \ge 0.$$

§2. Positive and copositive approximation

Pointwise estimates in the approximation of a nonnegative $f \in \mathbb{C}^r[-1,1] \cap \Delta^0$, there are of two types. The Timan–Brudnyi type estimates of the form

$$|f(x) - p_n(x)| \le C(r,k)\rho_n^r(x)\omega_k(f^{(r)}, \rho_n(x))_\infty, \quad 0 \le x \le 1, \quad n \ge N, \tag{2.1}$$

where $\rho_n(x) := \frac{1}{n^2} + \frac{1}{n}\varphi(x)$. Here $C(r,k)$ is a constant which depends only on r and k, and which is independent of f and n. And the Telyakovskiĭ–Gopengauz or interpolatory type estimates,

$$|f(x) - p_n(x)| \le C(r,k)\delta_n^r(x)\omega_k(f^{(r)}, \delta_n(x)), \quad 0 \le x \le 1, \quad n \ge N, \tag{2.2}$$

where $\delta_n(x) := \frac{1}{n}\varphi(x)$.

Dzyubenko [5] has shown that estimates of the form (2.1) are valid for positive approximation for all $n \ge N := r + k - 1$. Namely, for each $n \ge N := r + k - 1$, there exists a polynomial $p_n \in \Pi_n \cap \Delta^0$, for which (2.1) holds. In contrast, in a recent paper Gonska, Leviatan, Shevchuk and Wenz [9] have shown that (2.2) is valid only when either $r = 0$ and $k = 1, 2$, or if $k \le r$. Specifically they proved the following two complementing results, namely,

Theorem 2.1. *Let either $r = 0$ and $k = 1, 2$, or $1 \le k \le r$. If $f \in \mathbb{C}^r[-1,1] \cap \Delta^0$, then for each $n \ge N := 2[(r + k + 1)/2]$, there is a polynomial $p_n \in \Pi_n \cap \Delta^0$, such that*

$$|f(x) - p_n(x)| \le C(r)\delta_n^r(x)\omega_k(f^{(r)}, \delta_n(x)), \quad 0 \le x \le 1. \tag{2.3}$$

(Note that the case $r + k \le 2$ is due to DeVore and Yu [DY].) And

Theorem 2.2. *Let either $r = 0$ and $k > 2$, or $k > r \ge 1$. Then for each $n \ge 1$ and any constant $A > 0$, a function $f = f_{k,r,n,A} \in \mathbb{C}^r[-1,1] \cap \Delta^0$ exists, such that for any polynomial $p_n \in \Pi_n \cap \Delta^0$, there is a point $x \in [0,1]$, for which*

$$|f(x) - p_n(x)| > A\frac{(1-x)^{r/2}}{n^r}\omega_k\big(f^{(r)}, \frac{\sqrt{1-x}}{n}\big), \tag{2.4}$$

holds.

As was alluded to in the introduction the L_p-norm estimates for $0 < p < \infty$, do not behave like the case of the sup-norm. Denote by $W_p^r[-1,1]$, $0 < p \le \infty$, the Sobolev space of functions f such that $f^{(r-1)}$ is locally absolutely continuous in $(-1,1)$ and $f^{(r)} \in L_p[-1,1]$. If $f \in W_p^1[-1,1] \cap \Delta^0$, $1 \le p < \infty$, then we come close to (1.2), with an estimate due to Stojanova (see [9]).

$$E_n^{(0)}(f)_p \le \frac{C}{n} E_{n-1}(f')_p \le \frac{C(k)}{n} \omega_k^\varphi\big(f', \frac{1}{n}\big)_p, n \ge 1.$$

The constant $C(k)$ depends only on k and on p, we are going to suppress indicating the dependence on the latter.

However, if we merely assume that $f \in \Delta^0 \cap \mathbb{L}_p[-1,1]$, then Hu, Kopotun and Yu [10], proved that for $0 < p < \infty$, there is a constant C such that,

$$E_n^{(0)}(f)_p \le C\omega\big(f, \frac{1}{n}\big)_p,$$

but on the other hand for each $A > 0$, every $n \ge 1$, and any $0 < p < \infty$, there exists an $f := f_{A,n,p} \in \Delta^0 \cap \mathbb{L}_p[-1,1]$, such that

$$E_n^{(0)}(f)_p \ge A\omega_2(f,1)_p.$$

Stojanova also proved that for $f \in \Delta^0 \cap \mathbb{L}_p[-1,1]$, we always have the estimate

$$E_n^{(0)}(f)_p \le C(k)\tau_k\big(f, \frac{1}{n}\big)_p, \quad n \ge 1,$$

where $\tau_k(f,\cdot)_p$ are the averaged moduli of smoothness which were introduced by Sendov (see [10] for details and references).

We turn now to copositive approximation. Here we still have variations in the estimates for $p = \infty$ and for $1 \le p < \infty$, but in no case the behavior is as in unconstrained approximation. Recall that in this case we deal with a function which changes its sign at $Y_s \in \mathbb{Y}_s$.

For the sup-norm the estimates are due to Hu and Yu [13], and Kopotun [17], and negative results are due to Zhou [37]. We summarize their results in

Theorem 2.3. *Let $f \in \mathbb{C}[-1,1] \cap \Delta^0(Y_s)$. Then there exists a constant $C = C(Y_s)$ such that*

$$E_n^{(0)}(f, Y_s)_\infty \le C(Y_s)\omega_3^\varphi\big(f, \frac{1}{n}\big)_\infty, \quad n \ge 2, \tag{2.5}$$

and for each $n \ge 2$, there exists a polynomial $p_n \in \Pi_n \cap \Delta^0(Y_s)$, such that

$$|f(x) - p_n(x)| \le C(Y_s)\omega_3(f, \rho_n(x))_\infty.$$

Furthermore, if $f \in \mathbb{C}^1[-1,1] \cap \Delta^0(Y_s)$, then

$$E_n^{(0)}(f, Y_s)_\infty \le \frac{C(k, Y_s)}{n}\omega_k\big(f', \frac{1}{n}\big)_\infty, \quad n \ge k,$$

and for each $n \ge 2$, there exists a polynomial $p_n \in \Pi_n \cap \Delta^0(Y_s)$, such that

$$|f(x) - p_n(x)| \le \frac{C(k, Y_s)}{n}\omega_k(f', \rho_n(x))_\infty, \quad n \ge k.$$

Conversely, there is an $f \in \mathbb{C}^1[-1,1] \cap \Delta^0(\{0\})$, for which

$$\limsup_{n\to\infty} \frac{E_n^{(0)}(f, \{0\})_\infty}{\omega_4(f, \frac{1}{n})_\infty} = \infty.$$

Since in Section 3 we are going to discuss the dependence of the constants on the collection Y_s, especially in contrast to dependence on s alone, we mention that if in (2.5), we replace the third modulus of smoothness of f by its modulus of continuity, then Leviatan has proved that the inequality holds with a constant $C = C(s)$. In view of (2.3) one may ask whether it is possible to give some interpolatory estimates for copositive approximation. This question is completely open.

If $1 \le p < \infty$, then it was proved by Hu, Kopotun and Yu [10] that,

Theorem 2.4. *If $f \in \mathbb{L}_p[-1,1] \cap \Delta^0(Y_s)$, then*

$$E_n^{(0)}(f, Y_s)_p \le C(Y_s)\omega^\varphi\big(f, \frac{1}{n}\big)_p \quad n \ge 1,$$

and if $f \in W_p^1[-1,1] \cap \Delta^0(Y_s)$, then

$$E_n^{(0)}(f, Y_s)_p \le \frac{C(Y_s)}{n}\omega^\varphi\big(f', \frac{1}{n}\big)_p \quad n \ge 1.$$

Furthermore if $f \in W_p^2[-1,1] \cap \Delta^0(Y_s)$, *then*

$$E_n^{(0)}(f, Y_s)_p \leq \frac{C(k, Y_s)}{n^2} \omega_k^\varphi\big(f'', \frac{1}{n}\big)_p \quad n \geq k+1.$$

Conversely, for every $n \geq 1$ *and* $0 < p < \infty$, *and for any constants* $A > 0$ *and* $0 < \epsilon \leq 1$, *there is an* $f = f_{n,p,\epsilon,A} \in \mathbb{C}^\infty[-1,1]$, *satisfying* $xf(x) \geq 0$, $-1 \leq x \leq 1$, *such that for each* $p_n \in \Pi_n$, *for which* $p_n(0) \geq 0$, *we have*

$$\|f - p_n\|_{L_p[0,\epsilon]} > A\omega_2(f, 1)_p.$$

Also, there is a strictly increasing $f = f_{n,p,\epsilon,A} \in \mathbb{C}^\infty[-1,1]$, *satisfying* $f(0) = 0$ *such that for each* $p_n \in \Pi_n$, *for which* $p_n(0) = 0$, *and* $p_n(x) \geq 0$, $0 \leq x \leq \epsilon$, *we have*

$$\|f - p_n\|_{\mathbb{L}_p[0,\epsilon]} > A\omega_3(f', 1)_p.$$

In [10] there are some estimates involving the "tau" modulus but we will not detail them here. It should also pointed out that in [10] the authors introduce an interesting new concept of intertwining approximation which is related to both copositive approximation and one-sided approximation. We will not discuss this concept here and the interested reader should consult with that paper.

§3. Monotone and comonotone approximation

If $f \in \mathbb{L}_p[-1,1]$, $0 < p \leq \infty$, is nondecreasing, then the following is known (see [3] and [34]).

Theorem 3.1. *Let* $f \in \mathbb{L}_p[-1,1] \cap \Delta^1$. *Then for each* $n \geq 1$, *we have*

$$E^{(1)}(f)_p \leq C\omega_2^\varphi\big(f, \frac{1}{n}\big)_p, \tag{3.1}$$

where $C = C(p)$. (*The dependence on* p *is crucial only when* $p \to 0$.)
Conversely, if $k \geq 3$, *then for any* $A > 0$ *and* $n \geq 1$, *there exists a function* $f = f_{p,n,A} \in L_p[-1,1] \cap \Delta^1$, *such that*

$$E^{(1)}(f)_p \geq A\omega_k(f, 1)_p > 0.$$

If $1 \le p \le \infty$, then (3.1) readily implies that for $f \in W_p^1[-1,1] \cap \Delta^1$, we have

$$E^{(1)}(f)_p \le \frac{C}{n}\omega^\varphi\big(f', \frac{1}{n}\big)_p.$$

Thus, one may hope that for smooth functions it would be possible to obtain estimates involving the moduli of smoothness of the appropriate derivatives and this way have better rates of monotone approximation. This is true for the sup-norm but it is not so for any of the L_p-norms. Specifically, Shevchuk [30] and [31], has proved that

Theorem 3.2. *If $f \in \mathbb{C}^1[-1,1] \cap \Delta^1$, then for for each $k \ge 1$, there is a constant $C = C(k)$ such that*

$$E^{(1)}(f)_\infty \le \frac{C}{n}\omega_k\big(f', \frac{1}{n}\big)_\infty. \tag{3.2}$$

However, Kopotun [18] has shown that

Theorem 3.3. *Let $0 < p < \infty$, $k \ge 0$, and $\max\{1, 3-k\} \le \nu < 1 + 1/p$. Then for each $n \ge 1$ and $\epsilon > 0$, and every constant $A > 0$, there exists an $f = f_{p,k,\nu,n,\epsilon,A} \in \mathbb{C}^\infty[-1,1] \cap \Delta^1$, such that for any $p_n \in \Pi_n$ for which $p_n'(-1) \ge 0$, it follows that*

$$\|f - p_n\|_{\mathbb{L}_p[-1,-1+\epsilon]} > A\omega_k(f^{(\nu)}, 1)_p.$$

Note that in particular if $k \ge 2$, then one cannot replace in (3.2), the sup-norm by any of the L_p-norms, $0 < p < \infty$.

One may ask whether we may have (3.1), if we relax the requirement on the constant by allowing such a constant to depend on the function f (but not on n). Wu and Zhou [35] have shown that this is impossible with $k = 4 + [1/p]$. On the other hand, Shevchuk and Leviatan [26] have recently closed the gap when f is monotone and continuous, proving

Theorem 3.4. *If $f \in \mathbb{C}[-1,1] \cap \Delta^1$, then there exists a constant $C = C(f)$ such that*

$$E^{(1)}(f)_\infty \le C\omega_3^\varphi\big(f, \frac{1}{n}\big)_\infty, \quad n \ge 2.$$

In fact, when dealing with monotone continuous functions a lot more is known. For instance if $f \in \mathbb{C}^r[-1,1] \cap \Delta^1$, then for each $n \ge r + k - 1$, there exists a polynomial

$p_n \in \Delta^1$, for which the pointwise estimates (2.1) are valid. This has recently been shown by Dzyubenko [6]. However, the interpolatory estimates (2.2) are valid in very few cases, namely, only when $r+k \le 2$, the affirmative result being due to DeVore and Yu [4]. We now know (see [9]) that

Theorem 3.5. *If $r > 2$, then for each n there is a function $f = f_{r,n} \in W^r_\infty[-1,1] \cap \Delta^1$, such that for every polynomial $p_n \in \Pi_n \cap \Delta^1$, either*

$$\limsup_{x \to -1} \frac{|f(x) - p_n(x)|}{\varphi^r(x)} = \infty, \tag{3.3}$$

or

$$\limsup_{x \to 1} \frac{|f(x) - p_n(x)|}{\varphi^r(x)} = \infty.$$

This readily implies,

Corollary 3.6. *Let $r + k > 2$. Then for each n there exists a function $f = f_{r,k,n} \in \mathbb{C}^r[-1,1] \cap \Delta^1$, such that for every polynomial $p_n \in \Pi_n \cap \Delta^1$, either*

$$\limsup_{x \to -1} \frac{|f(x) - p_n(x)|}{\varphi^r(x)\omega_k(f^{(r)}, \varphi(x))} = \infty,$$

or

$$\limsup_{x \to 1} \frac{|f(x) - p_n(x)|}{\varphi^r(x)\omega_k(f^{(r)}, \varphi(x))} = \infty.$$

Here we have an opportunity to show the nature of the function yielding Theorem 3.5.

Proof of Theorem 3.5. Given n, set $b = n^{-2}$ and let

$$f(x) := \begin{cases} (b^r - (b - x - 1)^r)/r!, & -1 \le x \le -1 + b, \\ b^r/r!, & -1 + b < x \le 1. \end{cases}$$

Then $f \in W^r_\infty[-1,1] \cap \Delta^1$. Suppose that there is a nondecreasing polynomial p_n for which (3.3) fails. Then for that polynomial and some constant A, we have

$$|f(x) - p_n(x)| \le A\varphi^r(x) \le A(1+x)^{3/2}, \quad -1 \le x \le -1 + b,$$

where the right-hand inequality follows since $r > 2$. Hence $p_n(-1) = f(-1) = 0$ and $p_n'(-1) = f'(-1) = b^{r-1}/(r-1)!$. Since $p_n \in \Delta^1$, we have $\|p_n\| = p_n(1)$, so that applying Markov's inequality we conclude that

$$\frac{b^{r-1}}{(r-1)!} = p_n'(-1) \le n^2\|p_n\| = n^2 p_n(1),$$

or

$$p_n(1) \geq \frac{b^{r-1}}{(r-1)!n^2} = \frac{b^r}{(r-1)!}.$$

On the other hand,

$$f(1) = \frac{b^r}{r!} < \frac{b^r}{(r-1)!}.$$

Thus $f(1) \neq p_n(1)$, and (1.12) is satisfied. $\square$

We still can salvage something if we are willing to settle for interpolation at only one of the endpoints while approximating well throughout the interval. Namely, it is proved in [9] that

Theorem 3.7. *If $k \leq \max\{r, 2\}$, and $f \in \mathbb{C}^r[-1,1] \cap \Delta^1$, then for each $n \geq N := k+r-1$, there is a polynomial $p_n \in \Pi_n \cap \Delta^1$, such that*

$$|f(x) - p_n(x)| \leq C(r)\rho_n^r(x)\omega_k(f^{(r)}, \rho_n(x)), \quad -1 \leq x \leq 1,$$

and

$$|f(x) - p_n(x)| \leq C(r)\frac{(1-x)^{r/2}}{n^r}\omega_k\big(f^{(r)}, \frac{\sqrt{1-x}}{n}\big), \quad -1 \leq x \leq 1.$$

For all other pairs (k, r), Theorem 3.7 does not hold. In fact we have

Theorem 3.8. *If $k > \max\{r, 2\}$, then for each $n \geq 1$ and any constant $A > 0$, a function $f = f_{r,k,n,A} \in C^r[-1,1] \cap \Delta^1$ exists, such that for any polynomial $p_n \in \Pi_n \cap \Delta^1$, there is a point $x \in [-1,1]$ for which (2.4) holds.*

We conclude the part on monotone approximation with a result on simultaneous pointwise estimates due to Kopotun [14].

Theorem 3.9. *If $f \in C^1[-1,1] \cap \Delta^1$, then for every $n \geq 1$, a polynomial $p_n \in \Pi_n \cap \Delta^1$ exists, such that*

$$|f^{(i)}(x) - p_n^{(i)}(x)| \leq C\omega_{2-i}(f^{(i)}, \rho_n(x)), \quad i = 0, 1, \quad -1 \leq x \leq 1.$$

We now proceed to investigate the degree of comonotone approximation of a function $f \in \mathbb{L}_p[-1,1]$, $0 < p \leq \infty$, which changes its monotonicity at $Y_s \in \mathbb{Y}_s$. Thus for the remainder of this section $s \geq 1$ unless we specifically say otherwise.

Again we have only some results for $p < \infty$, and most of the recent developments are for estimates in the max-norm. Denote

$$d(Y_s) := \min_{0 \le i \le s} (y_i - y_{i+1}). \tag{3.4}$$

Then the following general estimates have been obtained by Kopotun and Leviatan [19].

Theorem 3.10. *Let $f \in \mathbb{L}_p[-1,1] \cap \Delta^1(Y_s)$, $0 < p \le \infty$. Then there exists a constant $C = C(s)$ such that for each $n \ge C/d(Y_s)$,*

$$E_n^{(1)}(f, Y_s)_p \le C\omega_2^{\varphi}\big(f, \frac{1}{n}\big)_p.$$

On the other hand Zhou [38] have shown that for every $0 < p \le \infty$ and each $s \ge 1$, there is a collection Y_s and a function $f \in \mathbb{L}_p[-1,1] \cap \Delta^1(Y_s)$, for which

$$\limsup_{n\to\infty} \frac{E_n^{(1,s)}(f)_p}{\omega_k(f, 1/n)_p} = \infty,$$

with $k = 3 + [1/p]$. Thus taking $p = \infty$, one sees that Theorem 3.10 is not valid with any $k \ge 3$, even with $C = C(f)$ and $N = N(f)$.

If $f \in \mathbb{C}[-1,1] \cap \Delta^1(Y_s)$, then we can say much more. We begin with the following results of Dzyubenko, Gilewicz and Shevchuk [7] (see also [36]).

Theorem 3.11. *Let $f \in \mathbb{C}^r[-1,1] \cap \Delta^1(Y_s)$. Then the estimates*

$$E_n^{(1)}(f, Y_s)_\infty \le \frac{C}{n^r}\omega_k\big(f^{(r)}, \frac{1}{n}\big)_\infty, \quad n \ge N, \tag{3.5}$$

is valid with $C = C(k,r,s)$ and $N = N(k,r,s)$, only when either $k = 1$, or $r > s$, or in the particular case $k = 2$ and $r = s$, moreover, in these cases one can always take $N = k + r - 1$. If $k = 2$ and $0 \le r < s$, or $k = 3$ and $1 \le r \le s$, or if $k > 3$ and $2 \le r \le s$, then the estimates hold either with $C = C(k,r,Y_s)$ and $N = k + r$, or with $C = C(k,r,s)$ and $N = N(k,r,Y_s)$, and they fail to hold with $C = C(k,r,s)$ and $N = N(k,r,s)$.
On the other hand, if either $r = 0$ or $r = 1$, then for each $s \ge 1$, there is a collection $Y_s \in \mathbb{Y}_s$ and a function $f \in \mathbb{C}^r[-1,1] \cap \Delta^1(Y_s)$, for which

$$\limsup_{n\to\infty} \frac{n^r E_n^{(1)}(f, Y_s)_\infty}{\omega_{3+r}(f^{(r)}, 1/n)} = \infty,$$

i.e., (3.5) *is not valid even with constants which depend on* f.

We found it easier to remember, especially when later on we compare to other types of estimates, to illustrate the above in an array in which $+$ in the (k,r) entry means that both constants C and N depend only on k, r and s; $\oplus$ means that one of the two constants depends on k and r and on the location of the points of change of monotonicity, namely on Y_s; while $-$ asserts that (3.5) is not valid at all.

$$
\begin{array}{cccccccc}
r & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
s+2 & + & + & + & + & + & \cdots \\
s+1 & + & + & + & + & + & \cdots \\
s & + & + & \oplus & \oplus & \oplus & \cdots \\
s-1 & + & \oplus & \oplus & \oplus & \oplus & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
2 & + & \oplus & \oplus & \oplus & \oplus & \cdots \\
1 & + & \oplus & \oplus & - & - & \cdots \\
0 & + & \oplus & - & - & - & \cdots \\
 & 1 & 2 & 3 & 4 & 5 & k
\end{array}
$$

Fig. 1, $s \geq 1$.

In particular the first column of the array implies that if $f \in W_\infty^r$, then

$$E_n^{(1)}(f, Y_s)_\infty \leq C(r,s)\frac{\|f^{(r)}\|_\infty}{n^r}, \quad n \geq r-1. \tag{3.6}$$

Pointwise estimates of the type (2.1), for comonotone approximation present new phenomena. If $s=1$, then when either $r \geq 2$; or in three special cases, $k=1$ and $r=0,1$; and $k=2$ and $r=1$; we have a polynomial $p_n \in \Pi_n \cap \Delta^1$ satisfying

$$|f(x) - p_n(x)| \leq C(r)\rho_n^r(x)\omega_k(f^{(r)}, \rho_n(x))_\infty, \quad 0 \leq x \leq 1, \quad n \geq k+r-1.$$

Two other pairs $k=2$ and $r=0$, and $k=3$ and $r=1$, yield (2.1) with $C = C(Y_1) = C(y_1)$, while for the remaining pairs, namely, $r=0$ and $k \geq 3$, and $r=1$ and $k \geq 4$, we have no estimate of the type (2.1). Thus the array is exactly the one we had in Fig. 1, for $s=1$. If on the other hand, $s>1$, then the array looks entirely different. To be specific, (2.1) holds with $C = C(r,k,s)$, only for $n \geq N = N(r,k,Y_s)$ so that the array is as follows.

$$
\begin{array}{ccccccc}
r & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
3 & \oplus & \oplus & \oplus & \oplus & \oplus & \cdots \\
2 & \oplus & \oplus & \oplus & \oplus & \oplus & \cdots \\
1 & \oplus & \oplus & \oplus & - & - & \cdots \\
0 & \oplus & \oplus & - & - & - & \cdots \\
 & 1 & 2 & 3 & 4 & 5 & k
\end{array}
$$

Fig. 2, $s > 1$.

Estimates involving the D–T moduli are similar to those of the ordinary moduli and yield the same array as Fig. 1. This raises the expectation of having an estimate analogous to (3.6) for functions in $\mathbb{B}^r$. However, this is not so except when f is monotone. Indeed, Leviatan and Shevchuk [27] have proved that

Theorem 3.12. *Let $s \ge 0$ and assume that $f \in \mathbb{B}^r \cap \Delta^1(Y_s)$, $r \ge 1$. Then*

$$E_n^{(1)}(f, Y_s)_\infty \le C(r, Y_s) \frac{\|\varphi^r f^{(r)}\|_\infty}{n^r}, \quad n \ge r-1,$$

and

$$E_n^{(1)}(f, Y_s)_\infty \le C(r, s) \frac{\|\varphi^r f^{(r)}\|_\infty}{n^r}, \quad n \ge N(r, Y_s).$$

Furthermore, if $f \in \mathbb{B}^r \cap \Delta^1(Y_s)$, with either $s = 0$ or $r = 1$, or $r = 3$ and $s = 1$, or $r > 2s+2$, then

$$E_n^{(1)}(f, Y_s)_\infty \le C(r) \frac{\|\varphi^r f^{(r)}\|_\infty}{n^r}, \quad n \ge r-1. \tag{3.7}$$

For all other cases (3.7) is not valid, that is, we have (see [24])

Theorem 3.13. *Given $s \ge 1$. Let the constant $A > 0$ be arbitrary and let $2 \le r \le 2s+2$, excluding the case $r = 3$ and $s = 1$. Then for any n, there exists a function $f = f_{r,s,n} \in \mathbb{B}^r$, which changes monotonicity s times in $[-1, 1]$, for which*

$$e_n^{(1,s)}(f)_\infty \ge A \|\varphi^r f^{(r)}\|_\infty.$$

See (1.1) for the definition of $e_n^{(1,s)}(f)_\infty$.

It is in order to investigate this phenomenon that we introduced the modified moduli $\omega_{k,r}^\varphi$. In fact, we recall that in (1.6), we have noted that $\omega_{0,r}^\varphi(f^{(r)}, t) = \|\varphi^r f^{(r)}\|_\infty$. Indeed

Leviatan and Shevchuk [27] have obtained the following estimates for these moduli. (The case of monotone approximation, i.e., $s = 0$ had been treated earlier by Dzyubenko, Listopad and Shevchuk [8].)

Theorem 3.14. *Let $s \geq 0$ and assume that $f \in \mathbb{C}_\varphi^r \cap \Delta^1(Y_s)$, with $r > 2$. Then*

$$E_n^{(1)}(f, Y_s)_\infty \leq \frac{C(k,r,Y_s)}{n^r}\omega_{k,r}^\varphi\left(f^{(r)}, \frac{1}{n}\right), \quad n \geq k+r-1, \tag{3.8}$$

and

$$E_n^{(1)}(f, Y_s)_\infty \leq \frac{C(k,r,s)}{n^r}\omega_{k,r}^\varphi\left(f^{(r)}, \frac{1}{n}\right), \quad n \geq N(k,r,Y_s). \tag{3.9}$$

Furthermore, if $f \in \mathbb{C}_\varphi^r \cap \Delta^1(Y_s)$, with $r > 2s+2$, then

$$E_n^{(1)}(f, Y_s)_\infty \leq \frac{C(k,r,s)}{n^r}\omega_{k,r}^\varphi\left(f^{(r)}, \frac{1}{n}\right), \quad n \geq k+r-1. \tag{3.10}$$

Remark. Obviously, when $s = 0$ there is no dependence on $Y_0 = \emptyset$ in (3.8) and (3.9) and hence the former is just (3.10). Also in this case, (3.10) is valid for $0 \leq r+k \leq 2$, as follows from (3.1).

To the contrary we have,

Theorem 3.15. *For $s \geq 1$, let $0 \leq r \leq 2s+2$, excluding the three cases $r+k \leq 1$. Then for any constant $A > 0$ and every $n \geq 1$, there is function $f := f_{k,r,s,n,A} \in \mathbb{C}_\varphi^r$ which changes monotonicity s times in $[-1,1]$, for which*

$$e_n^{(1,s)}(f)_\infty > A\omega_{k,r}^\varphi(f^{(r)}, 1).$$

Finally we have some cases where (3.8) is valid with a constant $C = C(f)$, others when even so, it is not valid, and a few which are still open. We summarize what we know due to Leviatan and Shevchuk [25] and [28].

Theorem 3.16. *If $f \in \Delta^1$, then there exist constants $C = C(f)$ and $N = N(f)$, and an absolute constant c, such that for all $0 \leq k+r \leq 3$,*

$$E_n^{(1)}(f)_\infty \leq C\omega_{k,r}^\varphi\left(f, \frac{1}{n}\right), \quad n \geq 2, \tag{3.11}$$

and

$$E_n^{(1)}(f)_\infty \leq c\omega_{k,r}^\varphi\left(f, \frac{1}{n}\right), \quad n \geq N.$$

Theorem 3.17. *Let $s \geq 0$. Then there is a collection $Y_s \in \mathbb{Y}_s$ and a function $f \in \mathbb{C}^2_\varphi \cap \Delta^1(Y_s)$, satisfying*

$$\limsup_{n\to\infty} \frac{n^2 E_n^{(1)}(f, Y_s)_\infty}{\omega^\varphi_{3,2}(f'', 1/n)} = \infty.$$

The reader may have noticed that there are (very few) cases in which we have not given a complete and clear answer as to whether (3.11) is valid with $C = C(f)$, when nothing better is known. It is clear that it is not easy to differentiate between all cases without the assistance of arrays, so again we summarize the results in three arrays, one for the monotone case, one for one change of monotonicity which is special, and the third for $s > 1$. In addition to the symbols $+$, $\oplus$ and $-$, which have already been used in Figs. 1 and 2, here we also have the symbol $\ominus$ which when appearing in entry (k, r) means that (3.8) and (3.9) do not hold but (3.10) holds with $C = C(f)$. We have indicated the still open cases by ?.

$$\begin{array}{cccccccc}
r & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
4 & + & + & + & + & + & + & \cdots \\
3 & + & + & + & + & + & + & \cdots \\
2 & + & \ominus & ? & - & - & - & \cdots \\
1 & + & + & \ominus & ? & - & - & \cdots \\
0 & & + & + & \ominus & - & - & \cdots \\
 & 0 & 1 & 2 & 3 & 4 & 5 & k
\end{array}$$

Fig. 3, $s = 0$ (the monotone case).

$$\begin{array}{cccccccc}
r & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
6 & + & + & + & + & + & + & \cdots \\
5 & + & + & + & + & + & + & \cdots \\
4 & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \cdots \\
3 & + & \oplus & \oplus & \oplus & \oplus & \oplus & \cdots \\
2 & \oplus & \oplus & ? & - & - & - & \cdots \\
1 & + & \oplus & \oplus & ? & - & - & \cdots \\
0 & & + & \oplus & - & - & - & \cdots \\
 & 0 & 1 & 2 & 3 & 4 & 5 & k
\end{array}$$

Fig. 4, $s = 1$.

$$
\begin{array}{c|ccccccc}
r & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
2s+4 & + & + & + & + & + & + & \cdots \\
2s+3 & + & + & + & + & + & + & \cdots \\
2s+2 & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
3 & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \cdots \\
2 & \oplus & \oplus & ? & - & - & - & \cdots \\
1 & + & \oplus & \oplus & ? & - & - & \cdots \\
0 & & + & \oplus & - & - & - & \cdots \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & k
\end{array}
$$

Fig. 5, $s > 1$.

Remark. Note that while in Fig. 3 the open cases ? are either $\ominus$ or $-$, in Figs. 4 and 5 they may also be $\oplus$'s.

We conclude this section with a result on simultaneous approximation in comonotone approximation, due to Kopotun [16].

Theorem 3.18. *If $f \in C^1[-1,1] \cap \Delta^1(Y_s)$, then there exists a constant $C = C(s)$ such that for every $n \geq \frac{C}{d(Y_s)}$, a polynomial $p_n \in \Pi_n \cap \Delta^1(Y_s)$ exists, simultaneously yielding,*

$$\|f - p_n\| \leq \frac{C}{n}\omega^\varphi\big(f', \frac{1}{n}\big)_\infty,$$

and

$$\|f' - p_n'\| \leq \frac{C}{d_0}\omega^\varphi\big(f', \frac{1}{n}\big)_\infty,$$

where $d_0 := \min\{\sqrt{1+y_s}, \sqrt{1-y_1}\}$.

§4. Convex and coconvex approximation

We turn to convex approximation. Linear approximation methods similar to the ones for monotone approximation yielded estimates involving second moduli of smoothness of various types, while On the negative side Shvedov [34] proved that it is impossible to get an estimate involving $\omega_4(f,1)_p$ with an absolute constant. See also [36] for related results.

In 1994–1996 the gap between the affirmative estimates and the negative ones was closed in a series of papers by DeVore, Hu, Kopotun, Leviatan and Yu (see [12], [14] and [1]) who proved using nonlinear methods,

Theorem 4.1. *Let $f \in \mathbb{L}_p[-1,1] \cap \Delta^2$, $0 < p \le \infty$. Then there is an absolute constant $C = C(p)$, so that for each $n \ge 2$*

$$E_n^{(2)}(f)_p \le C\omega_3^{\varphi}\big(f, \frac{1}{n}\big)_p.$$

For convex approximation in the sup norm of convex functions we know a little more. Kopotun [14] has obtained some pointwise estimates. He has proved

Theorem 4.2. *Let $f \in C^r[-1,1] \cap \Delta^2$, $0 \le r \le 2$. Then for each $n \ge 2$, a polynomial $p_n \in \Pi_n \cap \Delta^2$, exists such that*

$$|f^{(i)}(x) - p_n^{(i)}(x)| \le C\omega_{r-i}(f^{(i)}, \rho_n(x)), \quad 0 \le i \le r, \quad -1 \le x \le 1.$$

In fact for convex differentiable functions with at least two continuous derivatives, according to Shevchuk [32, p.148, Thm 17.2], Manya proved but has never published, the following estimates.

Theorem 4.3. *If $f \in C^r[-1,1] \cap \Delta^2$, $r \ge 2$, then for each $n \ge r+k-1$, there exists a polynomial $p_n \in \Pi_n \cap \Delta^2$, such that*

$$|f(x) - p_n(x)| \le C\rho_n^r(x)\omega_k(f^{(r)}, \rho_n(x))_\infty, \tag{4.1}$$

where $C = C(r,k)$. In particular

$$E_n^{(2)}(f)_\infty \le Cn^{-r}\omega_k(f^{(r)}, 1/n)_\infty, \quad n \ge r+k-1.$$

Clearly, by virtue of Shvedov's result [34], for $f \in C[-1,1] \cap \Delta^2$, one cannot, in general, achieve pointwise estimates of the type (4.1), where the right-hand side is $\omega_4(f, \rho_n(x))$. Very recently at a conference in Kiev, L. P. Yushenko, a student of Shevchuk announced proving that for $f \in C^1[-1,1] \cap \Delta^2$, one cannot, in general, even have estimates of the type (4.1) where the right-hand side is $\rho_n(x)\omega_3(f', \rho_n(x))$.

Estimates involving the modified D–T moduli are due to Kopotun [15]. They can be summarized in the following result.

Theorem 4.4. *Let $r, k \ge 0$. Then for every convex $f \in C^r_\varphi$*

$$E^{(2)}_n(f)_\infty \le Cn^{-r}\omega^\varphi_{k,r}\big(f^{(r)}, \frac{1}{n}\big), \quad n \ge r+k-1,$$

with $C = C(r,k)$, if and only if either $0 \le r+k \le 3$, or $r \ge 5$.

We know even less about coconvex approximation, and what we know is restricted to the sup-norm. Recent results of Kopotun, Leviatan and Shevchuk [21], yield

Theorem 4.5. *Let $f \in \mathbb{C}[-1,1] \cap \Delta^2(Y_s)$. Then there exists a constant $C = C(s)$ such that for each $n \ge \frac{C}{d(Y_s)}$,*

$$E^{(2)}_n(f, Y_s)_\infty \le C\omega^\varphi_3\big(f, \frac{1}{n}\big)_\infty,$$

where $d(Y_s)$ was defined in (3.4).

It is also possible to obtain simultaneous approximation of f and its first and second derivatives when they exist while coconvexly approximating f (see [16] and [20]), namely,

Theorem 4.6. *Let $f \in \mathbb{C}^r[-1,1] \cap \Delta^2(Y_s)$, $1 \le r \le 2$. Then there exists a constant $C = C(s)$ such that for each $n \ge \frac{C}{d(Y_s)}$, there exist polynomials $p_n \in \Pi_n \cap \Delta^2(Y_s)$, such that if $r = 1$, we simultaneously have*

$$\|f^{(i)} - p^{(i)}_n\|_\infty \le \frac{C}{n^{1-i}}\omega^\varphi_2\big(f', \frac{1}{n}\big)_\infty, \quad 0 \le i \le 1;$$

and if $r = 2$, we simultaneously have

$$\|f^{(i)} - p^{(i)}_n\|_\infty \le \frac{C}{n^{2-i}}\omega^\varphi\big(f'', \frac{1}{n}\big)_\infty, \quad 0 \le i \le 1,$$

and

$$\|f'' - p''_n\|_\infty \le \frac{C}{d_0}\omega^\varphi\big(f'', \frac{1}{n}\big)_\infty,$$

where d_0 was defined in Theorem 3.18.

§5. Relaxing the constraints

In an effort to improve the estimates beyond what we have seen, we [25], [29], have recently attempted to approximate a function $f \in C^r[-1,1]\Delta^1(Y_s)$ by polynomials which are comonotone with it in a major portion of the interval, but not necessarily in small neighborhoods of the points Y_s and the endpoints ± 1, in other words relaxing a little the comonotonicity requirements. To be specific, given Y_s, $s \geq 0$, we set

$$O(n, Y_s) := [-1,1] \cap \cup_{i=1}^{s} (y_i - \rho_n(y_i), y_i + \rho_n(y_i)),$$

and

$$O^*(n, Y_s) := O(n, Y_s) \cup [-1, -1 + 1/n^2] \cup [1 - 1/n^2, 1].$$

Then we have the following results (compare with Theorem 3.11).

Theorem 5.1. *For each natural number M, there is a constant $C = C(s, M)$, for which $f \in \mathbb{C}[-1,1] \cap \Delta^1(Y_s)$, then for every $n \geq 2$ a polynomial $P_n \in \Pi_n$ which is comonotone with f on $I \setminus O^*(Mn, Y_s)$ exists, (i.e.,*

$$p_n'(x) \prod_{i=1}^{s} (x - y_i) \geq 0, \quad x \in [-1,1] \setminus O^*(Mn, Y_s)),$$

such that

$$\|f - P_n\|_\infty \leq C \omega_3^\varphi (f, \frac{1}{n}).$$

If we assume that f is differentiable, then we do not need to relax the comonotonicity requirements near the endpoints. Namely, we have

Theorem 5.2. *For each $k \geq 1$ and any natural number M, there is a constant $C = C(k, s, M)$, for which if $f \in \Delta^1(Y_s) \cap \mathbb{C}^1[-1,1]$, then for every $n \geq k$, a polynomial $P_n \in \Pi_n$, which is comonotone with f on $I \setminus O(Mn, Y)$ exists, such that*

$$\|f - P_n\|_\infty \leq C \frac{1}{n} \omega_k^\varphi (f', \frac{1}{n})_\infty.$$

Further, if we relax the requirements near the endpoints, then we can have

Theorem 5.3. *For each $k \geq 1$ and any natural number M, there is a constant $C = C(k, s, M)$, for which if $f \in \Delta^1(Y_s) \cap \mathbb{C}^1_\varphi$, then for every $n \geq k$, a polynomial $P_n \in \Pi_n$ which is comonotone with f on $I \setminus O^*(Mn, Y)$ exists, such that*

$$\|f - P_n\|_\infty \leq C \frac{1}{n} \omega_{k,1}^\varphi (f', \frac{1}{n}).$$

We also have improved pointwise estimates (compare with Fig. 2 and the paragraph preceding it).

Theorem 5.4. *There are a natural number $M = M(s)$ and a constant $C(s)$ such that if $f \in \mathbb{C}^1[-1,1] \cap \Delta^1(Y_s)$, then for every $n \geq 2$, a polynomial $p_n \in \Pi_n$, which is comonotone with f on $[-1,1] \setminus O^*(Mn, Y_s)$ exists, such that*

$$|f(x) - p_n(x)| \leq C(s)\, \omega_3(f, \rho_n(x)), \quad -1 \leq x \leq 1.$$

Also

Theorem 5.5. *There are a natural number $M = M(s,k)$ and a constant $C = C(s,k)$ for which, if $f \in \mathbb{C}^1[-1,1] \cap \Delta^1(Y_s)$, then for each $n \geq k$, a polynomial $p_n \in \Pi_n$ which is comonotone with f on $[-1,1] \setminus O(Mn, Y)$ exists such that*

$$|f(x) - p_n(x)| \leq C(s,k)\, \rho_n(x) \omega_k(f', \rho_n(x)), \quad -1 \leq x \leq 1.$$

Remark. One should note one major difference between Theorems 5.1– 5.3 which yield norm estimates, and Theorems 5.4 and 5.5 which yield pointwise estimates. The excluded neighborhoods in the former theorems may be taken proportionally as small as we wish (the number M may be arbitrarily big), while in the latter theorems the neighborhoods may not be too small (there is a number $M = M(s)$ or $M = M(s,k)$, as the case may be, and it may not be too big) as can be seen from the following (see [25]).

Theorem 5.6. *For each $A \geq 1$ and any $n \geq 60A$, there exists a collection $Y_2^n := \{-1 < y_2^n < y_1^n < 1\}$, and a function $f_n \in \mathbb{C}[-1,1] \cap \Delta^1(Y_2^n)$, such that any polynomial $p_n \in \Pi_n$ which is comonotone with f_n on $[-1,1] \setminus O^*(27n, Y_2^n)$, necessarily satisfies*

$$\left\| \frac{f_n - p_n}{\omega(f_n, \rho_n(\cdot))_\infty} \right\| > A.$$

Finally, we cannot push the estimates to ω_4 by relaxing the comonotonicity requirements on the nth polynomial, on any set of positive measure which tends to 0 when $n \to \infty$. In order to state the results we need some notation. Given an $\epsilon > 0$ and a function $f \in \Delta^1(Y_s)$, we denote

$$E_n^{(1)}(f; \epsilon; Y) := \inf_{p_n} \|f - p_n\|,$$

where the infimum is taken over all polynomials $p_n \in \Pi_n$ satisfying

$$\text{meas}(\{x : P_n'(x)\Pi(x, Y_s) \geq 0\} \cap I) \geq 2 - \epsilon.$$

The following was proved by DeVore, Leviatan and Shevchuk [2], for monotone approximation and by the last two authors [29], when the function changes monotonicity.

Theorem 5.7. *Given Y_s. For each sequence $\overline{\epsilon} := \{\epsilon_n\}_{n=1}^{\infty}$, of nonnegative numbers tending to 0, there exists a function $f := f_{\overline{\epsilon}} \in \Delta^1(Y_s)$ such that*

$$\limsup_{n\to\infty} \frac{E_n^{(1)}(f; \epsilon_n; Y)}{\omega_4\left(f, \frac{1}{n}\right)_{\infty}} = \infty.$$

Following up on the above ideas Hu, Kopotun and Yu [11] have investigated the analogous nearly positive and copositive approximation, and two variances of nearly intertwining approximation in $\mathbb{L}_p[-1,1]$, $1 \leq p \leq \infty$. Again we will not discuss intertwining approximation here and the interested reader should consult that paper. The nearly copositive estimates they have obtained (compare with the statements following Theorem 2.2 and Theorem 2.4) are,

Theorem 5.8. *If $f \in \mathbb{L}_p[-1,1] \cap \Delta^0(Y_s)$, $1 \leq p < \infty$, then for each $n \geq 1$, there is a polynomial $p_n \in \Pi_n$ which is copositive with f in $[-1,1] \setminus O^*(n, Y_s)$, and such that*

$$\|f - p_n\|_p \leq C\omega_2^{\varphi}(f, \frac{1}{n})_p,$$

where $C = C(p, Y_s)$.
Furthermore, if $f \in W_p^1[-1,1] \cap \Delta^0(Y_s)$, $1 \leq p < \infty$, then

$$\|f - p_n\|_p \leq \frac{C}{n}\omega_k^{\varphi}(f', \frac{1}{n})_p,$$

where $C = C(p, k, Y_s)$.
Conversely, for each $1 \leq p < \infty$, any constant $A > 0$ and every $n \geq 1$, there exists a function $f := f_{p,n,A} \in \mathbb{L}_p[-1,1] \cap \Delta^0$, for which if a polynomial $p_n \in \Pi_n$ is nonnegative in $[-1 + 1/n^2, 1 - 1/n^2]$, then

$$\|f - p_n\|_p > A\omega_3(f, 1)_p.$$

Also, for each $1 < p < \infty$, any constant $A > 0$ and every $n \ge 1$, there exists a function $f := f_{p,n,A} \in \mathbb{L}_p[-1,1] \cap \Delta^0(Y_s)$, for which if a polynomial $p_n \in \Pi_n$ is copositive with it in $[-1,1] \setminus O^*(n, Y_s)$, then

$$\|f - p_n\|_p > A\omega_3\big(f, \frac{1}{n}\big)_p,$$

and if $p = 1$ we can achieve

$$\|f - p_n\|_1 > A\omega_4\big(f, \frac{1}{n}\big)_1.$$

(Note the gap between the affirmative and negative estimates in the case $p = 1$.)

If $p = \infty$, we do not have to deal with nonnegative functions, whence we assume $s \ge 1$. What we have is

Theorem 5.9. *If $f \in \mathbb{C}[-1,1] \cap \Delta^0(Y_s)$, $s \ge 1$, then for each $n \ge k-1$, there is a polynomial $p_n \in \Pi_n$ which is copositive with f in $[-1,1] \setminus O^*(n, Y_s)$, and such that*

$$\|f - p_n\|_\infty \le C\omega_k^\varphi\big(f, \frac{1}{n}\big)_\infty,$$

where $C = C(k, Y_s)$

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