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Using Laurent polynomial representation for the analysis of non-uniform binary subdivision schemes

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Non-uniform binary linear subdivision schemes, with finite masks, over uniform grids, are studied. A Laurent polynomial representation is suggested and the basic operations required for smoothness analysis are presented. As an example it is shown that the interpolatory 4-point scheme is C^1 with an almost arbitrary non-uniform choice of the free parameter.

1. Introduction

Starting with values $\{f_j^0\}_{j\in\mathbb{Z}}$ assigned to the integers, a binary subdivision scheme defines recursively values $\{f_j^k\}_{j\in\mathbb{Z}}$, respectively assigned to the binary points $\{2^{-k}j\}_{j\in\mathbb{Z}}$. The purpose of subdivision analysis is to study the convergence of such processes and to establish the existence of a limit function on $\mathbb{R}$ and its smoothness class. A general treatment of uniform subdivision can be found in [1–3,7,9,16,17]. Level-dependent subdivision schemes, where the scheme may vary from one refinement level to the other, are discussed in [13]. In the present work we analyze non-uniform binary subdivision schemes in which the scheme for defining the points may vary from point to point and from level to level. To present the problem for non-uniform schemes we first review one way of analyzing uniform binary subdivision schemes using a Laurent polynomial representation.

A uniform binary subdivision scheme, with a finite mask $\{p_i\}_{i=-m}^{n}$, is defined by

$$f_j^{k+1} = \sum_{i\in\mathbb{Z}} p_{j-2i} f_i^k. \tag{1.1}$$

Let us represent the sequence of values $\{f_j^k\}$ at level k by its Laurent series

$$F_k(z) = \sum_{j\in\mathbb{Z}} f_j^k z^j.$$

The above uniform binary subdivision scheme may be represented by a generating polynomial

$$p(z) = \sum_{i=-m}^{n} p_i z^i$$

defining the transformation from level k to level $k+1$ by the formal relation

$$F_{k+1}(z) = p(z) F_k\left(z^2\right). \tag{1.2}$$

$F_k(z^2) = \sum_{j\in\mathbb{Z}} f_j^k z^{2j}$ is interpreted as assigning the values f_j^k to the even points, $\{2^{-(k+1)} \cdot 2j\}$, on the $(k+1)$ mesh. The equality in (1.2) is defined by equalities in the coefficients of equal powers of z in both sides. A necessary condition [12] for the convergence of the uniform binary subdivision scheme (1.1) to a C^0 function is

$$\sum p_{2i} = \sum p_{2i+1} = 1. \tag{1.3}$$

These conditions can be expressed in terms of the generating polynomial as

$$\begin{cases} p(1) = 2, \\ p(-1) = 0, \end{cases} \tag{1.4}$$

and from here onwards it is assumed that (1.4) holds.

The analysis in [9] makes use of difference schemes and iterated schemes. In the language of Laurent polynomials this gives us a practical tool for establishing the smoothness class of the limit functions generated by a given scheme.

Theorem 1.1 (Proven in [9] and in [17] in terms of matrices). Let $p[z]$ be the Laurent polynomial representing a uniform binary subdivision scheme and assume

$$p(z) = \frac{(z+1)^{m+1}}{2^m \delta^m(z)},$$

where $\delta^m(z)$ is a finite Laurent polynomial. A sufficient condition for a C^m limit of the subdivision process is that there exists some $\ell > 0$ such that the ℓ-interated polynomial

$$\delta^{m,\ell}(z) = \prod_{i=0}^{\ell-1} \delta^m\left(z^{2^i}\right) = \sum_j \delta_j^{m,\ell} z^j \tag{1.5}$$

is contractive in $\|\cdot\|_\infty$, i.e.,

$$\sum_i \left|\delta_{2^\ell i+r}^{m,\ell}\right| < 1, \quad 0 \leqslant r < 2^\ell. \tag{1.6}$$

We note that $\delta^m(z)$ is the Laurent polynomial of the subdivision scheme for the differences of the m-divided differences. Namely, if Δ is the backward difference operator, $\Delta f_j^k = f_{j-1}^k - f_j^k$, then $2^{km}\Delta^m f_j^k/m!$ are the m-divided differences at level k

and $g_j^k = 2^{km}\Delta^{m+1} f_j^k/m!$ are the differences of the m-divided differences at level k. The subdivision scheme induced by $\delta^m(z)$ transforms the values $\{g_j^k\}_{j\in\mathbb{Z}}$ to the values $\{g_j^{k+1}\}_{j\in\mathbb{Z}}$ [7].

The motivation for the analysis of non-uniform binary subdivision schemes arises from some applications of the 4-point interpolatory scheme [8]. This scheme is defined by

$$\begin{cases} f_{2j}^{k+1} = f_j^k, \\ f_{2j+1}^{k+1} = \left(\frac{1}{2} + w\right)\left(f_j^k + f_{j+1}^k\right) - w\left(f_{j-1}^k + f_{j+2}^k\right), \end{cases} \tag{1.7}$$

where w is a constant tension parameter.

It is easy to check that the corresponding generating polynomial is

$$p(z) = \frac{1}{2z}(z+1)^2\bigl(1 + wb(z)\bigr), \tag{1.8}$$

where

$$b(z) = -2z^{-2}(z-1)^2\bigl(z^2+1\bigr).$$

The analysis of this scheme in [8,9] gives ranges of the tension parameter w for which the limit curve of the subdivision scheme is C^0 or C^1. In applications one may need to use different tension values at different parts of the curve and at different levels of the process. In general we consider the case $w = w_j^k$, i.e., a tension value depending on the point j and the level k, and we would like to find conditions on the values $\{w_j^k\}$ so that the limit curve is still C^0 or even C^1.

The subdivision process (1.1) generates limit functions of the form

$$f(x) = \sum_{j\in\mathbb{Z}} f_j^0 \phi(x-j), \tag{1.9}$$

where the function ϕ, termed the "basic limit function", is supported in $[-m, n]$ and it satisfies the refinement equation

$$\phi(x) = \sum_{i=-m}^{n} p_i \phi(2x-i). \tag{1.10}$$

Therefore, subdivision analysis is almost equivalent to the analysis of compact solutions of corresponding refinement equations. This direction, which is also motivated by wavelet analysis, is pursued in many works, e.g., [2,3], the main tool for this analysis being of course Fourier analysis.

2. Analysis of non-uniform binary subdivision schemes

The analysis tools for uniform binary subdivision schemes in [7,9,17], or the Fourier analysis approach in [2] do not seem to be appropriate for non-uniform schemes. In [4] de Boor studies the convergence of non-uniform corner cutting, and this problem is further investigated by Gregory and Qu [14] who presented a full C^1 analysis. In [10] an analysis of piecewise uniform subdivision schemes is presented. In what follows we suggest an adaptation of the the above Laurent polynomial tools for the analysis of general non-uniform binary subdivision schemes.

A general non-uniform linear binary subdivision scheme can be represented by a bi-infinite sequence of generating polynomials $\{p_{(j,k)}(z)\}$ where each $p_{(j,k)}$ is the polynomial representing the scheme generating the value f_j^k, $k \geqslant 1$, $j \in \mathbb{Z}$. That is,

$$f_j^{k+1} = \sum_{i\in\mathbb{Z}} p_{(j,k),j-2i} f_i^k, \tag{2.1}$$

where $p_{(j,k)}(z) = \sum_{m\in\mathbb{Z}} p_{(j,k),m} z^m$. We assume here that all the masks are finite, namely, $\{p_{(j,k)}(z)\}$ are all finite Laurent polynomials. In the uniform case there is one "basic limit function" which, together with its integer shifts and dyadic scalings, provides all the information on the limit function and the related multiresolution analysis. In the level-dependent case [13], there is a different "basic limit function" for each refinement level, while in the general non-uniform linear case we may define a "basic limit function" $\phi_{(j,k)}(x)$ per each dyadic point. Consequently there is a bi-infinite system of refinement equations satisfied by $\{\phi_{(j,k)}(x)\}$. We are not going to pursue this direction here.

Let $G(z) = \sum_{j\in\mathbb{Z}} g_i z^i$ be a Laurent series. We introduce the notation

$$\bigl[G(z)\bigr]_j = \biggl[\sum_{j\in\mathbb{Z}} g_i z^i\biggr]_j = g_j, \quad j \in \mathbb{Z}, \tag{2.2}$$

for the operator extracting the coefficient of z^j in a Laurent series. This simple operator is the main tool used in the definitions and the derivations in this work. Two useful simple properties of this operator are $[zG(z)]_j = [G(z)]_{j-1}$ and $[G(z^2)]_{2j} = [G(z)]_j$.

The transformation from level $k-1$ to level k by a general non-uniform scheme can be expressed by the formal relation

$$F_k(z) = \sum_{j\in\mathbb{Z}} \bigl[p_{(j,k)}(z) F_{k-1}\bigl(z^2\bigr)\bigr]_j z^j. \tag{2.3}$$

It is clear that if j is even (odd) then only the even (odd) terms of $p_{(j,k)}(z)$ contribute to the definition of the value f_j^k. However, in order to obtain a unified formulation, we let $p_{(j,k)}(z)$ contain both odd and even powers. We further assume that all the generating polynomials satisfy

$$\begin{cases} p_{(j,k)}(1) = 2, \\ p_{(j,k)}(-1) = 0. \end{cases} \tag{2.4}$$

These conditions, which are necessary for a C^0 limit in the uniform case, are certainly not necessary for a C^0 limit in the non-uniform case, but they are satisfied in some interesting examples of non-uniform schemes. The condition simply says that the scheme preserves the constant function.

Example. Consider the 4-point scheme (1.8) where w is replaced by w_j^k. The corresponding generating polynomials are

$$p_{(j,k)}(z) = \frac{1}{2z}(z+1)^2\bigl(1 + w_j^k b(z)\bigr). \tag{2.5}$$

In other words, at each new point $2^{-k}j$ the 4-point scheme is applied with possibly a different tension parameter w_j^k.

Let us now examine the applicability of the new notation (2.2)–(2.3) to the basic operations of backward differencing and to iterated subdivision. Backward differences of the values $\{f_j^k\}$ are represented by the Laurent series $(z-1)F_k(z)$. We would like to express the backward differencing at level k by the backward differencing at level $k-1$, as follows:

$$\begin{aligned}(z-1)F_k(z) &= \sum_{j\in\mathbb{Z}} \bigl\{ \bigl[p_{(j-1,k)}(z)F_{k-1}\bigl(z^2\bigr)\bigr]_{j-1} - \bigl[p_{(j,k)}(z)F_{k-1}\bigl(z^2\bigr)\bigr]_j \bigr\} z^j \\ &= \sum_{j\in\mathbb{Z}} \bigl[\bigl(zp_{(j-1,k)}(z) - p_{(j,k)}(z)\bigr)F_{k-1}\bigl(z^2\bigr)\bigr]_j z^j. \end{aligned} \tag{2.6}$$

Using (2.4) we have the factorization

$$zp_{(j-1,k)}(z) - p_{(j,k)}(z) = \bigl(z^2-1\bigr)\delta^1_{(j,k)}(z), \tag{2.7}$$

where $\delta^1_{(j,k)}(z)$ is a finite Laurent polynomial, and thus

$$(z-1)F_k(z) = \sum_{j\in\mathbb{Z}} \bigl[\delta^1_{(j,k)}(z)\bigl(z^2-1\bigr)F_{k-1}\bigl(z^2\bigr)\bigr]_j z^j. \tag{2.8}$$

Equation (2.8) expresses the transformation from differences at level $k-1$ to differences at level k using the polynomials $\{\delta^1_{(j,k)}(z)\}$. Viewing the polynomials $\{\delta^1_{(j,k)}(z)\}$ as a non-uniform subdivision scheme, the non-uniform subdivision scheme defined by the polynomials $\{p_{(j,k)}(z)\}$ is C^0 if the scheme $\{\delta^1_{(j,k)}(z)\}$ is contractive. If the scheme $\{2\delta^1_{(j,k)}(z)\}$ satisfies the conditions (2.4), then further difference schemes may be defined. An important special case is considered in the following lemma:

Lemma 2.1. Let

$$p_{(j,k)}(x) = \frac{(z+1)^{m+1}}{2^m}\bigl(a(z) + (z-1)^m b_j^k(z)\bigr), \tag{2.9}$$

with $a(1) = 1$, and define $\delta^0_{(j,k)}(z) \equiv p_{(j,k)}(z)$. Then all the non-uniform difference schemes $\{\delta^r_{(j,k)}(z)\}$, $1 \leqslant r \leqslant m+1$, defined recursively by

$$\delta^{r+1}_{(j,k)}(z) = \frac{2^r(z\delta^r_{(j-1,k)}(z) - \delta^r_{(j,k)}(z))}{z^2 - 1} \tag{2.10}$$

are finite Laurent polynomials. Furthermore, the scheme $\{\delta^{r+1}_{(j,k)}(z)\}$ maps the differences of the divided differences of order r of $\{f^{k-1}_j\}$ to the same differences at level k. Namely,

$$\frac{(z-1)^{r+1}}{2^{rk}} F_k(z) = \sum_{j\in\mathbb{Z}} \left[\delta^{r+1}_{(j,k)}(z) \frac{(z^2-1)^{r+1}}{2^{r(k-1)}} F_{k-1}\left(z^2\right)\right]_j z^j. \tag{2.11}$$

The proof follows by applying (2.7) and (2.8) recursively, using the special form (2.9) of $\{p_{(j,k)}(z)\}$.

Next, in analogy to (1.5), we look for the generating polynomials for the ℓ-iterated scheme: We introduce the notation $q_{(j,k,\ell)}(z)$ representing the rule for calculating the value $f^{k+\ell}_j$ using values at level k, $\{f^k_i\}$.

Lemma 2.2. Let $\{q_{(j,k)}(z)\}$ be the generating polynomials of a non-uniform binary subdivision scheme. Then the generating polynomials of the ℓ-iterated scheme, transforming values at level k directly to level $k+\ell$, are $\{q_{(j,k,\ell)}(z)\}$ defined recursively by

$$q_{(j,k,i+1)}(z) = \sum_m q_{(j,k+i,1),m} z^m q_{([(j-m)/2],k,i)}\left(z^2\right), \tag{2.12}$$

where

$$q_{(j,k,1)}(z) = q_{(j,k+1)}(z)$$

and

$$q_{(j,k,i)}(z) = \sum_m q_{(j,k,i),m} z^m, \quad i = 1, \ldots, \ell.$$

Proof. We would like to show that

$$F_{k+\ell}(z) = \sum_{j\in\mathbb{Z}} \left[q_{(j,k,\ell)}(z) F_k\left(z^{2^\ell}\right)\right]_j z^j. \tag{2.13}$$

By definition,

$$F_{k+i+1}(z) = \sum_{j\in\mathbb{Z}} \left[q_{(j,k+i,1)}(z) F_{k+i}\left(z^2\right)\right]_j z^j.$$

Hence, assuming

$$F_{k+i}(z)=\sum_{n\in\mathbb{Z}}\left[q_{(n,k,i)}(z)F_k\left(z^{2^i}\right)\right]_n z^n,$$

we have

$$\begin{aligned}
F_{k+i+1}(z)&=\sum_{j\in\mathbb{Z}}\left[q_{(j,k+i,1)}(z)\sum_{n\in\mathbb{Z}}\left[q_{(n,k,i)}(z)F_k\left(z^{2^i}\right)\right]_n z^{2n}\right]_j z^j\\
&=\sum_{j\in\mathbb{Z}}\sum_m q_{(j,k+i,1),m}\left[z^m\sum_{n\in\mathbb{Z}}\left[q_{(n,k,i)}\left(z^2\right)F_k\left(z^{2^{i+1}}\right)\right]_{2n}z^{2n}\right]_j z^j\\
&=\sum_{j\in\mathbb{Z}}\sum_m q_{(j,k+i,1),m}\left[\sum_{n\in\mathbb{Z}}\left[q_{(n,k,i)}\left(z^2\right)F_k\left(z^{2^{i+1}}\right)\right]_{2n}z^{2n}\right]_{j-m} z^j\\
&=\sum_{j\in\mathbb{Z}}\sum_m q_{(j,k+i,1),m}\left[q_{([(j-m)/2],k,i)}\left(z^2\right)F_k\left(z^{2^{i+1}}\right)\right]_{j-m}z^j\\
&=\sum_{j\in\mathbb{Z}}\left[\sum_m q_{(j,k+i,1),m}z^m q_{([(j-m)/2],k,i)}\left(z^2\right)F_k\left(z^{2^{i+1}}\right)\right]_j z^j.
\end{aligned}$$

Thus $q_{(j,k,i+1)}(z)$ defined by (2.12) does transform values from level k into values at level $k+i+1$. □

Definition 2.3. Using lemma 2.2, a non-uniform scheme represented by generating polynomials $\{q_{(j,k)}(z)\}$ is said to be contractive if there exists some integer ℓ and $\rho\in[0,1)$ such that

$$\sum_i\left|q_{(j,k,\ell),2^\ell i+r}\right|\leqslant\rho,\quad 0\leqslant r<2^\ell, \tag{2.14}$$

for any j and for any $k\geqslant K,\ K\geqslant 1$.

Theorem 2.4 (Sufficient conditions for C^m). Let $\{p_{(j,k)}(z)\}$ be the generating polynomials of a non-uniform binary subdivision scheme, and let $\{\delta^r_{(j,k)}(z)\}$, $1\leqslant r\leqslant m+1$, be defined by (2.10). Then the scheme $\{p_{(j,k)}(z)\}$ is C^m if the scheme defined by $\{\delta^{m+1}_{(j,k)}(z)\}$ is contractive.

3. The non-uniform 4-point scheme

In this section we apply the above analysis principles to the example of the non-uniform interpolatory 4-point scheme. The corresponding generating polynomials (2.5) can be written as

$$p_{(j,k)}(z)=\frac{(z+1)^2}{2}\left(\frac{1}{z}-2w_j^k z^{-3}(z-1)^2\left(z^2+1\right)\right). \tag{3.1}$$

The uniform case $\{w_j^k \equiv w\}$ is analyzed in [8], where it is shown that for any fixed $w \in I_1 \equiv (0, \frac{1}{8})$, the 4-point scheme produces C^1 limit functions. The special choice $w = \frac{1}{16}$ is fully investigated in [5,6]. The range of w for a C^1 limit is extended in [9] into $w \in I_2 \equiv (0, (\sqrt{5}-1)/8)$, which is still not the largest possible. Using the analysis in [10] it follows that applying the 4-point scheme different weights $\subset I_2$ in different subintervals still yields C^1 limit functions. For the non-uniform case we restrict the analysis to $\{w_j^k\} \subset I_1$.

Theorem 3.1. Let $\{w_j^k\}$ be chosen arbitrarily in $[\varepsilon, \frac{1}{8}-\varepsilon]$, for some fixed $0 < \varepsilon < \frac{1}{16}$. Then the non-uniform 4-point scheme defined by the polynomials (3.1) produces C^1 limit functions.

Proof. Let us view the corresponding difference schemes. The non-uniform scheme (3.1) is of the form (2.9) assumed in lemma 2.2. Hence, the difference schemes are defined by the finite Laurent polynomials $\{\delta^r_{(j,k)}(z)\}$, $r = 1, 2$ defined by (2.10). The explicit expressions are:

$$\delta^1_{(j,k)}(z) = (z+1)\left(\frac{1}{2z} + \left(zw^k_{j-1} - w^k_j\right)z^{-3}(z-1)\left(z^2+1\right)\right) \tag{3.2}$$

and

$$\delta^2_{(j,k)}(z) = \frac{1}{z} - 2\left(z^2 w^k_{j-2} - 2zw^k_{j-1} + w^k_j\right)z^{-3}\left(z^2+1\right). \tag{3.3}$$

By theorem 2.4, the non-uniform 4-point scheme would be C^1 if we could prove that the scheme $\{\delta^2_{(j,k)}(z)\}$ is contractive. Following [9], let us examine the iterated scheme $\{\delta^2_{(j,k,2)}(z)\}$. Using the recursive relation (2.12) we find that

$$\begin{aligned}
\delta^2_{(j,k,2)}(z) &= \sum_{m=-9}^{3} q_m z^m \\
&= z^{-9}\left[4w^{k+1}_{i+1}w^{k+2}_j\right] - z^{-8}\left[8w^{k+1}_{i+1}w^{k+2}_{j-1}\right] \\
&\quad - z^{-7}\left[2w^{k+1}_i\left(1 - 2w^{k+2}_{j-2} + 2w^{k+2}_j\right)\right] + z^{-6}\left[8w^{k+1}_i w^{k+2}_{j-1}\right] \\
&\quad + z^{-5}\left[4w^{k+1}_{i-1}\left(1 - w^{k+2}_{j-2} - w^{k+2}_j\right) - 2w^{k+2}_j + 4w^{k+1}_{i+1}w^{k+2}_j\right] \\
&\quad + z^{-4}\left[4w^{k+2}_{j-1}\left(1 + 2w^{k+1}_{i-1} - 2w^{k+1}_{i+1}\right)\right] \\
&\quad + z^{-3}\left[1 - 2w^{k+2}_{j-2} - 2w^{k+2}_j - 2w^{k+1}_{i-2}\left(1 + 2w^{k+2}_{j-2} - 2w^{k+2}_j\right)\right. \\
&\qquad\quad \left. - 2w^{k+1}_i\left(1 - 2w^{k+2}_{j-2} + 2w^{k+2}_j\right)\right] \\
&\quad + z^{-2}\left[4w^{k+2}_{j-1}\left(1 - 2w^{k+1}_{i-2} + 2w^{k+1}_i\right)\right] \\
&\quad + z^{-1}\left[4w^{k+1}_{i-1}\left(1 - w^{k+2}_{j-2} - w^{k+2}_j\right) - 2w^{k+2}_{j-2} + 4w^{k+1}_{i-3}w^{k+2}_{j-2}\right]
\end{aligned}$$

$$+ 8w_{i-1}^{k+1}w_{j-1}^{k+2} + z\big[-2w_{i-2}^{k+1}\big(1 - 2w_{j-2}^{k+2} + 2w_j^{k+2}\big)\big] - z^2\big[8w_{i-2}^{k+1}w_{j-1}^{k+2}\big]$$
$$+ z^3\big[4w_{i-3}^{k+1}w_{j-2}^{k+2}\big], \tag{3.4}$$

where $i = [j/2]$. To check if the scheme $\{p_{(j,k)}(z)\}$ is C^1 it is enough to prove that $\{\delta^2_{(j,k)}(z)\}$ is contractive, and in view of (2.14), it is enough to show that there exists $\rho \in [0, 1)$ such that each of the following four inequalities holds for any j and k:

$$\begin{aligned}
&|q_{-9}| + |q_{-5}| + |q_{-1}| + |q_3| \leqslant \rho,\\
&|q_{-8}| + |q_{-4}| + |q_0| \leqslant \rho,\\
&|q_{-7}| + |q_{-3}| + |q_1| \leqslant \rho, \quad \text{and}\\
&|q_{-6}| + |q_{-2}| + |q_2| \leqslant \rho.
\end{aligned}$$

To simplify the notation let us denote the parameters in (3.4) as $t_m = w_{i-4+m}^{k+1}$, $m = 1, \ldots, 5$, and $t_m = w_{j-8+m}^{k+2}$, $m = 6, 7, 8$. Now we have to show that above four inequalities are satisfied for some fixed $\rho \in [0, 1)$ for any $\{t_m\}_{m=1}^8 \subset [\varepsilon, \frac{1}{8} - \varepsilon]$. The four inequalities take the form

$$4t_5t_8 + \big|4t_3(1 - t_6 - t_8) - 2t_8 + 4t_5t_8\big| + \big|4t_3(1 - t_6 - t_8) - 2t_6 + 4t_1t_6\big|$$
$$+ 4t_1t_6 \leqslant \rho, \tag{3.5}$$
$$8t_5t_7 + 4t_7(1 + 2t_3 - 2t_5) + 8t_3t_7 \leqslant \rho, \tag{3.6}$$
$$2t_4(1 - 2t_6 + 2t_8) + \big|(1 - 2t_6 - 2t_8) - 2t_2(1 + 2t_6 - 2t_8) - 2t_4(1 - 2t_6 + 2t_8)\big|$$
$$+ 2t_2(1 - 2t_6 + 2t_8) \leqslant \rho, \tag{3.7}$$
$$8t_4t_7 + 4t_7(1 + 2t_4 - 2t_2) + 8t_2t_7 \leqslant \rho. \tag{3.8}$$

The inequalities (3.6) and (3.8) are easily satisfied with $\rho = \frac{3}{4}$. To handle (3.5) we consider all the cases

$$4t_3(1 - t_6 - t_8) \leqslant \text{or} > 2t_8 - 4t_5t_8$$

and

$$4t_3(1 - t_6 - t_8) \leqslant \text{or} > 2t_6 - 4t_1t_6,$$

and it follows that the inequality holds with $\rho = 1 - 10\varepsilon + 16\varepsilon^2$. In (3.7) we first observe that the expression within the absolute value sign can be written as

$$1 - 2t_6(1 + 2t_2 - 2t_4) - 2t_8(1 - 2t_2 + 2t_4) \geqslant \frac{3}{8} > 0.$$

Thus, the whole expression is bounded by $1 - 2t_6 - 2t_8 \leqslant 1 - 4\varepsilon$, and the four inequalities are satisfied with $\rho = 1 - 4\varepsilon$. $\square$

4. Guidelines for the multi-dimensional case

In this section we formulate the analysis principles for studying multi-dimensional non-uniform binary subdivision schemes over uniform square grids. Based upon the multidimensional theory in [7] the extension to the non-uniform case is quite straightforward, while the application to particular non-uniform cases becomes pretty involved. In the following we present an analysis of the so called *truncated tensor product* scheme [11], which is based upon the 4-point scheme. Using the standard multi-index notation $\mathbf{j} = (j_1, \ldots, j_d)$, $\mathbf{z} = (z_1, \ldots, z_d)$ and $\mathbf{z}^{\mathbf{j}} = z_1^{j_1} z_2^{j_2} \cdots z_d^{j_d}$, the multivariate notation and relations are as follows:

A general linear non-uniform d-variate binary subdivision scheme can be represented by a set of generating polynomials $\{p_{(\mathbf{j},k)}(\mathbf{z})\}$ where each $p_{(\mathbf{j},k)}$ is the polynomial representing the scheme generating the value $f_{\mathbf{j}}^k$, $k \geqslant 1$, $\mathbf{j} \in \mathbb{Z}^d$. That is,

$$f_{\mathbf{j}}^{k+1} = \sum_{\mathbf{i} \in \mathbb{Z}^d} p_{(\mathbf{j},k),\mathbf{j}-2\mathbf{i}} f_{\mathbf{i}}^k, \tag{4.1}$$

where $p_{(\mathbf{j},k)}(\mathbf{z}) = \sum_{\mathbf{m} \in \mathbb{Z}^d} p_{(\mathbf{j},k),\mathbf{m}} \mathbf{z}^{\mathbf{m}}$. As in the univariate case we use the notation

$$\left[\sum_{\mathbf{i} \in \mathbb{Z}^d} g_{\mathbf{i}} \mathbf{z}^{\mathbf{i}}\right]_{\mathbf{j}} = g_{\mathbf{j}}, \tag{4.2}$$

for the operator extracting the $\mathbf{j}$th coefficient of a Laurent series, and the values at level k are represented by $F_k(\mathbf{z}) = \sum_{\mathbf{j} \in \mathbb{Z}^d} f_{\mathbf{j}}^k \mathbf{z}^{\mathbf{j}}$. The transformation from level $k-1$ to level k is expressed by the formal relation

$$F_k(\mathbf{z}) = \sum_{\mathbf{j} \in \mathbb{Z}^d} \left[p_{(\mathbf{j},k)}(\mathbf{z}) F_{k-1}\left(\mathbf{z}^2\right)\right]_{\mathbf{j}} \mathbf{z}^{\mathbf{j}}. \tag{4.3}$$

We also denote $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{Z}^d$, $\{e_r\}_{r=1}^d$ the standard basis for $\mathbb{R}^d$ and $\mathbf{E} = \{\text{corners of } [-1,1]^d\}$. The necessary conditions for a C^0 limit in the uniform case are assumed here for any one of the generating polynomials:

$$\begin{cases} p_{(\mathbf{j},k)}(\mathbf{1}) = 2^d, \\ p_{(\mathbf{j},k)}|_{\mathbf{E} \setminus \mathbf{1}} = 0. \end{cases} \tag{4.4}$$

The main example we treat here is the non-uniform variant of the bivariate *truncated tensor product* scheme [11], which is defined by the generating polynomial

$$p(z_1, z_2) = \frac{1}{4 z_1 z_2} (z_1 + 1)^2 (z_2 + 1)^2 \left(1 + w\left(b(z_1) + b(z_2)\right)\right), \tag{4.5}$$

where $b(z)$ is the same as in (1.8). As shown in [15], this scheme is C^1 at least for the range $w \in (0, 0.122)$. For the non-uniform scheme we allow a different pair of tension parameters at each new point generated at each level. Namely, $\{u_{((i,j),k)}\}$ and $\{v_{((i,j),k)}\}$ as tension parameters in the x- and the y-directions, respectively, where (i, j)

denotes a generic point in $\mathbb{Z}^2$. The set of generating polynomials of the non-uniform *truncated tensor product* scheme is thus

$$p_{((i,j),k)}(z_1,z_2) = \frac{1}{4z_1z_2}(z_1+1)^2(z_2+1)^2\bigl(1+u_{((i,j),k)}b(z_1)+v_{((i,j),k)}b(z_2)\bigr). \tag{4.6}$$

To study the range of parameters for C^1 limits of this non-uniform scheme, we need to check the properties of the related difference schemes. In the multivariate case we consider a vector of first order backward differences at level k represented by vector Laurent polynomial

$$G_k(\mathbf{z}) \equiv (z_1-1, z_2-1, \ldots, z_d-1)^{\mathrm{T}} F_k(\mathbf{z}). \tag{4.7}$$

Using the same arguments as in [7] we can write

$$\begin{aligned} &z_r p_{(\mathbf{j}-e_r,k)}(\mathbf{z}) - p_{(\mathbf{j},k)}(\mathbf{z}) \\ &\quad = \bigl(\delta^{r,1}_{(\mathbf{j},k)}(\mathbf{z}), \delta^{r,2}_{(\mathbf{j},k)}(\mathbf{z}), \ldots, \delta^{r,d}_{(\mathbf{j},k)}(\mathbf{z})\bigr)\bigl(z_1^2-1, z_2^2-1, \ldots, z_d^2-1\bigr)^{\mathrm{T}}, \end{aligned} \tag{4.8}$$

where $\{\delta^{r,s}_{(\mathbf{j},k)}(\mathbf{z})\}$ are finite Laurent polynomials.

Following the univariate derivation, using (2.6)–(2.7), we find out here that

$$G_k(z) = \sum_{\mathbf{j}\in\mathbb{Z}^d} \bigl[D^1_{(\mathbf{j},k)}(\mathbf{z}) G_{k-1}\bigl(\mathbf{z}^2\bigr)\bigr]_{\mathbf{j}} \mathbf{z}^{\mathbf{j}}, \tag{4.9}$$

where $D^1_{(\mathbf{j},k)}(\mathbf{z})$ is the finite matrix Laurent polynomial with entries $\{\delta^{r,s}_{(\mathbf{j},k)}(\mathbf{z})\}^d_{r,s=1}$. The relation (4.9) is the multivariate analogue of (2.8). Following [7] again, it follows that the scheme $\{p_{(\mathbf{j},k)}(\mathbf{z})\}$ produces C^0 limit function if the matrix scheme $\{D^1_{(\mathbf{j},k)}(\mathbf{z})\}$ is contractive. To analyze higher smoothness we need to consider higher order difference schemes, and for their existence we need to assume some further conditions on the matrix polynomials $\{D^1_{(\mathbf{j},k)}(\mathbf{z})\}$. The vectors of higher differences get longer and the corresponding matrix schemes are of corresponding bigger sizes. Instead of pushing further the smoothness analysis for a general scheme in $\mathbb{R}^d$, we rather focus on the specific example of the above mentioned bivariate non-uniform *truncated tensor product* scheme. The generating polynomials of this scheme can be written as

$$\begin{aligned} p_{((i,j),k)}(z_1,z_2) = \frac{1}{4z_1z_2}\Bigl[&(z_1+1)^2(z_2+1)^2 - u_{((i,j),k)} z_1^{-2}\bigl(z_1^2-1\bigr)^2\bigl(z_1^2+1\bigr)(z_2+1)^2 \\ &- v_{((i,j),k)} z_2^{-2}\bigl(z_2^2-1\bigr)^2\bigl(z_2^2+1\bigr)(z_1+1)^2\Bigr]. \end{aligned} \tag{4.10}$$

To check C^1 we look for the scheme generating the vector of differences of first order divided differences, namely

$$H_k(z_1,z_2) = 2\bigl((z_1-1)^2, (z_1-1)(z_2-1), (z_2-1)^2\bigr)^{\mathrm{T}} F_k(z_1,z_2). \tag{4.11}$$

Using (2.6) twice we have

$$H_k(z_1,z_2) = \sum_{(i,j)\in\mathbb{Z}^2} \bigl[V_{((i,j),k)}(z_1,z_2) F_{k-1}(z_1,z_2)\bigr]_{(i,j)} z_1^i z_2^j, \tag{4.12}$$

where (with $\mathbf{z} = (z_1, z_2)$)

$$V_{((i,j),k)}(\mathbf{z}) = \begin{pmatrix} z_1^2 p_{((i-2,j),k)}(\mathbf{z}) - 2z_1 p_{((i-1,j),k)}(\mathbf{z}) + p_{((i,j),k)}(\mathbf{z}) \\ 2\big(z_1 z_2 p_{((i-1,j-1),k)}(\mathbf{z}) - z_1 p_{((i-1,j),k)}(\mathbf{z}) - z_2 p_{((i,j-1),k)}(\mathbf{z}) + p_{((i,j),k)}(\mathbf{z})\big) \\ z_2^2 p_{((i,j-2),k)}(\mathbf{z}) - 2z_2 p_{((i,j-1),k)}(\mathbf{z}) + p_{((i,j),k)}(\mathbf{z}) \end{pmatrix}.$$

For the special form of the scheme (4.10) we find that

$$V_{((i,j),k)}(\mathbf{z}) = 2A_{((i,j),k)}(\mathbf{z})\Big(\big(z_1^2-1\big)^2, \big(z_1^2-1\big)\big(z_2^2-1\big), \big(z_2^2-1\big)^2\Big)^{\mathrm{T}}, \tag{4.13}$$

where

$$A_{((i,j),k)}(\mathbf{z}) \equiv \frac{1}{2z_1 z_2}\big\{A^{r,s}_{((i,j),k)}(\mathbf{z})\big\}^3_{r,s=1}$$

is a finite 3×3 matrix Laurent polynomial. The non-zero entries $A^{r,s}_{((i,j),k)}(\mathbf{z})$ are given below:

$$\begin{aligned}
A^{1,1}_{((i,j),k)}(\mathbf{z}) &= (z_2+1)^2 - \big(z_1^2+1\big)(z_2+1)^2 z_1^{-2}\big(z_1^2 u_{((i-2,j),k)} - 2z_1 u_{((i-1,j),k)} \\
&\quad + u_{((i,j),k)}\big),\\
A^{1,3}_{((i,j),k)}(\mathbf{z}) &= -\big(z_2^2+1\big)(z_1+1)^2 z_2^{-2}\big(z_1^2 u_{((i-2,j),k)} - 2z_1 u_{((i-1,j),k)} + u_{((i,j),k)}\big),\\
A^{2,1}_{((i,j),k)}(\mathbf{z}) &= -\big(z_1^2+1\big)(z_2+1)^2 z_1^{-2}(z_1 z_2 u_{((i-1,j-1),k)} - z_1 u_{((i-1,j),k)} - z_2 u_{((i,j-1),k)} \\
&\quad + u_{((i,j),k)}),\\
A^{2,2}_{((i,j),k)}(\mathbf{z}) &= (z_1+1)(z_2+1),\\
A^{2,3}_{((i,j),k)}(\mathbf{z}) &= -\big(z_2^2+1\big)(z_1+1)^2 z_2^{-2}(z_1 z_2 v_{((i-1,j-1),k)} - z_1 v_{((i-1,j),k)} - z_2 v_{((i,j-1),k)} \\
&\quad + v_{((i,j),k)}),\\
A^{3,1}_{((i,j),k)}(\mathbf{z}) &= -\big(z_1^2+1\big)(z_2+1)^2 z_1^{-2}\big(z_2^2 u_{((i,j-2),k)} - 2z_2 u_{((i,j-1),k)} + u_{((i,j),k)}\big),\\
A^{3,3}_{((i,j),k)}(\mathbf{z}) &= (z_1+1)^2 - \big(z_2^2+1\big)(z_1+1)^2 z_2^{-2}\big(z_2^2 v_{((i,j-2),k)} - 2z_2 v_{((i,j-1),k)} \\
&\quad + v_{((i,j),k)}\big).
\end{aligned}$$

Combining (4.11)–(4.13) we have

$$H_k(z_1, z_2) = \sum_{(i,j)\in\mathbb{Z}^2} \big[A_{((i,j),k)}(z_1,z_2) H_{k-1}(z_1,z_2)\big]_{(i,j)} z_1^i z_2^j. \tag{4.14}$$

Thus $\{A_{((i,j),k)}(z_1, z_2)\}$ is the non-uniform matrix scheme generating the vector of differences of first order divided differences of the values generated by the non-uniform scheme $\{p_{((i,j),k)}(z_1, z_2)\}$. Therefore, the scheme $\{p_{((i,j),k)}(z_1, z_2)\}$ is a C^1 scheme if the scheme $\{A_{((i,j),k)}(z_1, z_2)\}$ is contractive. As in the univariate case we would like to find a range of parameters $\{u_{((i,j),k)}\}$ and $\{v_{((i,j),k)}\}$ for which the non-uniform *truncated tensor product* scheme is C^1. The result in lemma 2.2 for computing iterated schemes is easily translated into the multivariate matrix case using the multi-index notation, with $\mathbf{z} \in \mathbb{R}^d$ and $\mathbf{j}, \mathbf{m} \in \mathbb{Z}^d$, as follows:

Lemma 4.1. Let $\{Q_{(\mathbf{j},k)}(\mathbf{z})\}$ be the generating matrix polynomials of a non-uniform binary multivariate matrix subdivision scheme. Then the generating matrix polynomials of the ℓ-iterated scheme, transforming vectors at level k directly to level $k+\ell$, are $\{Q_{(\mathbf{j},k,\ell)}(\mathbf{z})\}$ defined recursively by

$$Q_{(\mathbf{j},k,i+1)}(\mathbf{z}) = \sum_{\mathbf{m}} Q_{(\mathbf{j},k+i,1),\mathbf{m}} \mathbf{z}^{\mathbf{m}} Q_{([(\mathbf{j}-\mathbf{m})/2],k,i)}\big(\mathbf{z}^2\big), \tag{4.15}$$

where

$$Q_{(\mathbf{j},k,1)}(\mathbf{z}) = Q_{(\mathbf{j},k+1)}(\mathbf{z})$$

and

$$Q_{(\mathbf{j},k,i)}(\mathbf{z}) = \sum_{\mathbf{m}} Q_{(\mathbf{j},k,i),\mathbf{m}} \mathbf{z}^{\mathbf{m}}, \quad i = 1, \dots, \ell.$$

These formulae can be used to check the contractivity of the difference matrix scheme $\{A_{((i,j),k)}(z_1, z_2)\}$ for verifying C^1 of the non-uniform *truncated tensor product* scheme (4.6). Considering the multitude of parameters involved, the application of this approach for checking contractivity seems quite frightening here. Hence, instead of a detailed algebraic proof, we performed a numerical simulation test. In this test we have checked, for some fixed values of ε, the norm of $\{A_{((i,j),k,2)}(z_1, z_2)\}$ for many sets of parameters $\{u_{((i,j),k)}\}$ and $\{v_{((i,j),k)}\}$ chosen at random in the interval $(\varepsilon, \frac{1}{8} - \varepsilon)$. Our conclusion is that the scheme is contractive at least for $\varepsilon = 0.02$, i.e., the parameters are in $(0.02, 0.105)$. In order to verify if the interval of parameters can be further stretched, one has to check the norms of $\{A_{((i,j),k,\ell)}(z_1, z_2)\}$ for $\ell > 2$.

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