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Derivation and Analysis of Green Coordinates

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Abstract. Green coordinates define a special representation of a point inside a closed polygon in terms of its vertices and the normals to its edges (faces). This representation has been found to be very useful for object manipulation in computer graphics. The mapping defined by Green coordinates is shown to be analytic. It has a closed form formula in 2D and 3D, and it can be extended analytically through a face of the polygon. In 2D the mapping is proved to be conformal.

Keywords. Conformal mapping, Green identities, barycentric coordinates, analytic continuation, quasiconformal mapping.

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1. Introduction

Recently, Lipman *et al.* [3] presented a method for creating controllable conformal mappings in $\mathbb{R}^2$ and quasi-conformal mappings in $\mathbb{R}^3$. Their technique is based on closed form formulae for representing a point inside a simplicial surface (to be defined shortly) as a linear combination of the vertices and the normals of the simplicial surface: Let P be an oriented simplicial surface, i.e. a closed polygon in 2D, or a closed polyhedron with triangular faces in 3D. That is $P = (\mathbb{V}, \mathbb{T})$, where $\mathbb{V} = \{v_i\}_{i \in I_{\mathbb{V}}} \subset \mathbb{R}^d$ are the vertices and $\mathbb{T} = \{t_j\}_{j \in I_{\mathbb{T}}}$ are the simplicial face elements $t_j = (v_{j_1}, \dots, v_{j_d})$, namely edges in case of polygons in 2D, triangles in case of triangular meshes in 3D. In what follows we use the term *cage* to address this simplicial surface P. Let us further denote by $n(t_j)$ the outward normal to the oriented simplicial face t_j ($\|n(t_j)\| = 1$). As stated above we aim at representing each interior point η of the cage P by a linear combination

$$\eta = F(\eta; P) = \sum_{i \in I_{\mathbb{V}}} \phi_i(\eta) v_i + \sum_{j \in I_{\mathbb{T}}} \psi_j(\eta) n(t_j). \tag{1}$$

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We refer to $\phi_i(\cdot)$ and $\psi_j(\cdot)$ by the term *Green Coordinates*. The name choice is due to the use of Green's third identity to derive the coordinates.

This representation can be seen as an extension of the so called "generalized barycentric coordinates" which represent a point inside a simplicial surface as an *affine* combination of the vertices of the simplicial surface [6, 7, 2],

$$\eta = F(\eta; P) = \sum_{i \in I_{\mathbb{V}}} \varphi_i(\eta) v_i, \tag{2}$$

the coefficients of the affine sum $\varphi_i(\cdot)$ are usually referred to by the term *coordinates*.

One interesting application of the above representation is defining mappings of the interior of P, P^{in}, induced by deforming the cage $P = (\mathbb{V}, \mathbb{T})$ into $P' = (\mathbb{V}', \mathbb{T}')$. We assume that P and P' have the same topological structure, and define the mapping by

$$\eta \mapsto F(\eta; P') = \sum_{i \in I_{\mathbb{V}}} \phi_i(\eta) v_i' + \sum_{j \in I_{\mathbb{T}}} \psi_j(\eta) s_j n(t_j'), \tag{3}$$

where v_i' and t_j' denote the vertices and simplicial faces of P', respectively. The scaling factors $\{s_j\}_{j \in I_{\mathbb{T}}}$ are essential for achieving important properties such as scale invariance. The definition of the scalars $\{s_j\}$ is explained later on, in particular, in 2D, it is simply $s_j = \|t_j'\| / \|t_j\|$, where $\|t_j\|$ is the length of t_j.

The current paper aims at providing some of the theoretical justifications to the claims made in the previous paper. In particular, we prove the conformality of the mapping $F(\cdot; P')$ for arbitrary P', derive the closed form formulae for the Green coordinates $\phi_i(\cdot)$ and $\psi_j(\cdot)$, and construct the unique analytical continuation of the mapping F outside the cage. For completeness of our discussion we provide here the definition and derivation similarly to [3].

2. Derivation of Green Coordinates

In this section we derive the Green Coordinates in $\mathbb{R}^d$. As argued in [3], shape-preservation cannot be achieved by affine combinations of the cage's vertices alone, and we suggest considering combinations of vertices and normals of the form (1), where the exact relation is coded in the coordinate functions $\{\phi_i\}$ and $\{\psi_j\}$ and the scalars $\{s_j\}$. Our derivation of these coordinate functions is based upon the theory of Green functions and upon the following Green's third integral identity: Let u be a harmonic function in a domain $D \subset \mathbb{R}^d$ enclosed by a piecewise-smooth boundary ∂D. A scalar function u is called harmonic if it is a solution to Laplace equation, i.e. $\Delta u = \nabla \cdot \nabla u = 0$. Further, let $G(\cdot, \cdot)$ be the fundamental solution of the Laplace equation in $\mathbb{R}^d$, that is $\Delta_\xi G(\xi, \eta) = \delta(\xi - \eta)$, where $\delta(\cdot)$ is the delta function, $\xi, \eta \in \mathbb{R}^d$. Then, for any $\eta \in D^{in} := interior(D)$,

$u(\eta)$ can be expressed by its boundary values and boundary normal derivatives as

$$u(\eta) = \int_{\partial D} \left(u(\xi) \frac{\partial_\xi G(\xi, \eta)}{\partial n} - G(\xi, \eta) \frac{\partial u(\xi)}{\partial n} \right) d\sigma_\xi, \tag{4}$$

where n is the oriented outward normal to ∂D, and $d\sigma_\xi = d\sigma$ is the volume element on ∂D.

The fundamental solutions of the Laplace equation in $\mathbb{R}^d$ are:

$$G(\xi, \eta) = \begin{cases} \dfrac{1}{(2-d)\omega_d} \|\xi - \eta\|^{2-d} & d \geq 3, \\ \dfrac{1}{2\pi} \log \|\xi - \eta\| & d = 2, \end{cases} \tag{5}$$

where ω_d is the volume of a unit sphere in $\mathbb{R}^d$.

Now let us take the domain D to be the domain enclosed by our cage P, and let $u(\eta) = \eta$, that is the coordinate functions, in (4). Note that here we take u as the vector function $u = \xi \colon \mathbb{R}^d \to \mathbb{R}^d$. Writing the integral as a sum of integrals over the cage's faces, and noting that on each face t_j the normal $n(t_j)$ is constant, we arrive at

$$\eta = \sum_{j \in I_{\mathbb{T}}} \left(\int_{t_j} \xi \frac{\partial G(\xi, \eta)}{\partial n} \, d\sigma - \int_{t_j} G(\xi, \eta) n(t_j) \, d\sigma \right), \qquad \eta \in D^{in}. \tag{6}$$

Denote by $\mathrm{N}\{v_i\}$ the union of all faces in the 1-ring neighborhood of vertex v_i, and let the function Γ_i be the piecewise-linear hat function defined on $\mathrm{N}\{v_i\}$, which is one at v_i, zero at all other vertices in the 1-ring and linear on each face. Then writing ξ as the (unique) barycentric combination in the simplicial face t_j, $\xi = \sum_{k=1}^{d} \Gamma_k(\xi) v_k$, where v_k are the vertices of the face t_j, we get from (6)

$$\eta = \sum_{i \in I_{\mathbb{V}}} \phi_i(\eta) v_i + \sum_{j \in I_{\mathbb{T}}} \psi_j(\eta) n(t_j), \qquad \eta \in D^{in}. \tag{7}$$

The coordinate functions ϕ_i and ψ_j are

$$\begin{aligned} \phi_i(\eta) &= \int_{\xi \in \mathrm{N}\{v_i\}} \Gamma_i(\xi) \frac{\partial G(\xi, \eta)}{\partial n} \, d\sigma, \qquad && i \in I_{\mathbb{V}}, \\ \psi_j(\eta) &= -\int_{\xi \in t_j} G(\xi, \eta) \, d\sigma, && j \in I_{\mathbb{T}}. \end{aligned} \tag{8}$$

To complete the construction of the mapping $\eta \mapsto F(\eta; P')$ defined by (3) we still need to define the scaling factors $\{s_j\}$. The definition of these factors is derived by the following properties, desirable for shape-preserving deformations:

(i) *Linear reproduction*: $\eta = F(\eta; P)$, for $\eta \in P^{in}$.
(ii) *Translation invariance*: $\sum_{i \in I_{\mathbb{V}}} \phi_i(\eta) = 1$, for $\eta \in P^{in}$.

(iii) *Rotation and scale invariance*: For an affine transformation which consists of a rotation with possible isotropic scale U, $F(\eta; UP) = U\eta$.
(iv) *Conformality*: For $d = 2$, the mapping $\eta \mapsto F(\eta; P')$ is holomorphic.
(v) *Smoothness*: $\{\phi_i(\eta)\}$, $\{\psi_j(\eta)\}$ are harmonic functions in P^{in}. Hence, they are C^∞ for $\eta \in P^{in}$.

Linear reproduction is the basic relation (7) we started with, we just need to take $s_j = 1$ if $t'_j = t_j$. This choice is also suitable for the second property, together with the relation $\sum_{i \in I_{\mathbb{V}}} \phi_i(\eta) = 1$ followed by applying (4) to the function $u(\eta) \equiv 1$. To ensure the third property we take $s_j = \|U\|_2$, and thus $Un(t_j) = s_j n(t'_j)$. The face t_j, together with the point $v_{j_1} + n(t_j)$, where v_{j_1} is a vertex in t_j, define a simplex S_j in $\mathbb{R}^d$, and similarly t'_j and $v'_{j_1} + s_j n(t'_j)$ define a simplex S'_j. In the case of a similarity (rotation and uniform scaling) map S we have $U(S_j) = S'_j$. In the general case we would like to define s_j so that the linear mapping taking S_j onto S'_j is least-distorting. In other words, s_j should represent the *stretch* the face t_j undergoes as the cage is deformed. In 2D ($d = 2$) this stretch is well defined, simply take

$$s_j = \frac{\|t'_j\|}{\|t_j\|}, \tag{9}$$

i.e. the exact stretch of the edge t_j. In higher dimensions, however, the stretch is not so evident and it cannot be described by a single scalar. Nevertheless, we find the following definition natural: In 3D, let σ_1, σ_2 be the singular values of the linear map taking t_j to t'_j. Then, to have a least-distorting map taking S_j onto S'_j we should define s_j as some average of σ_1 and σ_2. The choice that provided us with the desired quasi-conformality property is $s_j = \sqrt{({\sigma_1}^2 + {\sigma_2}^2)/2}$. Using computations presented in [4] for linear transformations between triangles in $\mathbb{R}^3$, one (t_j) with edges defined by the vectors u, v and the other (t'_j) by the corresponding vectors u', v', it turns out that

$$s_j = \frac{\sqrt{|u'|^2|v|^2 - 2(u' \cdot v')(u \cdot v) + |v'|^2|u|^2}}{\sqrt{8}\,\mathrm{area}(t_j)}. \tag{10}$$

Note that this final definition encapsulates and generalizes all of the above cases. As demonstrated by the examples throughout the chapter, the above definition of the factors s_j leads to 'least-distorting' deformations. However, in some cases, one may be interested in a distortion, such as stretching the object non-uniformly. Such effects may still be achieved by replacing the definitions (9) and (10) by the simple choice $s_j = 1$. Intermediate effects may be obtained by sliding the values of s_j between these two options.

Property (v) holds for any choice of $\{s_j\}$, and is due to the fact that for $\eta \in P^{in}$ $\{\phi_i\}$ and $\{\psi_j\}$ can be differentiated an infinite number of times under the integral sign. Furthermore, since the function $G(\cdot, \cdot)$ is symmetric and harmonic, it follows that $\{\phi_i\}$, $\{\psi_j\}$ are also harmonic functions. Finally, let us prove property (iv) in the case of $d = 2$, that is, the mapping $\eta \mapsto F(\eta; P')$ is pure conformal. Note that the proof shows that this mapping is holomorphic and does not guarantees that the Jacobian does not degenerate. However, in practice we have noticed degeneracies are rather rare and happen mainly when the cage is drastically deformed.

Theorem 1. *For $d = 2$ the deformation $\eta \mapsto F(\eta; P')$ defined by* (3)*, with the coordinates defined in* (8)*, is conformal in P^{in} for all P'.*

Proof. For the proof, assume the vertices $v_1, v_2, \ldots$ of the cage are ordered in a clockwise manner and denote $t_j = v_{j+1} - v_j$. Let us introduce the linear operator $\perp\colon \mathbb{R}^2 \to \mathbb{R}^2$ which will stand for counter-clockwise rotation of $\pi/2$ radians. Using this symbol, the deformation in 2D can be written as:

$$\eta \mapsto F(\eta; P') = \sum_{i \in I_{\mathbb{V}}} \phi_i(\eta) v_i' + \sum_{j \in I_{\mathbb{T}}} \psi_j(\eta)(t_j')^{\perp}.$$

We begin with three simple lemmas which form the basis of the proof.

Lemma 1. *Let u be a harmonic function defined in an open domain $D \subset \mathbb{R}^2$, then $f = u_y + iu_x$ is holomorphic.*

Proof. Directly from Cauchy-Riemann equations we obtain

$$\begin{aligned}(u_y)_x &= (u_x)_y,\\ (u_x)_x &= -(u_y)_y,\end{aligned}$$

where the first equality is due to the fact that partial derivatives of smooth functions commute. The second equality is due to the fact that u is harmonic. ■

Lemma 2. *Let $v \in \mathbb{C}$ be an arbitrary complex point. Then, if the map $h(z)+ir(z)$ is holomorphic then the map $ivh(z) - vr(z)$ is also holomorphic.*

Proof. The proof is immediate by multiplying $h + ir$ by iv. ■

An immediate corollary is the following result.

Corollary 1. *Let $v \in \mathbb{R}^2$ and let $h(x, y)$ and $r(x, y)$ be conjugate harmonic functions in $D \subset \mathbb{R}^2$. Then the mapping $f\colon D \mapsto \mathbb{R}^2$ defined by*

$$f(x, y) = v^{\perp} h(x, y) - vr(x, y)$$

is conformal.

Lemma 3. *Let $v_i \in \mathbb{V}$ be an arbitrary vertex of P. Denote by t_{i-1} and t_i the faces (edges in this case) $\overrightarrow{v_{i-1}v_i}$ and $\overrightarrow{v_i v_{i+1}}$, respectively. Then ϕ_i and $\psi_i - \psi_{i-1}$ are conjugate harmonic. In other words*

$$(\psi_i - \psi_{i-1}) + i\phi_i,$$

is holomorphic.

Before laying out the proof of this lemma, let us show that it implies that the map $\eta \mapsto F(\eta; P')$ is conformal (holomorphic). It is enough to consider two cages P', P'' which differ in only one vertex v_i. Then successive application of the following argument will constitute the proof. So, let P', P'' be such cages. Then, if t_{i-1} and t_i are the edges previous and following v_i, then

$$F(\eta; P'') - F(\eta; P') = \phi_i(\eta)(v_i'' - v_i') + \sum_{j=i-1,i} \psi_j(\eta)(t_j''^{\perp} - t_j'^{\perp}).$$

Next we note that since $\perp$ is a linear operator we get

$$\sum_{j=i-1,i} \psi_j(\eta)(t_j''^{\perp} - t_j'^{\perp}) = (v_i'' - v_i')^{\perp} \left(\psi_{i-1}(\eta) - \psi_i(\eta)\right).$$

Thus we have

$$(11)\quad H(\eta) \equiv F(\eta; P'') - F(\eta; P') = (v_i'' - v_i')\phi_i(\eta) + (v_i'' - v_i')^{\perp} \left(\psi_{i-1}(\eta) - \psi_i(\eta)\right).$$

Therefore, from Corollary 1 and Lemma 3 $H(\eta)$ is holomorphic. ■

Proof of Lemma 3. Denote by T the triangle with vertices $\overrightarrow{v_{i-1}, v_i, v_{i+1}}$, and denote by e the edge $\overrightarrow{v_{i+1}v_{i-1}}$, see Figure 1. First, let us assume that $\eta \notin T$.

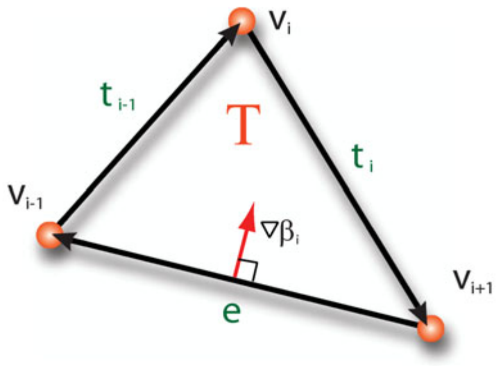

FIGURE 1. Illustration for the proof.

Denoting β_i to be the linear function over the triangle T, having the value of one at vertex v_i and the value zero at v_{i-1} and v_{i+1}, we note that

$$\phi_i(\eta) = \int_{t_{i-1}} \beta_i \frac{\partial G}{\partial n}\, d\sigma + \int_{t_i} \beta_i \frac{\partial G}{\partial n}\, d\sigma + \int_{e} \beta_i \frac{\partial G}{\partial n}\, d\sigma,$$

where

$$\int_e \beta_i \frac{\partial G}{\partial n}\, d\sigma = 0$$

since β_i is zero on e. Hence, using Green's first identity we get

$$\phi_i(\eta) = \int_{t_{i-1} \bigcup t_i \bigcup e} \beta_i \frac{\partial G}{\partial n}\, d\sigma = \int_T \left(\beta_i \Delta G + (\nabla \beta_i \cdot \nabla G)\right) dV.$$

Now since $\eta \notin T$, G is harmonic in T and therefore we get

$$\phi_i(\eta) = \int_{t_{i-1} \bigcup t_i \bigcup e} \beta_i \frac{\partial G}{\partial n}\, d\sigma = \int_T (\nabla_\xi \beta_i(\xi) \cdot \nabla_\xi G(\eta, \xi))\, dV. \tag{12}$$

We also claim that

$$\psi_{i-1}(\eta) = -\frac{1}{|t_{i-1}|} \int_{t_{i-1}} G\, d\sigma = -\int_{t_{i-1}} G \nabla \beta_i \cdot\, d\vec{\sigma},$$

where $d\vec{\sigma}$ is the line integral element. Indeed, since $\nabla\beta_i = -e^\perp/(2\,\mathrm{area}\{T\})$ pointing inside triangle T (note that for concave setting $\nabla\beta_i = e^\perp/(2\,\mathrm{area}\{T\})$ is also pointing inside triangle T),

$$\nabla\beta_i \cdot\, d\vec{\sigma} = \frac{-e^\perp}{2\,\mathrm{area}\{T\}} \cdot \frac{\dot{\sigma}}{|\dot{\sigma}|}\, d\sigma = \frac{|e| \sin \measuredangle(v_{i+1} v_{i-1} v_i)}{2\,\mathrm{area}\{T\}}\, d\sigma = \frac{1}{|t_{i-1}|}\, d\sigma.$$

Similarly we have

$$\psi_i(\eta) = -\frac{1}{|t_i|} \int_{t_i} G\, d\sigma = \int_{t_i} G \nabla \beta_i \cdot\, d\vec{\sigma}.$$

Note that the different sign is due to the fact that the direction of the vector t_{i-1} agrees with the direction of $\nabla\beta_i$ while the direction of the vector t_i is opposite (see Figure 1). Since $\nabla\beta_i \cdot d\vec{\sigma} = 0$ on e, we can write

$$\psi_{i-1}(\eta) - \psi_i(\eta) = -\int_{t_{i-1} \bigcup t_i \bigcup e} G \nabla\beta_i \cdot\, d\vec{\sigma}.$$

Next, let us write Green's Theorem in our notation, that is, for a vector field $Q(\eta)$ there exists

$$\int_{t_{i-1} \bigcup t_i \bigcup e} Q \cdot d\vec{\sigma} = \int_T \nabla \cdot (Q^\perp)\, dV.$$

Taking $Q = G\nabla\beta_i$ and noting that

$$\nabla \cdot (G\nabla\beta_i)^\perp = \nabla \cdot (G(\nabla\beta_i)^\perp) = \nabla G \cdot (\nabla\beta_i)^\perp,$$

we get

$$(\psi_i - \psi_{i-1})(\eta) = \int_T \nabla_\xi G(\eta, \xi) \cdot (\nabla_\xi \beta_i(\xi))^\perp\, dV. \tag{13}$$

We note that due to the symmetry of G

$$\nabla_\xi G(\xi, \eta) = \nabla_\eta G(\eta, \xi).$$

Now, since $\nabla\beta_i$ and $(\nabla\beta_i)^\perp$ are constant, orthogonal, positive oriented vectors, and due to the rotation invariance of the Laplace operators and Lemma 1 we have that for each fixed ξ

$$\nabla_\eta G(\eta, \xi) \cdot \nabla\beta_i, \qquad \nabla_\eta G(\eta, \xi) \cdot (\nabla\beta_i)^\perp$$

are conjugate harmonic. By integrating we get that identities (12) and (13) represent conjugate harmonic functions. Thus, we get that $\psi_i - \psi_{i-1}$ and ϕ_i define a holomorphic map.

In the case $\eta \in T$, let us add the point $w = (\eta + v_i)/2$ and denote the vectors $e_0 = \overrightarrow{wv_{i-1}}$, $e_1 = \overrightarrow{v_{i+1}w}$ and $e = \overrightarrow{v_i\eta}$. We also denote the two new triangles $T_0 = \overrightarrow{v_{i-1}v_iw}$ and $T_1 = \overrightarrow{v_iv_{i+1}w}$, see Figure 2.

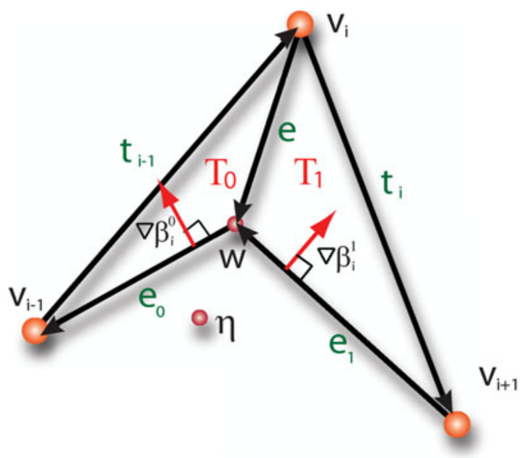

FIGURE 2. Illustration for the proof.

Furthermore, we define the functions β_i^0 and β_i^1 to be the linear functions which coincide with β_i on t_{i-1} and t_i, respectively, and are zero on e_0 and e_1, respectively. We note that

$$\phi_i = \int_{t_{i-1}\bigcup e\bigcup e_0} \beta_i^0 \frac{\partial G}{\partial n}\, d\sigma + \int_{t_i\bigcup e_1\bigcup -e} \beta_i^1 \frac{\partial G}{\partial n}\, d\sigma,$$

based upon the facts that the integral on e equals zero since $\partial G/\partial n = 0$ over e, and the integrals on e_0, e_1 equal zero since the corresponding functions β_i^0, β_i^1 vanish there. Next, using the Green's first identity on each of these closed integrals we get

$$\phi_i = \int_{T_0} \nabla\beta_i^0 \cdot \nabla G\, dV + \int_{T_1} \nabla\beta_i^1 \cdot \nabla G\, dV.$$

Similarly, we note that

$$\psi_i - \psi_{i-1} = \int_{t_{i-1}\bigcup e\bigcup e_0} G\nabla\beta_i^0 \cdot d\vec{\sigma} + \int_{t_i\bigcup e_1\bigcup -e} G\nabla\beta_i^1 \cdot d\vec{\sigma},$$

where we used the facts that the integrals on e and $-e$ cancel each other and the integrals on e_0 and e_1 vanish because $\nabla\beta_i^0 \cdot d\vec{\sigma} = 0$ on e_0 and $\nabla\beta_i^1 \cdot d\vec{\sigma} = 0$ on e_1, respectively. Then, using Green's Theorem again we get

$$\psi_i - \psi_{i-1} = \int_{T_0} \nabla G \cdot (\nabla\beta_i^0)^{\perp}\, dV + \int_{T_1} \nabla G \cdot (\nabla\beta_i^1)^{\perp}\, dV.$$

And we finish as above. ∎

3. Closed-form formulae for 2D and 3D

Interestingly, closed-form formulae can be derived for the dimensions $d = 2, 3$.

Throughout this section we fix η and calculate $\phi_i(\eta), i \in I_{\mathbb{V}}$ and $\psi_j(\eta), j \in I_{\mathbb{T}}$ in the relevant dimension.

3.1. The case $d = 2$. The derivation in this case is rather straightforward. Note that the Laplace fundamental solution in this case is

$$G(\xi, \eta) = \frac{-1}{2\pi} \log \|\xi - \eta\|$$

(see (5)). Let us first establish a formula for

$$\psi_j(\eta) = -\int_{\xi \in t_j} G(\xi, \eta)\, d\sigma.$$

Denote by $v_i, v_{i+1} \in \mathbb{V}$ the ordered two vertices which consist the edge t_j. Next, denote the vectors $a_i = v_{i+1} - v_i$ and $b_i = v_i - \eta$. Then, taking the parametrization $\gamma(t) = v_i + t a_i$, $t \in [0, 1]$ we get

$$\int_{\xi \in t_j} G(\xi, \eta)\, d\sigma = \frac{-1}{2\pi} \int_{t=0}^{1} \log \|b_i + t a_i\| \|a_i\|\, dt.$$

Therefore,

$$\psi_j(\eta) = \frac{\|a_i\|}{2\pi} \int_{t=0}^{1} \log(t^2 \|a_i\|^2 + 2t(a_i \cdot b_i) + \|b_i\|^2)\, dt,$$

and we use the relevant antiderivative:

$$\int^{T} \log(qt^2 + rt + s)\, dt = \log(qT^2 + rT + s) \left(T + \frac{r}{2q}\right) - \arctan\left(\frac{2qT + r}{\sqrt{4sq - r^2}}\right) \left(\frac{2q + r}{q\sqrt{4sq - r^2}}\right).$$

Next, for $\phi_i(\eta)$ denote by t_{j-1}, t_j the edges which are adjacent to vertex v_i, that is, t_{j-1} is the edge between v_{i-1} and v_i and t_j is the edge between v_i and v_{i+1}. Then,

$$\phi_i(\eta) = \sum_{k=j-1,j} \int_{\xi \in t_k} \Gamma_i(\xi) \frac{\partial G(\xi, \eta)}{\partial n}\, d\sigma.$$

For t_{j-1} we use the parametrization

$$\gamma(t) = a_{i-1} t + v_{i-1}, \qquad t \in [0, 1],$$

and get

$$\int_0^1 t\left(-\frac{a_{i-1}t+b_{i-1}}{2\pi\|a_{i-1}t+b_{i-1}\|^2}\cdot n(a_{i-1})\right)\|a_{i-1}\|\,dt$$
$$=\frac{-(b_{i-1}\cdot a_{i-1}^{\perp})}{2\pi}\int_0^1 \frac{tdt}{\|a_{i-1}\|^2t^2+2t(a_{i-1}\cdot b_{i-1})+\|b_{i-1}\|^2}.$$

For t_j we use the parametrization $\gamma(t)=a_it+v_i$, $t\in[0,1]$, and get

$$\int_0^1 (1-t)\left(-\frac{a_it+b_i}{2\pi\|a_it+b_i\|^2}\cdot n(a_i)\right)\|a_i\|\,dt$$
$$=\frac{-(b_i\cdot a_i^{\perp})}{2\pi}\int_0^1 \frac{(1-t)dt}{\|a_i\|^2t^2+2t(a_i\cdot b_i)+\|b_i\|^2}.$$

The relevant antiderivatives are:

$$\int^T \frac{t-1}{qt^2+rt+s}\,dt = \frac{1}{2q}\log(qT^2+rT+s)$$
$$-\arctan\left(\frac{2qT+r}{\sqrt{4sq-r^2}}\right)\left(\frac{2q+r}{q\sqrt{4sq-r^2}}\right),$$
$$\int^T \frac{t}{qt^2+rt+s}\,dt = \frac{1}{2q}\log(qT^2+rT+s)$$
$$-\arctan\left(\frac{2qT+r}{\sqrt{4sq-r^2}}\right)\left(\frac{r}{q\sqrt{4sq-r^2}}\right).$$

All the above is combined to yield an algorithm for calculating the coordinates $\phi_i(\eta),\psi_j(\eta)$ in 2D as given in Algorithm A.1 (see Appendix).

3.2. The case $d=3$. First, we establish the formulae for computing the

$$\psi_j(\eta) = -\int_{\xi\in t_j} G(\xi,\eta)\,d\sigma,$$

where $G(\xi,\eta)=-1/4\pi\|\xi-\eta\|$. Denote by v_i,v_{i+1},v_{i+2} the order set of vertices consisting the face t_j, and let p be the projection of the point η onto the plane defined by the face t_j. Then,

$$\|\xi-\eta\| = \sqrt{\|\eta-p\|^2+\|p-\xi\|^2}.$$

Since $\|\eta-p\|^2$ is a constant, in the integral we denote it by $c>0$. First, let us establish a formula for calculating the above integral over the triangle $\triangle_1$ with vertices (p,v_i,v_{i+1}). Denote the angles of $\triangle_1$ by $\alpha=\measuredangle(pv_iv_{i+1})$ and $\beta=\measuredangle(v_{i+1}pv_i)$. Using polar coordinates on the plane defined by t_j, with origin at p we arrive at

$$\begin{aligned}\int_{\xi\in\triangle_1} G(\xi,\eta)\,d\sigma &= \frac{-1}{4\pi}\int_{\xi\in\triangle_1}\frac{1}{\sqrt{c+\|p-\xi\|^2}}\,d\sigma\\ &= \frac{-1}{4\pi}\int_{\theta=0}^{\beta}\int_{r=0}^{R(\theta)}\frac{r}{\sqrt{c+r^2}}\,dr\,d\theta\\ &= \frac{-1}{4\pi}\int_{\theta=0}^{\beta}\left(\sqrt{c+R(\theta)^2}-\sqrt{c}\beta\right)\,d\theta,\end{aligned}$$

where from the law of sines

$$R(\theta) = \frac{\|\overrightarrow{pv_i}\|\sin(\alpha)}{\sin(\pi-\alpha-\theta)}.$$

Denote $\lambda = \|\overrightarrow{pv_i}\|^2\sin^2(\alpha)$ and $\delta = \pi-\alpha$. By translating the parameter θ we get

$$\int_{\theta=0}^{\beta}\sqrt{c+R(\theta)^2}\,d\theta = \int_{\varphi=\delta-\beta}^{\delta}\sqrt{c+\frac{\lambda}{\sin^2(\varphi)}}\,d\varphi.$$

The relevant antiderivative is

$$\int^{T}\sqrt{c+\frac{a}{\sin^2(t)}}\,dt = Q(a,c,\sin(T),\cos(T)),$$

where

$$\begin{aligned}Q(a,c,S,C) = \frac{-\operatorname{sign}(S)}{2}&\Bigg[2\sqrt{c}\ \arctan\left(\frac{\sqrt{c}C}{\sqrt{a+cS^2}}\right)\\ &+\sqrt{a}\log\left(\frac{2\sqrt{a}S^2}{(1-C)^2}\left(1-\frac{2cC}{c(1+C)+a+\sqrt{a^2+acS^2}}\right)\right)\Bigg].\end{aligned}$$

So at this point we know how to calculate the integral $\int_{\xi\in\triangle_1} G(\xi,\eta)\,d\sigma$. Clearly, we can use this formula also for $\triangle_2$ which is the triangle defined by the points (p, v_{i+1}, v_{i+2}) and $\triangle_3$ which is the triangle defined by (p, v_{i+2}, v_i). Therefore, we can calculate

$$\int_{\xi\in t_j} G(\xi,\eta)\,d\sigma = \sum_{i=1}^{3}\operatorname{sign}(\triangle_i)\int_{\xi\in\triangle_i} G(\xi,\eta)\,d\sigma,$$

where $\operatorname{sign}(\triangle_i)$ is the orientation sign of the triplet of vertices consisting triangle $\triangle_i$.

At this point we have closed formulae for calculating $\psi_j(\eta)$. Let us use these to derive formulae for $\phi_i(\eta)$. Denote by Υ the tetrahedron defined by the points $\eta, v_i, v_{i+1}, v_{i+2}$, and let $\triangle_1, \triangle_2, \triangle_3$ be the triangles defined by the points

(η, v_i, v_{i+1}), (η, v_{i+1}, v_{i+2}), (η, v_{i+2}, v_i), respectively. Using Green's third identity for the domain is Υ we get

$$\rho\eta = \int_{\partial\Upsilon} \xi \frac{\partial G}{\partial n}\, d\sigma - \int_{\partial\Upsilon} Gn\, d\sigma,$$

where ρ is some constant. To simplify things we translate η to the origin and hence the left-hand side of the equality is zero. Next, note that $\frac{\partial G}{\partial n} = 0$ on the triangles $\triangle_1, \triangle_2, \triangle_3$. Therefore, we get

$$\int_{t_j} \xi \frac{\partial G}{\partial n}\, d\sigma = \sum_{i=1}^{3} n_i \int_{\triangle_i} G\, d\sigma + n(t_j) \int_{t_j} G\, d\sigma,$$

where n_i is the outward normal vector to $\triangle_i$. Now, the right hand side can be easily calculated with the above formulae, and the left hand side equals

$$\int_{t_j} \xi \frac{\partial G}{\partial n}\, d\sigma = v_i \int_{t_j} \Gamma_i(\xi) \frac{\partial G}{\partial n}\, d\sigma + v_{i+1} \int_{t_j} \Gamma_{i+1}(\xi) \frac{\partial G}{\partial n}\, d\sigma + v_{i+2} \int_{t_j} \Gamma_{i+2}(\xi) \frac{\partial G}{\partial n}\, d\sigma.$$

In the case v_i, v_{i+1}, v_{i+2} are not co-planar we have

$$\int_{t_j} \Gamma_{i+k}(\xi) \frac{\partial G}{\partial n}\, d\sigma = \frac{n_{k+2} \cdot \left(\int_{t_j} \xi \frac{\partial G}{\partial n}\, d\sigma \right)}{n_{k+2} \cdot v_{i+k}}, \qquad k = 0, 1, 2.$$

In the case v_i, v_{i+1}, v_{i+2} are co-planar we have that $\partial G / \partial n = 0$ on t_j and therefore

$$\int_{t_j} \Gamma_{i+k}(\xi) \frac{\partial G}{\partial n}\, d\sigma = 0, \qquad k = 0, 1, 2.$$

This is combined into Algorithm A.2 (see Appendix) for calculating the coordinates $\phi_i(\eta), \psi_j(\eta)$ in 3D.

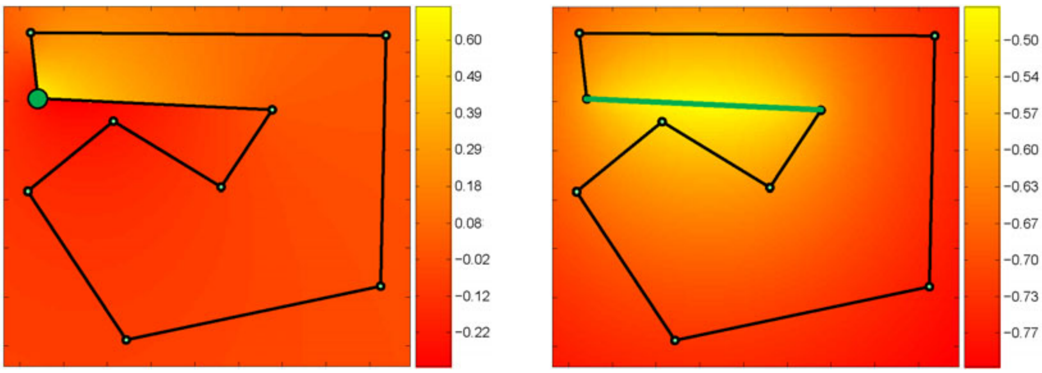

FIGURE 3. An illustration of the values of ϕ_i (left) for one vertex (marked in bold green point), and ψ_j (right) for one edge (marked in bold green line) in 2D.

4. Extending to the cage's exterior

The Green Coordinates defined by (3) and (8) are smooth in the interior of the cage P. However, each coordinate $\phi_i(\eta)$ has jump discontinuities along the edges (simplicial faces) meeting at v_i, see Figure 3. A natural question is whether the coordinates can be smoothly extended to the exterior of P. In 2D the Green Coordinates induce conformal transformations of the interior of P, and the above question addresses the analytic continuation of these conformal transformations through the boundaries of P.

In this section we derive the analytic continuation of the coordinates outside the cage, and show that it requires only a rather slight modification to the closed-form formulae at hand. Let us remark that the use of the term *analytic continuation* is twofold: In case $d = 2$ we refer to the classical meaning of extending the conformal (or analytic) complex maps. While in the case $d \geq 3$ we mean (real) analytic extension of harmonic functions (the coordinate functions ϕ_i, ψ_j are harmonic functions).

4.1. Extension through a face. Let us describe how the coordinate should be extended through some face $t_\ell \in \mathbb{T}$, $\ell \in I_{\mathbb{T}}$ of the cage, i.e. as η is moving outside the cage through that face. Let $i_1, \ldots, i_d \in$ be the indices of the vertices which consist the face t_ℓ. First, we note that Theorem 1 implies that the mapping $\eta \mapsto F(\eta; P')$ is conformal also for η outside the cage, which we denote by $\eta \in P^{ext}$. However, outside the cage we loose the important linear reproduction property (property (i), Section 2). In particular we have $F(\eta; P) = 0$ which is shown in the following lemma.

Lemma 4. *For $\eta \in P^{ext}$ there exists;*

$$\sum_{i \in I_{\mathbb{V}}} \phi_i(\eta) v_i + \sum_{j \in I_{\mathbb{T}}} \psi_j(\eta) n(t_j) = 0, \tag{14}$$

$$\sum_{i \in I_{\mathbb{V}}} \phi_i(\eta) = 0. \tag{15}$$

Proof. From the arguments given in Section 2, we have that

$$\sum_{i \in I_{\mathbb{V}}} \phi_i(\eta) v_i + \sum_{j \in I_{\mathbb{T}}} \psi_j(\eta) n(t_j) = \int_{\partial P} \left(u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) d\sigma,$$

where ∂P is the cage (piecewise linear) surface, $u(\xi) = \xi$, and the singularity η is exterior to the cage. Furthermore, Green's second identity implies that for harmonic u and G

$$\int_{\partial P} \left(u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) d\sigma = \int_{P^{in}} (u \Delta G - G \Delta u) \; dV = 0.$$

Hence the first statement follows. For the second statement, translate the origin by a constant vector $-e_1 = (-1, 0, \ldots, 0)^t \in \mathbb{R}^d$. Then from the above,

$$\sum_{i \in I_{\mathbb{V}}} \phi_i(\eta + e_1)(v_i + e_1) + \sum_{j \in I_{\mathbb{T}}} \psi_j(\eta + e_1) n(t_j) = 0.$$

Furthermore, we note that $\psi_j(\eta + e_1)$ and $\phi_i(\eta + e_1)$ based on the cage $P + e_1$ is equal to $\psi_j(\eta)$ and $\phi_i(\eta)$ based on the cage P. Therefore, subtracting the latter equality from the above equality implies the second statement. ■

Another point is that the coefficients $\phi_i(\cdot)$ are not continuous over the faces t_j of the cage. These observations prevent the use of ϕ, ψ, as defined in (8), outside the cage. In order to extend the coordinates smoothly to the exterior we take the following path. We note that from properties (i) and (ii) listed in Section 2, the coordinates $\phi_{i_1}(\eta), \ldots, \phi_{i_d}(\eta), \psi_\ell(\eta)$ where $\eta \in P^{in}$ satisfy

$$\text{(16a)} \quad \eta - \sum_{i \neq i_1, \ldots, i_d} \phi_i(\eta) v_i - \sum_{j \neq \ell} \psi_j(\eta) n(t_j) = \sum_{k=1}^{d} \phi_{i_k}(\eta) v_{i_k} + \psi_\ell(\eta) n(t_e),$$

$$\text{(16b)} \quad 1 - \sum_{i \neq i_1, \ldots, i_d} \phi_i(\eta) = \sum_{k=1}^{d} \phi_{i_k}(\eta).$$

This yields a linear system for the coefficients $\phi_{i_k}(\eta), k = 1, \ldots, d$, and $\psi_\ell(\eta)$. If the system is invertible then these "coordinates" are uniquely defined by all the other coordinates via the linear system. Let us prove that this system is invertible.

Lemma 5. *The linear system* (16) *for the coefficients* $\phi_{i_k}(\eta), k = 1, \ldots, d$, *and* $\psi_\ell(\eta)$ *is non-singular.*

Proof. Assume there exists a non-zero vector $w = (w_1, \ldots, w_{d+1})$ in the kernel of the system. From (16b) we have that

$$\text{(17)} \quad \sum_{k=1}^{d} w_k = 0.$$

From equation (16a) we have that

$$0 = \sum_{k=1}^{d} w_k v_{i_k} + w_{d+1} n(t_\ell) = \sum_{k \geq 2}^{d} w_k (v_{j_k} - v_{j_1}) + w_{d+1} n(t_\ell),$$

using (17). Now, noting that the vectors $v_{j_k} - v_{j_1}$, $k = 2, \ldots, d$, and $n(t_\ell)$ are independent the lemma follows. ■

By the above lemma we have that solving the system (16) for $\eta \in P^{in}$ reproduce the coordinates $\phi_{i_k}(\eta), k = 1, \ldots, d$, and $\psi_\ell(\eta)$. Therefore, it is natural to extend the coordinates crossing face t_ℓ by keeping the original definition for all the coordinates except $\phi_{i_k}(\eta), k = 1, \ldots, d$, and $\psi_\ell(\eta)$ and define the latter coordinates by the system of linear equations (16). In order to distinguish the newly defined coordinates outside the cage from the original ones (which are also defined everywhere on the plane) we denote the new ones with $\tilde{*}$. Note that $\tilde{\phi}_i(\eta) = \phi_i(\eta)$ and $\tilde{\psi}_j(\eta) = \psi_j(\eta)$ inside the cage. It is possible to simplify the system (16) as follows. By Lemma 4 we have that for $\eta \in P^{ext}$

$$\sum_{i \in I_{\mathbb{V}}} \phi_i(\eta) v_i + \sum_{j \in I_{\mathbb{T}}} \psi_j(\eta) n(t_j) = 0,$$

and $\sum_{i \in I_{\mathbb{V}}} \phi_i(\eta) = 0$. Plugging these into the equations (16a) and (16b) respectively, results in:

$$\eta = \sum_{k=1}^{d} \alpha_k v_{i_k} + \beta n(t_\ell) \tag{18a}$$

$$1 = \sum_{k=1}^{d} \alpha_k, \tag{18b}$$

where $\alpha_k = \tilde{\phi}_{i_k}(\eta) - \phi_{i_k}(\eta)$ and $\beta = \tilde{\psi}_\ell(\eta) - \psi_\ell(\eta)$ $\eta \in P^{ext}$. Furthermore, for a point η on the exact boundary of P we get the same equations where the right hand sides are multiplied by 1/2. We finally define the new coordinates $\tilde{\phi}_{i_k}(\eta), k = 1, \ldots, d$, and $\tilde{\psi}_\ell(\eta)$ for $\eta \in P^{ext}$ by

$$\tilde{\phi}_{i_k}(\eta) = \phi_{i_k}(\eta) + \alpha_k, \qquad k = 1, \ldots, d \tag{19a}$$

$$\tilde{\psi}_\ell(\eta) = \psi_\ell(\eta) + \beta. \tag{19b}$$

It is interesting to note that the system (18) has the following simple characterization of the solution α_k, β: from the second equation we see that $\sum_k \alpha_k v_{i_k}$ is an affine sum of the vertices which constitute the face t_ℓ. Therefore, the first equation represent the orthogonal decomposition of the point η to the sum of a point on the hyperplane defined by the face t_ℓ and the normal offset. Another observation is that (18) defines $\{\alpha_k\}$ and β as the unique affine coordinates of the point η in the *simplex* defined by the vertices $\{v_{i_k}\}$ of the face t_ℓ plus the vertex $v_{i_1} + n(t_\ell)$: $\eta = L_\ell(\eta; P)$ where

$$L_\ell(\eta; P) = (\alpha_1 - \beta) v_{i_1} + \sum_{k=2}^{d} \alpha_k v_{i_k} + \beta \left(v_{i_1} + n(t_\ell) \right). \tag{20}$$

Altogether, the deformation outside the cage has the form

$$\begin{aligned} \tilde{F}(\eta; P') &= \sum_{i \in} \phi_i(\eta) v'_i + \sum_{j \in I_{\mathbb{T}}} \psi_j(\eta) s_j n(t'_j) + \sum_{k=1}^{d} \alpha_k v'_{i_k} + \beta s_\ell n(t'_\ell) \\ &= F(\eta; P') + L_\ell(\eta; P'). \end{aligned} \tag{21}$$

4.1.1. Properties in the case $d = 2$. A special case is $d = 2$ where the α_k, β can be written as follows:

$$\begin{aligned}\alpha_1 &= 1 - \alpha_2 \\ \alpha_2 &= \frac{(\eta - v_{i_1}) \cdot (v_{i_2} - v_{i_1})}{\|v_{i_2} - v_{i_1}\|^2} \\ \beta &= \frac{(\eta - v_{i_1}) \cdot n(t_\ell)}{\|v_{i_2} - v_{i_1}\|}.\end{aligned}$$

Plugging this into (21) we get

$$\tilde{F}(\eta; P') = \sum_{i\in} \phi_i(\eta) v'_i + \sum_{j \in I_{\mathbb{T}}} \psi_j(\eta) s_j n(t'_j) \\ + v'_{i_1} + \alpha_2(v'_{i_2} - v'_{i_1}) + \beta s_\ell n(t'_\ell). \tag{22}$$

By Theorem 1 we see that the sum $\sum_{i\in} \phi_i(\eta) v'_i + \sum_{j\in I_{\mathbb{T}}} \psi_j(\eta) s_j n(t'_j)$ represents a conformal mapping also for $\eta \in P^{ext}$. The new addition here is the function

$$L_\ell(\eta; P') = v'_{i_1} + \alpha_2(v'_{i_2} - v'_{i_1}) + \beta s_\ell n(t'_\ell).$$

Lemma 6. *$L_\ell(\eta; P')$ for $\eta \in P^{ext}$ is the unique linear conformal mapping taking the edge $\overrightarrow{v_{i_1} v_{i_2}}$ to the edge $\overrightarrow{v'_{i_1} v'_{i_2}}$.*

Proof. By substituting $\eta = v_{i_1}$, $\eta = v_{i_2}$ in $L_\ell(\eta; P')$ we get that $L_\ell(v_{i_1}; P') = v'_{i_1}$ and $L_\ell(v_{i_2}; P') = v'_{i_2}$, respectively. Also, we can write $L_\ell(\cdot; P')$ in the following form:

$$L_\ell(\eta; P') = v'_{i_1} + \frac{\|v'_{i_2} - v'_{i_1}\|}{\|v_{i_2} - v_{i_1}\|} \Bigg(\frac{(\eta - v_{i_1}) \cdot (v_{i_2} - v_{i_1})}{\|v_{i_2} - v_{i_1}\|} \frac{v'_{i_2} - v'_{i_1}}{\|v'_{i_2} - v'_{i_1}\|} \\ + (\eta - v_{i_1}) \cdot n(t_\ell) n(t_\ell) \Bigg). \tag{23}$$

And this shows that L_ℓ is conformal. The uniqueness is obvious from counting the degrees of freedom of 2D linear conformal mapping. ∎

Next, we can now prove that we have actually accomplished an analytic continuation of the mapping F through the face (edge) t_ℓ.

Theorem 2. *In the case $d = 2$, fixing an edge t_ℓ and defining the coordinates $\tilde{\phi}_{i_k}(\eta)$, $k = 1, 2$, and $\tilde{\psi}_\ell(\eta)$ by (16), we get that for $\eta \in P^{ext}$, $F(\eta; P') + L_\ell(\eta; P')$ is the unique analytic continuation of the conformal mapping $F(\eta; P')$ through the edge t_ℓ.*

Proof. We see from equation (22) and Lemma 6 that for $\eta \in P^{ext}$ the mapping $\eta \mapsto \tilde{F}(\eta; P')$ is conformal. Furthermore, from the linear system (16) and Lemma 5 we see that $\tilde{F}$ is continuous through face t_ℓ, that is $\tilde{F}(\eta; P') = F(\eta; P')$

for $\eta \in t_\ell$. By Schwarz Theorem in complex analysis we have that two conformal mappings continuous on a common line are analytic continuations of each other. The uniqueness of analytic continuation is due to the fact that an analytic function which is zero on an open set is everywhere zero. ■

Maximal region of conformality. An important question is what is the maximal region of conformality and do we have control on the location of singularities? We show two results: first, that for general P' one cannot expect an analytic continuation of the coordinates to the whole embedding space. That is, there is no *entire* function $\bar{F}$ such that $\bar{F}(\eta; P') = F(\eta; P')$ for $\eta \in P^{in}$ for general P'. However, and this is some remedy, it is possible to place the singularities in a rather flexible manner, as shown in the following theorem. Note that the following theorem is only for the case $d = 2$ but a similar result can be readily proven for $d > 2$.

Theorem 3.

(i) *There is no entire function $\bar{F}$ such that $\bar{F}(\eta; P') = F(\eta; P')$ for $\eta \in P^{in}$ for general P'.*
(ii) *Let P^{ext} be subdivided into disjoint domains O_k, $k \in K$, $P^{ext} = \bigcup_{k \in K} \bar{O}_k$ ($\bar{O}_k$ is the closure of O_k), such that for every $j \in I_{\mathbb{T}}$, t_j is contained in some $\bar{O}_k$, that is $t_j \subset \bar{O}_k$. Assuming for each $k \in K$ one extends F to O_k through a specific face $t_k \in O_k$. Then $\tilde{F}$ is analytic in $\bigcup_{k \in K} O_k$ except in all the faces $t_j \in O_k$ which do not satisfy $t'_j = L_k(t_j; P')$.*

Proof. For (i) assume in negation that there exists such continuation $\bar{F}$. By Theorem 2 we have that the unique continuation through edge t_j is

$$\bar{F}(\eta; P') = F(\eta; P') + L_j(\eta; P').$$

Now, since the function $\eta \mapsto F(\eta; P')$ is also conformal everywhere outside the cage, that is, for $\eta \in P^{ext}$, and since $L_j(\cdot; P')$, $j \in I_{\mathbb{T}}$ are entire functions, it follows by the uniqueness of analytic continuation that

$$L_1(\cdot; P') \equiv L_2(\cdot; P') \equiv \ldots \equiv L(\cdot).$$

That is, all the linear conformal transformations $L_j(\cdot; P')$ coincide. This is obviously not true for a general P', which proves (i).

For (ii), we have by Theorem 2 that $\tilde{F}$ is analytic through all faces $t_k \in O_k$. Furthermore, the extension in O_k is $\tilde{F}(\eta; P') = F(\eta; P') + L_k(\eta; P')$. Therefore for any other $t_j \in O_k$ which satisfies $t'_j = L_k(t_j; P')$, by Lemma 6 and Theorem 2, we have that the extension in O_k is also analytic through t_j. ■

4.1.2. Properties in the case $d > 2$. In the case of higher dimension $d > 2$, we don't have conformality, and therefore the continuation is in the sense of real analyticity. A function $f(x)$ is called *real analytic* in some domain $\Omega \subset \mathbb{R}^d$ if for every $x_0 \in \Omega$ it can be expressed by a power series $f(x) = \sum_\nu c_\nu (x - x_0)^\nu$

in some neighborhood of x_0. Note that we are using the multi-index notation $\nu = (\nu_1, \ldots, \nu_d)$, $x^\nu = x_1^{\nu_1} \cdots x_d^{\nu_d}$. The reason real analyticity give rise to a unique extension in its domain of definition is the following lemma coming from the classical theory of real analytic functions [5].

Lemma 7. *Let f be a real analytic function defined over a connected domain Ω such that $f = 0$ on some open subset. Then $f = 0$ in Ω.*

In the following we will show that the extended coordinates $\tilde{\phi}_i, \tilde{\psi}_j$ are real analytic in their domain of definition. This will be accomplished by another classical result from harmonic function theory (for the proof see [5]).

Lemma 8. *If f is harmonic on domain Ω, then it is real analytic in Ω.*

Let us show next that the extended functions $\tilde{\phi}_i, \tilde{\psi}_j$ are harmonic in their domain of definition.

Theorem 4. *The extended coordinate functions $\tilde{\phi}_i, \tilde{\psi}_j$ through a face t_ℓ are harmonic in their domain of definition.*

Proof. As noted in Section 2, ϕ_i, ψ_j are harmonic functions in the interior of the cage. From the same reason they are harmonic also outside the cage. The coordinates $\tilde{\phi}_{i_k}, \tilde{\psi}_\ell$, $k = 1, \ldots, d$, coincide with $\phi_, \psi_j$ in the interior of the cage and are hence harmonic there. At the exterior of the cage it can be seen from equations (19) that $\tilde{\phi}_{i_k}, \tilde{\psi}_\ell$ for $k = 1, \ldots, d$, equals the corresponding ϕ_{i_k}, ψ_ℓ plus the terms $\alpha_k = \alpha_k(\eta)$ and $\beta = \beta(\eta)$ which in view of (18) are linear functions of the coordinates of η, hence are harmonic also outside the cage. Obviously all other $\tilde{\phi}_i, \tilde{\psi}_j$ equals ϕ_i, ψ_j correspondingly and also harmonic outside. Finally, we note that from definition (16) of $\tilde{\phi}_i, \tilde{\psi}_j$ and Lemma 5, plus the fact that the coefficients of the system (16a) are C^∞ functions, that these coordinates are also C^∞ functions. Therefore, by continuity from both sides of the face t_ℓ we get that the defined coordinates functions $\tilde{\phi}_{i_k}, \tilde{\psi}_\ell$, $k = 1, \ldots, d$, are harmonic also through the face t_ℓ. ∎

Combining the above we can prove the uniqueness of the proposed extension in dimensions $d > 2$.

Theorem 5. *Fixing a face t_ℓ and defining the coordinates $\tilde{\phi}_{i_k}(\eta), k = 1, \ldots, d$, and $\tilde{\psi}_\ell(\eta)$ by (16) results in the unique real analytic continuation of the harmonic coordinate functions ϕ_i, ψ_j through the face t_ℓ.*

Proof. From Theorem 4 we have that the extended coordinates $\tilde{\phi}_i, \tilde{\psi}_j$ are harmonic in their domain of definition. Lemma 8 implies that harmonic functions are real analytic and Lemma 7 implies the continuation is unique and therefore since $\tilde{\phi}_i, \tilde{\psi}_j$ and ϕ_i, ψ_j coincide in the interior of the cage we have that $\tilde{\phi}_i, \tilde{\psi}_j$ furnish the unique continuation. ∎

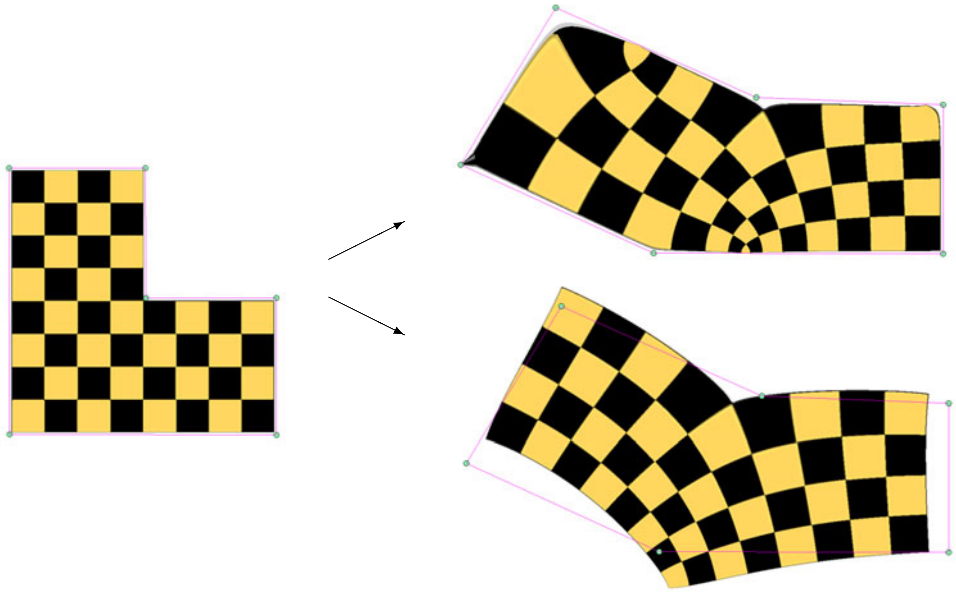

Figure 4. Comparison of Schwarz-Christoffel mapping (top) and Green coordinates mapping (bottom). Note that Green coordinates have lower distortion but are not interpolatory.

This paper presents several theoretical justifications to the paper by Lipman *et al.* [3]. In [3] the Green coordinates are used to create shape-preserving free-form space deformation. We believe that there exist more applications to this type of generalization of barycentric coordinates. As to open theoretical questions, we observed that in 3D the mapping F is near-conformal or quasi-conformal; proving some bound on the distortion would be interesting.

Figure 4 compares the conformal mappings created by the Green coordinates and the Schwarz-Christoffel formula [1]. We have employed Driscoll and Trefethen toolbox for computing the Schwarz-Christoffel mapping. Note that we have placed the conformal center of the mapping near the right lower vertex of the polygons P and P'. It is clear that Green coordinates have lower distortion than the Schwarz-Christoffel mapping, however it is not onto the image cage P'. An interesting question would be: how far is the image of F from P'. An initial result in this direction can be understood from formula (11). Assume that in a cage P the two edges t_{j-1}, t_j emanating from vertex v_j are of the same length. Further assume that the deformed cage P' is identical to P except for vertex v_j which is moved to a new position v_j'. Then, formula (11) states that a point η inside the cage P is mapped by the rule:

$$F(\eta; P') = \eta + (v_j' - v_j)\phi_j(\eta) + (v_j' - v_j)^{\perp}(\psi_{j-1}(\eta) - \psi_j(\eta)).$$

Now, we are interested understanding the image of the point $\eta = v_j$ under the mapping $F(\cdot; P')$. To that end, let us look at $\eta \to v_j$, where η is moving along the path of the angle bisector emanating at vertex v_j. Since η is on the bisector and t_{j-1} and t_j are of the same length we have that $\psi_{j-1}(\eta) = \psi_j(\eta)$. So we have

$$F(v_j; P') = v_j + \lim_{\eta \to v_j} \phi_j(\eta).$$

Using the closed form formulae from Section 3 it is possible to calculate this limit explicitly. Denote by 2κ the interior angle at vertex v_j, then

$$\lim_{\eta \to v_j} \phi_j(\eta) = \frac{\pi}{2} + \frac{1}{\pi} \arctan(|cot(\kappa)|).$$

Hence we see that $F(v_j; P') \to v_j'$ as $\kappa \to 0$, and for example, for $\kappa = \pi/4$, we see that $F(v_j; P') = v_j + 0.75(v_j' - v_j)$.

Appendix A. Algorithms

A.1. 2D Green coordinates algorithm.

Input: cage $P = (\mathbb{V}, \mathbb{T})$, set of points $\Lambda = \{\eta\}$
Output: 2D GC $\phi_i(\eta), \psi_j(\eta)$, $i \in I_{\mathbb{V}}$, $j \in I_{\mathbb{T}}$, $\eta \in \Lambda$

`/* Initialization */`

set all $\phi_i = 0$ and $\psi_j = 0$

`/* Coordinate computation */`

foreach *point* $\eta \in \Lambda$ **do**

  foreach *face* $j \in I_{\mathbb{T}}$ *with vertices* v_{j_1}, v_{j_2} **do**

    $a := v_{j_2} - v_{j_1}$; $b := v_{j_1} - \eta$

    $Q := a \cdot a$; $S := b \cdot b$; $R := 2a \cdot b$

    $BA := b \cdot \|a\| n(t_j)$; $SRT := \sqrt{4SQ - R^2}$

    $L0 := \log(S)$; $L1 := \log(S + Q + R)$

    $A0 := \dfrac{\arctan(R/SRT)}{SRT}$; $A1 := \dfrac{\arctan((2Q+R)/SRT)}{SRT}$

    $A10 := A1 - A0$; $L10 := L1 - L0$

    $$\psi_j(\eta) := -\frac{\|a\|}{4\pi}\left[\left(4S - \frac{R^2}{Q}\right) A10 + \frac{R}{2Q} L10 + L1 - 2\right]$$

    $$\phi_{j_2}(\eta) := \phi_{j_2}(\eta) - \frac{BA}{2\pi}\left[\frac{L10}{2Q} - A10 \frac{R}{Q}\right]$$

    $$\phi_{j_1}(\eta) := \phi_{j_1}(\eta) + \frac{BA}{2\pi}\left[\frac{L10}{2Q} - A10\left(2 + \frac{R}{Q}\right)\right]$$

  end

end

A.2. 3D Green coordinates algorithm.

```
Input: cage P = (V, T), set of points Λ = {η}
Output: 3D GC φ_i(η), ψ_j(η), i ∈ I_V, j ∈ I_T, η ∈ Λ
/* Initialization                                                   */
set all φ_i = 0 and ψ_j = 0

/* Coordinate computation                                           */
foreach point η ∈ Λ do
    foreach face j ∈ I_T with vertices v_{j1}, v_{j2}, v_{j3} do
        foreach ℓ = 1,2,3 do
            v_{jℓ} := v_{jℓ} − η
        p := (v_{j1} · n(t_j)) n(t_j)
        foreach ℓ = 1,2,3 do
            s_ℓ := sign(((v_{jℓ} − p) × (v_{jℓ+1} − p)) · n(t_j))
            I_ℓ := GCTriInt(p, v_{jℓ}, v_{jℓ+1}, 0);    II_ℓ := GCTriInt(0, v_{jℓ+1}, v_{jℓ}, 0)
            q_ℓ := v_{jℓ+1} × v_{jℓ};                    N_ℓ := q_ℓ / ||q_ℓ||
        I := −|Σ_{k=1}^{3} s_k I_k|;   ψ_j(η) := −I;   w := n(t_j) I + Σ_{k=1}^{3} N_k II_k
        if ||w|| > ε then
            foreach ℓ = 1,2,3 do
                φ_{jℓ}(η) := φ_{jℓ}(η) + (N_{ℓ+1} · w) / (N_{ℓ+1} · v_{jℓ})
    end
end
```

Procedure **GCTriInt**(p,v_1,v_2,η)

$$\alpha := \arccos\left(\frac{(v_2-v_1)\cdot(p-v_1)}{\|v_2-v_1\|\|p-v_1\|}\right); \quad \beta := \arccos\left(\frac{(v_1-p)\cdot(v_2-p)}{\|v_1-p\|\|v_2-p\|}\right)$$

$$\lambda := \|p-v_1\|^2\sin(\alpha)^2; \quad c := \|p-\eta\|^2$$

foreach $\theta = \pi-\alpha, \pi-\alpha-\beta$ **do**

$S := \sin(\theta)$; $C := \cos(\theta)$

$$I_\theta := \frac{-\operatorname{sign}(S)}{2}\left[2\sqrt{c}\arctan\left(\frac{\sqrt{c}C}{\sqrt{\lambda+S^2c}}\right) + \sqrt{\lambda}\log\left(\frac{2\sqrt{\lambda}S^2}{(1-C)^2}\left(1-\frac{2cC}{c(1+C)+\lambda+\sqrt{\lambda^2+\lambda cS^2}}\right)\right)\right]$$

return $\frac{-1}{4\pi}\left|I_{\pi-\alpha} - I_{\pi-\alpha-\beta} - \sqrt{c}\beta\right|$

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