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Piecewise L-splines of order 4: Interpolation and L^2 error bounds for splines in tension

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Abstract

Piecewise L-splines are generalizations of L-splines, in the sense that they satisfy different differential equations in different mesh intervals. Prenter attempted in [P.M. Prenter, Piecewise L-Splines, Numer. Math. 18 (2) (1971) 243–253] to obtain results on piecewise L-splines by generalizing the results of Schultz and Varga on L-splines in [M.H. Schultz, R.S. Varga, L-Splines, Numer. Math. 10 (1967) 345–369]. We show that the results of Prenter are erroneous, and provide correct ones for piecewise L-splines of order 4. We prove the existence and uniqueness of such interpolants and establish the first and second integral relations. In addition we obtain new L^2 error bounds for the special case of splines in tension with variable tension parameters.

Keywords: Exponential splines; Interpolation; Piecewise L-splines; Splines in tension; L^2 error bounds

1. Introduction

Fitting a smooth function to interpolate samples taken from a univariate function at selected points is a classical problem in numerical analysis, which is relevant to many applications. A popular interpolant is the *polynomial spline*, formed by joining polynomials of fixed degree m together at the interpolation points to obtain a C^{m-1} function. *Cubic splines* ($m = 3$) are used in many applications, due to their smoothness, low computational complexity and their

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energy minimizing property. Experiments show that the interpolatory cubic spline can exhibit undesirable oscillations between the interpolation points. Such artifacts can be eliminated by using the *spline in tension*, a C^2 interpolant, which is a linear combination of the functions $\{1, x, \sinh \rho x, \cosh \rho x\}$ in each mesh interval [2,8,9]. Here ρ, the *tension parameter*, is a nonnegative constant, possibly different in different mesh intervals. Increasing the tension parameter corresponding to a mesh interval tightens the spline in that interval, thus removing unnecessary oscillations there. Specifically, as ρ grows to infinity the spline in tension converges to the linear interpolant on the corresponding mesh interval, while as ρ tends to zero, it converges to a cubic polynomial there [9]. In this sense, splines in tension are generalizations of cubic splines. For sufficiently large tension parameters the spline in tension can mimic any monotonicity and convexity behavior that is present in the interpolated data [5]. Tension parameter selection algorithms are suggested in [2]. Splines in tension are powerful interpolants, because their smoothness and approximation orders equal those of cubic splines, while they can be controlled to a much larger extent.

Cubic splines and splines in tension with uniform tension (i.e., with equal tension parameters) are special cases of L-splines. A smooth interpolant $s(x)$ at the points $\{x_i\}_{i=0}^N$, $x_0 < x_1 < \cdots < x_N$ is called an L-*spline* of order $2n$ if it satisfies $L^*Ls = 0$ in each interval $[x_{i-1}, x_i]$, where

$$Lu(x) = \sum_{j=0}^{n} a_j(x) D^j u(x) \tag{1}$$

is a differential operator of degree n with $a_j \in C^n[x_0, x_N]$ for each j and $a_n(x) > 0$ on $[x_0, x_N]$, and L^* is its formal adjoint [1],

$$L^*v(x) = \sum_{j=0}^{n} (-1)^j D^j \{a_j(x)v(x)\}.$$

Important examples are the polynomial spline of degree $2n - 1$, corresponding to the differential operator $L = D^n$, and the spline in tension with uniform tension ρ, which is an L-spline of order 4 with $L = D^2 - \rho D$. Schultz and Varga obtained in [7] existence and uniqueness results for L-spline interpolants, as well as error bounds with respect to the L^∞ and the L^2 norms.

The family of L-splines is limited because it excludes splines satisfying different differential equations in different mesh intervals. For the investigation of such splines the concept of piecewise L-splines is required. A smooth interpolant $s(x)$ at the points $\{x_i\}_{i=0}^N$, $x_0 < x_1 < \cdots < x_N$ is called a *piecewise L-spline* of order $2n$ if it satisfies $L_i^* L_i s = 0$ in the interval $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, N$, where each L_i is an nth order differential operator of the form (1) defined on $[x_{i-1}, x_i]$. The spline in tension with non-uniform tension is an example of a piecewise L-spline which is not an L-spline. In [4] Prenter attempted to treat piecewise L-splines by generalizing the results in [7], but obtained incorrect results when summing up the formula (derived in [1])

$$\int_{x_{i-1}}^{x_i} \{vL_i u - uL_i^* v\} = P(u(x_i), v(x_i)) - P(u(x_{i-1}), v(x_{i-1})),$$

where $P(u, v) = \sum_{j=0}^{n-1} D^{n-j-1}u(x) \sum_{k=0}^{j} (-1)^k D^k \{a_{n-j+k}(x)v(x)\}$, and ignoring the fact that $P(u(x_i-), v(x_i-)) \neq P(u(x_i+), v(x_i+))$ for piecewise L-splines. A similar mistake was repeated in [3, equation (6.7)]. To the best of our knowledge these mistakes have not been detected in the literature.

In the first part of this paper we correct the erroneous results in [4] for a certain family of piecewise L-splines of order 4. We prove existence and uniqueness and obtain the first and second integral relations. These results are used in the second part of the paper to obtain L^2 error bounds for interpolation using splines in tension. Error bounds for derivatives are also obtained. Although L^2 approximation orders are derived in [4] for general piecewise L-splines, these bounds are not valid, since their proof is based on previous erroneous claims. Thus our L^2 error bounds are new for the case of splines in tension with non-uniform tension. We comment that the resulting L^2 approximation orders are identical to the known L^∞ ones [5].

2. Preliminaries

For a positive integer m let $K_m[a,b]$ denote the collection of all real-valued functions f defined on $[a,b]$ such that $f \in C^{m-1}[a,b]$, and such that $D^{m-1}f \equiv f^{(m-1)}$ is absolutely continuous on $[a,b]$, with $D^m f \in L^2[a,b]$. For a positive integer N let $\Delta : a = x_0 < x_1 < \cdots < x_N = b$ be a partition of $[a,b]$ with knots $\{x_i\}_{i=0}^N$. For each $i = 1, 2, \ldots, N$ let L_i be a normal differential operator (i.e. $L_i^* L_i = L_i L_i^*$) defined on $K_2[x_{i-1}, x_i]$ by

$$L_i u(x) = a_{i2}(x)u''(x) + a_{i1}(x)u'(x) + a_{i0}(x)u(x),$$

where $a_{i0}, a_{i1}, a_{i2} \in C^2[x_{i-1}, x_i]$ and $a_{i2}(x) > 0$ on $[x_{i-1}, x_i]$. Suppose further that

$$a_{i2}(x_i) = a_{i+1,2}(x_i) \quad \text{and} \quad a'_{i2}(x_i) = a'_{i+1,2}(x_i), \quad i = 1, 2, \ldots, N-1. \tag{2}$$

We denote such a set by $\mathscr{L}(\Delta) = \{L_1, L_2, \ldots, L_N\}$. Associated with each L_i is its formal adjoint

$$L_i^* v(x) = (a_{i2}(x)v(x))'' - (a_{i1}(x)v(x))' + a_{i0}(x)v(x). \tag{3}$$

Integration by parts gives the following relation (Green's formula) for any $u, v \in K_2[x_{i-1}, x_i]$:

$$\int_{x_{i-1}}^{x_i} v(x)\, L_i u(x) \mathrm{d}x - \int_{x_{i-1}}^{x_i} u(x) L_i^* v(x) \mathrm{d}x = P(u(x_i), v(x_i)) - P(u(x_{i-1}), v(x_{i-1})),$$

where

$$P(u(x), v(x)) = \sum_{j=0}^{1} D^{1-j} u(x) \sum_{k=0}^{j} (-1)^k D^k \{a_{i,2-j+k}(x) v(x)\}$$

is the *bilinear concomitant* (see [1], page 124). This formula can be simplified to

$$\int_{x_{i-1}}^{x_i} v(x)\, L_i u(x) \mathrm{d}x - \int_{x_{i-1}}^{x_i} u(x) L_i^* v(x) \mathrm{d}x$$
$$= \left\{ a_{i2}(x)u'(x)v(x) + u(x)(a_{i1}(x)v(x) - (a_{i2}(x)v(x))') \right\} \Big|_{x_{i-1}}^{x_i}. \tag{4}$$

Definition 1. A real-valued function s defined on $[a,b]$ is called a **piecewise L-spline of order 4** for $\mathscr{L}(\Delta)$ if $s \in C^2[a,b]$ and for each $i = 1, 2, \ldots, N s$ satisfies $s \in K_4[x_{i-1}, x_i]$, $L_i^* L_i s = 0$ almost everywhere on $[x_{i-1}, x_i]$.

For a fixed $\mathscr{L}(\Delta)$ the class of all piecewise L-splines is denoted by $\mathrm{SP}(\Delta, \mathscr{L})$. In this paper we investigate functions $s \in \mathrm{SP}(\Delta, \mathscr{L})$ satisfying

$$\begin{aligned} &s(x_i) = f(x_i), \quad i = 0, 1, \ldots, N, \\ &s'(a) = f'(a) \quad \text{and} \quad s'(b) = f'(b), \end{aligned} \tag{5}$$

for $f \in C^1[a, b]$, termed hereafter *interpolants to* f.

3. Existence and uniqueness

The following existence and uniqueness result shows that the set $\mathrm{SP}(\Delta, \mathscr{L})$ is appropriate for interpolation. In [4] Prenter states a similar theorem for piecewise L-splines of arbitrary order but gives an erroneous proof.

Theorem 2. *For* $f \in C^1[a, b]$*, there exists a unique* $s \in \mathrm{SP}(\Delta, \mathscr{L})$ *interpolating* f.

Proof. For each $i = 1, 2, \ldots, N$, $L_i^* L_i s = 0$ almost everywhere on $[x_{i-1}, x_i]$. Therefore on $[x_{i-1}, x_i]$

$$s(x) = C_{i1} v_{i1}(x) + C_{i2} v_{i2}(x) + C_{i3} v_{i3}(x) + C_{i4} v_{i4}(x),$$

where $v_{i1}, v_{i2}, v_{i3}, v_{i4} \in K_4[x_{i-1}, x_i]$ are linearly independent functions spanning the null space of $L_i^* L_i$. Thus finding s is equivalent to solving for the $4N$ unknowns $\{C_{ij}\}$. The requirement that $s \in C^2[a, b]$ yields $3(N-1)$ equations and the interpolation conditions yield $N+3$ equations. We therefore have a total of $4N$ equations in $4N$ unknowns. Thus it is sufficient to show that there exists a unique piecewise L-spline in $\mathrm{SP}(\Delta, \mathscr{L})$ which interpolates the zero function.

Clearly the zero spline from $\mathrm{SP}(\Delta, \mathscr{L})$ interpolates the zero function. Assume, to the contrary, the existence of another such spline, denoted by s. We claim that

$$\sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \left[(L_i s)^2 + (L_i^* s)^2 \right] = 0. \tag{6}$$

Indeed, with $v = L_i s$ and $u = s$ in (4) we have

$$\begin{aligned} \int_{x_{i-1}}^{x_i} (L_i s)^2 &= \int_{x_{i-1}}^{x_i} s\, L_i^* L_i s + \{a_{i2}\, s'\, L_i s + s(a_{i1} L_i s - (a_{i2} L_i s)')\} \,|_{x_{i-1}}^{x_i} \\ &= a_{i2}\, s'\, L_i s \,|_{x_{i-1}}^{x_i}, \end{aligned}$$

where the last equality follows since $s \in \mathrm{SP}(\Delta, \mathscr{L})$ and $s(x_{i-1}) = s(x_i) = 0$. Similarly we get from (4) and from the normality of L_i,

$$\begin{aligned} \int_{x_{i-1}}^{x_i} (L_i^* s)^2 &= \int_{x_{i-1}}^{x_i} s\, L_i L_i^* s - \{a_{i2}\, (L_i^* s)'\, s + (L_i^* s)(a_{i1}\, s - (a_{i2}\, s)')\} \,|_{x_{i-1}}^{x_i} \\ &= (a_{i2}\, s)'\, L_i^* s \,|_{x_{i-1}}^{x_i} = a_{i2}\, s'\, L_i^* s \,|_{x_{i-1}}^{x_i}. \end{aligned}$$

Therefore

$$\sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \left[(L_i s)^2 + (L_i^* s)^2 \right] = \sum_{i=1}^{N} a_{i2}\, s'\, (L_i s + L_i^* s) \,|_{x_{i-1}}^{x_i}.$$

By (3) and the fact that $s(x_{i-1}) = s(x_i) = 0$,

$$(L_i s + L_i^* s) \,|_{x_{i-1}}^{x_i} = (a_{i2} s'' + a_{i1} s' + 2a_{i2}' s' + a_{i2} s'' - a_{i1} s') \,|_{x_{i-1}}^{x_i} = 2(a_{i2}' s' + a_{i2} s'') \,|_{x_{i-1}}^{x_i}$$

and we obtain

$$\sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \left[(L_i s)^2 + (L_i^* s)^2 \right] = \sum_{i=1}^{N} 2a_{i2}\, s' \,(a'_{i2} s' + a_{i2} s'')\,|_{x_{i-1}}^{x_i} . \tag{7}$$

By the continuity requirements (2) and the fact that $s \in C^2[a, b]$ and $s'(a) = s'(b) = 0$ we conclude that the sum on the right-hand side of (7) is zero, so (6) is proved. Therefore

$$\int_{x_{i-1}}^{x_i} (L_i s)^2 = 0$$

and $L_i s = 0$ on $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, N$. It follows that for each $i = 1, 2, \ldots, N$,

$$s(x) = d_{i1} u_{i1}(x) + d_{i2} u_{i2}(x)$$

on $[x_{i-1}, x_i]$, where u_{i1} and u_{i2} are linearly independent functions in $K_2[x_{i-1}, x_i]$ spanning the null space of L_i. Starting with

$$s(x) = d_{11} u_{11}(x) + d_{12} u_{12}(x)$$

on $[x_0, x_1]$, and using $s(x_0) = s'(x_0) = 0$, we conclude that $s \equiv 0$ on $[x_0, x_1]$. The continuity requirements imply that $s(x_1+) = s'(x_1+) = 0$, so similarly $s \equiv 0$ on $[x_1, x_2]$. Continuing this argument across the knots we find that $s \equiv 0$ on $[a, b]$, contrary to our assumption. Thus $s \equiv 0$ is the only spline in $\mathrm{SP}(\Delta, \mathscr{L})$ interpolating the zero function. This completes the proof of the theorem. □

Remark 1. The proof in [4] is similar, but based on the identity $\sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (L_i s)^2 = 0$ instead of (6). The derivation of the above equality in [4] is erroneous.

4. Integral relations

The first and the second integral relations are important identities in the theory of splines. For L-splines these relations are derived in [7]. For piecewise L-splines erroneous formulas are given in [4]. Here we present the correct first and second integral relations for piecewise L-splines from $\mathrm{SP}(\Delta, \mathscr{L})$.

Theorem 3. *For $f \in K_2[a, b]$ and $s \in \mathrm{SP}(\Delta, \mathscr{L})$ interpolating f, let $e = f - s$. Then the following* ***first integral relation*** *holds:*

$$\sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (L_i f)^2 = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (L_i e)^2 + \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (L_i s)^2 + 2 \sum_{i=1}^{N-1} a_{i2}(x_i) \mathrm{e}'(x_i)\, (L_i - L_{i+1}) s(x_i). \tag{8}$$

Proof. For each $i = 1, 2, \ldots, N$, $L_i f = L_i e + L_i s$, so

$$\int_{x_{i-1}}^{x_i} (L_i f)^2 = \int_{x_{i-1}}^{x_i} (L_i e)^2 + 2 \int_{x_{i-1}}^{x_i} L_i s\, L_i e + \int_{x_{i-1}}^{x_i} (L_i s)^2.$$

With $v = L_i s$ and $u = e$ in (4) we have

$$\int_{x_{i-1}}^{x_i} L_i s\, L_i e = \int_{x_{i-1}}^{x_i} e\, L_i^* L_i s + \left\{a_{i2} e' L_i s + e(a_{i1} L_i s - (a_{i2} L_i s)')\right\}\Big|_{x_{i-1}}^{x_i}$$
$$= a_{i2} e' L_i s\,\Big|_{x_{i-1}}^{x_i},$$

where the last equality follows since $s \in \mathrm{SP}(\Delta, \mathscr{L})$ and $e(x_{i-1}) = e(x_i) = 0$. By (2) and the fact that $e'(a) = e'(b) = 0$ we obtain

$$\sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} L_i s\, L_i e = \sum_{i=1}^{N-1} a_{i2}(x_i) e'(x_i)(L_i - L_{i+1}) s(x_i) \tag{9}$$

and the proof is complete. □

Remark 2. In the proof of the first integral relation in [4] and in [3, equation (6.7)], it is claimed that the sum in (9) vanishes. This is incorrect, as is demonstrated by the following example. Consider the function $f(x) = x^2$ on the interval $[-1, 1]$, the partition $\Delta : -1 < 0 < 1$ and $L_1 = D^2 + 5D$, $L_2 = D^2 + D$. In this case the sum (9) has one term only which equals $-4s'(0)^2$. It is easy to verify that for this special case $s'(0) \neq 0$.

For smoother functions another integral relation holds.

Theorem 4. *For $f \in K_4[a, b]$ and $s \in \mathrm{SP}(\Delta, \mathscr{L})$ interpolating f, let $e = f - s$. Then the following* ***second integral relation*** *holds:*

$$\sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (L_i e)^2 = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} e\, L_i^* L_i f + \sum_{i=1}^{N-1} a_{i2}(x_i) e'(x_i)\, (L_i - L_{i+1}) e(x_i). \tag{10}$$

Proof. With $v = L_i e$ and $u = e$ in (4) we have for $i = 1, 2, \ldots, N$

$$\int_{x_{i-1}}^{x_i} (L_i e)^2 = \int_{x_{i-1}}^{x_i} e L_i^* L_i e + \left\{a_{i2}\, e' L_i e + e(a_{i1} L_i e - (a_{i2} L_i e)')\right\}\Big|_{x_{i-1}}^{x_i}$$
$$= \int_{x_{i-1}}^{x_i} e\, L_i^* L_i f + a_{i2} e' L_i e\,\Big|_{x_{i-1}}^{x_i},$$

where the last equality follows since $s \in \mathrm{SP}(\Delta, \mathscr{L})$ and $e(x_{i-1}) = e(x_i) = 0$. By (2) and the fact that $e'(a) = e'(b) = 0$ we obtain

$$\sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (L_i e)^2 = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} e\, L_i^* L_i f + \sum_{i=1}^{N-1} a_{i2}(x_i) e'(x_i)\, (L_i - L_{i+1}) e(x_i)$$

and the proof is complete. □

5. L^2 error bounds for splines in tension

In this section we investigate the special class of piecewise L-splines called *splines in tension*.

Definition 5. Given a partition Δ of the interval $[a, b]$ and a sequence of nonnegative parameters $\{\rho_i\}_{i=1}^N$, a real-valued function s defined on $[a, b]$ is called a **spline in tension** if $s \in C^2[a, b]$ and $s^{(4)} - \rho_i^2 s'' = 0$ on $[x_{i-1}, x_i]$ for each $i = 1, 2, \ldots, N$.

Since $s^{(4)} - \rho_i^2 s'' = (D^2 - \rho_i D)(D^2 + \rho_i D)s$, s can be interpreted as a piecewise L-spline from $\mathrm{SP}(\Delta, \mathscr{L})$, where

$$L_i = D^2 + \rho_i D, \quad i = 1, 2, \ldots, N. \tag{11}$$

In what follows all operators L_i are of the form (11). For fixed Δ and $\rho = \{\rho_i\}_{i=1}^N$, the class of all splines in tension is denoted by $\mathrm{ST}(\Delta, \rho)$. Given $f \in C^1[a, b]$, a spline in tension s is an *interpolant to* f provided that

$$s(x_i) = f(x_i), \quad i = 0, 1, \ldots, N,$$
$$s'(a) = f'(a) \quad \text{and} \quad s'(b) = f'(b).$$

We start by establishing a few preliminary results.

Lemma 6. *If $f \in K_1[a, b]$ and $f(a) = f(b) = 0$, then*

$$\int_a^b f^2 \leq \frac{(b-a)^2}{\pi^2} \int_a^b (f')^2.$$

Proof. This inequality is called the Rayleigh–Ritz inequality. It is proved in [6, page 5] for $f \in C^1[a, b]$, but the same arguments apply also to $f \in K_1[a, b]$. □

Lemma 7. *For $f \in K_1[a, b]$ and $s \in \mathrm{ST}(\Delta, \rho)$ interpolating f, let $e = f - s$. Then*

$$\int_a^b \mathrm{e}^2 \leq \frac{\bar{\Delta}^2}{\pi^2} \int_a^b (e')^2,$$

where $\bar{\Delta} = \max_{1 \leq i \leq N}(x_i - x_{i-1})$.

Proof. Note that $e(x_i) = 0$ for $i = 0, 1, \ldots, N$. Applying Lemma 6 to e yields

$$\int_{x_{i-1}}^{x_i} \mathrm{e}^2 \leq \frac{(x_i - x_{i-1})^2}{\pi^2} \int_{x_{i-1}}^{x_i} (e')^2 \leq \frac{\bar{\Delta}^2}{\pi^2} \int_{x_{i-1}}^{x_i} (e')^2$$

for $i = 1, 2, \ldots, N$. Summing these inequalities with respect to i from 1 to N completes the proof. □

Lemma 8. *For $f \in K_2[a, b]$ and $s \in \mathrm{ST}(\Delta, \rho)$ interpolating f, let $e = f - s$. Then*

$$\sum_{i=1}^N \int_{x_{i-1}}^{x_i} \left[(L_i e)^2 - 2\rho_i e' e''\right] \leq \frac{1}{2} \sum_{i=1}^N \int_{x_{i-1}}^{x_i} \left[(L_i f)^2 + (L_i^* f)^2\right]. \tag{12}$$

Moreover, for $f \in K_4[a, b]$,

$$\sum_{i=1}^N \int_{x_{i-1}}^{x_i} \left[(L_i e)^2 - 2\rho_i e' e''\right] = \sum_{i=1}^N \int_{x_{i-1}}^{x_i} e\, L_i^* L_i f. \tag{13}$$

Proof. First we prove (12). By the first integral relation (8),

$$\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}\left[(L_i e)^2 - 2\rho_i e' e''\right] \le \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}\left[(L_i f)^2 - 2\rho_i e' e''\right] - 2\sum_{i=1}^{N-1} e'(x_i)(\rho_i - \rho_{i+1})s'(x_i). \tag{14}$$

Denote $\tilde{\mathscr{L}}(\Delta) = \{L_1^*, L_2^*, \ldots, L_N^*\}$ and let $\tilde{s}$ be the piecewise L-spline from $\mathrm{SP}(\Delta, \tilde{\mathscr{L}})$ interpolating f. Since $L_i^* L_i = L_i L_i^* = (L_i^*)^* L_i^*$ for each i, $s \equiv \tilde{s}$ and we may apply the first integral relation to $\tilde{s}$ and to $\tilde{\mathscr{L}}(\Delta)$ to obtain

$$\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}\left[(L_i^* e)^2 + 2\rho_i e' e''\right] \le \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}\left[(L_i^* f)^2 + 2\rho_i e' e''\right] - 2\sum_{i=1}^{N-1} e'(x_i)(-\rho_i + \rho_{i+1})s'(x_i). \tag{15}$$

Because $(L_i e)^2 - 2\rho_i e' e'' = (e'')^2 + \rho_i^2 (e')^2$ and $(L_i^* e)^2 + 2\rho_i e' e'' = (e'')^2 + \rho_i^2 (e')^2$, the left-hand sides of (14) and (15) are equal. Taking the average of (14) and (15) proves (12).

To prove (13), we apply the second integral relation (10) to s,

$$\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}(L_i e)^2 = \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i} e\, L_i^* L_i f + \sum_{i=1}^{N-1} e'(x_i)^2(\rho_i - \rho_{i+1}). \tag{16}$$

But

$$\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i} 2\rho_i e' e'' = \sum_{i=1}^{N}\rho_i \int_{x_{i-1}}^{x_i}\left((e')^2\right)' = \sum_{i=1}^{N}\rho_i (e')^2 \,\big|_{x_{i-1}}^{x_i} = \sum_{i=1}^{N-1} e'(x_i)^2(\rho_i - \rho_{i+1}). \tag{17}$$

Subtracting (17) from (16) proves (13). □

We now derive L^2 error bounds for interpolation with splines in tension. The L^2 norm of a real-valued function f such that f^2 is Lebesgue integrable on $[a, b]$ is

$$\|f\|_2 = \left(\int_a^b f^2(x)\mathrm{d}x\right)^{1/2}.$$

We derive two types of error bounds, one for functions in $K_2[a, b]$ and one for smoother functions in $K_4[a, b]$.

Theorem 9. *For $f \in K_2[a, b]$ and $s \in \mathrm{ST}(\Delta, \rho)$ interpolating f,*

$$\|f - s\|_2 \le \frac{2\bar{\Delta}^2}{\pi\sqrt{\pi^2 + 4\underline{\rho}^2\bar{\Delta}^2}}\left(\frac{1}{2}\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}\left[(L_i f)^2 + (L_i^* f)^2\right]\right)^{1/2}, \tag{18}$$

$$\|f' - s'\|_2 \le \frac{2\bar{\Delta}}{\sqrt{\pi^2 + 4\underline{\rho}^2\bar{\Delta}^2}} \left(\frac{1}{2} \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \left[(L_i f)^2 + (L_i^* f)^2 \right] \right)^{1/2}, \tag{19}$$

and

$$\|f'' - s''\|_2 \le \left(\frac{1}{2} \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \left[(L_i f)^2 + (L_i^* f)^2 \right] \right)^{1/2}, \tag{20}$$

where $\underline{\rho} = \min_{1\le i\le N} |\rho_i|$.

Proof. We start by proving (20). Let $e = f - s$. Then

$$\int_{x_{i-1}}^{x_i} (e'')^2 = \int_{x_{i-1}}^{x_i} (L_i e)^2 - \int_{x_{i-1}}^{x_i} 2\rho_i e' e'' - \int_{x_{i-1}}^{x_i} \rho_i^2 (e')^2$$

for $i = 1, 2, \ldots, N$. Summing these equalities with respect to i from 1 to N and using (12) gives

$$\begin{aligned} \int_a^b (e'')^2 &= \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \left[(L_i e)^2 - 2\rho_i e' e'' \right] - \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \rho_i^2 (e')^2 \\ &\le \frac{1}{2} \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \left[(L_i f)^2 + (L_i^* f)^2 \right], \end{aligned} \tag{21}$$

from which (20) follows.

To prove (19) note that $e(x_i) = 0$ for $i = 0, 1, \ldots, N$. Applying Rolle's theorem to e we obtain points $\xi_i \in (x_{i-1}, x_i)$ such that $e'(\xi_i) = 0$ for $i = 1, 2, \ldots, N$. Define $\xi_0 = a$ and $\xi_{N+1} = b$ and note that $e'(\xi_0) = e'(\xi_{N+1}) = 0$ by the interpolation conditions. Applying Lemma 6 to e' gives

$$\pi^2 \int_{\xi_{i-1}}^{\xi_i} (e')^2 \le (\xi_i - \xi_{i-1})^2 \int_{\xi_{i-1}}^{\xi_i} (e'')^2 \tag{22}$$

for $i = 1, 2, \ldots, N + 1$. Let $\rho(x)$ be a step function with $\rho(x) = \rho_i$ on $[x_{i-1}, x_i)$, $i = 1, 2, \ldots, N - 1$ and $\rho(x) = \rho_N$ on $[x_{N-1}, x_N]$. Adding $(\xi_i - \xi_{i-1})^2 \int_{\xi_{i-1}}^{\xi_i} \rho^2 (e')^2$ to both sides of (22) we obtain

$$\left(\pi^2 + (\xi_i - \xi_{i-1})^2 \underline{\rho}^2 \right) \int_{\xi_{i-1}}^{\xi_i} (e')^2 \le (\xi_i - \xi_{i-1})^2 \int_{\xi_{i-1}}^{\xi_i} \left[(e'')^2 + \rho^2 (e')^2 \right] \tag{23}$$

or

$$\begin{aligned} \int_{\xi_{i-1}}^{\xi_i} (e')^2 &\le \frac{(\xi_i - \xi_{i-1})^2}{\pi^2 + (\xi_i - \xi_{i-1})^2 \underline{\rho}^2} \int_{\xi_{i-1}}^{\xi_i} \left[(e'')^2 + \rho^2 (e')^2 \right] \\ &\le \frac{4\bar{\Delta}^2}{\pi^2 + 4\underline{\rho}^2 \bar{\Delta}^2} \int_{\xi_{i-1}}^{\xi_i} \left[(e'')^2 + \rho^2 (e')^2 \right], \end{aligned}$$

where the last inequality readily follows from $\xi_i - \xi_{i-1} \le 2\Delta$. Summing these inequalities with respect to i from 1 to $N + 1$ gives

$$\int_a^b (e')^2 \le \frac{4\bar{\Delta}^2}{\pi^2 + 4\underline{\rho}^2 \bar{\Delta}^2} \int_a^b \left[(e'')^2 + \rho^2 (e')^2 \right].$$

Because $(L_i e)^2 = (e'')^2 + 2\rho_i e' e'' + \rho_i^2 (e')^2$,

$$\int_a^b (e')^2 \le \frac{4\bar{\Delta}^2}{\pi^2 + 4\underline{\rho}^2 \bar{\Delta}^2} \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \left[(L_i e)^2 - 2\rho_i e' e'' \right]. \tag{24}$$

Combining this with (12) proves (19).

Finally by Lemma 7 and (19)

$$\int_a^b \mathrm{e}^2 \le \frac{\bar{\Delta}^2}{\pi^2} \int_a^b (e')^2 \le \frac{4\bar{\Delta}^4}{\pi^2(\pi^2 + 4\underline{\rho}^2 \bar{\Delta}^2)} \frac{1}{2} \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \left[(L_i f)^2 + (L_i^* f)^2 \right],$$

which proves (18). □

Remark 3. It is easy to verify that the results of Theorem 9 hold with $\frac{1}{2}\sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} ((L_i f)^2 + (L_i^* f)^2)$ replaced by $\sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (L_i f)^2$ if $\rho_1 = \rho_2 = \cdots = \rho_N$. Yet, this is not the case when the $\{\rho_i\}$ are different, as is demonstrated by the following example. Consider the function $f(x) = x\,(x-1)^2\,(x+1)^2$ on the interval $[-1, 1]$, the partition $\Delta : -1 < 0 < 1$ and $\rho_1 = 0$, $\rho_2 = 1$. Then $s \equiv 0$ on $[-1, 1]$ and it is readily verified that (20) does not hold with the above change.

If f is smoother we can prove a finer result:

Theorem 10. *For $f \in K_4[a, b]$ and $s \in \mathrm{ST}(\Delta, \rho)$ interpolating f,*

$$\|f - s\|_2 \le \frac{4\bar{\Delta}^4}{\pi^2(\pi^2 + 4\underline{\rho}^2 \bar{\Delta}^2)} \left(\sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (L_i^* L_i f)^2 \right)^{\frac{1}{2}}, \tag{25}$$

$$\|f' - s'\|_2 \le \frac{4\bar{\Delta}^3}{\pi(\pi^2 + 4\underline{\rho}^2 \bar{\Delta}^2)} \left(\sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (L_i^* L_i f)^2 \right)^{\frac{1}{2}}, \tag{26}$$

and

$$\|f'' - s''\|_2 \le \frac{2\bar{\Delta}^2}{\pi\sqrt{\pi^2 + 4\underline{\rho}^2 \bar{\Delta}^2}} \left(\sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (L_i^* L_i f)^2 \right)^{\frac{1}{2}}. \tag{27}$$

Proof. We start by proving (25). Let $e = f - s$. By Lemma 7 and (24)

$$\int_a^b \mathrm{e}^2 \le \frac{\bar{\Delta}^2}{\pi^2} \int_a^b (e')^2 \le \frac{4\bar{\Delta}^4}{\pi^2(\pi^2 + 4\underline{\rho}^2 \bar{\Delta}^2)} \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \left[(L_i e)^2 - 2\rho_i e' e'' \right].$$

Using (13) and Cauchy–Schwarz inequality, we get

$$\int_a^b \mathrm{e}^2 \le \frac{4\bar{\Delta}^4}{\pi^2(\pi^2 + 4\underline{\rho}^2 \bar{\Delta}^2)} \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} e\, L_i^* L_i f$$

$$\le \frac{4\bar{\Delta}^4}{\pi^2(\pi^2 + 4\underline{\rho}^2 \bar{\Delta}^2)} \left(\int_a^b \mathrm{e}^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (L_i^* L_i f)^2 \right)^{\frac{1}{2}},$$

from which (25) follows.

To prove (26) we use (24) with (13) and obtain

$$\int_a^b (e')^2 \le \frac{4\bar{\Delta}^2}{\pi^2 + 4\underline{\rho}^2\bar{\Delta}^2} \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} e\, L_i^* L_i f.$$

By Cauchy–Schwarz inequality and (25),

$$\int_a^b (e')^2 \le \frac{4\bar{\Delta}^2}{\pi^2 + 4\underline{\rho}^2\bar{\Delta}^2} \left(\int_a^b \mathrm{e}^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (L_i^* L_i f)^2\right)^{\frac{1}{2}}$$
$$\le \frac{4\bar{\Delta}^2}{\pi^2 + 4\underline{\rho}^2\bar{\Delta}^2} \frac{4\bar{\Delta}^4}{\pi^2(\pi^2 + 4\underline{\rho}^2\bar{\Delta}^2)} \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (L_i^* L_i f)^2,$$

which yields (26).

Finally by (21) and (13)

$$\int_a^b (e'')^2 \le \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \left[(L_i e)^2 - 2\rho_i e' e''\right] = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} e L_i^* L_i f.$$

Cauchy–Schwarz inequality and (25) lead to

$$\int_a^b (e'')^2 \le \left(\int_a^b \mathrm{e}^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (L_i^* L_i f)^2\right)^{\frac{1}{2}}$$
$$\le \frac{4\bar{\Delta}^4}{\pi^2(\pi^2 + 4\underline{\rho}^2\bar{\Delta}^2)} \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (L_i^* L_i f)^2,$$

which completes the proof. □

Remark 4. Cubic splines correspond to the case $\rho_1 = \rho_2 = \cdots = \rho_N = 0$, since then $L_i = D^2$ and $L_i^* L_i s = s^{(4)} = 0$ is the differential equation that cubic splines satisfy in each mesh interval. Therefore L^2 error bounds for interpolation with cubic splines can be obtained from Theorems 9 and 10 as a special case by setting $\rho_i = 0$ for each i. Such bounds and similar ones can be found in [6].

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