

Analytic Geometry - Prof. David Levin

1. Basic definitions :

Let $u, v \in \mathbb{R}^3$, $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$.

For $t \in \mathbb{R}$, $u + tv = (u_1 + tv_1, u_2 + tv_2, u_3 + tv_3)$.

Norm : $\|u\| = (u_1^2 + u_2^2 + u_3^2)^{\frac{1}{2}}$. Distance : $d(u, v) = \|u - v\|$.

Triangle inequality : $d(u, v) \leq d(u, w) + d(w, v)$.

A parametric line through u and v : $p(t) = u + t(v - u)$, $-\infty < t < \infty$.

Distance on the line: $d(p(t_1), p(t_2)) = \|p(t_1) - p(t_2)\| = |t_1 - t_2| \|v - u\|$.

$p(t)$, $0 < t < 1$ divides the line segment between u and v in ratio $\frac{t}{1-t}$.

Scalar product : a, b vectors in \mathbb{R}^3 , $(a, b) = \sum_{i=1}^3 a_i b_i = ab^t$.

Angle θ between a and b : $\cos(\theta) = (a, b) / \|a\| \|b\|$. $\|a\| = (a, a)^{\frac{1}{2}}$.

$a \perp b \iff (a, b) = 0$; Cauchy-Schwartz inequality : $|(a, b)| \leq \|a\| \|b\|$.

Vector product : $a \times b = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right)$.

$(a \times b, a) = (a \times b, b) = 0$ i.e. $a \times b \perp a$, $a \times b \perp b$. $a \times b = -b \times a$.

$\|a \times b\|^2 = \|a\|^2 \|b\|^2 - (a, b)^2$, implying $\|a \times b\| = \|a\| \|b\| \sin(\theta)$.

If $a \times b \neq 0$ then $a, b, a \times b$ form a basis for \mathbb{R}^3 .

A plane through u , orthogonal to a : $(x, a) = (u, a) = D$, $a_1 x_1 + a_2 x_2 + a_3 x_3 = D$.

A plane through u, v, w :

Parametric representation : $p(s, t) = u + s(v - u) + t(w - u)$.

Plane equation : $(x, a) = (u, a)$ with $a = (v - u) \times (w - u)$.

2. Distances :

Between a point q and a parametric line $p(t) = u + tw$: $d = \frac{\|(q-u) \times w\|}{\|w\|}$.

Distance between a point q and a plane $(x, a) = (u, a)$: $d = \frac{|(q-u, a)|}{\|a\|}$.

Distance between two (non-parallel) lines $u + tw$ and $p + sq$: $d = \frac{|(p-u, w \times q)|}{\|w \times q\|}$.

3. Angles and Intersections :

Between a line $u + tw$ and a plane $(x, a) = D$: $\sin(\phi) = \frac{|(a, w)|}{\|a\| \|w\|}$.

Between two planes $(x, a) = D$ and $(x, b) = E$: $\cos(\theta) = \frac{|(a, b)|}{\|a\| \|b\|}$.

The line of intersection between the planes : If $a \times b \neq 0$ then $a \times b$ is the direction of that line. To find a point p on both planes we express p as

$$p = \alpha a + \beta b + \gamma(a \times b). \text{ Thus } \begin{cases} (p, a) = \alpha(a, a) + \beta(b, a) = D \\ (p, b) = \alpha(b, a) + \beta(b, b) = E \end{cases} .$$

The intersection line is $\alpha^* a + \beta^* b + t(a \times b)$.

4. Linear transformations : $Tx = Ax$

Here $x \in \mathbb{R}^3$ is a column vector $x = (x_1, x_2, x_3)^t$, and A a 3×3 matrix.

The columns of A are $Te_i = Ae_i \equiv e'_i$, $i = 1, \dots, 3$.

If A is an orthogonal matrix then $a = A^t x$ is the representation of x in the orthogonal system e'_1, e'_2, e'_3 , i.e., $x = a_1 e'_1 + a_2 e'_2 + a_3 e'_3$. Clearly, $x = Aa$.

Isometry : $\|u - v\| = \|Au - Av\|$. A is isometry iff A is orthogonal ($A^t A = I$).

2×2 orthogonal matrices :

Rotation : $T_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$; Reflection : $R_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$.

3×3 orthogonal matrices :

A orthogonal has eigenvalue $\lambda = \pm 1$ if $\text{Det}(A) = \pm 1$, $Av = \pm v$.

If $\text{Det}(A) = \pm 1$ then $A = V \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} V^t$, $V^t V = I$, $\text{Det}(V) = 1$.

$\text{Det}(A) = 1 \implies$ rotation around v ,

$\text{Det}(A) = -1 \implies$ rotation around v + reflection across the plane $(x, v) = 0$.

5. **Quadratic curves in \mathbb{R}^2 :** $q(x_1, x_2) = ax_1^2 + 2bx_1x_2 + dx_2^2 + 2\alpha x_1 + 2\beta x_2 + \ell = 0$.

$$q(v) = v^t A v + 2(v, \rho) + \ell = 0, \quad v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}, \quad \rho = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Case 1: $\det(A) \neq 0$: Defining $v_0 = -A^{-1}\rho$ and using an orthonormal transformation P such that $A = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^t$ we get

$$\lambda_1 u_1^2 + \lambda_2 u_2^2 + q(v_0) = 0, \quad \text{where } u = P^t(v - v_0) \quad \text{and } v = Pu + v_0.$$

Case 2: $\det(A) = 0$. Here we use P such that $A = P \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} P^t$ and get

$$\lambda u_1^2 + 2\alpha' u_1 + 2\beta' u_2 + l = 0, \quad \text{where } u = P^t v \quad \text{and } \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = P^t \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

6. **Quadratic surfaces in \mathbb{R}^3 :**

$$q(v) = v^t A v + 2(v, \rho) + \ell = 0, \quad v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \rho = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.$$

Case 1: $\det(A) \neq 0$: Defining $v_0 = -A^{-1}\rho$ and using an orthonormal transformation P such that $A = P \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} P^t$ we get

$$\lambda_1 u_1^2 + \lambda_2 u_2^2 + \lambda_3 u_3^2 + q(v_0) = 0, \quad \text{where } u = P^t(v - v_0) \quad \text{and } q(v_0) = (v_0, \rho) + \ell.$$

Case 2: $\det(A) = 0$. Here we use P such that $A = P \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} P^t$

$$\lambda_1 u_1^2 + \lambda_2 u_2^2 + 2\alpha' u_1 + 2\beta' u_2 + 2\gamma' u_3 + l = 0, \quad \text{where } u = P^t v \quad \text{and } \begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \end{pmatrix} = P^t \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.$$

Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Cone: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$.

Elliptic Paraboloid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$.

Hyperbolic Paraboloid: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$.

One-sheet Hyperboloid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.

Two-sheet Hyperboloid: $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.

Cylindrical surfaces:

Elliptic: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; **Hyperbolic:** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; **Parabolic:** $y^2 = 2px$.

7. **Tangent plane to the quadratic surface (or curve) $q(v) = v^t Av + 2(v, \rho) + \ell = 0$, at a point $v = v_0$ on it:**

$$v^t Av_0 + (v + v_0, \rho) + \ell = 0 .$$

8. **Geometry on the unit sphere: $S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$.**

For $P, Q \in S^2$, the distance on S^2 : $d(P, Q) = \cos^{-1}(P, Q)$.

A line between $P, Q \in S^2$: $x(t) = \cos(t)P + \sin(t)Q'$ where $Q' = \frac{Q - (Q, P)P}{\|Q - (Q, P)P\|}$.

The angle α (near A) of a triangle formed by $A, B, C \in S^2$:

$$\cos(\alpha) = (A_B, A_C) \text{ where } A_B = \frac{B - (A, B)A}{\|B - (A, B)A\|} \text{ and } A_C = \frac{C - (A, C)A}{\|C - (A, C)A\|} .$$

For a triangle T with angles α, β, γ and side lengths a, b, c :

The area is: $S(T) = \alpha + \beta + \gamma - \Pi$.

Cosine Theorem: $\cos(\alpha) = \frac{\cos(a) - \cos(b)\cos(c)}{\sin(b)\sin(c)}$.

Sine Theorem: $\frac{\sin(\alpha)}{\sin(a)} = \frac{\sin(\beta)}{\sin(b)} = \frac{\sin(\gamma)}{\sin(c)}$.

9. **Projective Geometry :**

Desargues Theorem:

If two triangles are perspective from a point then the intersection points of corresponding edges lie on a common line.

Pappus Theorem:

If points A, B , and C are on one line and A', B' and C' are on another line then the points of intersection of the lines AC' and $C'A$, AB' and BA' , and BC' and CB' lie on a common line.