Introduction to Numerical Analysis

1. Interpolation :

Lagrange : $p_n(x) = \sum_{k=0}^{n} f(x_k)\ell_k(x)$, $\ell_k(x) = \prod_{i\neq k} \frac{(x-x_i)}{(x_k-x_i)}$.

Newton : $p_n(x) = \sum_{k=0}^{n} f[x_0,...,x_k] \prod_{j=0}^{k-1}(x-x_j)$.

$f(x) - p_n(x) = f[x_0,...,x_n,x]\Psi_n(x)$, $\Psi_n(x) = \prod_{j=0}^{n}(x-x_j)$.

For $x_i = x_0 + ih$, $x = x_0 + sh$, $p_n(x) = \sum_{k=0}^{n} \binom{s}{k}\Delta^k f_0$, $\binom{s}{k} = \frac{s(s-1)...(s-k+1)}{k!}$,

and the error bound : $|f(x) - p_n(x)| \le \frac{1}{4(n+1)} \max|f^{(n+1)}|h^{n+1}$.

The error for interpolation at Chebyshev points in [-1,1] :

$$|f(x) - p_n(x)| \le \frac{1}{(n+1)!} \max|f^{(n+1)}|2^{-n} .$$

The divided difference :

$$f[x_0,...,x_k] = \begin{cases} \frac{1}{x_k-x_0}(f[x_1,...,x_k] - f[x_0,...,x_{k-1}]) , & \textbf{if } x_0 \neq x_k , \\ f^{(k)}(x_0)/k! , & \textbf{if } x_0 = x_1 = ... = x_k. \end{cases}$$

$$f[x_0,...,x_k] = f^{(k)}(\xi)/k! .$$

For $x_i = x_0 + ih$, $f[x_i,\ldots,x_{i+k}] = \frac{\Delta^k f_i}{k!h^k}$ **where** $\Delta f_i = f_{i+1} - f_i$.

Derivative of the divided difference : $\frac{d}{dx}f[x_0,...,x_k,x] = f[x_0,...,x_k,x,x]$.

2. Integration : $\int_a^b f(x)dx = I(f) = I(p_k) + \int_a^b f[x_0,...,x_k,x]\Psi_k(x)dx$.

If $\int_a^b \Psi_k(x)dx = 0$ **then** $I(f) = I(p_k) + \int_a^b f[x_0,...,x_k,x_{k+1},x]\Psi_{k+1}(x)dx$.

Mid-point rule : $I(f) = (b-a)f(\frac{a+b}{2}) + f''(\eta)\frac{(b-a)^3}{24}$.

Trapezoidal rule : $I(f) = \frac{b-a}{2}[f(a) + f(b)] - f''(\eta)\frac{(b-a)^3}{12}$.

Simpson's rule : $I(f) = \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)] - \frac{f^{(4)}(\eta)}{90}(\frac{b-a}{2})^5$.

Corrected trapezoidal rule :

$I(f) = \frac{b-a}{2}[f(a) + f(b)] + \frac{(b-a)^2}{12}[f'(a) - f'(b)] + f^{(4)}(\eta)\frac{(b-a)^5}{720}$.

Composite trapezoidal rule :

$I(f) = \frac{h}{2}(f_0 + f_N) + h\sum_{i=1}^{N-1} f_i - \frac{f''(\eta)}{12}h^2(b-a)$.

Composite Simpson's rule :

$I(f) = \frac{h}{6}[f_0 + f_N + 2\sum_{i=1}^{N-1} f_i + 4\sum_{i=1}^{N} f_{i-\frac{1}{2}}] - \frac{f^{(4)}(\eta)}{180}(\frac{h}{2})^4(b-a)$.

Gaussian rule : $I(f) = \sum_{i=1}^{k} A_i f(x_i) + \frac{f^{(2k)}(\eta)}{(2k)!\alpha_k^2}\int_a^b p_k^2(x)w(x)dx$,

where $p_k(x) = \alpha_k \prod_{i=1}^{k}(x - x_i)$ **is orthogonal to** x^j, $j = 0, \ldots, k-1$.

3. **Orthogonal polynomials - weight function** $w(x)$ **on** $[a,b]$, $h_n = \int_a^b w(x)p_n^2(x)dx$.

   a. **Legendre :** $w(x) = 1$, $[a,b] = [-1,1]$, $h_n = \frac{2}{2n+1}$,

$$P_0 = 1 \ , \ P_1(x) = x \ , \quad P_{n+1}(x) = \frac{2n+1}{n+1}xP_n(x) - \frac{n}{n+1}P_{n-1}(x) \ .$$

   b. **Chebyshev :** $w(x) = (1-x^2)^{-\frac{1}{2}}$, $[a,b] = [-1,1]$, $h_0 = \pi$, $h_n = \pi/2$, $n > 0$,

$$T_0 = 1 \ , \ T_1(x) = x \ , \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \ .$$

   The roots of $T_n(x)$ **:** $x_i = cos(\frac{2i-1}{2k}\pi)$, $T_n(x) = 2^{n-1}\prod_{i=1}^{k}(x - x_i)$.

   c. **Laguerre :** $w(x) = e^{-x}$, $[a,b) = [0,\infty)$, $h_n = 1$,

$$L_0 = 1 \ , \ L_1(x) = 1 - x \ , \quad L_{n+1}(x) = (\frac{2n+1}{n+1} - \frac{x}{n+1})L_n(x) - \frac{n}{n+1}L_{n-1}(x) \ .$$

   d. **Hermite :** $w(x) = e^{-x^2}$, $(a,b) = (-\infty,\infty)$, $h_n = \sqrt{\pi}\ 2^n n!$,

$$H_0 = 1 \ , \ H_1(x) = 2x \ , \quad H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \ .$$

4. **Differentiation :** $f'(x) = p_n'(x) + f[x_0, ..., x_n, x, x]\Psi_n(x) + f[x_0, ..., x_n, x]\Psi_n'(x)$.

5. **Non-linear Equations :** $f(x) = 0$

   Secant method : $x_{n+1} = \frac{f(x_n)x_{n-1} - f(x_{n-1})x_n}{f(x_n) - f(x_{n-1})}$; $e_{n+1} = -\frac{f[x_{n-1},x_n,\xi]}{f[x_{n-1},x_n]}e_n e_{n-1}$.

   Newton's method : $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$; $e_{n+1} = -\frac{f[x_n,x_n,\xi]}{f[x_n,x_n]}e_n^2 = -\frac{1}{2}\frac{f''(\eta_n)}{f'(x_n)}e_n^2$.

   Newton's method for a system : $f(x,y) = 0$; $g(x,y) = 0$.

$$x_{n+1} = x_n + [\frac{-f \cdot g_y + g \cdot f_y}{f_x \cdot g_y - f_y \cdot g_x}]_{(x_n,y_n)} \ ,$$

$$y_{n+1} = y_n + [\frac{-g \cdot f_x + f \cdot g_x}{f_x \cdot g_y - f_y \cdot g_x}]_{(x_n,y_n)} \ .$$

6. **Bernstein Approx. :** $p_n(x)=\sum_{k=0}^{n} f(\frac{k}{n})B_k^n(x)$ **,** $B_k^n(x)=\binom{n}{k}x^k(1-x)^{n-k}$.

$\sum_{k=0}^{n} B_k^n(x)=1$, $\sum_{k=0}^{n}\frac{k}{n}B_k^n(x)=x$, $\sum_{k=0}^{n}(\frac{k}{n}-x)^2B_k^n(x)=\frac{x(1-x)}{n}$.

If $n\geq \max_{x\in[0,1]}|f(x)|\frac{1}{\varepsilon\delta^2}$ **then** $\max_{x\in[0,1]}|f(x)-p_n(x)|<\varepsilon$.

7. **Least-squares Approximations :**

$\langle g,h\rangle=\int_a^b w(x)g(x)h(x)dx$, $\langle g,h\rangle_N=\sum_{n=1}^{N} w(x_n)g(x_n)h(x_n)$, $\|g\|^2=\langle g,g\rangle$.

Normal equations : $\sum_{i=1}^{k} c_i\,\langle\phi_i,\phi_j\rangle=\langle f,\phi_j\rangle$, $j=1,\ldots,k$.

Orthogonal polynomials - construction by moments $m_j=\int_a^b w(x)x^jdx$.

$$p_n(x)=\begin{vmatrix} 1 & x & x^2 & \ldots & x^n \\ m_0 & m_1 & \ldots & \ldots & m_n \\ m_1 & m_2 & \ldots & \ldots & m_{n+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ m_{n-1} & m_n & \ldots & \ldots & m_{2n-1} \end{vmatrix}$$

8. **Fourier Series :** $f(x)=\sum_{j=-\infty}^{\infty}\widehat{f}(j)e^{ijx}$, $\widehat{f}(j)=\frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ijx}dx$,

$f(x+2\pi)=f(x)$, $e^{ix}=cos(x)+isin(x)$, $\langle g,h\rangle=\frac{1}{2\pi}\int_0^{2\pi} g(x)\overline{h}(x)dx$,

$\langle g,h\rangle_N=\frac{1}{N}\sum_{n=0}^{N-1} g(\frac{2\pi n}{N})\overline{h}(\frac{2\pi n}{N})$, $\widehat{f}_N(j)=\frac{1}{N}\sum_{n=0}^{N-1} f(\frac{2\pi n}{N})e^{-i\frac{2\pi n}{N}j}$.

Parseval's relation : $\frac{1}{2\pi}\int_0^{2\pi}|f(x)|^2=\sum_{j=-\infty}^{\infty}|\widehat{f}(j)|^2$.

9. **Systems of linear equations** $Ax=b$ **:**

$$\frac{1}{\|A\|\|A^{-1}\|}\frac{\|r\|}{\|b\|}\leq\frac{\|e\|}{\|x\|}\leq\|A\|\|A^{-1}\|\frac{\|r\|}{\|b\|} \quad , \; e=x-\overline{x} \quad , \; r=Ax-A\overline{x} \; .$$

$$\|A\|_\infty=\max_{\|x\|\neq 0}\frac{\|Ax\|_\infty}{\|x\|_\infty}=\max_{1\leq i\leq n}\sum_{j=1}^{n}|a_{ij}| \; , \quad A=\{a_{ij}\}_{i,j=1}^{n} \; .$$

$Ax=b\iff x=Bx+c$ **where** $B=I-GA$ **,** $c=Gb$ **,** $G\approx A^{-1}$ **.**

Error estimates for the iterative process $x^{(m+1)}=Bx^{(m)}+c$ **:**

$$\|x-x^{(m)}\|\leq\frac{\|B\|^m}{1-\|B\|}\|x^{(1)}-x^{(0)}\| \; ; \quad \frac{\|x-x^{(m)}\|}{\|x\|}\leq\|B\|^m \quad if\;\; x^{(0)}=0.$$

The power method : Let $|\lambda_1|>|\lambda_2|\geq\;\ldots\;\leq|\lambda_n|$ **, then**

$$|\lambda_1|=\lim_{m\to\infty}\frac{\|A^{m+1}z\|}{\|A^mz\|} \; , \quad \lambda_1=\lim_{m\to\infty}\frac{(A^{m+1}z)_j}{(A^mz)_j} \; .$$

10. Splines :

$B_i^{(k)}(x) = (\cdot - x)_+^k [x_{i-m_1}, ..., x_{i+m_2}](x_{i+m_2} - x_{i-m_1})$ **where**

$$(x)_+^k = \begin{cases} x^k & x \geq 0 \ , \\ 0 & x < 0 \ . \end{cases}$$

and, if $k = 2m \Rightarrow m_1 = m$**,** $m_2 = m + 1$ **and if** $k = 2m - 1 \Rightarrow m_1 = m_2 = m$**.**

B-Splines for equidistant knots $x_i = ih$ **:**

Quadratic B-Splines : $B_i^{(2)}(x) = B_1^{(2)}(x - (i-1)h)$ **:**

$$B_1^{(2)}(x) = \begin{cases} \frac{1}{2}u^2 & 0 \leq u = x/h \leq 1 \ , \\ \frac{1}{2} + u - u^2 & 0 \leq u = x/h - 1 \leq 1 \ , \\ \frac{1}{2}(1-u)^2 & 0 \leq u = x/h - 2 \leq 1 \ . \end{cases}$$

Cubic B-Splines : $B_i^{(3)}(x) = B_2^{(3)}(x - (i-2)h)$ **:**

$$B_2^{(3)}(x) = \begin{cases} \frac{1}{6}u^3 & 0 \leq u = x/h \leq 1 \ , \\ \frac{1}{6}(1 + 3u + 3u^2 - 3u^3) & 0 \leq u = x/h - 1 \leq 1 \ , \\ \frac{1}{6}(4 - 6u^2 + 3u^3) & 0 \leq u = x/h - 2 \leq 1 \ , \\ \frac{1}{6}(1-u)^3 & 0 \leq u = x/h - 3 \leq 1 \ . \end{cases}$$

Error estimates for cubic spline interpolation, $S_3(ih) = f(ih) \ , \ i = 0, \ldots, N$**, with derivative boundary conditions** $S_3'(0) = f'(0)$ **,** $S_3'(Nh) = f'(Nh)$ **:**

$$|f(x) - S_3(x)| \leq max|f^{(4)}|\frac{5h^4}{384} \ , \ x \in [0, Nh] \ ,$$

$$|f'(ih) - S_3'(ih)| \leq max|f^{(5)}|\frac{h^4}{60} \ , \ i = 1, \ldots, N-1 \ .$$

11. Extrapolation :

$I(f) = A_h(f) + ch^r + o(h^r)$ **,** $I(f) = \frac{q^r A_h(f) - A_{qh}(f)}{q^r - 1} + o(h^r)$ **.**

12. Linear map : $x \in [c, d] \longleftrightarrow t \in [a, b] \ : \ t = a + \frac{b-a}{d-c}(x - c)$ **.**

$x \in [-1, 1] \longleftrightarrow t \in [a, b] \ : \ t = \frac{a+b}{2} + \frac{b-a}{2}x \ , \ x = \frac{a+b}{a-b} + \frac{2}{b-a}t \ .$