The Subdivision Experience

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Dedicated to the memory of John A. Gregory

Abstract. Subdivision started as a tool for efficient computation of spline functions, and is now an independent subject with many applications. It is used for developing new methods for curve and surface design, for approximation, for generating wavelets and multiresolution analysis and also for solving some classes of functional equations. This paper reviews recent new directions and new developments in subdivision analysis. Extensions of the uniform stationary binary subdivision process and their analysis are presented. These include non-stationary subdivision, non-uniform subdivision, integral subdivision, distributional subdivision and the convergence of the above in the strong and in the weak sense.

§1. Introduction

The *stationary uniform binary subdivision scheme* is a process which recursively defines a sequence of sets of control points $\{P_\alpha^k : \alpha \in \mathbb{Z}^s\}_{k\in\mathbb{Z}_+} \subset \mathbb{R}^d$ by a rule of the form

$$P_\alpha^{k+1} = \sum_{\beta\in\mathbb{Z}^s} a_{\alpha-2\beta} P_\beta^k = \sum_{\beta\in\alpha+2\mathbb{Z}^s} a_\beta P_{(\alpha-\beta)/2}^k , \tag{1.1}$$

which is denoted formally by $P^{k+1} = S_a P^k$. Here we assume that the set $\mathrm{supp}(a) = \{\alpha : a_\alpha \neq 0\}$ is finite.

The geometric interpretation to this sequence of sets of control points is given by considering a corresponding sequence of geometrical entities

$$F_k(t) = \sum_{\alpha\in\mathbb{Z}^s} P_\alpha^k \psi(2^k t - \alpha) , \tag{1.2}$$

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where $\psi(t) \in C(\mathbb{R}^s)$ is of compact support and satisfies

$$\sum_{\alpha \in \mathbb{Z}^s} \psi(\cdot - \alpha) = 1 \ , \tag{1.3}$$

$$m||c||_\infty \leq || \sum_{\alpha \in \mathbb{Z}^s} c_\alpha \psi(\cdot - \alpha)||_\infty \leq M||c||_\infty, \quad c \in \ell_\infty(\mathbb{Z}^s) \ . \tag{1.4}$$

For $s = 1$, $d = 2$ or 3, $F_k(t)$ is a curve, while for $s = 2$, $d = 3$ it is a surface. For $s = 1$ and the choice $\psi(t) = 1 - |t|$, $t \in (-1, 1)$ and zero otherwise, $F_k(t)$ is the control polygon through the control points. Stationary uniform binary subdivision schemes received a lot of attention in the literature in recent years. A general analysis of such schemes can be found in [3, 8, 9, 12, 13, 15, 19, 20, 21]. The subdivision scheme S_a is termed *uniformly convergent* if for the initial set of control points in $\mathbb{R}$, $f^0 = \delta = \{\delta_{\alpha,0}\}_{\alpha \in \mathbb{Z}^s}$, $\{F_k(t)\}$ converges uniformly to a non-zero function ϕ. This property of the scheme is independent of the choice of ψ in (1.2) [3] (see also [12]) and it implies that

$$\lim_{k \to \infty} \sup_{2^{-k}\alpha \in \Omega} ||P_\alpha^k - F_\infty(2^{-k}\alpha)||_\infty = 0 \ , \tag{1.5}$$

for any compact $\Omega \subset \mathbb{R}^s$, where $F_\infty(t) = \sum_{\alpha \in \mathbb{Z}^s} P_\alpha^0 \phi(t - \alpha)$.

A necessary condition on the mask a for the corresponding subdivision process $\{S_a\}$ to be uniformly convergent is [3]:

$$\sum_{\beta \in \mathbb{Z}^s} a_{\alpha - 2\beta} = 1, \quad \alpha_i \in \{0, 1\}, \quad i = 1, \ldots, s$$

which implies that $\sum_{\alpha \in \mathbb{Z}^s} a_\alpha = 2^s$.

We are interested in the smoothness properties of ϕ which determine the smoothness of the components of $F_\infty(t)$. A scheme is termed C^m if $\phi \in C^m$, and it is termed H^γ with $\gamma \in (0, 1)$, if $\phi \in H^\gamma$, namely if

$$|\phi(t) - \phi(\tau)| \leq A|t - \tau|^\gamma \ . \tag{1.6}$$

Thus, the analysis of the convergence of the scheme and the properties of ϕ requires the study of the scalar case $d = 1$.

A uniformly convergent $\{S_a\}$ defines its basic function ϕ, which satisfies the functional (refinement) equation

$$\phi(x) = \sum_{\alpha \in \mathbb{Z}^s} a_\alpha \phi(2x - \alpha) \ . \tag{1.7}$$

This is the case, since $(S_a\delta)_\alpha = a_\alpha$, $\alpha \in \mathbb{Z}^s$, and $\phi(2\cdot)$ is the limit function for the initial data δ in the second level of the subdivision process. Also it is easy to verify that the support of ϕ is contained in the convex hull of the points in

$\mathbb{Z}^s$ constituting the support of the mask a [3]. Equation (1.7) indicates that the spaces $V_k = \text{span}\{\phi(2^k \cdot -\alpha) : \alpha \in \mathbb{Z}^s\}$ are nested, namely

$$V_0 \subset V_1 \subset V_2 \cdots , \tag{1.8}$$

which is necessary for multiresolution analysis.

Example – B-splines. A typical example of a stationary uniform binary subdivision scheme is provided by schemes generating B-splines with equally-spaced knots. The B-spline with integer knots of order m (degree $m-1$ and smoothness C^{m-2}) is the basic function of a scheme defined by the mask [5]

$$a_\alpha = 2^{-m+1}\binom{m}{\alpha}, \quad \alpha = 0, 1, \ldots, m .$$

In this paper we review several different extensions and generalizations of stationary uniform binary subdivision schemes, and the notion of their convergence. Some of these extensions have geometric advantage and others are of analytical interest. First we discuss briefly the extensions considered in the paper, and list several more, some of which have not yet been investigated.

1. Non-stationary schemes. The mask applied at level k of the subdivision depends on the level:

$$P_a^{k+1} = \sum_{\beta \in \mathbb{Z}^s} a_{\alpha-2\beta}^{(k)} P_\beta^k, \quad \alpha \in \mathbb{Z}^s . \tag{1.9}$$

Here a necessary condition for convergence is $\lim\limits_{k\to\infty} \sum_{\beta\in\mathbb{Z}^s} a_{\alpha-2\beta}^{(k)} = 1$ for $\alpha_i \in \{0,1\}$, $i = 1, \ldots, s$ [16, 17].

2. Non-uniform schemes. The mask depends on the location of the defined point. In the simplest case $\mathbb{R}^s$ is divided into disjoint domains, with different rules in these domains:

$$P_\alpha^{k+1} = \sum_{\beta \in \mathbb{Z}^s} a_{\alpha-2\beta}^{(\ell)} P_\beta^k, \quad 2^{-k-1}\alpha \in \Omega_\ell , \tag{1.10}$$

where $\Omega_\ell \cap \Omega_j = \emptyset$ if $j \neq \ell$ and $\cup_\ell \Omega_\ell = \mathbb{R}^s$. The supports of the masks $\{a^{(\ell)}\}$ are uniformly bounded [14].

A more general non-uniform scheme is [2]

$$P_\alpha^{k+1} = \sum_{\beta \in 2\mathbb{Z}^s + \alpha} a_{\alpha,\beta} P_\beta^k, \quad \alpha \in \mathbb{Z}^s . \tag{1.11}$$

Here $a_{\alpha,\beta}$ has the property that $a_{\alpha,\beta} = 0$ for $\alpha - 2\beta \notin J$, where J is a finite subset of $\mathbb{Z}^s$, which is termed the "support" of the masks. The matrix $A = \{a_{\alpha\beta}\}$ is termed two-slanted.

3. Non-uniform and non-stationary schemes. These schemes are obtained from (1.10) and (1.11) by adding dependence of the masks on the subdivision level [2].

4. Integral schemes. All schemes discussed up to now have masks defined on discrete sets of points. Consider a mask which is a function of compact support in $L_1(I\!R^s)$ and sets of control points depending on a continuous parameter. The process becomes

$$P^{k+1}(t) = \int_{I\!R^s} a(t-2\tau)P^k(\tau)d\tau \ , \tag{1.12}$$

and defines a sequence of vectors with components in $C(I\!R^s)$, starting with such a vector $P^0(t)$. The geometrical entities are given by

$$F_k(t) = P^k(2^k t) \ , \tag{1.13}$$

and one can define the process directly in terms of the geometrical entities as

$$F_{k+1} = 2^{ks}a(2^{k+1}\cdot) * F_k = 2^{-s}\int_{I\!R^s} F_k(\cdot - 2^{-k-1}\tau)a(\tau)d\tau \ , \tag{1.14}$$

with $F_0(t) = P^0(t)$. Equation (1.14) indicates that the process is equivalent to repeated averaging over exponentially decreasing neighborhoods if $\int_{\mathbf{R}^s} a(\tau)d\tau = 2^s$, which is indeed a necessary condition for the convergence of the process [10].

There is a way to unify the representation of discrete and integral subdivision schemes. It is achieved by first extending the notion of convergence.

5. Weak convergence. The subdivision scheme

$$f^{k+1}_\alpha = \sum_{\beta\in Z\!\!Z^s} a_{\alpha-2\beta}f^k_\beta, \quad \alpha\in Z\!\!Z^s, \quad k\in Z\!\!Z_+ \ , \tag{1.16}$$

is *weakly convergent* if for every $g\in C^\infty_0$ and $f^0=\delta$, the following limit exists [10]:

$$\lim_{k\to\infty} 2^{-ks}\sum_{\alpha\in Z\!\!Z^s} f^k_\alpha g(2^{-k}\alpha) \ . \tag{1.17}$$

The weak limit of the subdivision scheme is the distribution F_∞ defined by

$$\langle F_\infty, g\rangle = \lim_{k\to\infty} 2^{-ks}\sum_{\alpha\in Z\!\!Z^s} f^k_\alpha g(2^{-k}\alpha), \quad g\in C^\infty_0 \ . \tag{1.18}$$

6. Distributional schemes. The distributional schemes include the discrete and the integral schemes as special cases. A distributional scheme has the form as in (1.14), $F_{k+1} = 2^{ks}a(2^{k+1}\cdot)*F_k$, with $\{F_k\}$ a sequence of distributions and

$a(\cdot)$ a distribution of compact support satisfying $\langle a, 1\rangle = 2^s$, where 1 denotes any C_0^∞ function which is 1 on supp(a). For $a \in L_1(\mathbb{R}^s)$ and $F_0 \in C(\mathbb{R}^s)$ it is an integral scheme, while for $F_0 = \delta(\cdot)$, and $a = \sum_{\alpha\in\mathbb{Z}^s} a_\alpha \delta(\cdot - \alpha)$, the distribution F_k has the form

$$F_k = \sum_{\alpha\in\mathbb{Z}^s} f_\alpha^k \delta(2^k \cdot - \alpha) \ , \tag{1.19}$$

with f^k obtained by (1.16). The limit distribution of $\{F_k\}$ is the weak limit of the discrete scheme (1.16) given by (1.18)[10].

Note that in the distributional case, $\delta(\cdot)$ denotes the Dirac distribution $\langle \delta(\cdot), g\rangle = g(0)$.

7. Non-binary distributional schemes. These schemes have a similar form to (1.14)

$$F_{k+1} = p^{ks} a(p^{k+1}\cdot) * F_k \ , \tag{1.20}$$

with $p \in \mathbb{R}$, $p > 1$, and $a(\cdot)$ a distribution of compact support satisfying $\langle a, 1\rangle = p^s$. For $p \in \mathbb{Z}_+\backslash\{0,1\}$, $a = \sum_{\alpha\in\mathbb{Z}^s} a_\alpha \delta(\cdot - \alpha)$ and $F_0 = \delta(\cdot)$, we get the discrete p-nary scheme

$$F_k = \sum_{\alpha\in\mathbb{Z}^s} f_\alpha^k \delta(p^k \cdot - \alpha) \ , \tag{1.21}$$

where

$$f_\alpha^{k+1} = \sum_{\beta\in\mathbb{Z}^s} a_{\alpha - p\beta} f_\beta^k \ . \tag{1.22}$$

Another case of interest is when p is rational, namely $p = r/q$, $r, q \in \mathbb{Z}_+\backslash\{0\}$, $r > q$, and $a(\cdot)$ is the distribution

$$a = \sum_{\alpha\in\mathbb{Z}^s} a_\alpha \delta(\cdot - \alpha/q), \quad \sum_{\alpha\in\mathbb{Z}^s} a_\alpha = p^s \ . \tag{1.23}$$

Then for $F_0 = \delta(\cdot)$, we get the discrete scheme

$$F_k = \sum_{\alpha\in\mathbb{Z}^s} f_\alpha^k \delta\Big(p^k \cdot - \frac{\alpha}{q^k}\Big) \ , \tag{1.24}$$

$$f_\alpha^{k+1} = \sum_{\beta\in I_\alpha^k} a_{\frac{\alpha - r\beta}{q^k}} f_\beta^k \ , \tag{1.25}$$

where $I_\alpha^k = \{\beta : \alpha - r\beta \text{ is divisible by } q^k\}$.

Note that the set $\mathcal{F}^k = \{f_\alpha^k : \alpha \in q^k\mathbb{Z}^s\}$ defines the set $\mathcal{F}^{k+1}$ by the rule

$$f_{\alpha q^{k+1}}^{k+1} = \sum_{\beta\in\mathbb{Z}^s} a_{\alpha q - r\beta} f_{\beta q^k}^k, \alpha \in \mathbb{Z}^s \ , \tag{1.26}$$

and that $f_{\alpha q^k}^k$ is attached to the parameter value αp^{-k}. Thus (1.26) gives rise to the rational schemes introduced in [1].

A more general non-binary scheme can be defined in terms of a sequence of monotonically increasing reals $\{p_k\}_{k\in\mathbb{Z}_+}$ with $p_0 = 1$. This scheme has the form

$$F_{k+1} = p^s_{k+1} a(p_{k+1}\cdot) * F_k \ , \tag{1.27}$$

with $a(\cdot)$, a compactly supported distribution satisfying $\langle a, 1\rangle = 1$.

The scheme (1.27) can be further generalized into a non-stationary scheme of the form

$$F_{k+1} = a_k * F_k, \quad k \in \mathbb{Z}_+ \ , \tag{1.28}$$

where $a_k(\cdot)$ is a distribution supported in $p^{-1}_{k+1}\Omega$, with $\Omega \subset \mathbb{R}^s$ compact, and $\lim_{k\to\infty}\langle a_k, 1\rangle = 1$.

In the following sections we review results about convergence and smoothness of some of the types of schemes presented above. We also consider several interesting examples, and discuss properties of spaces spanned by the shifts of the corresponding basic functions.

§2. Non-Stationary Discrete Schemes

A converging non-stationary scheme $\{S_{a^{(k)}}\}_{k\in\mathbb{Z}_+}$ defines a system of refinement equations [17]:

$$\phi_\ell = \sum_{\alpha\in\mathbb{Z}^s} a^{(k)}_\alpha \phi_{\ell+1}(2\cdot - \alpha), \quad \ell \in \mathbb{Z}_+ \ , \tag{2.1}$$

where ϕ_ℓ is the basic function of the non-stationary scheme $\{S_{a^{(k+\ell)}}\}_{k\in\mathbb{Z}_+}$.

Thus the spaces $V_\ell = \text{span}\{\phi_\ell(2^\ell\cdot - \alpha) : \alpha \in \mathbb{Z}^s\}$, $\ell \in \mathbb{Z}_+$ are nested (satisfy (1.8)), and can be used for multiresolution analysis. The advantage of this set-up over the nested spaces generated by stationary schemes, is the possibility to get $\{\phi_\ell\}$ of compact support and infinitely smooth [10].

Three classes of non-stationary schemes together with typical examples are reviewed in this section. A mask is termed stationary if it determines a stationary C^0 scheme. The first class of schemes consists of schemes defined by stationary masks only.

2.1. Schemes defined by stationary masks

The simplest convergence result for non-stationary schemes is a direct consequence of a convergence result for stationary schemes. In what follows $f^k = S_{a^{(k-1)}} \cdots S_{a^{(0)}} f^0$, $k \in \mathbb{Z}_+$ and $e^{(i)} = \delta_{i,j}$, $i, j = 1, \ldots, s$.

Theorem 2.1 (Convergence [17]). *Let $\{S_{a^{(k)}}\}_{k\in\mathbb{Z}_+}$ be a non-stationary scheme such that each $a^{(k)}$ satisfies the conditions*

$$\sum_{\beta\in\mathbb{Z}^s} a^{(k)}_{\alpha-2\beta} = 1, \quad \alpha_i \in \{0,1\}, \quad i = 1,\ldots,s \ . \tag{2.2}$$

Let $S^\Delta_{a^{(k)}}$ be the subdivision rule with a matrix mask mapping the set of vectors

$$\Delta f^k = \{\Delta f^k_j = (f^k_{j+e^{(i)}} - f^k_j, \quad i = 1,\ldots,s) \ : \ j \in \mathbb{Z}^s\} \ , \tag{2.3}$$

into the set of vectors Δf^{k+1}. *(The existence of* $S^{\Delta}_{a^{(k)}}$ *is guaranteed by (2.2).) If there exists* $\mu \in (0,1)$ *such that for all* $k \in \mathbb{Z}_+$

$$||S^{\Delta}_{a^{(k)}} \Delta f||_\infty \le \mu ||\Delta f||_\infty, \quad f \in \ell^\infty(\mathbb{Z}^s) \ , \tag{2.4}$$

where $||\Delta f||_\infty = \sup_{j \in \mathbb{Z}^s} ||\Delta f_j||_\infty$, *then* $\{S_{a^{(k)}}\}_{k \in \mathbb{Z}_+}$ *is* H^ν, $\nu = \min\{-\log_2 \mu, 1\}$.

Note that a necessary and sufficient condition for a mask a to be stationary is the contractivity of the stationary scheme $\{S^{\Delta}_a\}$ as an operator on the space $\{\Delta f : f \in \ell^\infty(\mathbb{Z}^s)\}$(see e.g. [12]). Hence by (2.4) $a^{(k)}$ is stationary.

Theorem 2.2 (Smoothness [17]). *Let* $\{S_{a^{(k)}}\}_{k \in \mathbb{Z}_+}$ *be such that for some* $\lambda \in \mathbb{Z}^s$ *and* $k_0 \in \mathbb{Z}_+$

$$\sum_{\alpha \in \mathbb{Z}^s} a^{(k)}_\alpha z^\alpha = (1 + z^\lambda) \sum_{\alpha \in \mathbb{Z}^s} b^{(k)}_\alpha z^\alpha, \quad k \ge k_0 \ . \tag{2.5}$$

If $\{S_{b^{(k)}}\}_{k \in \mathbb{Z}_+}$ *is convergent then* $\{S_{a^{(k)}}\}_{k \in \mathbb{Z}_+}$ *is convergent and the corresponding basic functions denoted by* ϕ_b *and* ϕ_a, *respectively, are related by*

$$\partial^\lambda \phi_a = \sum_{i=1}^{s} \lambda_i \frac{\partial}{\partial x_i} \phi_a = \phi_b - \phi_b(\cdot - \lambda) \in C(\mathbb{R}^s) \ . \tag{2.6}$$

Note that $\phi_a \in C^1(\mathbb{R}^s)$ if (2.6) holds for s linearly independent vectors $\lambda^1, \ldots, \lambda^s \in \mathbb{Z}^s$.

The first example we present is typical to nonstationary schemes with $\mathrm{supp}(a^{(k)})$ growing linearly with k.

2.2. The up-function

Let $s = 1$ and let $a^{(k)}$ be the stationary mask generating the B-spline of order k,

$$a^{(k)}_i = \frac{1}{2^{k-1}} \binom{k}{i}, \quad i = 0, \ldots, k, \quad k \in \mathbb{Z}_+ \ . \tag{2.7}$$

By Theorems 2.1 and 2.2 the functions $\{\phi_\ell\}$, with ϕ_ℓ the basic function of $\{S_{a^{(k+\ell)}}\}_{k \in \mathbb{Z}_+}$, are in $C_0^\infty(\mathbb{R})$ with $\mathrm{supp}\, \phi_\ell = (0, \ell + 2)$. As shown in [10] the function ϕ_0 is the Rvachev's up-function [22]. Also, by (2.1) the functions $\{\phi_\ell\}$ generate a nested sequence of shift invariant spaces $\{V_k\}_{k \in \mathbb{Z}_+}$.

The above example is interesting enough to deserve detailed analysis. Here we cite the major properties of the sequence of nested spaces generated by it.

Theorem 2.3 (Stability [18]). *The function $\phi_j(2^j\cdot)$ is a stable generator of $V_j = \mathrm{span}\{\phi_j(2^j \cdot - \alpha) : \alpha \in \mathbb{Z}\}$, namely, there exist constants $0 < R_j < T_j$ such that for each $d \in \ell^2(\mathbb{Z})$*

$$R_j||d||_{\ell^2(\mathbb{Z})} \leq || \sum_{\alpha\in\mathbb{Z}} d_\alpha \phi_j(2^j \cdot - \alpha)||_{L_2(\mathbb{R})} \leq T_j||d||_{\ell^2(\mathbb{Z})} \ , \qquad (2.8)$$

with $T_k/R_k = O\left(\left(\frac{\pi}{2}\right)^k\right)$. Moreover $\{\phi_j(\cdot - \alpha) : \alpha \in \mathbb{Z}\}$ are linearly independent.

Theorem 2.4 (Spectral approximation order [18]). *Let*

$$E_k(f) = \inf\{||f - v||_{L_2(\mathbb{R})} : v \in V_k\} \ .$$

Then for any $r \in \mathbb{Z}_+$ and $f \in W_2^r(\mathbb{R})$ (the Sobolev space of order r)

$$\lim_{k\to\infty} 2^{kr} E_k(f) = 0 \quad \text{as } k \to \infty \ . \qquad (2.9)$$

For the sequence of nested spaces $\{V_k\}_{k\in\mathbb{Z}_+}$ one can construct a sequence of prewavelets $\{\psi_k\}_{k\in\mathbb{Z}_+}$ such that

$$V_k \oplus \mathrm{span}\{\psi_k(\cdot - \alpha) : \alpha \in 2^{-k}\mathbb{Z}\} = V_{k+1} \ . \qquad (2.10)$$

These prewavelets are given in terms of their Fourier transforms:

Theorem 2.5 (Properties of prewavelets [18]). *Let $\rho_k = \phi_k(2^k\cdot)$ and let $\tau_k(\omega) = \sum_{\alpha\in 2^{k+1}\mathbb{Z}} |\hat{\rho}_k(\omega + 2\pi\alpha)|^2$. Then*

$$\widehat{\psi}_k(\omega) = \frac{1}{2} e^{-i\omega/2^{k+1}} a_k(e^{-i\omega/2^{k+1}})\tau_k(\omega)\widehat{\rho}_{k+1}(\omega) \ , \qquad (2.11)$$

and $|\,\mathrm{supp}\,\psi_k| = 2|\,\mathrm{supp}\,\rho_k| = \frac{k+2}{2^{k-1}}$. Also, ψ_k generates a stable basis of the orthogonal complement of V_j in V_{j+1}.

2.3. Asymptotically stationary schemes

An interesting class of non-stationary schemes is that of "asymptotically stationary schemes" [16, 17]. The masks defining such a scheme $a^{(k)} = \{a_\alpha^{(k)} : \alpha \in \mathbb{Z}^s\}$, $k \in \mathbb{Z}_+$ satisfy the two conditions

(i) $\mathrm{supp}(a^{(k)}) \subset [0, n]^s$, $k \in \mathbb{Z}_+$ for some $n \in \mathbb{Z}_+^s$.

(ii) There exists a limiting mask a^* supported in $[0, n]^s$ such that

$$\lim_{k\to\infty} \max_{\alpha\in[0,n]^s} |a_\alpha^{(k)} - a_\alpha^*| = 0 \ . \qquad (2.12)$$

Example – the exponential splines. Univariate exponential B-splines with integer knots are basic functions of asymptotically stationary schemes, with masks $a^{(k)}$, $k \in \mathbb{Z}_+$ of the form

$$\frac{1}{2}\sum_{\alpha\in\mathbb{Z}} a_\alpha^{(k)} z^\alpha = \prod_{i=0}^{\ell}(1+e^{2^{-k-1}\gamma_i}z)/2, \quad k \in \mathbb{Z}_+ . \tag{2.13}$$

For $\ell \geq 1$, the scheme generates a $C^{\ell-1}$ piecewise exponential function in the span of $\{e^{\gamma_i x} : i = 0, \ldots, \ell\}$ with integer knots and support $(0, \ell+1)$ (see [2, 16, 17]). The scheme is asymptotically stationary since

$$a_j^{(k)} = \frac{1}{2^\ell} \sum_{0\leq i_1<\cdots<i_j\leq \ell} \exp\{2^{-k-1}\sum_{r=1}^{j}\gamma_{i_r}\} = \frac{1}{2^\ell}\binom{\ell+1}{j} + 0(2^{-k}) . \tag{2.14}$$

The limiting mask is that of a B-spline of order $\ell+1$.

Similarly in the multivariate case, the sequence of masks

$$\frac{1}{2^s}\sum_{\alpha\in\mathbb{Z}^s} a_\alpha^{(k)} z^\alpha = \prod_{i=0}^{\ell}(1+\exp\{2^{-k-1}\gamma_i\}z^{\lambda^i})/2 ,$$

with directions $\Lambda = \{\lambda^0, \ldots, \lambda^\ell\} \subset \mathbb{Z}^s$, $\ell \geq s$, determines a C^0 scheme with an exponential box-spline as its basic function, if any ℓ vectors from Λ span $\mathbb{R}^s$ and there is a subset of s vectors in Λ with determinant ± 1 [6, 17]. The limiting mask is stationary generating the polynomial box-spline with directions Λ [3].

Theorem 2.6 (Convergence and smoothness [17]). *Let $\{a^{(k)}\}$ satisfy conditions (i), (ii) with a^* a stationary mask, and let*

$$\theta_k = \max\{|a_\alpha^{(k)} - a_\alpha^*| : \alpha \in \mathbb{Z}^s\}, \quad k \in \mathbb{Z}_+ .$$

If for some $m \geq 0$, $\sum_{k\in\mathbb{Z}_+}\theta_k 2^{km} < \infty$ and $\{S_{a^*}\}$ is C^m, then $\{S_{a^{(k)}}\}_{k\in\mathbb{Z}_+}$ is C^m. If $\{S_{a^*}\}$ is H^ν, $\nu \in (0,1)$, and $\theta_k \leq A2^{-k\mu}$, $k \in \mathbb{Z}_+$ for some $\mu \in (0,1)$, then the scheme $\{S_{a^{(k)}}\}_{k\in Z_+}$ is H^γ with $\gamma = \min(\nu,\mu)$.

For asymptotically stationary schemes there is an analogous result to Theorem 2.2.

Theorem 2.7 (Smoothness [17]). *Let $\{a^{(k)}\}_{k\in\mathbb{Z}_+}$ be a sequence of masks satisfying* $\mathrm{supp}(a^{(k)}) \subset [0,n]^s$, *and*

$$a_k(z) = \sum_{\alpha\in\mathbb{Z}^s} a_\alpha^{(k)} z^\alpha = \prod_{i=1}^{s}(1+r_{k,i}z^{\lambda^i})b_k(z), \quad k \geq k_0 , \tag{2.15}$$

with $\lambda^1, \ldots, \lambda^s \in \mathbb{Z}^s$ a basis of $\mathbb{R}^s$. If , moreover, for each $i = 1, \ldots, s$, the polynomials $\{a_k(z)/(1+r_{k,i}z^{\lambda^i})\}_{k\in\mathbb{Z}_+}$ determine a C^m scheme and

$$\lim_{k\to\infty}(r_{k,i}^2 - 1)2^k = c_i < \infty, \quad \sum_{k=k_0}^{\infty} 2^k|r_{k+1,i} - r_{k,i}| < \infty , \tag{2.16}$$

then $\{S_{a^{(k)}}\}_{k\in\mathbb{Z}_+}$ *is a* C^{m+1} *scheme.*

Example – exponential B-splines revisited. For the exponential B-spline scheme (2.13) with $\ell \geq 1$, S_{a^*} generates B-splines of order $\ell+1$ hence S_{a^*} is $C^{\ell-1}$. By (2.14), $\{a^{(k)}\}_{k\in\mathbb{Z}_+}$ satisfies the conditions of Theorem 2.6 with $m = 0$. Hence the scheme $\{S_{a^{(k)}}\}_{k\in\mathbb{Z}_+}$ is converging uniformly to a C^0 function. The stronger result in Theorem 2.7 allows us to conclude that $\{S_{a^{(k)}}\}_{k\in\mathbb{Z}_+}$ is $C^{\ell-1}$. Indeed, a repeated application of Theorem 2.7 with $r_k = \exp(2^{-k-1}\gamma_j)$, $k \in \mathbb{Z}_+$ for $j = 0,\ldots,\ell-2$, yields the desired property of the scheme, since $r_{k+1}^2 = r_k$, $1 - r_k^2 = 0(2^{-k})$.

2.4. Schemes with C_0^∞ basic functions

A general class of univariate ($s = 1$) non-stationary schemes generating compactly supported C^∞ basic functions is studied in [4]. The interesting examples are the C_0^∞ cardinal interpolatory basic functions ($\phi|_{\mathbb{Z}} = \delta$) and the C_0^∞ orthonormal wavelets. Typical to the masks of these examples is the linear growth of their support with the subdivision level. In this sense it is a generalization of the process leading to the up-function studied in Subsection 2.2, where the Fourier transforms $a_k(\omega) = (1/2)\sum_{\alpha\in\mathbb{Z}} a_\alpha^{(k)} e^{-i\alpha\omega}$, $k \in \mathbb{Z}_+$ of the masks $\{a^{(k)}\}_{k\in\mathbb{Z}_+}$, given by (2.7) are of the form

$$a_k(\omega) = \exp(-i(k+1)(\omega/2)) \cos^{k+1}(\omega/2), \quad k \in \mathbb{Z}_+ \ . \tag{2.17}$$

Theorem 2.8 (Convergence and smoothness [4]). *Let $a(\omega)$ be a 2π-periodic function satisfying*

$$a(0) = 1, \quad a(\omega) = \cos^r(\omega/2)q(\omega), \quad \sup_{\omega\in[0,2\pi]} \prod_{j=1}^{m} |q(2^{j-1}\omega)| < 2^{rm} \ , \tag{2.18}$$

for some $r > 0$ and $m \in \mathbb{Z}_+ \setminus \{0\}$, with q bounded and Hölder continuous at the origin. Let $\{a^{(k)}\}_{k\in\mathbb{Z}_+}$ be a sequence of masks of finite support with Fourier transforms $a_k(\omega)$ satisfying the four requirements

(i) $||a_k||_\infty \leq C$, $k \in \mathbb{Z}_+$.

(ii) $|a_k(\omega)| \leq (1+\mu_k)|a(\omega)|^k$, $k \in \mathbb{Z}_+$, $\sum_{k\in\mathbb{Z}_+} |\mu_k| < \infty$.

(iii) $\sum_{k\in\mathbb{Z}_+} \deg(a_k) 2^{-k} < \infty$.

(iv) $\sum_{k\in\mathbb{Z}_+} |a_k(0) - 1| < \infty$.

Then the subdivision scheme $\{S_{a^{(k)}}\}_{k\in\mathbb{Z}_+}$ is uniformly convergent and its basic function ϕ_0 is a compactly supported C^∞ function. Moreover the sequence of functions

$$F_k = \sum_{\alpha\in\mathbb{Z}} f_\alpha^k \psi(2^k \cdot -\alpha), \quad f^k = S_{a^{(k-1)}} \cdots S_{a^{(0)}}\delta, \quad k \in \mathbb{Z}_+ \ , \tag{2.19}$$

with $\psi = \sin(2\pi\cdot)/(2\pi\cdot)$, *and the sequences of their derivatives, converge uniformly to* ϕ_0 *and its derivatives:*

$$\lim_{k\to\infty} ||F_k^{(j)} - \phi_0^{(j)}||_\infty = 0, \quad j \in \mathbb{Z}_+ \ . \tag{2.20}$$

Note that requirement (iii) guarantees the finite support of the basic function ϕ_0 of the scheme $\{S_{a^{(k)}}\}_{k\in\mathbb{Z}_+}$, with

$$\deg(a_k) = \max\{\alpha : a_\alpha^{(k)} \neq 0\} - \min\{\alpha : a_\alpha^{(k)} \neq 0\} \ .$$

This is the case since

$$|\operatorname{supp}\phi_0| \le \sum_{k\in\mathbb{Z}_+} 2^{-k-1}\deg(a_k) \ .$$

Approximation orders of the shift invariant spaces of the multiresolution analysis determined by $\{a^{(k)}\}_{k\in\mathbb{Z}_+}$ are studied in [4], and similar results to those in Theorem 2.4 for approximation in the L_2 – norm are obtained for approximation in Sobolev norms..

Theorem 2.9 (Spectral approximation in Sobolev norms [4]). *Let* $\{a^{(k)}\}_{k\in\mathbb{Z}_+}$ *satisfy the assumptions of Theorem 2.8 and the additional condition*

$$|a_k(\omega)| \ge (1-\nu_k)|A(\omega)|^k \ ,$$

where $|\nu_k| < 1$, $\sum_{k\in\mathbb{Z}_+} |\nu_k| < \infty$, $A(0) = 1$ *and* $A(\cdot)$ *is Hölder continuous at the origin. Then the spaces* $\{V_j\}_{j\in\mathbb{Z}_+}$ *generated by this subdivision have density order* r *in* H^s *for all* $r \ge s$, *namely for* $r \ge s$ *and* $f \in H^r = \{g : \|g\|_r = \int_{\mathbb{R}} |\widehat{g}(\omega)|^2(1+|\omega|^{2r})d\omega < \infty\}$

$$\inf_{v\in V_j} \|f - v\|_s \le C2^{j(s-r)}\|f\|_r\epsilon(f,j) \ ,$$

with $0 \le \epsilon(f,j) \le 1$ *and* $\lim_{j\to\infty}\epsilon(f,j) = 0$. *Moreover, there exists* $t \in (0,\pi]$ *such that the sequence of the* L_2*-projections of* S_jf *into* V_j, $j \in \mathbb{Z}_+$, *with* $(S_jf)^\wedge = \chi_{[-2^jt,2^jt]}\widehat{f}$, *constitutes an approximation scheme which achieves density order* r *in* H^s *for all* $r \ge s$.

Example – C_0^∞ **cardinal interpolatory basic functions.** The construction is based on a family of stationary univariate interpolatory subdivision schemes with masks $\{a^{(n)}\}_{n\in\mathbb{Z}_+}$, introduced in [11]. The stationary scheme with the mask $a^{(n)}$ reproduces polynomials of degree $\le 2n+1$. The non-zero elements of $a^{(n)}$ are

$$a_0^{(n)} = 1, \quad a_{2\alpha+1}^{(n)} = a_{-2\alpha-1}^{(n)} = \ell_\alpha(0), \quad \alpha = 0,1,\dots,n \ , \tag{2.21}$$

where $\ell_\alpha(x) \in \pi_{2n+1}$ satisfies $\ell_\alpha\big((2\beta+1)/2\big) = \delta_{\alpha,\beta}$, $\beta = -n-1, -n, \dots, n$.

The Fourier transform of the mask (2.21) is [11]

$$a_n(\omega) = \cos^{2n+2}(\omega/2)\sum_{j=0}^{n}\binom{n-1-j}{j}y^j, \quad y = \sin^2(\omega/2) \ . \tag{2.22}$$

Thus $a_n(0) = 1$, $\deg(a_n) = 4n+2$, and requirements (iii) and (iv) of Theorem 2.8 hold.

It is shown in [4] that $|a_n(\omega)| \le \big(a(\omega)\big)^n$ where

$$a(\omega) = \begin{cases} 1, & |\omega| \le \frac{\pi}{2} \ , \\ \sin^2\omega, & \frac{\pi}{2} \le |\omega| < \pi \ . \end{cases} \tag{2.23}$$

Hence $||a_n||_\infty \le 1$, and requirements (i) and (ii) of Theorem 2.8 are valid. Defining $a(\omega) = \cos^2(\omega/2)q(\omega)$ we get from (2.23) that $q(\omega) < q(\pi) = 4$ for $|\omega| < \pi$, and hence (2.18) holds with $r = 2$ and $m = 2$. Thus the non-stationary scheme $\{S_{a^{(k)}}\}_{k\in\mathbb{Z}_+}$ with $a^{(k)}$ given by (2.21) or (2.22) generates a C_0^∞ interpolatory basic function ϕ_0 satisfying $\phi_0|_{\mathbb{Z}} = \delta$. In fact, for any $j \in \mathbb{Z}_+$, the basic function ϕ_j corresponding to the scheme $\{S_{a^{(k+j)}}\}_{k\in\mathbb{Z}_+}$ is C_0^∞ and interpolatory. Also, $\{\phi_j\}$ generates multiresolution analysis, with the property of spectral approximation in Sobolev norms.

It should be noted that although $a^{(n)}$ in this example is stationary for $n \in \mathbb{Z}_+$ [11], Theorem 2.1 does not apply, since

$$||S^{\Delta}_{a^{(n)}}||_\infty > 1, \quad n \ne 0 \ .$$

§3. Non-Uniform Discrete Schemes

First we consider the case $s = 1$ and non-uniformity according to domain as in (1.10). Let $\Omega_j = [x_j, x_{j+1})$, $j \in J \subset \mathbb{Z}$, with $\{x_j\}_{j\in J} \subset \mathbb{Z}$ satisfying $x_{j+1} - x_j \ge L > 0$, $j \in J$, and $\bigcup_{j\in J}\Omega_j = \mathbb{R}$. We are concerned with non-uniform schemes of the form

$$f_\alpha^{k+1} = \sum_\beta a^{(\ell)}_{\alpha-2\beta} f^k_\beta, \quad \alpha 2^{-k-1} \in \Omega_\ell \ , \tag{3.1}$$

where $a^{(\ell)}$ is the mask applicable in Ω_ℓ.

To each point x_j there corresponds an index set I_j and a matrix A_j of the form

$$(A_j)_{\alpha\beta} = \begin{cases} a^{(j-1)}_{\alpha-2\beta}, & \alpha < 0 \\ a^{(j)}_{\alpha-2\beta}, & \alpha \ge 0 \end{cases} \qquad (\alpha,\beta) \in I_j \ . \tag{3.2}$$

The index set I_j is given by $I_j = \{\alpha : \alpha + 1 \in \widetilde{I}_j \quad \text{or} \quad \alpha - 1 \in \widetilde{I}_j\}$, where $\widetilde{I}_j$ is the smallest set of integers including zero with the property

$$(-1)^\ell\alpha \ge \ell, \quad \alpha \in \widetilde{I}_j, \quad a^{(j-\ell)}_{\alpha\beta} \ne 0 \Rightarrow \beta \in \widetilde{I}_j \quad \text{for} \quad \ell = 0,1 \ . \tag{3.3}$$

The cardinality of I_j is denoted by $|I_j|$.

Theorem 3.1 (Convergence and smoothness [14]). *Let $m \in \mathbb{Z}_+$, and let $\{S_{a^{(j)}}\}$ be a C^m stationary scheme for $j \in J$. If for all $j \in J$ the eigenvalues of A_j, $\{\lambda_i^{(j)}\}_{i=1}^{|I_j|}$ have the form*

$$\lambda_i^{(j)} = 2^{-i}, \quad i = 0, \ldots, m, \quad |\lambda_i| < 2^{-m}, \quad i = m+1, \ldots, |I_j| \,, \tag{3.4}$$

then the scheme (3.1) is C^m.

Results on the convergence of non-uniform schemes of the type (1.11) with non-negative masks and of several generalizations are obtained in [2]. Here we cite one result which applies to non-uniform and non-stationary schemes.

Theorem 3.2 (Convergence [2]). *Let $\gamma, \eta \in \mathbb{Z}_+^s$ such that $\gamma_i < \eta_i$, $i = 1, \ldots, s$, and let $\{A^{(k)}\}_{k \in \mathbb{Z}_+}$ be a sequence of stochastic matrices with the property*

$$A_{\alpha\beta}^{(k)} \neq 0 \quad \textit{only if} \quad \gamma_i \leq \alpha_i - 2\beta_i \leq \eta_i, \quad i = 1, \ldots, s \,. \tag{3.5}$$

If moreover, $A_{\alpha\beta}^{(k)} \geq \varepsilon > 0$ for $\alpha_i - 2\beta_i \in \{\gamma_i, \gamma_i + 1, \eta_i - 1, \eta_i\}$, $i = 1, \ldots, s$, then the scheme

$$P^{k+1} = A^{(k)} P^k, \quad k \in \mathbb{Z}_+ \,,$$

is uniformly convergent.

§4. Integral Subdivision and Weak Convergence

The discrete binary subdivision process generates points $\{P_\alpha^{k+1}\}$ at level $k+1$ by a linear operation on $\{P_\alpha^k\}$, where the operator is a two slanted matrix [2]. Viewing $\{P_\alpha^k\}$ as values of a function F_k at the points $\{2^{-k}\alpha\}$ respectively, the slanted matrix operator can be viewed as a discrete convolution operator. In the following we discuss the natural extension of the discrete binary subdivision to integral binary subdivision where the entities are functions instead of sequences of points, and the operators are convolutions. Here the mask is a function $a \in L^1(\mathbb{R}^s)$ of compact support, and starting with an initial 'control' function $F_0 : \mathbb{R}^s \to \mathbb{R}^d$ we recursively define a sequence of functions $\{F_k\}$ by the rule

$$F_{k+1} = 2^{ks} a(2^{k+1} \cdot) * F_k = 2^{ks} \int_{\mathbb{R}^s} F_k(\tau) a\big(2^{k+1}(t - \tau)\big) dt, \quad k \in \mathbb{Z}_+ \,. \tag{4.1}$$

As shown in the introduction, the rule (4.1) is derived from (1.12) by the substitution $F_k(t) = P^k(2^k t)$. It is interesting to note that the two slanted nature of the discrete subdivision, which is still apparent as the two slanted convolution (1.12), is absent when we consider the function $F_k(t) = P^k(2^k t)$. In (4.1) the operator is a simple convolution operator on F_k defined by a weight function whose support is exponentially decreasing with k. A necessary condition for the convergence of the integral subdivision process is that $\int_{\mathbf{R}^s} a(\tau) d\tau = 2^s$, which simply means that $2^{ks} a(2^{k+1} \cdot) \to \delta(\cdot)$ as $k \to \infty$ [10].

Example – the up-function. The up-function derived in Section 2 by a non-stationary subdivision scheme is also the natural example for integral subdivision. We consider $s = d = 1$ and the mask

$$a(x) = \chi_{[-1,1]}(x) = \begin{cases} 1, & -1 \le x \le 1 \ , \\ 0, & \text{otherwise} \ . \end{cases} \tag{4.2}$$

Starting the integral subdivision process with $F_1 = a(2\cdot)$ (corresponding to the initial distribution $F_0 = \delta(\cdot)$), the sequence $\{F_k\}$ is uniformly convergent to the up-function, up $(\cdot) \in C_0^\infty$ [10].

As in the discrete case, the integral subdivision process is related to a dilation equation which is now an integral functional equation

$$F(x) = \int_{\mathbb{R}^s} a(t)F(2x-t)dt = (a * F)(2x) \ . \tag{4.3}$$

For example, the up-function satisfies the functional equation

$$\text{up}(x) = \int_{-1}^{1} \text{up}(2x-t)dt \ . \tag{4.4}$$

The convergence of integral subdivision processes, the nature of the limit functions and the relation to the integral functional equation, are discussed in [10] for $s = 1$ and in [7] for $s \ge 1$. The following theorems and example are straightforward extensions of some of the results in [10] to the case $s \ge 1$. The analysis is based on Fourier methods and the results depend strongly on the decay rate of the Fourier transform $\widehat{a}(\lambda)$ of a as $|\lambda| \to \infty$. Here $|\lambda| = \sum_{i=1}^{s} |\lambda_i|$.

Theorem 4.1 (Convergence [10]) . *If the infinite product* $\prod_{j=1}^{\infty}\left(2^{-s}\widehat{a}(\lambda 2^{-j})\right)$ *is convergent in* $L^1(\mathbb{R}^s)$ *to* $\widehat{F}$, *and if we apply the integral subdivision process (1.5) starting with* $F_1 = a(2\cdot)$, *then* $\{F_k\}$ *converges uniformly on* $\mathbb{R}^s$ *to* F *which is a solution of the integral functional equation (4.3).*

The condition $\widehat{a}(0) = 2^s$ is evidently a necessary condition for the convergence of the above infinite product, and for the existence of an L^1 solution of (4.3) with $\widehat{F}(0) \ne 0$. The limit function F in Theorem 4.1 is termed "the basic function" of the integral subdivision (4.1).

Theorem 4.2 (Uniform convergence to C^∞ functions [10]). *Let* a *be of finite support and satisfy for some* $r > 0$,

$$\int_{\mathbb{R}^s} a(t)dt = 2^s, \quad \int_{\mathbb{R}^s} |a(t)|dt \le 2^{s+1}, \quad |\widehat{a}(\lambda)| = o(|\lambda|^{-r}) \quad \text{as } |\lambda| \to \infty \ . \tag{4.5}$$

Then the integral subdivision process starting with $F_1 = a(2\cdot)$ *is a sequence which converges uniformly on* $\mathbb{R}^s$ *to a* C_0^∞ *function* F. *Moreover any derivative of* $\{F_k\}$ *is a sequence which converges uniformly on* $\mathbb{R}^s$ *to the corresponding derivative of* F.

The asymptotic behavior of $\widehat{a}(\lambda)$ as $\lambda \to 0$ and as $|\lambda| \to \infty$ gives us indication on the continuity of the limit function F:

Theorem 4.3 (Smoothness of limit function [10]). *Let $\widehat{a}$ satisfy the two conditions*

$$|\widehat{a}(\lambda) - 2^s| = 0(|\lambda|^\gamma) \quad \text{as} \quad |\lambda| \to 0, \quad \gamma > 0 , \tag{4.6}$$

and

$$|a(\lambda)| < q \leq 2^{-n-s+1-\varepsilon} \quad \text{for} \quad |\lambda| > R, \quad \varepsilon > 0 . \tag{4.7}$$

Then the integral subdivision process converges weakly to $F \in C^n(I\!R^s)$.

Corollary 4.4. *If $\widehat{a}$ satisfies (4.6) and also*

$$|\widehat{a}(\lambda)| = o(1) \quad \text{as} \quad |\lambda| \to \infty , \tag{4.8}$$

then the integral subdivision converges weakly to $F \in C^\infty(I\!R^s)$.

In the up-function example $\widehat{a}(\lambda) = 2\frac{\sin\lambda}{\lambda}$ and a satisfies the conditions of Theorem 4.2. Thus the uniform convergence and the C^∞ property follow.

The relation between the integral subdivision and the non-stationary subdivision which appears in the case of the up-function is not a coincidence. The following theorem reveals that if the mask $a(\cdot)$ of the integral subdivision process (or the integral functional equation) is itself a basic function of a discrete uniform binary subdivision (stationary or not), then the basic function of the integral subdivision process can be obtained as the limit of a non-stationary discrete subdivision process.

Theorem 4.5 (Relation between integral and non-stationary subdivisions). *Let $a(2\cdot) \in L^1(I\!R^s)$ be the basic function of a non-stationary subdivision process with coefficients $\{c_\beta^{(k)}\}$ satisfying $\sum_{\beta\in\mathbb{Z}^s} c_\beta^{(k)} = 2^s$. Let $c_k(\omega) = 2^{-s}\sum_{\beta\in\mathbb{Z}^s} c_\beta^{(k)} e^{-i\beta\omega}$, and define another non-stationary subdivision process with coefficients $\{b_\alpha^{(k)}\}$ determined by the trigonometric equality*

$$b_k(\omega) = 2^{-s}\sum_{\alpha\in\mathbb{Z}^s} b_\alpha^{(k)} e^{-i\alpha\omega} = \prod_{\ell=0}^{k} c_\ell(\omega) . \tag{4.9}$$

If the non-stationary process, $\{S_{b^{(k)}}\}_{k\in\mathbb{Z}_+}$, converges weakly to an L^1 limit function, then its basic function is the basic function of the integral subdivision process with the mask a, and it is a solution of the integral functional equation with the mask a.

Example – box-up functions. Consider an integral subdivision process with a mask $a = B(\cdot/2)$, where $B(x)$ is a box-spline function in $I\!R^s$ centered at the origin and normalized such that $\int_{\mathbf{R}^s} B(x)dx = 1$. Clearly this a satisfies condition (4.5) of Theorem 4.2, and the integral subdivision process with the initial function $F_1 = a$ is uniformly convergent to $F_B \in C^\infty(I\!R^s)$. Recalling the up-function example, we term F_B a box-up function. Directly from the infinite convolution process and the non-negativity of B, it follows that

$$\operatorname{supp} F_B = \operatorname{supp} B(\cdot/2) .$$

Also, as in the case of the up-function, the integer shifts of F_B, $\{F_B(\cdot - \alpha) : \alpha \in \mathbb{Z}^s\}$ are linearly independent if $\{B(\cdot - \alpha) : \alpha \in \mathbb{Z}^s\}$ are, and both spaces reproduce the same subspace of polynomials. Furthermore, since box-splines based on a set of directions which includes a subset with determinant ± 1, are basic functions of a discrete stationary subdivision process [3], we get by Theorem 4.5 that the box-up function F_B can also be computed by a discrete non-stationary subdivision with easily computable masks.

As is shown in the Introduction, by allowing the mask and the initial F_0 in (4.1) to be distributions of δ-type, the integral subdivision scheme (4.1) gives rise to the discrete subdivision scheme (1.1), and the notion of weak (distributional) convergence becomes natural. The following is a strong result on weak convergence of discrete schemes:

Theorem 4.6 (Weak convergence [10]). *The discrete uniform stationary binary subdivision process (1.1) with a mask a satisfying* $\sum_{\alpha \in \mathbb{Z}^s} a_\alpha = 2^s$, *is always weakly convergent. Its basic function is the weak solution of the functional equation (1.7).*

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