

Incidences between points and lines on two- and three-dimensional varieties*

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Abstract

Let P be a set of m points and L a set of n lines in \mathbb{R}^4 , such that the points of P lie on an algebraic three-dimensional surface of degree D that does not contain hyperplane or quadric components, and no 2-flat contains more than s lines of L . We show that the number of incidences between P and L is

$$I(P, L) = O\left(m^{1/2}n^{1/2}D + m^{2/3}n^{1/3}s^{1/3} + nD + m\right),$$

for some absolute constant of proportionality. This significantly improves the bound of the authors [33], for arbitrary sets of points and lines in \mathbb{R}^4 , when D is not too large. Moreover, when D and s are constant, we get a *linear* bound. The same bound holds when the three-dimensional surface is embedded in any higher dimensional space.

For the proof of this bound, we revisit certain parts of [33], combined with the following new incidence bound, for which we present a direct and fairly simple proof. Let P be a set of m points and L a set of n lines in \mathbb{R}^d , for $d \geq 3$, which lie in a common two-dimensional algebraic surface of degree D (assumed to be $\ll n^{1/2}$) that does not contain any 2-flat, so that no 2-flat contains more than s lines of L (here we require that the lines of L also be contained in the surface). Then the number of incidences between P and L is

$$I(P, L) = O\left(m^{1/2}n^{1/2}D^{1/2} + m^{2/3}D^{2/3}s^{1/3} + m + n\right).$$

When $d = 3$, this improves the bound of Guth and Katz [14] for this special case, when $D \ll n^{1/2}$. Moreover, the bound does not involve the term $O(nD)$. This term arises in most standard approaches, and its removal is a significant aspect of our result. Again, the bound is linear when $D = O(1)$.

Finally, we also obtain (slightly weaker) variants of both results over the complex field. For two-dimensional varieties, when the ambient space is \mathbb{C}^3 (or any higher-dimensional \mathbb{C}^d), under the same assumptions as above, we have

$$I(P, L) = O\left(m^{1/2}n^{1/2}D^{1/2} + m^{2/3}D^{2/3}s^{1/3} + D^3 + m + n\right).$$

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For three-dimensional varieties, embedded in \mathbb{C}^4 (or any higher-dimensional \mathbb{C}^d), under the same assumptions as above, we have

$$I(P, L) = O\left(m^{1/2}n^{1/2}D + m^{2/3}n^{1/3}s^{1/3} + D^6 + m + nD\right).$$

These new incidence bounds are among the very few bounds, known so far, that hold over the complex field. The bound for two-dimensional (resp., three-dimensional) varieties coincides with the bound in the real case when $D = O(m^{1/3})$ (resp., $D = O(m^{1/6})$).

1 Introduction

Let P be a set of m distinct points and L a set of n distinct lines in \mathbb{R}^d or in \mathbb{C}^d . Let $I(P, L)$ denote the number of incidences between the points of P and the lines of L ; that is, the number of pairs (p, ℓ) with $p \in P$, $\ell \in L$, and $p \in \ell$. If all the points of P and all the lines of L lie in a common 2-flat, then, in the real case, the classical Szemerédi–Trotter theorem [39] yields the worst-case tight bound

$$I(P, L) = O\left(m^{2/3}n^{2/3} + m + n\right). \tag{1}$$

The same bound also holds in the complex plane, as shown later by Tóth [40] and Zahl [41].

This bound clearly also holds in \mathbb{R}^d and in \mathbb{C}^d , for any d , by projecting the given lines and points onto some generic 2-flat. Moreover, the bound will continue to be worst-case tight by placing all the points and lines in a common 2-flat, in a configuration that yields the planar lower bound.

In the 2010 groundbreaking paper of Guth and Katz [14], an improved bound for the real case has been derived for $I(P, L)$, for a set P of m points and a set L of n lines in \mathbb{R}^3 , provided that not too many lines of L lie in a common plane. Specifically, they showed:¹

Theorem 1.1 (Guth and Katz [14]). *Let P be a set of m distinct points and L a set of n distinct lines in \mathbb{R}^3 , and let $s \leq n$ be a parameter, such that no plane contains more than s lines of L . Then*

$$I(P, L) = O\left(m^{1/2}n^{3/4} + m^{2/3}n^{1/3}s^{1/3} + m + n\right).$$

This bound (or, rather, an alternative formulation thereof) was a major step in the derivation of the main result of [14], which was an almost-linear lower bound on the number of distinct distances determined by any finite set of points in the plane, a classical problem posed by Erdős in 1946 [7]. Guth and Katz’s proof uses several nontrivial tools from algebraic and differential geometry, most notably the Cayley–Salmon–Monge theorem on osculating lines to algebraic surfaces in \mathbb{R}^3 , and various properties of ruled surfaces. All this machinery comes on top of the major innovation of Guth and Katz, the introduction of the *polynomial partitioning technique*.

¹We skip over certain subtleties in their bound: They also assume that no *regulus* contains more than $s = \sqrt{n}$ input lines, but then they are able also to bound the number of intersection points of the lines. Moreover, if one also assumes that each point is incident to at least three lines then the term m in the bound can be dropped.

For the purpose of the analysis in this paper, it is important to recall, right away, that the polynomial partitioning technique holds only over the reals. This will be the major stumbling block that we will face as we handle the complex case. We overcome (or rather bypass) it by exploiting the fact that all the lines are also contained in the given variety; see below for details.

In four dimensions, and for the real case, the authors established in [33] a sharper and (almost) optimal bound. More precisely, they have shown:

Theorem 1.2. *Let P be a set of m distinct points and L a set of n distinct lines in \mathbb{R}^4 , and let $s \leq q \leq n$ be parameters, such that (i) no hyperplane or quadric contains more than q lines of L , and (ii) no 2-flat contains more than s lines of L . Then,*

$$I(P, L) \leq 2^{c\sqrt{\log m}} \left(m^{2/5} n^{4/5} + m \right) + A \left(m^{1/2} n^{1/2} q^{1/4} + m^{2/3} n^{1/3} s^{1/3} + n \right), \quad (2)$$

for suitable (absolute) constant parameters A and c . Moreover, except for the factor $2^{c\sqrt{\log m}}$, the bound is tight in the worst case, for all m and n and suitable ranges of q and s . For certain ranges of m and n the bound holds without that factor.

Our results. In this work we “contest” the leading terms $O(m^{1/2}n^{3/4})$ (for $d = 3$) and $2^{c\sqrt{\log m}}m^{2/5}n^{4/5}$ (for $d = 4$), and show that in many situations they can be significantly improved. See item (2) in the discussion in Section 5 for further elaboration of this issue. A major feature of this work is that, in the setups considered here, the analysis can also be carried over to the complex domain, except for a small penalty that we pay for bypassing the polynomial partitioning technique, which, as noted, only holds over the reals.

Concretely, we assume that the points of P lie on some algebraic variety, and derive significantly improved bounds when the degree of the variety is not too large. In the former case we assume that the points and the lines lie on a two-dimensional variety, which is allowed to be embedded in any \mathbb{R}^d , for $d \geq 3$. In the latter case we assume that the points (but not necessarily the lines) lie on a three-dimensional variety, embedded in any \mathbb{R}^d , for $d \geq 4$. In the former (resp., latter) case we also assume that the variety contains no plane (resp., no hyperplane or quadric). Thus, in addition to improving the respective bounds in Theorems 1.1 and 1.2, for the special cases under consideration, and in extending them to the complex domain, we obtain an extra bonus by extending the results to two-dimensional and three-dimensional varieties embedded in any higher dimension.

Points on a two-dimensional variety. We derive two closely related results, one that holds over the real field and one that holds also over the complex field. It is simplest to think of the variety as embedded in \mathbb{R}^3 or in \mathbb{C}^3 . The real case is a special case of the setup of Guth and Katz [14], where there is no need to use the polynomial partitioning method, because we assume that the points and lines all lie in a common surface (the zero set of a polynomial) of degree D . This very assumption is also the one that lets us derive the (slightly weaker) version that holds over \mathbb{C} , thereby constituting a significant progress over the existing theory of incidences in three (and higher) dimensions. (To be more precise, over the reals we do apply the polynomial partitioning technique (as a step in the application of the Guth-Katz bound), but only to a small subset of the lines.)

Concretely, our first main result, for this setup, is the following theorem.

Theorem 1.3. (a) The real case: *Let P be a set of m distinct points and L a set of n distinct lines in \mathbb{R}^d , for any $d \geq 3$, and let $2 \leq s \leq D$ be two integer parameters, so that all the points and lines lie in a common two-dimensional algebraic variety V of degree D that does not contain any 2-flat, and so that no 2-flat contains more than s lines of L . Then*

$$I(P, L) = O\left(m^{1/2}n^{1/2}D^{1/2} + m^{2/3}D^{2/3}s^{1/3} + m + n\right). \quad (3)$$

(b) The complex case: *Under exactly the same assumptions, when the ambient space is \mathbb{C}^d , for any $d \geq 3$, we have*

$$I(P, L) = O\left(m^{1/2}n^{1/2}D^{1/2} + m^{2/3}D^{2/3}s^{1/3} + D^3 + m + n\right). \quad (4)$$

The assumption that s is at most D can be dropped, because, for any 2-flat π , the intersection $\pi \cap V$ is a one-dimensional plane algebraic curve of degree at most D in π (this holds since V does not contain any 2-flat), and can therefore contain at most D lines.

We also have the following easy and interesting corollary.

Corollary 1.4. *Let P be a set of m distinct points and L a set of n distinct lines in \mathbb{R}^d or in \mathbb{C}^d , for any $d \geq 3$, such that all the points and lines lie in a common two-dimensional algebraic variety of constant degree that does not contain any 2-flat. Then $I(P, L) = O(m + n)$, where the constant of proportionality depends on the degree of the surface.*

For $d = 3$, the corollary can also be derived, for the real case, from the analysis in Guth and Katz [14], using a somewhat different approach. Moreover, although not explicitly stated there, it seems that the argument in [14] also works over \mathbb{C} . As a matter of fact, the corollary can also be extended (with a different bound though) to the case where the containing surface may have planar components. See a remark to that effect in Corollary 5.1 in the concluding section.

We also exploit the proof technique of Theorem 1.3 to derive an upper bound of $O(nD)$ on the number of 2-rich points determined by a set of n lines contained in a variety, as above, in both the real and complex cases. See Section 5 for details.

We note that if we only assume that the points lie on V but the lines are arbitrary, we will incur the term $O(nD)$, which we are trying to avoid. In the case of three-dimensional varieties, discussed below, we cannot avoid this term even when the lines lie on the variety.

The significance of Theorem 1.3 is fourfold:

(a) First and foremost, the theorem yields a new incidence result for points and lines on a two-dimensional variety *over the complex field*, in three and higher dimensions. Incidence results over the complex domain are rather rare. They include (as already mentioned) Tóth's extension of the Szemerédi-Trotter bound to the complex plane [40], which was the only result of that kind that predated the introduction of the algebraic machinery by Guth and Katz, and several more recent works [34, 36, 37, 41] (where the latter work [41] provides an alternative algebraic derivation of Tóth's bound).

(b) In the real three-dimensional case, the bound improves the Guth–Katz bound when $D \ll n^{1/2}$, for two-dimensional varieties V that do not contain planes. Note that the

threshold $n^{1/2}$ is a natural one because, as is well known and easy to show, any set of n lines in \mathbb{R}^3 admit a polynomial of degree $O(n^{1/2})$ whose zero set contains all the lines; a simple modification of the construction applies in higher dimensions too. (Of course the comparison is far from perfect, because this polynomial may have many linear components, in which case our bound does not apply. Still, it offers some basis for evaluating the quality of our bound.) In three dimensions, this threshold is in fact larger than the standard degree $O(m^{1/2}/n^{1/4})$ used in the analysis of Guth and Katz [14], when $m < n^{3/2}$.

(c) Another significant feature of our bound is that it does not contain the term nD , which arises naturally in [14] and other works, and seems to be unavoidable when P is an arbitrary set of points. When D is not a constant, this becomes a crucial feature of the new bound, which has already been exploited in the analysis in [33], and is also used in the second main result of this paper, Theorem 1.5 below.² See an additional discussion of this feature at the end of the paper.

(d) It offers a sharper point-line incidence bound in arbitrary dimensions, for the special case assumed in the theorem (which again holds over the complex field too).

Theorem 1.3(a) has been used, as one of the key tools, in the analysis in our paper [33] on incidences between points and lines in four dimensions (see below). In this application, the absence of the term nD is a crucial feature of our result, which was required in the scenario considered in [33].

The proof of Theorem 1.3 makes extensive use of several properties of ruled surfaces in \mathbb{R}^3 or in \mathbb{C}^3 . While these results exist as folklore in the literature, and short proofs are provided for some of them, e.g., in [14], we include here detailed and rigorous proofs thereof, making them more accessible to the combinatorial geometry community. Other recent expositions include Guth’s recent survey [11] and book [12], and a survey by Kollár [19].³

Points on a three-dimensional variety. Our second main result deals with the case where the points lie on a three-dimensional variety, embedded in \mathbb{R}^4 or in \mathbb{C}^4 , or in any higher dimension. Similar to the three-dimensional case discussed above, we have to be careful here too, because hyperplanes and 3-quadrics (in \mathbb{R}^4 , and, a posteriori, in \mathbb{C}^4 too) admit “too many” incidences in the worst case. That is, by a generalization of Elekes’s construction [5], there exists a configuration of m points and n lines on a 3-flat with $\Theta(m^{1/2}n^{3/4})$ incidences. More recently, Solomon and Zhang [35] established an analogous statement for three-dimensional quadrics, when $n^{9/8} < m < n^{3/2}$. Concretely, for such values of m and n , they have constructed a quadric $S \subset \mathbb{R}^4$, a set P of m points on S , and a set L of n lines contained in S , so that (i) the number of lines in any common 2-flat is $O(1)$, (ii) the number of lines in any hyperplane is $O(n/m^{1/3})$, and (iii) the number of incidences between the points and lines is $\Omega(m^{2/3}n^{1/2})$, which is asymptotically larger than the corresponding bound in (2) (for $s = O(1)$), when $n^{9/8} \ll m \leq n^{3/2}$.

In other words, when studying incidences with points on a variety in \mathbb{R}^4 or in \mathbb{C}^4 , the cases where the variety is (or contains) a hyperplane or a quadric are special and do not yield sharper bounds. (In the real case, the case of a hyperplane puts us back in \mathbb{R}^3 , where

²Although the bound in Theorem 1.5 does contain the term nD , it still crucially relies on the absence of this term in the bound for two-dimensional varieties.

³While there is (naturally) some overlap between these surveys and our exposition, the main technical properties that we present do not seem to be rigorously covered in the other works.

the best bound is Guth and Katz's in Theorem 1.1, and the case of a quadric reduces to the same setup via a suitable generic projection onto \mathbb{R}^3 .) In the second main result of this paper, we show that if all the points lie on a three-dimensional algebraic variety of degree D without 3-flat or 3-quadric components, and if no 2-flat contains more than s lines, then, if D and s are not too large, the bound becomes significantly smaller. Moreover, here too we get a real version and a complex version of the theorem. Specifically, we show:

Theorem 1.5. (a) The real case: *Let P be a set of m distinct points and L a set of n distinct lines in \mathbb{R}^d , for any $d \geq 4$, and let s and D be parameters, such that (i) all the points of P lie on a three-dimensional algebraic variety of degree D , without 3-flat or 3-quadric components, and (ii) no 2-flat contains more than s lines of L . Then*

$$I(P, L) = O\left(m^{1/2}n^{1/2}D + m^{2/3}n^{1/3}s^{1/3} + nD + m\right). \quad (5)$$

When D and s are constants, we get the linear bound $O(m + n)$.

(b) The complex case: *Under exactly the same assumptions, when the ambient space is \mathbb{C}^d , for any $d \geq 4$, we have*

$$I(P, L) = O\left(m^{1/2}n^{1/2}D + m^{2/3}n^{1/3}s^{1/3} + D^6 + nD + m\right). \quad (6)$$

Remarks. (1) Note that for $D < \min\{m^{1/2}/n^{1/4}, n^{1/4}\}$, our bound for the real case is sharper than the bound of Guth and Katz [14] (note that $m^{1/2}/n^{1/4}$ is the degree of the partitioning polynomial used in the analysis of [14] for $m \leq n^{3/2}$). On the other hand, when $D > \min\{m^{1/2}/n^{1/4}, n^{1/4}\}$, our bound is not the best possible. Indeed, in this case we can project P and L onto some generic 3-flat, and apply instead the bound of [14] to the projected points and lines (we also show that a generic choice of the image 3-flat also ensures that no 2-flat contains more than s of the projected lines), which is sharper than ours for these values of D . In the complex case, we also need to assume that D is small enough so that the term D^6 does not dominate the other terms, so as to make the bound look like the bound in the real case. 6

(2) As already noted earlier, here we do not have to insist that the lines of L be contained in the variety. A line not contained in a variety of degree D can intersect it at most D times, so the number of incidences with such lines is at most nD . (The actual argument that yields this term is more involved, because we apply the argument to a variety of larger degree; see below for details.) For two-dimensional varieties, since we want to avoid this term, we require the lines to be contained in the variety. Here, since we allow this term, the lines can be arbitrary.

(3) The assumption that the points of P lie on a variety is not as restrictive as it might sound, because, in four dimensions, one can always construct a polynomial f of degree $O(m^{1/4})$ whose zero set contains all the given points, or alternatively, a polynomial f of degree $O(n^{1/3})$ whose zero set fully contains all the given lines. The assumptions become restrictive, and the bound becomes more interesting, when D is significantly smaller. In addition, the constructed polynomial f could be one for which $Z(f)$ does contain 3-flats or 3-quadrics, so another restrictive aspect of our assumptions is that they exclude these situations.

Similar to the case of a two-dimensional variety, we also have the following easy and interesting corollary (it does not hold when V contains 3-flats or 3-quadratics, in light of the lower bound constructions in [33, 35]).

Corollary 1.6. *Let P be a set of m distinct points and L a set of n distinct lines in \mathbb{R}^d or in \mathbb{C}^d , for any $d \geq 4$, such that all the points and lines lie in a common three-dimensional algebraic variety V of constant degree that does not contain any 3-flats or 3-quadratics, and no 2-flat contains more than $O(1)$ lines of L . Then $I(P, L) = O(m + n)$, where the constant of proportionality depends on the degree of V .*

Theorem 1.3 is a key technical ingredient in the proof of Theorem 1.5. The proofs of both theorems are somewhat technical, and use a battery of sophisticated tools from algebraic geometry. Some of these tools are borrowed and adapted from our previous work [33]. Other tools involve properties of ruled surfaces, which, as already said, are established here rigorously, for the sake of completeness. Since most of the presentation and derivation of these results is within the scope of algebraic and differential geometry, we delegate most of this part to an appendix, keeping only a brief overview in the following section.

The proof of Theorem 1.3 is then presented in Section 3, and the proof of Theorem 1.5 is presented in Section 4. The concluding Section 5 discusses our results, establishes a few consequences thereof, and raises several related open problems.

2 Algebraic preliminaries and ruled surfaces

In this section we present a brief overview of ruled surfaces and some of the algebraic machinery. Full details are given in the appendix.

Ruled surfaces. We say that a real (resp., complex) surface V is *ruled by real* (resp., *complex*) *lines* if every point $p \in V$ is incident to a real (resp., complex) line that is fully contained in V . The surface is *doubly ruled* if each point is incident to two distinct lines that are fully contained in V . See the appendix for a slightly looser definition and more details, including the not-so-extensive literature on ruled surfaces. The recent survey [11] and book [12], as well as an exposition in [19] make up for this sparsity.

Flecnodes in three dimensions and the Cayley–Salmon–Monge Theorem. We first recall the classical theorem of Cayley and Salmon, also due to Monge. Consider a polynomial $f \in \mathbb{C}[x, y, z]$ of degree $D \geq 3$. A *flecnode* of f is a point p on the zero set $Z(f)$ of f , for which there exists a line that is incident to p and *osculates* to $Z(f)$ at p to order three. That is, if the direction of the line is v then $f(p) = 0$, and $\nabla_v f(p) = \nabla_v^2 f(p) = \nabla_v^3 f(p) = 0$, where $\nabla_v f, \nabla_v^2 f, \nabla_v^3 f$ are, respectively, the first, second, and third-order derivatives of f in the direction v . The *flecnode polynomial* of f , denoted FL_f , is the polynomial obtained by eliminating v , via resultants, from these three homogeneous equations (where p is regarded as a fixed parameter). As shown in Salmon [27, Chapter XVII, Section III], the degree of FL_f is at most $11D - 24$. By construction, the flecnode polynomial of f vanishes on all the flecnodes of f , and in particular on all the lines fully contained in $Z(f)$.

Theorem 2.1 (Cayley and Salmon [27], Monge [23]). *Let $f \in \mathbb{C}[x, y, z]$ be a polynomial of degree $D \geq 3$. Then $Z(f)$ is ruled by (complex) lines if and only if $Z(f) \subseteq Z(\text{FL}_f)$.*

The notions of flecnodes and of the flecnode polynomial can be extended to four dimensions, as done in [33]. Informally, the *four-dimensional flecnode polynomial* FL_f^4 of a 4-variate polynomial f is defined analogously to the three-dimensional variant FL_f , and captures the property that a point on $Z(f)$ is incident to a line that osculates to $Z(f)$ up to the *fourth* order. It is obtained by eliminating the direction v of the osculating line, using resultants, from the four homogeneous equations given by the vanishing of the first four terms of the Taylor expansion of $f(p + tv)$ near p . Clearly, FL_f^4 vanishes identically on every line that is fully contained in $Z(f)$. As in the three-dimensional case, its degree can be shown to be $O(D)$.

Landsberg [21] derives an analog of Theorem 2.1 that holds for three-dimensional surfaces (see [33, Theorem 2.11]). Specifically, Landsberg’s theorem asserts that if FL_f^4 vanishes identically on $Z(f)$, then $Z(f)$ is ruled by (possibly complex) lines. We will discuss this in more detail in Section 4. These theorems, in three and four dimensions, play an important role in the proofs of the main theorems.

Reguli. A *regulus* is the surface (in \mathbb{R}^3 or \mathbb{C}^3) spanned by all lines that meet three pairwise skew lines in 3-space.⁴ For an elementary proof that a doubly ruled surface over \mathbb{R} must be a regulus, we refer the reader to Fuchs and Tabachnikov [8, Theorem 16.4]. We give in the appendix a proof of the following more general characterization of doubly ruled surfaces in \mathbb{R}^3 or \mathbb{C}^3 .

Lemma 2.2. *Let V be an irreducible ruled surface in \mathbb{R}^3 or in \mathbb{C}^3 which is not a plane, and let $\mathcal{C} \subset V$ be an algebraic curve, such that every non-singular point $p \in V \setminus \mathcal{C}$ is incident to exactly two lines that are fully contained in V . Then V is a regulus.*

Singly ruled surfaces. Ruled surfaces that are neither planes nor reguli are called *singly ruled* surfaces (a terminology justified by Theorem 2.3, given below). A line ℓ , fully contained in an irreducible singly ruled surface V , such that every point of ℓ is “doubly ruled”, i.e., every point on ℓ is incident to another line fully contained in V , is called an *exceptional* line of V . A point $p_V \in V$ that is incident to infinitely many lines fully contained in V is called an *exceptional* point of V .

The following result is another folklore result in the theory of ruled surfaces, used in many studies (such as Guth and Katz [14]). It justifies the terminology “singly-ruled surface”, by showing that the surface is generated by a one-dimensional family of lines, and that each point on the surface, with the possible exception of points lying on some curve, is incident to exactly one generator. It also shows that there are only finitely many exceptional lines; the property that their number is at most two (see [14]) is presented later. We give (in the appendix) a detailed and rigorous proof, to make our presentation as self-contained as possible; we are not aware of any similarly detailed argument in the literature.

Theorem 2.3. (a) *Let V be an irreducible ruled two-dimensional surface of degree $D > 1$ in \mathbb{R}^3 (or in \mathbb{C}^3), which is not a regulus. Then, except for finitely many exceptional lines, the lines that are fully contained in V are parameterized by an irreducible algebraic curve Σ_0 (in the parametric Plücker space \mathbb{P}^5 that represents lines in 3-space; see the appendix), and*

⁴Technically, in some definitions (cf., e.g., Edge [4, Section I.22]) a regulus is a one-dimensional family of generator lines of the actual surface, i.e., a curve in the Plücker or Grassmannian space of lines, but we use here the alternative notion of the surface spanned by these lines.

thus yield a 1-parameter family of generator lines $\ell(t)$, for $t \in \Sigma_0$, that depend continuously on the real or complex parameter t . Moreover, if $t_1 \neq t_2$, and $\ell(t_1) \neq \ell(t_2)$, then there exist sufficiently small and disjoint neighborhoods Δ_1 of t_1 and Δ_2 of t_2 , such that all the lines $\ell(t)$, for $t \in \Delta_1 \cup \Delta_2$, are distinct.

(b) There exists a one-dimensional curve $\mathcal{C} \subset V$, such that any point p in $V \setminus \mathcal{C}$ is incident to exactly one line fully contained in V .

Exceptional lines on a singly ruled surface. In view of Theorem 2.3, every point on a singly ruled surface V is incident to at least one generator. Hence an exceptional (non-generator) line is a line $\ell \subset V$ such that every point on ℓ is incident to a generator (which is different from ℓ). We establish in the appendix the following property.

Lemma 2.4. *Let V be an irreducible ruled surface in \mathbb{R}^3 or in \mathbb{C}^3 , which is neither a plane nor a regulus. Then (i) V contains at most two exceptional lines, and (ii) V contains at most one exceptional point.*

Generic projections preserve non-planarity. In the analysis in Section 3, the goal is to project \mathbb{R}^d onto some generic 3-flat so that non-coplanar triples of lines do not project to coplanar triples. This is easily achieved by repeated applications of the following technical result, reducing the dimension one step at a time.

Lemma 2.5. *Let ℓ_1, ℓ_2, ℓ_3 be three non-coplanar lines in \mathbb{R}^d , for $d \geq 4$. Then, under a generic projection of \mathbb{R}^d onto some hyperplane H , the respective images $\ell_1^*, \ell_2^*, \ell_3^*$ of these lines are still non-coplanar.*

Real vs. complex. With a few exceptions, our analysis holds over the complex field too, and therefore most of the unfolding analysis is carried out over \mathbb{C} , the main reason being that most of the basic tools in algebraic geometry only hold over \mathbb{C} (or other algebraically closed fields). There are a few places, though, where the analysis holds only over the reals, such as in the (explicit or implicit) applications of the polynomial partitioning technique of Guth and Katz [14]. Passing to the complex domain (and sometimes also to the projective setting) does not pose any difficulties for upper bounds in real incidence problems—every real incidence will be preserved, and at worst we will be counting additional incidences, on the non-real portion of the extended varieties. With this understanding, and with the appropriate caution, we will move freely between the real and complex domains, as convenient, staying mostly in the complex domain.

3 Proof of Theorem 1.3

In most of the analysis in this section, we will consider the case $d = 3$. The reduction from an arbitrary dimension to $d = 3$ will be presented at the end of the section.

We will prove both parts of the theorem “hand in hand”, bifurcating (in a significant manner) only towards the end of the analysis.

For a point p on an irreducible singly ruled surface V , which is not the exceptional point of V (see Section 2 and the appendix for its definition), we let $\Lambda_V(p)$ denote the number of

generator lines incident to p and fully contained in V (so if p is incident to an exceptional line, we do not count that line in $\Lambda_V(p)$). We also put $\Lambda_V^*(p) := \max\{0, \Lambda_V(p) - 1\}$. Finally, if V is a cone and p_V is its exceptional point (that is, apex), we put $\Lambda_V(p_V) = \Lambda_V^*(p_V) := 0$. We also consider a variant of this notation, where we are also given a finite set L of lines (where not all lines of L are necessarily contained in V), which does not contain any of the (at most two) exceptional lines of V . For a point $p \in V$, we let $\lambda_V(p; L)$ denote the number of lines in L that pass through p and are fully contained in V , with the same provisions as above, namely that we do not count incidences with exceptional lines, nor do we count incidences occurring at an exceptional point, and put $\lambda_V^*(p; L) := \max\{0, \lambda_V(p; L) - 1\}$. If V is a cone with apex p_V , we put $\lambda_V(p_V; L) = \lambda_V^*(p_V; L) = 0$. We clearly have $\lambda_V(p; L) \leq \Lambda_V(p)$ and $\lambda_V^*(p; L) \leq \Lambda_V^*(p)$, for each point p .

Lemma 3.1. *Let V be an irreducible singly ruled two-dimensional surface of degree $D > 1$ in \mathbb{R}^3 or in \mathbb{C}^3 . Then, for any line ℓ , except for the (at most) two exceptional lines of V , we have*

$$\begin{aligned} \sum_{p \in \ell \cap V} \Lambda_V(p) &\leq D && \text{if } \ell \text{ is not fully contained in } V, \\ \sum_{p \in \ell \cap V} \Lambda_V^*(p) &\leq D && \text{if } \ell \text{ is fully contained in } V. \end{aligned}$$

Proof. To streamline the analysis and avoid degenerate situations that might arise over the reals, we confine ourselves to the complex case; as already mentioned, the incidence bounds that we will obtain will automatically hold over the reals too. We note that the difference between the two cases arises because we do not want to count ℓ itself—the former sum would be infinite when ℓ is fully contained in V . Note also that if V is a cone and $p_V \in \ell$, we ignore in the sum the infinitely many lines incident to p_V and contained in V .

The proof is a variant of an observation due to Salmon [27] and repeated in Guth and Katz [14] over the real numbers, and later in Kollár [19] over the complex field and other general fields.

By Theorem 2.3(a), excluding the exceptional lines of V , the set of lines fully contained in V can be parameterized as a (real or complex) 1-parameter family of generator lines $\ell(t)$, represented by the irreducible curve $\Sigma_0 \subseteq F(V)$. Let $V^{(2)}$ denote the locus of points of V that are incident to at least two generator lines fully contained in V . By Theorem 2.3(b), $V^{(2)}$ is contained in some one-dimensional curve $\mathcal{C} \subset V$.

Let $p \in V \cap \ell$ be a point incident to k generator lines of V , other than ℓ , for some $k \geq 1$. In case V is a cone, we assume that $p \neq p_V$. Denote the generator lines incident to p (other than ℓ , if $\ell \subset V$, in which case it is assumed to be a generator) as $\ell_i = \ell(t_i)$, for $t_i \in \Sigma_0$ and for $i = 1, \dots, k$. (If ℓ_i is a singular point of $F(V)$, it may arise as $\ell(t_i)$ for several values of t_i , and we pick one arbitrary such value.) Let π be a generic plane containing ℓ , and consider the curve $\gamma_0 = V \cap \pi$, which is a plane curve of degree D . Since $V^{(2)} \subseteq \mathcal{C}$ is one-dimensional, a generic choice of π will ensure that $V^{(2)} \cap \pi$ is a discrete set (since ℓ is non-exceptional, it too meets $V^{(2)}$ in a discrete set).

There are two cases to consider: If ℓ is fully contained in V (and is thus a generator), then γ_0 contains ℓ . In this case, let γ denote the closure of $\gamma_0 \setminus \ell$; it is also a plane algebraic curve, of degree at most $D - 1$. Otherwise, we put $\gamma := \gamma_0$. By Theorem 2.3(a), we can take, for each $i = 1, \dots, k$, a sufficiently small open (real or complex) neighborhood Δ_i along Σ_0

containing t_i , so that, for any $1 \leq i < j \leq k$, all the lines $\ell(t)$, for $t \in \Delta_i \cup \Delta_j$, are distinct. Put $V_i := \bigcup_{t \in \Delta_i} \ell(t)$. Recall that $V_i \cap \pi$ is either a simple arc or a union of simple arcs meeting at p (depending on whether or not ℓ_i is a regular point of Σ_0); in the latter case, take γ_i to be any one of these arcs. Each of the arcs γ_i is incident to p and is contained in γ . Moreover, since π is generic, the arcs γ_i are all distinct. Indeed, for any $i \neq j$, and any point $q \in \gamma_i \cap \gamma_j$, there exist $t_i \in \Delta_i, t_j \in \Delta_j$ such that $\ell(t_i) \cap \pi = \ell(t_j) \cap \pi = q$, and $\ell(t_i) \neq \ell(t_j)$ (by the properties of these neighborhoods). Therefore, any point in $\gamma_i \cap \gamma_j$ is incident to (at least) two distinct generator lines fully contained in V . Again, the generic choice of π ensures that $\gamma_i \cap \gamma_j \subseteq V^{(2)}$ is a discrete set, so, in particular, γ_i and γ_j are distinct.

We have therefore shown that (i) if ℓ is not contained in V then p is a singular point of γ of multiplicity at least k (for $k \geq 2$; when $k = 1$ the point does not have to be singular), and (ii) if ℓ is contained in V then p is singular of multiplicity at least $k + 1$. We have $k \geq \Lambda_V(p)$ (resp., $k \geq \Lambda_V^*(p)$) if ℓ is not fully contained (resp., is fully contained) in V . As argued at the beginning of Section 6, the line ℓ can intersect γ in at most D points, *counted with multiplicity*, and the result follows. \square

We also need the following result, established by Guth and Katz [13]; see also [6]. It is an immediate consequence of the Cayley–Salmon–Monge theorem (Theorem 2.1) and a suitable extension of Bézout’s theorem for intersecting surfaces (see Fulton [9, Proposition 2.3]).

Proposition 3.2 (Guth and Katz [14]). *Let V be an irreducible two-dimensional variety in \mathbb{C}^3 of degree D . If V fully contains more than $11D^2 - 24D$ lines then V is ruled by (complex) lines.*

Corollary 3.3. *Let V be an irreducible two-dimensional variety in \mathbb{C}^3 of degree D that does not contain any planes. Then the number of lines that are fully contained in the union of the non-ruled components of V is $O(D^2)$.*

Proof. Let V_1, \dots, V_k denote those irreducible components of V that are not ruled by lines. By Proposition 3.2, for each i , the number of lines fully contained in V_i is at most $11\deg(V_i)^2 - 24\deg(V_i)$. Summing over $i = 1, \dots, k$, the number of lines fully contained in the union of the non-ruled components of V is at most $\sum_{i=1}^k 11\deg(V_i)^2 = O(D^2)$. \square

The following theorem, which we believe to be of independent interest in itself, is the main technical ingredient of our analysis. Note that it holds over both real and complex fields.

Theorem 3.4. *Let V be a possibly reducible two-dimensional algebraic surface of degree $D > 1$ in \mathbb{R}^3 or in \mathbb{C}^3 , with no linear components. Let P be a set of m distinct points on V and let L be a set of n distinct lines fully contained in V . Then there exists a subset $L_0 \subseteq L$ of at most $O(D^2)$ lines, such that the number of incidences between P and $L \setminus L_0$ satisfies*

$$I(P, L \setminus L_0) = O\left(m^{1/2}n^{1/2}D^{1/2} + m + n\right). \quad (7)$$

Remark. An important feature of the theorem, already noted for the more general Theorem 1.3, and discussed in more detail later on, is that the bound in (7) avoids the term nD , which arises naturally in many earlier works, e.g., when bounding the number of incidences

between points on V and lines not crossing V . This is significant when D is large—see below.

Proof. As in the proof of Lemma 3.1, we only work over \mathbb{C} , and the results are then easily transported to the real case too. Consider the irreducible components W_1, \dots, W_k of V . By Corollary 3.3, the number of lines contained in the union of the non-ruled components of V is $O(D^2)$, and we place all these lines in the exceptional set L_0 . In what follows we thus consider only ruled components of V . For simplicity, continue to denote them as W_1, \dots, W_k , and note that $k \leq D/2$.

We further augment L_0 as follows. We first dispose of lines of L that are fully contained in more than one ruled component W_i . We claim that their number is $O(D^2)$. Indeed, for any pair W_i, W_j of distinct components, the intersection $W_i \cap W_j$ is a curve of degree (at most) $\deg(W_i)\deg(W_j)$, which can therefore contain at most $\deg(W_i)\deg(W_j)$ lines (by the generalized version of Bézout’s theorem [9, Proposition 2.3], already mentioned in connection with Proposition 3.2). Since $\sum_{i=1}^k \deg(W_i) \leq D$, we have

$$\sum_{i \neq j} \deg(W_i)\deg(W_j) \leq \left(\sum_i \deg(W_i) \right)^2 = O(D^2),$$

as claimed. We add to L_0 all the $O(D^2)$ lines in L that are contained in more than one ruled component, and all the exceptional lines of all singly ruled components. The number of lines of the latter kind is at most $2k \leq 2 \cdot (D/2) = D$, so the size of $|L_0|$ is still $O(D^2)$. Hence, each line of $L_1 := L \setminus L_0$ is fully contained in a unique (singly or doubly) ruled component of V , and is a generator of that component.

The strategy of the proof is to consider each line ℓ of L_1 , and to estimate the number of its incidences with the points of P in an indirect manner, via Lemma 3.1, applied to ℓ and to each of the ruled components W_j of V . We recall that ℓ is fully contained in a unique component W_i , and treat that component in a somewhat different manner than the treatment of the other components.

In more detail, we proceed as follows. We first ignore, for each singly ruled *conic* component W_i , the incidences between its apex (exceptional point) p_{W_i} and the lines of L_1 that are contained in W_i . We refer to these incidences as *conical incidences* and to the other incidences as *non-conical*. When we talk about a line ℓ incident to another line ℓ' at a point p , we will say that ℓ is *conically incident* to ℓ' (at p) if p is the apex of some conic component W_i and ℓ' is fully contained in W_i (and thus incident to p). In all other cases, we will say that ℓ is *non-conically incident* to ℓ' (at p). (Note that this definition is asymmetric in ℓ and ℓ' ; in particular, ℓ does not have to lie in the cone W_i .) We note that the number of conical point-line incidences is at most n , because each line of L_1 is fully contained in a unique component W_i , so it can be involved in at most one conical incidence (at the apex of W_i , when W_i is a cone).

We next prune away points $p \in P$ that are non-conically incident to at most three lines of L_1 . (Note that p might be an apex of some conic component(s) of V ; in this case p is removed if it is incident to at most three lines of L_1 that are not contained in any of these components.) We lose $O(m)$ (non-conical) incidences in this process. Let P_1 denote the subset of the remaining points.

Lemma 3.5. *Each line $\ell \in L_1$ is non-conically incident, at points of P_1 , to at most $4D$ other lines of L_1 .*

Proof. Fix a line $\ell \in L_1$ and let W_i denote the unique ruled component that fully contains ℓ . Let W_j be any of the other ruled components. We estimate the number of lines of L_1 that are non-conically incident to ℓ and are fully contained in W_j .

If W_j is a regulus, there are at most four such lines, since ℓ meets the quadratic surface W_j in at most two points, each incident to exactly two generators (and to no other lines contained in W_j). In this case, we write the bound 4 as $\deg(W_j) + 2$. Assume then that W_j is singly ruled. By Lemma 3.1, we have

$$\sum_{p \in \ell \cap W_j} \lambda_{W_j}(p; L_1) \leq \sum_{p \in \ell \cap W_j} \Lambda_{W_j}(p) \leq \deg(W_j).$$

Note that, by definition, the above sum counts only non-conical incidences (and only with generators of W_j , but the exceptional lines of W_j have been removed from L_1 anyway).

We sum this bound over all components $W_j \neq W_i$, including the reguli. Denoting the number of reguli by ρ , which is at most $D/2$, we obtain a total of

$$\sum_{j \neq i} \deg(W_j) + 2\rho \leq D + 2\rho \leq 2D.$$

Consider next the component W_i containing ℓ . Assume first that W_i is a regulus. Each point $p \in P_1 \cap \ell$ can be incident to at most one other line of L_1 contained in W_i (the other generator of W_i through p). Since p is in P_1 , it is non-conically incident to at least $3 - 2 = 1$ other line of L_1 , contained in some other ruled component of V . That is, the number of lines that are (non-conically) incident to ℓ and are contained in W_i , which apriorily can be arbitrarily large, is nevertheless at most the number of other lines (not contained in W_i) that are non-conically incident to ℓ , which, as shown above, is at most $2D$.

If W_i is not a regulus, Lemma 3.1 implies that

$$\sum_{p \in \ell \cap W_i} \Lambda_{W_i}^*(p) \leq \deg(W_i) \leq D,$$

where again only non-conical incidences are counted in this sum (and only with generators). That is, the number of lines of L_1 that are non-conically incident to ℓ (at points of P_1) and are contained in W_i is at most D . Adding the bound for W_i , which has just been shown to be either D or $2D$, to the bound $2D$ for the other components, the claim follows. \square

To proceed, choose a threshold parameter $\xi \geq 3$, to be determined shortly. Each point $p \in P_1$ that is non-conically incident to at most ξ lines of L contributes at most ξ (non-conical) incidences, for a total of at most $m\xi$ incidences. (Recall that the overall number of conical incidences is at most n .) For the remaining non-conical incidences, let ℓ be a line in L_1 that is incident to t points of P_1 , so that each such point p is non-conically incident to at least $\xi + 1$ lines of L_1 (one of which is ℓ). It then follows from Lemma 3.5 that $t \leq 4D/\xi$. Hence, summing this over all $\ell \in L_1$, we obtain a total of at most $4nD/\xi$ incidences. We can now bring back the removed points of $P \setminus P_1$, since the non-conical incidences that they are involved in are counted in the bound $m\xi$. That is, we have

$$I(P, L_1) \leq m\xi + n + \frac{4nD}{\xi}.$$

We now choose $\xi = (nD/m)^{1/2}$. For this choice to make sense, we want to have $\xi \geq 3$, which will be the case if $9m \leq nD$. In this case we get the bound $O(m^{1/2}n^{1/2}D^{1/2} + n)$. If $9m > nD$ we take $\xi = 3$ and obtain the bound $O(m)$. Combining these bounds, and adding the at most n conical incidences, the theorem follows. \square

The final stretch: The real case. It remains to bound the number $I(P, L_0)$ of incidences involving the lines in L_0 . We remark that, in both the real and the complex cases, no special properties need to be assumed for the lines of L_0 ; the only thing that matters is that their number is small. We have $|L_0| = O(D^2)$. In the real case, we estimate $I(P, L_0)$ using Guth and Katz's bound ([14]; see Theorem 1.1), recalling that no plane contains more than s lines of L_0 . We thus obtain

$$\begin{aligned} I(P, L_0) &= O\left(m^{1/2}|L_0|^{3/4} + m^{2/3}|L_0|^{1/3}s^{1/3} + m + |L_0|\right) \\ &= O\left(m^{1/2}n^{1/2}D^{1/2} + m^{2/3}D^{2/3}s^{1/3} + m + n\right). \end{aligned} \tag{8}$$

Combining the bounds in Theorem 3.4 and in (8) yields the asserted bound on $I(P, L)$.

We remark that in the first term in (8) we have estimated $L_0^{3/4}$ by $O(n^{1/2}D^{1/2})$ instead of the sharper estimate $O(D^{3/2})$. This is because the term $O(m^{1/2}n^{1/2}D^{1/2})$ appears in Theorem 3.4 anyway, so the sharper estimate has no effect on the overall asymptotic bound.

The final stretch: The complex case. We next estimate $I(P, L_0)$ in the complex case. Again, we have $n_0 := |L_0| = O(D^2)$. We may ignore the points of P that are incident to fewer than three lines of L_0 , as they contribute altogether only $O(m)$ incidences. Continue to denote the set of surviving points as P . A point p that is incident to at least three lines of L_0 is either a singular point of V (when not all its incident lines are coplanar) or a flat point of V (see the Appendix, and also Guth and Katz [14]).

We need the following lemma, adapted (with almost the same proof, which we omit) from a similar lemma that was established in our earlier work [33] for the four-dimensional case (see also Section 4).

Lemma 3.6 ([33, Lemma 2.15]). *Let $f \in \mathbb{C}[x, y, z]$ be an irreducible polynomial. If a line $\ell \subset Z(f)$ is flat, then the tangent plane $T_p Z(f)$ is fixed for all the non-singular points $p \in \ell$.*

We decompose V into its irreducible components, and assign each point $p \in P$ (resp., line $\ell \in L_0$) to the first component that (fully) contains it. Similar to what has been observed above, the number of ‘‘cross-incidences’’, between points and lines assigned to different components of V , is $O(n_0 D) = O(D^3)$. We therefore assume, as we may, that V is irreducible (over \mathbb{C}), and write $V = Z(f)$, for an irreducible trivariate polynomial f of degree D .

We call a line *flat* if all its non-singular points are flat. As argued in the Appendix and in earlier works (see, e.g., [6, 14]), since $Z(f)$ is not a plane, there exists a certain polynomial Π satisfying (i) $\deg(\Pi) = 3D - 4$, (ii) $Z(f, \Pi)$ is a curve, and (iii) the flat points of P and the flat lines of L_0 are contained in $Z(f, \Pi)$.

All this implies, arguing as in previous works [6, 14], that a line $\ell \in L_0$ that is non-singular and non-flat contains at most $4D - 4$ points of P (each of which is either singular or

flat). We prune away these lines from L_0 , losing at most $(4D - 4)|L_0| = O(D^3)$ incidences with the points of P . Continue to denote the subset of surviving lines as L_0 .

Therefore, it remains to bound the number of incidences between the surviving points and the surviving lines, each of which is either singular or flat. Write P as the union of the subset P_f of flat points and the subset P_s of singular points. Similarly, write L_0 as the union of the subset L_f of flat lines and the subset L_s of singular lines. A singular line contains no flat points, and a flat line contains at most $D - 1$ singular points. Thus,

$$I(P, L_0) \leq I(P_f, L_f) + I(P_s, L_s) + n_0 D.$$

By Lemma 3.6, all the non-singular points of a flat line have the same tangent plane. Assign each point $p \in P_f$ (resp., line in L_f) to its tangent plane $T_p Z(f)$ (resp., $T_p Z(f)$ for some non-singular point $p \in P_f \cap \ell$; we only consider lines in L_f that are incident to at least one point in P_f). We have therefore partitioned the points in P_f and the lines in L_f among planes in some finite set $H = \{h_1, \dots, h_k\}$, and we only need to count incidences between points and lines assigned to the same plane. Within each $h \in H$, we have a set $P_h \subseteq P_f$ of m_h points in h , and a set $L_h \subseteq L_f$ of n_h lines contained in h . Using the planar bound (1), which also holds in the complex plane, the number of incidences within h is $O(m_h^{2/3} n_h^{2/3} + m_h + n_h)$. Summing these bounds over $h \in H$, and using Hölder's inequality, and the fact that $n_h \leq s$ for each h , we obtain a total of $O(m^{2/3} n_0^{1/3} s^{1/3} + m + n_0)$ incidences.

Bounding incidences involving singular points and lines. Bounding the number of incidences between the singular points and lines is done via degree reduction. Assuming, without loss of generality, that f_x does not vanish identically on $Z(f)$, the points of P_s and the lines of L_s are then (fully) contained in $Z(f_x)$, and $\deg(f_x) \leq D - 1$. We thus construct a sequence of partial derivatives of f that are not identically zero on $Z(f)$. For this we assume, as we may, that f , and each of its derivatives, are square-free; whenever this fails, we replace the corresponding derivative by its square-free counterpart before continuing to differentiate. Without loss of generality, assume that this sequence is f, f_x, f_{xx} , and so on. Denote the j -th element in this sequence as f_j , for $j = 0, 1, \dots$ (so $f_0 = f, f_1 = f_x$, and so on). Assign each point $p \in P$ to the first polynomial f_j in the sequence for which p is non-singular; more precisely, we assign p to the first f_j for which $f_j(p) = 0$ but $f_{j+1}(p) \neq 0$ (recall that $f_0(p)$ is always 0 by assumption. Similarly, assign each line ℓ to the first polynomial f_j in the sequence for which ℓ is fully contained in $Z(f_j)$ but not fully contained in $Z(f_{j+1})$ (again, by assumption, there always exists such a polynomial f_j). If ℓ is assigned to f_j then it can only contain points p that were assigned to some f_k with $k \geq j$. Indeed, if ℓ contained a point p assigned to f_k with $k < j$ then $f_{k+1}(p) \neq 0$ but ℓ is fully contained in $Z(f_{k+1})$, since $k + 1 \leq j$; this is a contradiction that establishes the claim.

Fix a line $\ell \in L$, which is assigned to some f_j . An incidence between ℓ and a point $p \in P$, assigned to some f_k , for $k > j$, can be charged to the intersection of ℓ with $Z(f_{j+1})$ at p (by construction, p belongs to $Z(f_{j+1})$). The number of such intersections is at most $\deg(f_{j+1}) \leq D - j - 1 \leq D$, so the overall number of incidences of this sort, over all lines $\ell \in L$, is $O(n_0 D) = O(D^3)$. It therefore suffices to consider only incidences between points and lines that are assigned to the same zero set $Z(f_i)$.

The reductions so far have produced a finite collection of up to D polynomials, each of degree at most D , so that the points of P are *partitioned* among the polynomials and so

are the lines of L , and we only need to bound the number of incidences between points and lines assigned to the *same* polynomial. Moreover, for each j , all the points assigned to f_j are non-singular, by construction. For each j , let P_j and L_j denote the subsets of P and of L_0 , respectively, that are assigned to f_j , and put $m_j := |P_j|$ and $n_j := |L_j|$. We have $\sum_j m_j \leq m$ and $\sum_j n_j \leq n_0$.

We would like to apply the preceding analysis to P_j and L_j , but we face the technical issue that f_j might be reducible and have some linear factors. (The theorem does not require the variety to be irreducible, but forbids it to have linear components.)

We therefore proceed as follows. We first consider only those points and lines that are (fully) contained in some nonlinear component of $Z(f_j)$. We apply the preceding analysis to these sets, and obtain the incidence bound

$$O\left(m_j^{2/3} n_j^{1/3} s^{1/3} + m_j + n_j D\right).$$

Summing these bounds over all f_j 's, using Hölder's inequality, we get the overall bound

$$O\left(m^{2/3} n_0^{1/3} s^{1/3} + m + n_0 D\right).$$

To bound the number of incidences involving points and lines in the linear components of $Z(f_j)$, for any fixed j , we reason as we did above, when handling the flat points and lines. That is, we order arbitrarily the planar components of $Z(f_j)$, assign each point to first component that contains it, and assign each line to the first component that fully contains it. As $Z(f_j)$ has at most D such components, the number of ‘‘cross incidences,’’ between points and lines assigned to different components, is $O(n_j D)$. For the number of ‘‘same component’’ incidences, we use the Tóth-Zahl extension of the Szemerédi-Trotter bound [40, 41] in each plane, and sum them up, exactly as in the case of flat points and lines, and get a total bound of

$$O\left(m_j^{2/3} n_j^{1/3} s^{1/3} + m_j + n_j D\right),$$

and, summing these bounds over all f_j 's, as above, we get the overall bound

$$I(P_s, L_s) = O\left(m^{2/3} n_0^{1/3} s^{1/3} + m + n_0 D\right),$$

thereby completing the proof of part (b) (for the case where the ambient space is three-dimensional).

Reduction to three dimensions. To complete the analysis, we need to consider the case where V is a two-dimensional variety embedded in \mathbb{R}^d , for $d > 3$.

Let H be a generic 3-flat, and denote by P^* , L^* , and V^* the respective projections of P , L , and V onto H . Since H is generic, we may assume that all the projected points in P^* are distinct, and so are all the projected lines in L^* . Clearly, every incidence between a point of P and a line of L corresponds to an incidence between the projected point and line. Since no 2-flat contains more than s lines of L , and H is generic, repeated applications of Lemma 2.5 imply that no plane in H contains more than s lines of L^* .

One subtle point is that the set-theoretic projection V^* of V does not have to be a real algebraic variety (in general, it is only a semi-algebraic set), but it is always contained in a

two-dimensional real algebraic variety \tilde{V} , which we call, as we did in an earlier work [32], the *algebraic projection* of V ; it is the zero set of all polynomials belonging to the ideal of polynomials vanishing on V , after eliminating variables in the complementary space of H (this is also known as an *elimination ideal* of V ; see Cox et al. [3] for details), and is equal to the Zariski closure of V^* . Since the closure of a projection does not increase the original degree (see, e.g., Harris [15]), $\deg(\tilde{V}) \leq D$. That \tilde{V} does not contain a 2-flat follows by a suitable adaptation of the argument in [32, Lemma 2.1] (which is stated there for $d = 4$ over the reals), that applies for general d and over the complex field too.

In conclusion, we have $I(P, L) \leq I(P^*, L^*)$, where P^* is a set of m points and L^* is a set of n lines, all contained in the two-dimensional algebraic variety \tilde{V} , embedded in 3-space, which is of degree at most D and does not contain any plane, and no plane contains more than s lines of L^* . The preceding analysis thus implies that the bound asserted in the theorem applies in any dimension $d \geq 3$. \square

4 Proof of Theorem 1.5

In most of this section we assume that the ambient space is four-dimensional, and work, with a few exceptions, over the complex field. The reduction from higher dimensions to four dimensions is handled as the reduction to three dimensions just discussed.

We exploit the following useful corollary of Theorem 1.3 (recall that we are now in four dimensions). Note that in the bound given below, the term nD does not appear yet.

Corollary 4.1. *Let f and g be two 4-variate polynomials, over \mathbb{R} or \mathbb{C} , of degree $O(D)$, such that $Z(f, g)$ is two-dimensional over \mathbb{C} . Let P be a set of m points and L a set of n lines, such that all the points of P and all the lines of L are (fully) contained in the union of the irreducible components of $Z(f, g)$ that are not 2-flats. Assume also that no 2-flat contains more than s lines of L . Then we have*

(a) in the real case:

$$I(P, L) = O\left(m^{1/2}n^{1/2}D + m^{2/3}D^{4/3}s^{1/3} + m + n\right), \quad (9)$$

(b) and in the complex case:

$$I(P, L) = O\left(m^{1/2}n^{1/2}D + m^{2/3}D^{4/3}s^{1/3} + D^6 + m + n\right). \quad (10)$$

Proof. Let $Z(f, g) = \bigcup_{i=1}^s V_i$ be the decomposition of $Z(f, g)$ into its irreducible components. By the generalized version of Bézout's theorem [9], we have $\sum_{i=1}^s \deg(V_i) \leq \deg(f)\deg(g) = O(D^2)$. Assume that V_1, \dots, V_k are the components that are not 2-flats, for some $k \leq s$, and let W denote their union. As just observed, $\deg(W) = O(D^2)$. Applying to W Theorem 1.3(a) (over the reals) or Theorem 1.3(b) (over the complex) thus completes the proof. \square

Remark. Corollary 4.1(a) is significant when $D \ll n^{1/4}$. For larger values of D , we can project P , L , and $Z(f, g)$ onto some generic 3-flat, and apply the incidence bound of Guth and Katz, as given in Theorem 1.1, within that 3-flat. When $D \geq n^{1/4}$, the resulting bound is better than the one in (9).

The proof of Theorem 1.5 revisits the proof of Theorem 1.2, as presented in [33], and applies Corollary 4.1 as a major technical tool. The only difference between the real and the complex cases is in the application of that corollary. Except for this application, we will work over the complex domain, but the analysis carries over, in a straightforward manner, to the real case too.

Each line $\ell \in L$ that is not fully contained in V contributes at most D incidences, for a total of $O(nD)$ incidences. We thus assume, as we may, that all the lines of L are contained in V . Let $V = \bigcup_{i=1}^t V_i$ be the decomposition of V into its irreducible components, and assign each point $p \in P$, (resp., line $\ell \in L$) to the first V_i that contains it (resp., fully contains it; such a V_i always exists). It is easy to verify that points and lines that are assigned to different V_i 's contribute at most nD incidences. Indeed, any such incidence (p, ℓ) can be charged to an intersection point of ℓ with the component V_i that p is assigned to, and thus there are at most $\sum_i \deg(V_i) = D$ such incidences for each ℓ , for an overall number of $O(nD)$ such incidences. Therefore, it suffices to establish the bound in (5) or in (6) for the number of incidences between points and lines assigned to the same component. The exponents in the first two terms in either of these bounds are favorable, in the sense that they sum up to 1 (ignoring the factors D and $s^{1/3}$), thereby allowing us to estimate the sum of the resulting bounds, over the components of V , via Hölder's inequality, to obtain the bounds in Theorem 1.5 (see below for details). We thus assume that V is irreducible, and write $V = Z(f)$, for some real or complex irreducible polynomial f of degree D .

We assume for now that P consists exclusively of *non-singular* points of the *irreducible* variety $Z(f)$. The treatment of the singular points, similar to the handling of singular points in the proof of Theorem 1.3(b), will be given towards the end of the proof.

We recall the definition of the *four-dimensional flecnode polynomial* FL_f^4 of f , as given in Section 2. That is, FL_f^4 vanishes at each *flecnode* $p \in Z(f)$, namely, points p for which there exists a line that is incident to p and osculates to $Z(f)$ up to the fourth order. It is obtained by eliminating (in a standard manner, using resultants) the direction v of the osculating line from the four homogeneous equations given by the vanishing of the first four terms of the Taylor expansion of $f(p + tv)$ near p (v is a point in projective 3-space, and its elimination from these four equations yields a single polynomial equation in p). Clearly, FL_f^4 vanishes identically on every line of L , and thus also on P (assuming that each point of P is incident to at least one line of L). As in the three-dimensional case, its degree can be shown to be $O(D)$.

If FL_f^4 does not vanish identically on $Z(f)$, then $Z(f, \text{FL}_f^4) := Z(f) \cap Z(\text{FL}_f^4)$ is a two-dimensional variety that contains P and all the lines of L , and is of degree $\deg(f) \cdot \deg(\text{FL}_f^4) = O(D^2)$ (so we are in the setup assumed in Corollary 4.1). The other possibility is that FL_f^4 vanishes identically on $Z(f)$, and then a theorem of Landsberg [21] (see also [33] for details) implies that $Z(f)$ is ruled by (real or complex) lines. (Landsberg's theorem is a generalization of Theorem 2.1, the classical Cayley–Salmon–Monge theorem [23, 27] in three dimensions.)

First case: $Z(f, \text{FL}_f^4)$ is two-dimensional. Put $g = \text{FL}_f^4$ and apply Corollary 4.1 to f and g . In the real case we obtain the bound

$$O\left(m^{1/2}n^{1/2}D + m^{2/3}D^{4/3}s^{1/3} + m + n\right),$$

and in the complex case we obtain the bound

$$O\left(m^{1/2}n^{1/2}D + m^{2/3}D^{4/3}s^{1/3} + D^6 + m + n\right),$$

over all components of $Z(f, \text{FL}_f^4)$ that are not 2-flats.

Incidences within 2-flats fully contained in $Z(f, \text{FL}_f^4)$. From this point on in the proof, there is no distinction between the real and complex cases, so we work over the complex domain. The strategy here is to distribute the points of P and the lines of L among the 2-flats that contain them (lines not contained in any 2-flat are fully contained in some other component of $Z(f, \text{FL}_f^4)$ and are dealt with, as above, within that component). See also below for a more detailed account of this strategy. Points that belong to at most two such 2-flats get duplicated at most twice, and we bound the number of incidences with these points by applying the planar bound (1) (which, as we recall, also holds in the complex plane [40, 41]) to each 2-flat separately, and sum up the bounds, to get $O(m^{2/3}n^{1/3}s^{1/3} + m + n)$, using Hölder’s inequality, combined with the assumption that no 2-flat contains more than s lines of L .

Extending to four dimensions (and in the complex domain) the notation for the three-dimensional case from Guth and Katz [13] (see the appendix, [6], and also Pressley [24] and Ivey and Landsberg [17] for more basic references), we call a non-singular point p of $Z(f)$ *linearly flat*, if it is incident to at least three distinct 2-flats that are fully contained in $Z(f)$ (and thus also in the tangent hyperplane $T_p Z(f)$). Linearly flat points can then be shown to be *flat*, meaning that the second fundamental form of f vanishes at them (see the appendix and [33]). This property, at a point p , can be expressed by several polynomials of degree $3D - 4$ vanishing at p (see [33, Section 2.5]). We call a line *flat* if all its non-singular points are flat. Each line of L that is not flat contains at most $O(D)$ flat points, and thus the non-flat lines contribute a total of at most $O(nD)$ incidences with flat points, so we assume in what follows that the points of P and the lines of L are all flat. Since $Z(f)$ is not a hyperplane, the second fundamental form does not vanish identically on $Z(f)$ (this property holds in any dimension; see, e.g. [17, Exercise 3.2.12.2]), and it then follows from the characterization of flat points that there exists a certain polynomial Π satisfying (i) $\deg(\Pi) = 3D - 4$, (ii) $Z(f, \Pi)$ is two-dimensional, and (iii) the (flat) points of P and the (flat) lines of L are contained in $Z(f, \Pi)$.

In this case we partition the points of P among their tangent hyperplanes, and denote the resulting set of hyperplanes by H . Similar to Lemma 3.6, we have the property that all the (flat) points that lie on the same flat line have the *same* tangent hyperplane to $Z(f)$ (see [33, Lemma 2.15]). We use this property to obtain a partition of the points and lines into distinct (tangent) hyperplanes, so that it suffices to bound the number of incidences within each hyperplane in H .

For each $h \in H$, we have a set $P_h \subseteq P$ of m_h points in h , a set $L_h \subseteq L$ of n_h lines contained in h , and a set F_h of 2-flats contained in $Z(f, \text{FL}_f^4) \cap h$; the 2-flats of F_h are the 2-flat components of $Z(f, \text{FL}_f^4) \cap h$ (ignoring singularities, which are treated later, these 2-flats are necessarily contained in the corresponding tangent hyperplanes). Notice that each 2-flat in F_h is also contained in the two-dimensional surface $Z(f) \cap h$, which is of degree D , so, by the generalized version of Bézout’s theorem [9], we have $|F_h| \leq D$. We assign each point $p \in P_h$ (resp., line $\ell \in L_h$) to the first 2-flat in F_h that (fully) contains it. Similar to what has been observed above, the number of “cross-incidences”, between points and

lines assigned to different 2-flats, within h is at most $n_h D$, for a total, over the hyperplanes $h \in H$, of at most nD incidences. Again, using the planar bound (1) and Hölder's inequality, the numbers of incidences within the 2-flats of h sum up to $O(m_h^{2/3} n_h^{2/3} + m_h + n_h D)$, and, summing over the hyperplanes $h \in H$, using Hölder's inequality once again, and the fact that $n_h \leq s$, we obtain a total of $O(m^{2/3} n^{1/3} s^{1/3} + m + nD)$ incidences.

Remark. The novel feature of this step of the proof, as compared with the analogous argument used in [33], is that the number of 2-flats in F_h , for any fixed h , is at most D . This allows us to bound the number of incidences within each hyperplane $h \in H$ separately, so that, within each such hyperplane, instead of using the Guth-Katz bound (in the real case), we partition the points and lines among at most D planes, and then use the Szemerédi-Trotter bound in the real case, or the Tóth-Zahl bound in the complex case. The fact that there are at most D planes within each hyperplane $h \in H$ guarantees that the number of “cross-incidences” (within h) is at most $n_h D$, for a total of nD incidences (notice that, in contrast, the *total* number of planes (over all $h \in H$) can be arbitrarily large).

Second case: $Z(f)$ is ruled by (complex) lines. We next consider the case where the four-dimensional flecnode polynomial FL_f^4 vanishes identically on $Z(f)$. By Landsberg's theorem [21] mentioned above, this implies that $Z(f)$ is ruled by (complex) lines.

As in the three-dimensional case treated in the appendix (see, e.g., the proof of Theorem 2.3), we denote by Σ_p^3 (resp., Σ_p), for $p \in Z(f)$, the set of all lines that are incident to p and osculate to $Z(f)$ to order 3 at p (resp., are contained in $Z(f)$). We put $\Sigma^3 := \bigcup_{p \in Z(f)} \Sigma_p^3$, and $\Sigma := \bigcup_{p \in Z(f)} \Sigma_p$. (Σ is the *Fano variety* of (lines contained in) $Z(f)$, now represented in a higher-dimensional projective space.)

In [33], we proved that, for each $p \in P$, either $|\Sigma_p| \leq 6$ or Σ_p is infinite. In the interest of completeness, we recall here the outline of this argument. The analysis provides an algebraic characterization of points p of the latter kind, which uses an auxiliary polynomial $U = U(p; u_0, u_1, u_2, u_3)$, called the *u-resultant*, defined in terms of f and its derivatives at p (see [33] and also [3] for details), where (u_0, \dots, u_3) denotes the direction of a line incident to p (in homogeneous coordinates). The polynomial U is of degree $O(D)$ in p and is a homogeneous polynomial of degree six in u . The characterization is that Σ_p^3 is infinite if and only if $U(p; u_0, u_1, u_2, u_3) \equiv 0$, as a polynomial of u , at p . In the complementary case, Bézout's theorem [9] can be used to show that there are only six lines in Σ_p^3 , and thus at most six lines in Σ_p . Pruning away points $p \in P$ with $|\Sigma_p^3| \leq 6$ (the number of incidences involving these points is at most $6m = O(m)$), we may then assume that Σ_p^3 is infinite for every $p \in P$.

If $U(p; u_0, u_1, u_2, u_3)$ does not vanish identically (as a polynomial in u_0, u_1, u_2, u_3) at every point $p \in Z(f)$, then at least one of its coefficients, call it c_U , which is a polynomial in p , of degree $O(D)$, does not vanish identically on $Z(f)$. In this case, as U vanishes identically at every point of P (as a polynomial in u_0, u_1, u_2, u_3), we have $P \subset Z(f, c_U)$, which is a two-dimensional variety. The machinery developed in the first case can then be applied here (with $g = c_U$), and the bounds and properties derived for that case hold here too.

We may therefore assume that $U(p; u_0, u_1, u_2, u_3) \equiv 0$ at every non-singular point $p \in Z(f)$ (as a polynomial in u_0, u_1, u_2, u_3). By the aforementioned characterization via *u*-resultants, it follows that Σ_p^3 is then infinite at each such point.

We now use another theorem of Landsberg [17, Theorem 3.8.7]: Let f be a polynomial over $\mathbb{P}^4(\mathbb{C})$, such that there exists an irreducible component $\Sigma_0^3 \subset \Sigma^3 = \Sigma^3(Z(f))$ with the property that, for every point p in a Zariski-open set⁵ $\mathcal{O} \subset Z(f)$, $\dim \Sigma_{0,p}^3 \geq 1$, where $\Sigma_{0,p}^3$ is the set of lines in Σ_0^3 incident to p . Landsberg's theorem then asserts that, for every point $p \in \mathcal{O}$, all lines in $\Sigma_{0,p}^3$ are fully contained in $Z(f)$; that is, for each $p \in \mathcal{O}$, $\Sigma_{0,p}^3$ is equal to the set $\Sigma_{0,p}$ of lines incident to p and fully contained in $Z(f)$.

Since Σ_p^3 is infinite at each non-singular point $p \in Z(f)$, its dimension is ≥ 1 at each such point. As shown in [33], the main condition in Landsberg's theorem, about the existence of a component Σ_0^3 of Σ^3 with the required properties, is satisfied too. One can then argue that the conclusion of Landsberg's theorem holds at *every* point of $Z(f)$; see Lemma 6.1 in the appendix for a similar claim concerning two-dimensional surfaces. That is, $Z(f)$ is *infinitely ruled* by (complex) lines, in the sense that each point $p \in Z(f)$ is incident to infinitely many (complex) lines that are fully contained in $Z(f)$, and, moreover, $\Sigma_{0,p}^3 = \Sigma_{0,p}$ at each p . That is, Σ_0^3 is contained in Σ . Denoting this set as Σ_0 , it is shown (in full detail) in [33] that the union of the lines in Σ_0 is equal to $Z(f)$, and that $\dim(\Sigma_0) \geq 3$.

Severi's theorem. The following theorem was already used in [33], and we make a similar use thereof here too. It has been obtained by Severi [29] in 1901. A variant of this result has also been obtained by Segre [28]; see also the more recent works [22, 25, 26]. We state here a special case of the theorem that we need.

Theorem 4.2 (Severi's Theorem [29]; special case). *Let $X \subset \mathbb{P}^4(\mathbb{C})$ be a three-dimensional irreducible variety, and let Σ_0 be a component of maximal dimension of the Fano variety $\Sigma = \Sigma(X)$ of X , such that the lines of Σ_0 cover X . Then the following holds. (i) If $\dim(\Sigma_0) = 4$, then X is a hyperplane. (ii) If $\dim(\Sigma_0) = 3$, then either X is a quadric, or X is ruled by 2-flats.*

Informally, $\dim(\Sigma_0) = 3$ corresponds to the case where X is infinitely ruled by lines of Σ_0 : There are four degrees of freedom to specify a line in Σ_0 , three to specify $p \in X$, and one to specify the line in $\Sigma_{0,p}$. (We can assume that $\Sigma_{0,p}$ is one-dimensional, because if it were two-dimensional, then X would have been a hyperplane.) However, one degree of freedom has to be removed, to account for the fact that the same line (being contained in X) arises at each of its points. Severi's theorem asserts, again informally, that in this case the infinite family of lines of $\Sigma_{0,p}$ must form a 2-flat, unless X is a quadric or a hyperplane. Moreover, by [33, Theorem 3.9] (whose proof is based on Theorem 6.2, given in the appendix), Σ_0 has maximal dimension.

Applying the second case in Severi's theorem to $Z(f)$, which is justified by the preceding arguments, we conclude that either $Z(f)$ is a quadric or it is ruled by 2-flats. The cases where $Z(f)$ is a quadric or a hyperplane are ruled out by our assumption, so we only need to consider the case where $Z(f)$ is ruled by (complex) 2-flats.

The case where $Z(f)$ is ruled by 2-flats. Handling this last step is somewhat intricate; it resembles the analysis of flat points and lines in the first case, where here points and lines are partitioned among the ruling 2-flats. In this case, every point $p \in Z(f)$ is incident to at least one 2-flat $\tau_p \subset Z(f)$. Let D_p denote the set of 2-flats that are incident to p and are contained in $Z(f)$.

⁵See the appendix and Cox et al. [3, Section 4.2] for further details.

For a non-singular point $p \in Z(f)$, if $|D_p| > 2$, then p is a (linearly flat and thus) flat point of $Z(f)$. Recall that we have bounded the number of incidences involving flat points (and lines) by partitioning them among a finite number of containing hyperplanes, and by bounding the incidences within each hyperplane. Lines incident to fewer than $3D - 4$ points of P have been pruned away, losing only $O(nD)$ incidences, and the remaining lines are all flat. Repeating this argument here, noticing that here too, the number of 2-flats contained in a hyperplane is at most D , we obtain the bound

$$O\left(m^{2/3}n^{1/3}s^{1/3} + m + nD\right).$$

In what follows we therefore assume that all points of P are non-singular and non-flat (call these points, as in [33], *ordinary* for short), and therefore $|D_p| = 1$ or 2 , for each such p . Put $H_1(p)$ (resp., $H_1(p), H_2(p)$) for the 2-flat (resp., two 2-flats) in D_p , when $|D_p| = 1$ (resp., $|D_p| = 2$).

Clearly, each line in L , containing at least one ordinary point $p \in Z(f)$, is fully contained in at most two 2-flats fully contained in $Z(f)$ (namely, the 2-flats of D_p).

Assign each ordinary point $p \in P$ to *each* of the at most two 2-flats in D_p , and assign each line $\ell \in L$ that is incident to at least one ordinary point to the at most two 2-flats that fully contain ℓ and are fully contained in $Z(f)$ (it is possible that ℓ is not assigned to any 2-flat—see below). Changing the notation, enumerate these 2-flats, over all ordinary points $p \in P$, as U_1, \dots, U_k , and, for each $i = 1, \dots, k$, let P_i and L_i denote the respective subsets of points and lines assigned to U_i , and let m_i and n_i denote their cardinalities. We then have $\sum_i m_i \leq 2m$ and $\sum_i n_i \leq 2n$, and the total number of incidences within the 2-flats U_i (excluding lines not assigned to any 2-flat) is at most $\sum_{i=1}^k I(P_i, L_i)$. This incidence count can be obtained exactly as in the first case of the analysis, with the aid of Hölder's inequality, and yields the bound

$$\sum_{i=1}^k I(P_i, L_i) = O\left(m^{2/3}n^{1/3}s^{1/3} + m + n\right).$$

As noted, this bound does not take into account incidences involving lines which are not contained in any of the 2-flats U_i (and are therefore not assigned to any such 2-flat). It suffices to consider only lines of this sort that are non-singular and non-flat, since singular or flat lines are only incident to singular or flat points, and we assumed above that all the points of P are ordinary points. If ℓ is a non-singular and non-flat line, and is not fully contained in any of the U_i , we call it a *piercing line* of $Z(f)$. We need the following lemma from [33].

Lemma 4.3 ([33, Lemma 3.13]). *Let $p \in Z(f)$ be an ordinary point. Then p is incident to at most one piercing line.*

Therefore, each ordinary point $p \in P$ is incident to at most one piercing line, and the total contribution of incidences involving ordinary points and piercing lines is at most m .

In conclusion, combining the bounds that we have obtained for the various subcases of the second case, we get the desired bound in (5).

Incidences involving singular points of $Z(f)$. The forthcoming reasoning is very similar to the handling of singular points and lines in the proof of Theorem 1.3(b), although it is somewhat more involved because we need to ensure that the resulting polynomials that we construct be irreducible; we present the analysis in detail, for the sake of clarity.

In the analysis presented so far, we have assumed that the points of P are non-singular points of $Z(f)$. To reduce the general setup to this situation we proceed as follows; an identical reduction has also been used in [33]. We only handle lines that are fully contained in $Z(f)$, because the other lines contribute at most $O(nD)$ incidences. We construct a sequence of partial derivatives of f that are not identically zero on $Z(f)$. For this we assume, as we may, that f , and each of its derivatives, are square-free; whenever this fails, we replace the corresponding derivative by its square-free counterpart before continuing to differentiate. Without loss of generality, assume that this sequence is obtained by always differentiating with respect to x , and denote the j -th element in this sequence as f_j , for $j = 0, 1, \dots$. That is, $f_0 = f$, $f_1 = f_x$, with repeated factors removed, f_2 is the x -derivative of f_1 , again with repeated factors removed, and so on. Assign each point $p \in P$ to the first polynomial f_j in the sequence for which p is non-singular; more precisely, we assign p to the first f_j for which $f_j(p) = 0$ but $f_{j+1}(p) \neq 0$ (recall that $f_0(p)$ is always 0 by assumption). Similarly, assign each line ℓ to the first polynomial f_j in the sequence for which ℓ is fully contained in $Z(f_j)$ but not fully contained in $Z(f_{j+1})$ (again, by assumption, there always exists such a j). If ℓ is assigned to f_j then it can only contain points p that were assigned to some f_k with $k \geq j$. Indeed, if ℓ contained a point p assigned to f_k with $k < j$ then $f_{k+1}(p) \neq 0$ but ℓ is fully contained in $Z(f_{k+1})$, since $k + 1 \leq j$; this is a contradiction that establishes the claim.

Fix a line $\ell \in L$, which is assigned to some f_j . An incidence between ℓ and a point $p \in P$, assigned to some f_k , for $k > j$, can be charged to the intersection of ℓ with $Z(f_{j+1})$ at p (by construction, p belongs to $Z(f_{j+1})$). The number of such intersections is at most $\deg(f_{j+1}) \leq D - j - 1 \leq D$, so the overall number of incidences of this sort, over all lines $\ell \in L$, is $O(nD)$. It therefore suffices to consider only incidences between points and lines that are assigned to the same zero set $Z(f_i)$.

The reductions so far have produced a finite collection of up to $O(D)$ polynomials, each of degree at most D , so that the points of P are *partitioned* among the polynomials and so are the lines of L , and each point p is non-singular with respect to the polynomial it is assigned to, and we only need to bound the number of incidences between points and lines assigned to the *same* polynomial. This is not the end yet, because the various (reduced forms of the) partial derivatives might be reducible, which we want to avoid. Thus, in a final decomposition step, we split each derivative polynomial f_j into its irreducible factors, and reassign the points and lines that were assigned to $Z(f_j)$ to the various factors, by the same “first come first served” rule used above. The overall number of incidences that are lost in this process is again $O(nD)$. The overall number of polynomials is $O(D^2)$, as can easily be checked. Note also that the last decomposition step preserves non-singularity of the points in the special sense defined above; that is, as is easily verified, a point $p \in Z(f_j)$ with $f_{j+1}(p) \neq 0$, continues to be a non-singular point of the irreducible component it is reassigned to.

We now fix one such final polynomial, call it f_j , denote its degree by D_j (which is upper bounded by the original degree D), and denote by P_j and L_j the subsets of the original sets of points and lines that are assigned to f_j , and by m_j and n_j their respective cardinalities.

We now may assume that P_j consists exclusively of *non-singular* points of the *irreducible* variety $Z(f_j)$. The preceding analysis yields the bound (5) for each j

$$I(P_j, L_j) = O\left(m_j^{1/2} n_j^{1/2} D_j + m_j^{2/3} n_j^{1/3} s^{1/3} + n_j D_j + m_j\right).$$

Summing these bounds, upper bounding D_j by D , and using Hölder's inequality for the first two terms, we get the bound (5) for the entire sets P and L . This completes the proof for the case where the variety containing P and the lines of L are embedded in \mathbb{R}^4 .

Reduction to the four-dimensional case. To complete the analysis, we need to consider the case where V is a three-dimensional variety embedded in \mathbb{R}^d , for $d > 4$. The analysis follows closely the one at the end of the proof of Theorem 1.3, in Section 3.

Concretely, let H be a generic 4-flat, and denote by P^* , L^* , and V^* the respective projections of P , L , and V onto H . Since H is generic, we may assume that all the projected points in P^* are distinct, and so are all the projected lines in L^* . Clearly, every incidence between a point of P and a line of L corresponds to an incidence between the projected point and line. Since no 2-flat contains more than s lines of L , and H is generic, repeated applications of Lemma 2.5 imply that no 2-flat in H contains more than s lines of L^* .

As in Section 3, the set-theoretic projection V^* of V does not have to be a real algebraic variety, so we use instead the algebraic projection \tilde{V} of V that contains V^* . That \tilde{V} does not contain a hyperplane or quadric follows by a suitable adaptation of the preceding argument (see [32, Lemma 2.1]). The case of a hyperplane is straightforward (reasoning as in the preceding section). For quadrics we have:

Claim. Let X be a three-dimensional real algebraic variety in \mathbb{R}^d , for $d \geq 5$, such that a generic (real) algebraic projection of V on \mathbb{R}^4 is a quadric. Then X is a quadric in \mathbb{R}^d .

Proof. Assume to the contrary that X is not a quadric in \mathbb{R}^d . This implies that, for a generic $(d-2)$ -flat h , the curve $C_h = X \cap h$ is not a quadratic curve. For any 2-flat $g \subset h$, let $C_{h,g}$ denote the projection of C_h onto g . This implies that for a generic choice of g and a $(d-2)$ -flat h satisfying $g \subset h \subset \mathbb{R}^d$, the curve $C_{h,g}$ is not a quadratic planar curve (that is, a conic section) in g . Next, by taking a suitable rotation of the coordinate frame, we may assume that g is the x_1x_2 -flat, and h is the $x_1x_2 \dots x_{d-2}$ -flat. In these coordinates, it is easy to verify that $C_{h,g}$ can be obtained by first projecting X onto the $x_1x_2x_{d-1}x_d$ -flat, and then cutting it with the x_1x_2 -plane. But the projection of X onto the $x_1x_2x_{d-1}x_d$ -flat (which is actually a generic 4-flat) is a 3-quadric by assumption, and then cutting it with any 2-flat yields a quadratic planar curve, a contradiction that completes the proof. \square

In conclusion, we have $I(P, L) \leq I(P^*, L^*)$, where P^* is a set of m points and L^* is a set of n lines, all contained in the three-dimensional algebraic variety \tilde{V} , embedded in 4-space, which is of degree at most D and does not contain any hyperplane or quadric component, and no 2-flat contains more than s lines of L^* . The preceding analysis thus implies that the bound asserted in the theorem applies in any dimension $d \geq 4$. \square

5 Discussion

(1) As already emphasized, most of the analysis in the proof of Theorem 1.3(a) is carried

out over the complex domain. The only place where the proofs of (a) and (b) bifurcate is in the final step. Over the reals we bound $I(P, L_0)$ using the bound of Guth and Katz (which only holds over the reals, because of the polynomial partitioning that it employs), as a “black box”. Over the complex field, we use a variant of the analysis to bypass this step, and obtain more or less the same bound, except for the additional term $O(D^3)$, which becomes insignificant for $D = O(m^{1/3})$, say.

For three-dimensional varieties, the proof of both parts of Theorem 1.5 are more or less the same, with the main difference being the application of the real or complex version of Corollary 4.1. Another difference is in the application of the planar point-line incidence bound—the bound is the same in both cases, but the sources (Szemerédi-Trotter or Tóth-Zahl) are different.

The derivation of fairly sharp point-line incidence bounds over the complex domain in higher dimensions constitutes, in our opinion, significant progress in this theory.

(2) In view of the lower bound constructions in [14, 33, 35], the new bounds in Theorems 1.3 and 1.5 do not hold without the assumption that the points lie on a variety of relatively small degree. We also note that, for a three-dimensional variety, we also get rid of the term $m^{1/2}n^{1/2}q^{1/4}$; this term may arise only when we consider points on hyperplanes or quadrics, but in our case the variety does not contain any such components. Therefore, our theorems indicate that these terms may *only* arise if the variety contains such components.

(3) As mentioned in the introduction, Corollary 1.4 can be extended to the case where V , which is of constant degree D , also contains planes. Here too, we assume that no plane contains more than s lines of L , but this time it is not necessarily the case that $s \leq D$.

Let π_1, \dots, π_k denote the planar components of V , where $k \leq D = O(1)$. For each $i = 1, \dots, k$, the number of incidences within π_i , namely, between the set P_i of points contained in π_i and the set L_i of lines fully contained in π_i , in both real and complex cases, is

$$I(P_i, L_i) = O\left(|P_i|^{2/3}|L_i|^{2/3} + |P_i| + |L_i|\right) = O\left(m^{2/3}s^{2/3} + m + s\right).$$

Summing these bounds over the $k = O(1)$ planes, we get the same asymptotic bound for the overall number of the incidences within these planes. Any other incidence between a point p lying in one of these planes π_i and a line ℓ not contained in π_i can be uniquely identified with the intersection of ℓ with π_i . The overall number of such intersections is at most $nk = O(n)$. This leads to the following extension of Corollary 1.4.

Corollary 5.1. *Let P be a set of m distinct points and L a set of n distinct lines in \mathbb{R}^d or in \mathbb{C}^d , for any $d \geq 3$, and let $s \leq n$ be a parameter, such that all the points and lines lie in a common two-dimensional algebraic surface of constant degree, and no 2-flat contains more than s lines of L . Then*

$$I(P, L) = O\left(m^{2/3}s^{2/3} + m + n\right),$$

where the constant of proportionality depends on the degree of the surface.

(4) As already noted, one of the significant achievements of the analysis in Theorem 1.3 is that the bound there does not include the term $O(nD)$. Such a term arises naturally, when

one considers incidences between points lying in some irreducible component of V and lines not contained in that component. These incidences can be bounded by nD , by charging them, as above, to line-component intersections. When D is large, eliminating the term nD can be crucial for the analysis, as demonstrated in our earlier work [33].

(5) Another interesting challenge is to establish a similar bound for $I(P, L)$, for the case where the points of P lie on a two-dimensional variety V , but the lines need not be contained in V . A trivial extension of the proof adds the term $O(nD)$ to the bound. The challenge is to avoid this term (if possible); see also Remark (4) above.

(6) Similar to item (2), Theorem 1.5(a) can be extended to the case where V also contains hyperplane and quadric components, albeit only for the real case. Here, as in Theorem 1.2, we add the condition that no hyperplane or quadric contains more than q lines of L .

Let H_1, \dots, H_k denote the hyperplane and quadric components of V , where $k \leq D$. Assign, whenever applicable, each point (resp., line) to the first H_i that (fully) contains it. As observed above, the number of “cross-incidences” is $O(nD)$. By [33, Proposition 3.6], the total number of incidences within the hyperplanes and quadrics H_i , for $i = 1, \dots, k$, is

$$O(m^{1/2}n^{1/2}q^{1/4} + m^{2/3}n^{1/3}s^{1/3} + m + n).$$

This leads to the following extension of Theorem 1.5(a).

Corollary 5.2. *Let P be a set of m distinct points and L a set of n distinct lines in \mathbb{R}^d , for any $d \geq 4$, and let $s \leq q \leq n$ be parameters, such that all the points and lines lie in a common three-dimensional algebraic surface of degree D , and assume that (i) no 3-flat or 3-quadric contains more than q lines of L , and (ii) no 2-flat contains more than s lines of L . Then*

$$I(P, L) = O\left(m^{1/2}n^{1/2}(D + q^{1/4}) + m^{2/3}n^{1/3}s^{1/3} + nD + m\right). \quad (11)$$

(7) An interesting offshoot of Lemma 3.5 is the following result.

Proposition 5.3. *Let V be a possibly reducible two-dimensional algebraic surface of degree $D > 1$ in \mathbb{R}^3 or in \mathbb{C}^3 , with no plane or regulus components, and let L be a set of n distinct lines fully contained in V . Then the number of 2-rich points (points incident to at least two lines of L) is $O(nD)$.*

Proof. Partition L into the subsets L_1 and L_0 , as in the proof of Theorem 1.3. (Recall that L_0 is the set of all lines that are either contained in non-ruled components of V , or contained in more than one component, or are exceptional lines on ruled components.) By Lemma 3.5, each line of L_1 is non-conically incident to only $O(D)$ other lines of L_1 , for a total of $O(nD)$ 2-rich points of this sort. Note that we now carry out the analysis without pruning any point (we do not want to do that), because V does not contain any plane or regulus component.

The number of 2-rich points that are exceptional points is at most the number of irreducible components of V , that is, at most D , so this number is negligible.

The number of lines in L_0 is $O(D^2)$. Let h be a plane or a regulus. The number of lines of L_0 contained in h is at most $\deg(V \cap h) \leq 2D = O(\sqrt{|L_0|})$ (this holds if we assume, as

we may, that $|L_0| = \Theta(D^2)$). It therefore follows from Guth and Katz [14] that the number of 2-rich points involving the lines of L_0 is $O(|L_0|^{3/2}) = O(|L_0|D)$.

It remains to consider 2-rich points that are intersection points of a line in L_0 and a line in L_1 . By construction, each line $\ell \in L_1$ is contained in precisely one (ruled) component W of V . If $p \in \ell$ is also incident to a line $\ell' \in L_0$ then, again by construction, ℓ' is fully contained in another component W' of V , which does not fully contain ℓ . Hence ℓ intersects W' in at most $\deg(W')$ points (one of which is p), for a total of at most $\deg(V) = D$ points. Therefore, the number of 2-rich points involving one line in L_1 and another in L_0 is at most $n_1 D$.

As we have exhausted all cases, the assertion follows. \square

(8) Challenging directions for further research are (a) to bound the number of incidences between points and lines on $(d - 1)$ -dimensional varieties in \mathbb{R}^d (or in higher dimensions), for $d \geq 5$, (b) to bound the number of 2-rich points in a finite set of lines contained in such a variety, and (c) to bound the number of incidences between points on a variety and k -flats (under suitable restrictions) in three, four, or higher dimensions.

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6 Algebraic tools and ruled surfaces

In this section we review the preliminary algebraic (and differential) geometry infrastructure needed for our analysis, and then go on to establish the properties of ruled surfaces that we will use. These properties are considered folklore in the literature; having failed to find rigorous proofs of them (except for several short proofs or proof sketches for some of them), we provide here such proofs for the sake of completeness. Some of the notions covered in this section are also discussed in our study [33] on point-line incidences in four dimensions.

Singularity. The notion of singularities is a major concept, treated in full generality in algebraic geometry (see, e.g., Kunz [20, Theorem VI.1.15] and Cox et al. [3]). Here we only recall some of their properties, and only for a few special cases that are relevant to our analysis.

Let V be a two-dimensional variety in \mathbb{R}^3 or \mathbb{C}^3 of degree D , given as the zero set $Z(f)$ of some trivariate polynomial f . Assuming f to be square-free, a point $p \in Z(f)$ is *singular* if $\nabla f(p) = 0$. For any point $p \in Z(f)$, let

$$f(p+x) = f_\mu(x) + f_{\mu+1}(x) + \dots$$

be the Taylor expansion of f near p , where f_j is the j -th order term in the expansion (which is a homogeneous polynomial of x of degree j), and where we assume that there are no terms of order (i.e., degree) smaller than μ . (The terms f_j also depend on p , which we regard as fixed in the present discussion.) In general, we have $f_1(x) = \nabla f(p) \cdot x$, $f_2(x) = \frac{1}{2}x^T H_f(p)x$, where H_f is the Hessian matrix of f , and the higher-order terms are similarly defined, albeit with more involved expressions.

If p is singular, we have $\mu \geq 2$. In this case, we say that p is a singular point of $V = Z(f)$ of *multiplicity* $\mu = \mu_V(p)$. For any point $p \in Z(f)$, we call the hypersurface $Z(f_\mu)$ the *tangent cone* of $Z(f)$ at p , and denote it by $C_p Z(f)$. If $\mu = 1$, then p is non-singular and the tangent cone coincides with the (well-defined) tangent plane $T_p Z(f)$ to $Z(f)$ at p . We denote by V_{sing} the locus of singular points of V . This is a subvariety of dimension at most 1; see, e.g., Solymosi and Tao [36, Proposition 4.4]. We say that a line ℓ is a *singular line* for V if all of its points are singular points of V .

Similarly, let γ be a one-dimensional algebraic curve in \mathbb{R}^2 or in \mathbb{C}^2 , specified as $Z(f)$, for some bivariate square-free polynomial f . Then $p \in Z(f)$ is *singular* if $\nabla f(p) = 0$. The *multiplicity* μ of a point $p \in \gamma$ is defined as in the three-dimensional case, and we denote it as $\mu_\gamma(p)$; the multiplicity is at least 2 when p is singular. The singular locus γ_{sing} of γ is now a discrete set. Indeed, the fact that f is square-free guarantees that f has no common factor with any of its first-order derivatives, and Bézout's Theorem (see, e.g., [3, Theorem 8.7.7]) then implies that the common zero set of f , f_x , f_y , and f_z is a (finite) discrete set.

Still in two dimensions, a line ℓ , not fully contained in the curve γ , can intersect it in at most D points, *counted with multiplicity*. To define this concept formally, as in, e.g., Beltrametti [2, Section 3.4], let ℓ be a line and let $p \in \ell \cap \gamma$, such that ℓ is not contained in the tangent cone of γ at p . The *intersection multiplicity* of γ and ℓ at p is the smallest order of a nonzero term of the Taylor expansion of f at p in the direction of ℓ . As it happens, the intersection multiplicity is also equal to $\mu_\gamma(p)$ (informally, this is the number of branches of γ that ℓ crosses at p , counted with multiplicity; see [3, Section 8.7] for a treatment on the intersection multiplicity in the plane). The intersection between a line ℓ and a curve γ (not containing ℓ) consists of at most $\deg(\gamma)$ points, counted with their intersection multiplicities.

Assume that V is irreducible. By Guth and Katz [13] (see also Elekes et al. [6, Corollary 2]), the number of singular lines contained in V is at most $D(D-1)$.

Flatness. We say that a non-singular point $x \in V$ is *flat* if the second-order Taylor expansion of f at x vanishes on the tangent plane $T_x V$, or alternatively, if the *second fundamental form* of V vanishes at x (see, e.g., Pressley [24]). As argued, e.g., in Elekes et

al. [6], if x is a non-singular point of V and there exist three lines incident to p that are fully contained in V (this property is captured by calling p a *linearly flat* point) then x is a flat point. Following Guth and Katz [13], Elekes et al. [6, Proposition 6] proved that a non-singular point $x \in V$ is flat if and only if certain three polynomials, each of degree at most $3D - 4$, vanish at p . A non-singular line ℓ is said to be *flat* if all of its non-singular points are flat. By Guth and Katz [13] (see also Elekes et al. [6, Proposition 7]), the number of flat lines fully contained in V is at most $D(3D - 4)$, unless V is a plane.

As in the proof of Theorem 1.5, the notions of linear flatness and flatness can be extended to any higher dimension. For example, for a three-dimensional surface V in \mathbb{R}^4 or in \mathbb{C}^4 , which is the zero set of some polynomial f of degree D , a non-singular point $x \in V$ is said to be *linearly flat*, if it is incident to at least three distinct 2-flats that are fully contained in $V = Z(f)$ (and thus also in the tangent hyperplane $T_p Z(f)$). Linearly flat points can then be shown to be *flat*, meaning that the second fundamental form of f vanishes at them. This property, at a point p , can be expressed by several polynomials of degree $3D - 4$ vanishing at p (see [33, Section 2.5]). As in the three-dimensional case, the second fundamental form vanishes identically on $Z(f)$ if and only if $Z(f)$ is not a hyperplane. This property holds in any dimension; see, e.g. [17, Exercise 3.2.12.2]). As in three dimensions, we call a line fully contained in V *flat* if all its non-singular points are flat.

Ruled surfaces. For a modern approach to ruled surfaces, there are many references; see, e.g., Hartshorne [16, Section V.2], or Beauville [1, Chapter III]; see also Salmon [27] and Edge [4] for earlier treatments of ruled surfaces. Three relevant very recent additions are the survey [11] and book [12] of Guth, as well as a survey in Kollár [19], where this topic is addressed in detail.

We say that a real (resp., complex) surface V is *ruled by real* (resp., *complex*) *lines* if every point p in a Zariski-open⁶ dense subset of V is incident to a real (complex) line that is fully contained in V . This definition has been used in several recent works, see, e.g., [14, 19]; it is a slightly weaker condition than the classical condition where it is required that *every* point of V be incident to a line contained in V (e.g., as in [27]). Nevertheless, similarly to the proof of Lemma 3.4 in Guth and Katz [14], a limit argument implies that the two definitions are in fact equivalent. We give, in Lemma 6.1 below, a short algebraic proof of this fact, for the sake of completeness.

Flecnodes in three dimensions and the Cayley-Salmon-Monge Theorem. We first recall the classical theorem of Cayley and Salmon, also due to Monge. Consider a polynomial $f \in \mathbb{C}[x, y, z]$ of degree $D \geq 3$. A *flecnode* of f is a point $p \in Z(f)$ for which there exists a line that is incident to p and *osculates* to $Z(f)$ at p to order three. That is, if the direction of the line is v then $f(p) = 0$, and $\nabla_v f(p) = \nabla_v^2 f(p) = \nabla_v^3 f(p) = 0$, where $\nabla_v f, \nabla_v^2 f, \nabla_v^3 f$ are, respectively, the first, second, and third-order derivatives of f in the direction v (compare with the definition of singular points, as reviewed earlier, for the explicit forms of $\nabla_v f$ and $\nabla_v^2 f$). The *flecnode polynomial* of f , denoted FL_f , is the polynomial obtained by eliminating v from these three homogeneous equations (where p is regarded as a fixed parameter). As shown in Salmon [27, Chapter XVII, Section III], the

⁶The Zariski closure of a set Y is the smallest (by containment) algebraic variety V that contains Y . Y is Zariski closed if it is equal to its closure (and is therefore a variety), and is (relatively) Zariski open if its complement (within a given variety) is Zariski closed. See Cox et al. [3, Section 4.2] for further details.

degree of FL_f is at most $11D - 24$. By construction, the flecnode polynomial of f vanishes on all the flecnodes of f , and in particular on all the lines fully contained in $Z(f)$.

Theorem 2.1. (Cayley and Salmon [27], Monge [23]) *Let $f \in \mathbb{C}[x, y, z]$ be a polynomial of degree $D \geq 3$. Then $Z(f)$ is ruled by (complex) lines if and only if $Z(f) \subseteq Z(\text{FL}_f)$.*

(Note that the correct formulation of Theorem 2.1 is over \mathbb{C} ; earlier applications, over \mathbb{R} , as the one in Guth and Katz [14], require some additional arguments to establish their validity; see Katz [18] for a discussion of this issue.)

Lemma 6.1. *Let $f \in \mathbb{C}[x, y, z]$ be an irreducible polynomial such that there exists a nonempty Zariski open dense set in $Z(f)$ so that each point in the set is incident to a line that is fully contained in $Z(f)$. Then FL_f vanishes identically on $Z(f)$, and $Z(f)$ is ruled by lines.*

Proof. Let $U \subset Z(f)$ be the set assumed in the lemma. By assumption and definition, FL_f vanishes on U , so U , and its Zariski closure, are contained in $Z(f, \text{FL}_f)$. Since U is open, it must be two-dimensional. Indeed, otherwise its complement would be a (nonempty) two-dimensional subvariety of $Z(f)$ (a Zariski closed set is a variety). In this case, the complement must be equal to $Z(f)$, since f is irreducible, which is impossible since U is nonempty. Hence $Z(f, \text{FL}_f)$ is also two-dimensional, and thus, by the same argument just used, must be equal to $Z(f)$. Theorem 2.1 then implies that $Z(f)$ is ruled by (complex) lines, as claimed. \square

The notions of flecnodes and of the flecnode polynomial can be extended to four dimensions, as done in [33]. Informally, the *four-dimensional flecnode polynomial* FL_f^4 of f is defined analogously to the three-dimensional variant FL_f , and captures the property that a point on $Z(f)$ is incident to a line that osculates to $Z(f)$ up to the *fourth* order. It is obtained by eliminating the direction v of the osculating line from the four homogeneous equations given by the vanishing of the first four terms of the Taylor expansion of $f(p + tv)$ near p . Clearly, FL_f^4 vanishes identically on every line that is fully contained in $Z(f)$. As in the three-dimensional case, its degree can be shown to be $O(D)$.

Landsberg [21] derives an analog of Theorem 2.1 that holds for three-dimensional surfaces (see [33, Theorem 2.11]). Specifically, Landsberg's theorem asserts that if FL_f^4 vanishes identically on $Z(f)$, then $Z(f)$ is ruled by (possibly complex) lines. We will discuss this in more detail in Section 4. These theorems, in three and four dimensions, play an important role in the proofs of the main theorems.

Theorem of the fibers and related tools. The main technical tool for the analysis is the following so-called *Theorem of the Fibers*. Both Theorem 6.2 and Theorem 6.3 hold (only) for the complex field \mathbb{C} .

Theorem 6.2 (Harris [15, Corollary 11.13]). *Let X be a projective variety and $\pi : X \rightarrow \mathbb{P}^d$ be a homogeneous polynomial map (i.e., the coordinate functions $x_0 \circ \pi, \dots, x_d \circ \pi$ are homogeneous polynomials); let $Y = \pi(X)$ denote the image of X . For any $p \in Y$, let $\lambda(p) = \dim(\pi^{-1}(\{p\}))$. Then $\lambda(p)$ is an upper semi-continuous function of p in the Zariski topology on Y ; that is, for any m , the locus of points $p \in Y$ such that $\lambda(p) \geq m$ is Zariski closed in Y . Moreover, if $X_0 \subset X$ is any irreducible component, $Y_0 = \pi(X_0)$ its image, and λ_0 the minimum value of $\lambda(p)$ on Y_0 , then*

$$\dim(X_0) = \dim(Y_0) + \lambda_0.$$

We also need the following theorem and lemma from Harris [15].

Theorem 6.3 (Harris [15, Proposition 7.16]). *Let $f : X \rightarrow Y$ be the map induced by the standard projection map $\pi : \mathbb{P}^d \rightarrow \mathbb{P}^r$ (which retains r of the coordinates and discards the rest), where $r < d$, where $X \subset \mathbb{P}^d$ and $Y \subset \mathbb{P}^r$ are projective varieties, X is irreducible, and Y is the image of X (which is also irreducible). Then the general fiber⁷ of the map f is finite if and only if $\dim(X) = \dim(Y)$. In this case, the number of points in a general fiber of f is constant.*

In particular, when Y is two-dimensional (and $d > r \geq 2$ are arbitrary), there exist an integer c_f and an algebraic curve $\mathcal{C}_f \subset Y$, such that for any $y \in Y \setminus \mathcal{C}_f$, we have $|f^{-1}(y)| = c_f$. With the notations of Theorem 6.3, the set of points $y \in Y$, such that the fiber of f over y is not equal to c_f is a Zariski closed proper subvariety of Y . For more details, we refer the reader to Shafarevich [30, Theorem II.6.4], and to Hartshorne [16, Exercise II.3.7].

Lemma 6.4 (Harris [15, Theorem 11.14]). *Let $\pi : X \rightarrow Y$ be a polynomial map between two projective varieties X, Y , with $Y = \pi(X)$ irreducible. Suppose that all the fibers $\pi^{-1}(\{p\})$ of π , for $p \in Y$, are irreducible and of the same dimension. Then X is also irreducible.*

Reguli. We rederive here the following (folklore) characterization of *doubly ruled* surfaces in \mathbb{R}^3 or \mathbb{C}^3 , namely, irreducible algebraic surfaces, each of whose points is incident to at least two distinct lines that are fully contained in the surface. Recall that a *regulus* is the surface spanned by all lines that meet three pairwise skew lines in 3-space.⁸ For an elementary proof that a doubly ruled surface must be a regulus, we refer the reader to Fuchs and Tabachnikov [8, Theorem 16.4]. Their proof however is analytic and works only over the reals.

Lemma 2.2. *Let V be an irreducible ruled surface in \mathbb{R}^3 or in \mathbb{C}^3 which is not a plane, and let $\mathcal{C} \subset V$ be an algebraic curve, such that every non-singular point $p \in V \setminus \mathcal{C}$ is incident to exactly two lines that are fully contained in V . Then V is a regulus.⁹*

Proof. As mentioned above (see also [13]), the number of singular lines in V is finite (it is smaller than $\deg(V)^2$). For any non-singular line ℓ , fully contained in V , but not in \mathcal{C} , the union of lines U_ℓ intersecting ℓ and fully contained in V is a subvariety of V (see Sharir and Solomon [31, Lemma 8] for the easy proof). By assumption, each non-singular point in $\ell \setminus \mathcal{C}$ is incident to another line (other than ℓ) fully contained in V , and thus U_ℓ is the union of infinitely many lines, and is therefore two-dimensional. Since V is irreducible, it follows that $U_\ell = V$. Next, pick any triple of non-singular and non-concurrent lines ℓ_1, ℓ_2, ℓ_3 that are contained in V and intersect ℓ at distinct non-singular points of $\ell \setminus \mathcal{C}$. There has to exist such a triple, for otherwise we would have an infinite family of concurrent (or parallel) lines incident to ℓ and contained in V (where the point of concurrency lies outside ℓ), and the

⁷The meaning of this statement is that the assertion holds for the fiber at any point outside some lower-dimensional exceptional subvariety.

⁸Technically, in some definitions (cf., e.g., Edge [4, Section I.22]) a regulus is a one-dimensional family of generator lines of the actual surface, i.e., a curve in the Plücker or Grassmannian space of lines, but we use here the alternative notion of the surface spanned by these lines.

⁹Over \mathbb{R} , a regulus is either a hyperbolic paraboloid or a one-sheeted hyperboloid. Over \mathbb{C} , balls (equivalent to hyperboloids) and paraboloids (equivalent to hyperbolic paraboloids) are also reguli, and are indeed doubly ruled by complex lines. Not all quadrics are reguli, though: for example, the cylinder $y = x^2$ is not a regulus.

plane that they span would then have to be contained in (the irreducible) V , contrary to assumption. See Figure 1 for an illustration. The argument given for ℓ applies equally well to ℓ_1 , ℓ_2 , and ℓ_3 (by construction, neither of them is fully contained in \mathcal{C}), and implies that $U_{\ell_1} = U_{\ell_2} = U_{\ell_3} = V$.

Assume that there exists some line $\tilde{\ell} \subset V$ intersecting ℓ_1 at some non-singular point $p \in \ell_1 \setminus \mathcal{C}$, and that $\tilde{\ell} \cap \ell_2 = \emptyset$. (We treat lines here as projective varieties, so this assumption means that $\tilde{\ell}$ and ℓ_2 are skew to one another; parallel lines are considered to be intersecting.) Since $p \in \ell_1 \subset V = U_{\ell_2}$, there exists some line $\hat{\ell}$ intersecting ℓ_2 , such that $\hat{\ell} \cap \ell_1 = \{p\}$. Hence there exist three distinct lines, namely $\ell_1, \tilde{\ell}$ and $\hat{\ell}$, that are incident to p and fully contained in V . Since p is non-singular, it must be a flat point (as mentioned above; see [6]). Repeating this argument for $3\deg(V)$ non-singular points $p \in \ell_1$, it follows that ℓ_1 contains at least $3\deg(V)$ flat points, and is therefore, by the properties of flat points noted earlier, a flat line. As is easily checked, ℓ_1 can be taken to be an arbitrary non-singular line among those incident to ℓ , so it follows that every non-singular point on V is flat, and therefore, as shown in [6, 13], V is a plane, contrary to assumption.

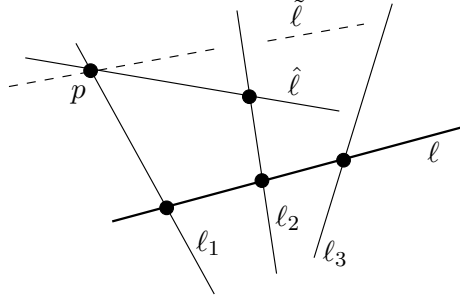


Figure 1: The structure of U_ℓ in the proof of Lemma 2.2.

Therefore, every non-singular line that intersects ℓ_1 at a non-singular point also intersects ℓ_2 , and, similarly, it also intersects ℓ_3 . This implies that the intersection of V and the surface R generated by the lines intersecting ℓ_1, ℓ_2 , and ℓ_3 is two-dimensional, and is therefore equal to V , since V is irreducible. Since ℓ_1, ℓ_2 and ℓ_3 are pairwise skew, $R = V$ is a regulus, as asserted. \square

Real vs. complex. Expanding on the comment at the end of Section 2, let us elaborate about the field over which the variety V is defined. Most of the basic algebraic geometry tools have been developed over the complex field \mathbb{C} , and some care has to be exercised when applying them over the reals. A major part of the theory developed in this section is of this nature. For example, both Theorems 6.2 and 6.3 hold only over the complex field. As another important example, one of the main tools at our disposal is the Cayley–Salmon–Monge theorem (Theorem 2.1), whose original formulation also applies only over \mathbb{C} . Expanding on a previously made comment, we note that even when V is a variety defined as the zero set of a real polynomial f , the vanishing of the flecnode polynomial FL_f only guarantees that the set of complex points of V is ruled by complex lines.

(A very simple example that illustrates this issue is the unit sphere σ , given by $x^2 + y^2 + z^2 = 1$, which is certainly not ruled by real lines, but the flecnode polynomial of $f(x, y, z) = x^2 + y^2 + z^2 - 1$ vanishes on σ (since the equation $\nabla_v^3 f(p) = 0$ that participates

in its construction is identically zero for any quadratic polynomial f). This is the condition in the Cayley–Salmon–Monge theorem that guarantees that σ is ruled by (complex) lines, and indeed it does, as is easily checked; in fact, for the same reason, every quadric is ruled by complex lines.)

This issue has not been directly addressed in Guth and Katz [14], although their theory can be adjusted to hold for the real case too, as noted later in Katz [18].

This is just one example of many similar issues that one must watch out for. It is a fairly standard practice in algebraic geometry that handles a real algebraic variety V , defined by real polynomials, by considering its complex counterpart $V_{\mathbb{C}}$, namely the set of complex points at which the polynomials defining V vanish. The rich toolbox that complex algebraic geometry has developed allows one to derive various properties of $V_{\mathbb{C}}$, but some care might be needed when transporting these properties back to the real variety V , as the preceding note concerning the Cayley–Salmon–Monge theorem illustrates. Fortunately, though, passing to the complex domain (and sometimes also to the projective setting) does not pose any difficulties for deriving upper bounds in incidence problems—every real incidence will be preserved, and at worst we will be counting additional incidences, on the non-real portion of the extended varieties. With this understanding, and with the appropriate caution, we will move freely between the real and complex domains, as convenient.

We note that most of the results developed in Section 3 of this paper also apply over \mathbb{C} , except for one crucial step (where we resort to the application of the result of Guth and Katz [14], which holds only over the reals), due to which we do not know how to extend Theorem 1.3 to the complex domain. Nevertheless, we can derive a weaker variant of it for the complex case—see a remark to that effect in Section 5.

Lines on a variety. In preparation for the key technical Theorem 2.3, given below, we make the following comments. Lines in three dimensions are parameterized by their *Plücker coordinates*, as follows (see, e.g., Griffiths and Harris [10, Section 1.5]). For two points $x, y \in \mathbb{P}^3$, given in projective coordinates as $x = (x_0, x_1, x_2, x_3)$ and $y = (y_0, y_1, y_2, y_3)$, let $\ell_{x,y}$ denote the (unique) line in \mathbb{P}^3 incident to both x and y . The Plücker coordinates of $\ell_{x,y}$ are given in projective coordinates in \mathbb{P}^5 as $(\pi_{0,1}, \pi_{0,2}, \pi_{0,3}, \pi_{2,3}, \pi_{3,1}, \pi_{1,2})$, where $\pi_{i,j} = x_i y_j - x_j y_i$. Under this parameterization, the set of lines in \mathbb{P}^3 corresponds bijectively to the set of points in \mathbb{P}^5 lying on the *Klein quadric* given by the quadratic equation $\pi_{0,1}\pi_{2,3} + \pi_{0,2}\pi_{3,1} + \pi_{0,3}\pi_{1,2} = 0$ (which is indeed always satisfied by the Plücker coordinates of a line).

Given a surface V in \mathbb{P}^3 , the set of lines fully contained in V , represented by their Plücker coordinates in \mathbb{P}^5 , is a subvariety of the Klein quadric, which is denoted by $F(V)$, and is called the *Fano variety* of V ; see Harris [15, Lecture 6, page 63] for details, and [15, Example 6.19] for an illustration, and for a proof that $F(V)$ is indeed a variety. The Plücker coordinates are continuous, in the sense that if one takes two points ℓ, ℓ' on the Klein quadric that are near each other, the lines in \mathbb{P}^3 that they correspond to are also near to one another, in an obvious sense whose precise details are omitted here.

We note again that our analysis is carried out in the complex projective setting, which makes it simpler, and facilitates the application of numerous tools from algebraic geometry that are developed in this setting. The passage from the complex projective setup back to the real affine one is straightforward—the former is a generalization of the latter.

Given a plane π by a homogeneous equation $A_0x_0 + A_1x_1 + A_2x_2 + A_3x_3 = 0$, and a line ℓ not fully contained in π , given in Plücker coordinates as $(\pi_{0,1}, \pi_{0,2}, \pi_{0,3}, \pi_{2,3}, \pi_{3,1}, \pi_{1,2})$, their point of intersection is given in homogeneous coordinates by $(A \cdot m, A \times m - A_0d)$, where $d = (\pi_{0,1}, \pi_{0,2}, \pi_{0,3})$, $m = (\pi_{2,3}, \pi_{3,1}, \pi_{1,2})$, and where \cdot stands for the scalar product, and \times for the vector product; see, e.g., [38, p. 29]. This, together with the continuity argument stated above, implies that, if the Fano variety $F(V)$ is one-dimensional, and ℓ is a line represented by a non-singular point of $F(V)$, then the cross section of the union of the lines that lie near ℓ in $F(V)$ with a generic plane π is a simple arc. When ℓ is a singular point of $F(V)$, then the cross section of the union of the lines that lie near ℓ in $F(V)$ with a generic plane π is a union of simple arcs meeting at $\ell \cap \pi$ where some of these arcs might appear with multiplicity; the number of these arcs is determined by the multiplicity of the singularity of ℓ .

Singly ruled surfaces. Ruled surfaces that are neither planes nor reguli are called *singly ruled* surfaces (a terminology justified by Theorem 2.3, given below). A line ℓ , fully contained in an irreducible singly ruled surface V , such that every point of ℓ is “doubly ruled”, i.e., every point on ℓ is incident to another line fully contained in V , is called an *exceptional line*¹⁰ of V . A point $p_V \in V$ that is incident to infinitely many lines fully contained in V is called an *exceptional point* of V .

The following result is another folklore result in the theory of ruled surfaces, used in many studies (such as Guth and Katz [14]). It justifies the terminology “singly-ruled surface”, by showing that the surface is generated by a one-dimensional family of lines, and that each point on the surface, with the possible exception of points lying on some curve, is incident to exactly one generator. It also shows that there are only finitely many exceptional lines; the property that their number is at most two (see [14]) is presented later. We give a detailed and rigorous proof, to make our presentation as self-contained as possible; we are not aware of any similarly detailed argument in the literature.

Theorem 2.3. (a) *Let V be an irreducible ruled two-dimensional surface of degree $D > 1$ in \mathbb{R}^3 (or in \mathbb{C}^3), which is not a regulus. Then, except for finitely many exceptional lines, the lines that are fully contained in V are parameterized by an irreducible algebraic curve Σ_0 in the Plücker space \mathbb{P}^5 , and thus yield a 1-parameter family of generator lines $\ell(t)$, for $t \in \Sigma_0$, that depend continuously on the real or complex parameter t . Moreover, if $t_1 \neq t_2$, and $\ell(t_1) \neq \ell(t_2)$, then there exist sufficiently small and disjoint neighborhoods Δ_1 of t_1 and Δ_2 of t_2 , such that all the lines $\ell(t)$, for $t \in \Delta_1 \cup \Delta_2$, are distinct.*

(b) *There exists a one-dimensional curve $C \subset V$, such that any point p in $V \setminus C$ is incident to exactly one generator line of V .*

Remark. For a detailed description of the algebraic representation of V by generators, as in part (a) of the theorem, see Edge [4, Section II].

Proof. Assume first that we are working over \mathbb{C} . Consider the Fano variety $F(V)$ of V , as defined above. We claim that all the irreducible components of $F(V)$ are at most one-dimensional. Informally, if any component Σ_0 of $F(V)$ were two-dimensional, then the set

¹⁰In Guth and Katz [14], a line ℓ fully contained in an irreducible singly ruled surface V , is called *exceptional* if it contains infinitely many “doubly ruled” points, each incident to another line fully contained in V . Our definition appears to be stricter, but, as the proof below will reveal, the two notions are equivalent.

$\{(p, \ell) \in V \times F(V) \mid p \in \ell\}$ would be three-dimensional, so, “on average”, the set of lines of $F(V)$ incident to a point $p \in V$ would be one-dimensional, implying that most points of V are incident to infinitely many lines that are fully contained in V , which can happen only when V is a plane (or a non-planar cone, which cannot arise with a non-singular point p as an apex), contrary to assumption.

To make this argument formal, consider the set (already mentioned above)

$$W := \{(p, \ell) \mid p \in \ell, \ell \in F(V)\} \subset V \times F(V),$$

and the two projections

$$\Psi_1 : W \rightarrow V, \quad \Psi_2 : W \rightarrow F(V)$$

to the first and second factors of the product $V \times F(V)$, respectively.

W can formally be defined as the zero set of suitable homogeneous polynomials; briefly, with the Plücker parameterization of lines in \mathbb{P}^3 , and putting the point p into homogeneous coordinates, the condition $p \in \ell$ can be expressed as the vanishing of two suitable homogeneous polynomials, and the other defining polynomials of W are those that define the projective variety $F(V)$. Therefore, W is a projective variety.

Consider an irreducible component Σ_0 of $F(V)$ (which is also a projective variety); put

$$W_0 := \Psi_2^{-1}(\Sigma_0) = \{(p, \ell) \in W \mid \ell \in \Sigma_0\}.$$

Since W and Σ_0 are projective varieties, so is W_0 . As is easily verified, $\Psi_2(W_0) = \Sigma_0$ (that is, Ψ_2 is surjective). We claim that W_0 is irreducible. Indeed, for any $\ell \in \Sigma_0$, the fiber of the map $\Psi_2|_{W_0} : W_0 \rightarrow \Sigma_0$ over ℓ is $\{(p, \ell) \mid p \in \ell\}$ which is (isomorphic to) a line, and is therefore irreducible of dimension one. As Σ_0 is irreducible, Lemma 6.4 implies that W_0 is also irreducible, as claimed.

For a point $p \in \Psi_1(W_0)$, consider the set $\Sigma_{0,p} = \Psi_1|_{W_0}^{-1}(\{p\})$, put $\lambda(p) = \dim(\Sigma_{0,p})$, and let $\lambda_0 := \min_{p \in \Psi_1(W_0)} \lambda(p)$. By the Theorem of the Fibers (Theorem 6.2), applied to the map $\Psi_1|_{W_0} : W_0 \rightarrow V$, we have

$$\dim(W_0) = \dim(\Psi_1(W_0)) + \lambda_0. \tag{12}$$

We claim that $\lambda_0 = 0$. In fact, $\lambda(p) = 0$ for all points $p \in V$, except for at most one point. Indeed, if $\lambda(p) \geq 1$ for some point $p \in V$, then $\Sigma_{0,p}$ is (at least) one-dimensional, and V , being irreducible, is thus a cone with apex at p ; since V can have at most one apex, the claim follows. Hence $\lambda_0 = 0$, and therefore

$$\dim(W_0) = \dim(\Psi_1(W_0)) \leq \dim(V) = 2. \tag{13}$$

Next, assume, for a contradiction, that $\dim(\Sigma_0) = 2$. For a point (i.e., a line in \mathbb{P}^3) $\ell \in \Psi_2(W_0)$, the set $\Psi_2|_{W_0}^{-1}(\{\ell\}) = \{(p, \ell) \mid p \in \ell\}$ is one-dimensional (the equality follows from the way W_0 is defined). Conforming to the notations in the Theorem of the Fibers, we have $\mu(\ell) := \dim(\Psi_2|_{W_0}^{-1}(\{\ell\})) = 1$, and thus $\mu_0 := \min_{\ell \in \Psi_2(W_0)} \mu(\ell) = 1$. Also, by assumption, $\dim(\Psi_2(W_0)) = \dim(\Sigma_0) = 2$. By the Theorem of the Fibers, applied this time to $\Psi_2|_{W_0} : W_0 \rightarrow \Sigma_0$, we thus have

$$\dim(W_0) = \dim(\Psi_2(W_0)) + \mu_0 = 3, \tag{14}$$

contradicting Equation (13). Therefore, every irreducible component of $F(V)$ is at most one-dimensional, as claimed.

Let Σ_0 be such an irreducible component, and let $W_0 := \Psi_2^{-1}(\Sigma_0)$, as above. As argued, for every $p \in V$, the fiber of $\Psi_1|_{W_0}$ over p is non-empty and finite, except for at most one point p (the apex of V if V is a cone). Since W_0 is irreducible, Theorem 6.3 implies that there exists a Zariski open set $\mathcal{O} \subseteq V$, such that for any point $p \in \mathcal{O}$, the fiber of $\Psi_1|_{W_0}$ over p has fixed cardinality c_f . Put $\mathcal{C} := V \setminus \mathcal{O}$. Being the complement of a Zariski open subset of the two-dimensional irreducible variety V , \mathcal{C} is (at most) a one-dimensional variety. If $c_f \geq 2$, then, by Lemma 2.2, V is a regulus. Otherwise, $c_f = 1$ (c_f cannot be zero for a ruled surface), meaning that, for every $p \in V \setminus \mathcal{C}$, there is exactly one line ℓ , such that $(p, \ell) \in W_0$, i.e., Σ_0 contains exactly one line incident to p and contained in V .

Moreover, we observe that the union of lines of Σ_0 is the entire variety V . Indeed, by Equations (12) and (13), we have $\dim(W_0) = \dim(\Psi_1(W_0)) = 2$. That is, the variety $\Psi_1(W_0)$, which is the union of the lines of Σ_0 , must be the entire variety V , because it is two-dimensional and is contained in the irreducible variety V .

To recap, we have proved that if Σ_0 is a one-dimensional component of $F(V)$, then the union of lines that belong to Σ_0 covers V , and that there exists a one-dimensional subvariety (a curve) $\mathcal{C} \subset V$ such that, for every $p \in V \setminus \mathcal{C}$, Σ_0 contains exactly one line incident to p and contained in V .

Since V is a ruled surface, some component of $F(V)$ has to be one-dimensional, for otherwise we would only have a finite number of lines fully contained in V . We claim that there is exactly one irreducible component of $F(V)$ which is one-dimensional. Indeed, assume to the contrary that Σ_0, Σ_1 are two (distinct) one-dimensional irreducible components of $F(V)$. As we observed, the union of lines parameterized by Σ_0 (resp., Σ_1) covers V . Let $\mathcal{C}_0, \mathcal{C}_1 \subset V$ denote the respective excluded curves, so that, for every $p \in V \setminus \mathcal{C}_0$ (resp., $p \in V \setminus \mathcal{C}_1$) there exists exactly one line in Σ_0 (resp., Σ_1) that is incident to p and contained in V .

Next, notice that the intersection $\Sigma_0 \cap \Sigma_1$ is a subvariety strictly contained in the irreducible one-dimensional variety Σ_0 (since Σ_0 and Σ_1 are two distinct irreducible components of $F(V)$), so it must be zero-dimensional, and thus finite. Let \mathcal{C}_{01} denote the union of the finitely many lines in $\Sigma_0 \cap \Sigma_1$, and put $\mathcal{C} := \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_{01}$. For any point $p \in V \setminus \mathcal{C}$, there are two (distinct) lines incident to p and fully contained in V (one belongs to $\Sigma_{0,p}$ and the other to $\Sigma_{1,p}$). Lemma 2.2 (with \mathcal{C} as defined above) then implies that V is a regulus, contrary to assumption.

In other words, the unique one-dimensional irreducible component Σ_0 of $F(V)$ serves as the desired 1-parameter family of generators for V . (The local parameterization of Σ_0 can be obtained, e.g., by using a suitable Plücker coordinate to represent its lines.) In addition to Σ_0 , there is a finite number of zero-dimensional components (i.e., points) of $F(V)$. They correspond to a finite number of lines, fully contained in V , and not parameterized by Σ_0 . Since the union of the lines in Σ_0 covers V , any of these additional lines ℓ is *exceptional*, since each point on ℓ is also incident to a generator (different from ℓ), and is thus “doubly ruled”.

This establishes part (a) of the theorem, when V is defined over \mathbb{C} . We remark that Guth and Katz [14, Corollary 3.6] argue that there are at most two such exceptional lines, so there are at most two zero-dimensional components of $F(V)$. For the sake of completeness,

we sketch a proof of our own of this fact, in Lemma 2.4 below.

If V is defined over \mathbb{R} , we proceed as above, i.e., consider instead the complex variety $V_{\mathbb{C}}$ corresponding to V . As we have just proven, the unique one-dimensional irreducible component Σ_0 of $F(V)$ (regarded as a complex variety) is a (complex) 1-parameter family of generators for the set of complex points of V . Since V is real, the (real) Fano variety of V consists of the real points of $F(V)$, i.e., it is $F(V) \cap \mathbb{P}^5(\mathbb{R})$. As we have mentioned above, the (complex) $F(V)$ is the union of Σ_0 with at most two other points. If $\Sigma_0|_{\mathbb{R}} := \Sigma_0 \cap \mathbb{P}^5(\mathbb{R})$ were zero-dimensional, the real $F(V)$ would also be discrete, as there is only one one-dimensional component Σ_0 , so V would fully contain only finitely many (real) lines, contradicting the assumption that V is ruled by real lines. Therefore, $\Sigma_0|_{\mathbb{R}}$ is a one-dimensional irreducible component of the real Fano variety of V . (It is irreducible, since otherwise the complex Σ_0 would be reducible too, as is easily checked.)

Summarizing, we have shown that there exists exactly one irreducible one-dimensional component Σ_0 of $F(V)$, and a corresponding one-dimensional subvariety $\mathcal{C} \subset V$, such that, for each point $p \in V \setminus \mathcal{C}$, Σ_0 contains exactly one line that is incident to p (and contained in V). In addition to Σ_0 , $F(V)$ might also contain up to two zero-dimensional (i.e., singleton) components, whose elements are the exceptional lines mentioned above. Let \mathcal{D} denote the union of \mathcal{C} and of the at most two exceptional lines; \mathcal{D} is clearly a one-dimensional subvariety of V . Then, for any point $p \in V \setminus \mathcal{D}$, there is exactly one line incident to p and fully contained in V , as claimed. This establishes part (b), and thus completes the proof of the theorem. \square

Exceptional lines on a singly ruled surface. In view of the proofs of Theorem 2.3 and Lemma 6.1, every point on a singly ruled surface V is incident to at least one generator. Hence an exceptional (non-generator) line is a line $\ell \subset V$ such that every point on ℓ is incident to a generator (which is different from ℓ).

Lemma 2.4. *Let V be an irreducible ruled surface in \mathbb{R}^3 or in \mathbb{C}^3 , which is neither a plane nor a regulus. Then (i) V contains at most two exceptional lines, and (ii) V contains at most one exceptional point.*

Proof. (i) We use the property, established in [31] and already used in the proof of Lemma 2.2, that for a line ℓ fully contained in V , the union $\tau(\ell)$ of the lines that meet ℓ and are fully contained in V is a variety in the complex projective space $\mathbb{P}^3(\mathbb{C})$. Moreover, if ℓ is an exceptional line of V , then it follows by [31, Lemma 8] that $\tau(\ell) = V$. (Indeed, $\tau(\ell)$ must be two-dimensional, since otherwise it would consist of only finitely many lines. Since V is irreducible, $\tau(\ell)$ must then be equal to V .)

If V contained three exceptional lines, ℓ_1, ℓ_2 and ℓ_3 , then V would have to be either a plane or a regulus. Indeed, otherwise, by Theorem 2.3 (whose proof does not depend on the number of exceptional lines), there would exist a one-dimensional curve $\mathcal{C} \subset V$ (that includes $\ell_1 \cup \ell_2 \cup \ell_3$), such that every point $p \in V \setminus \mathcal{C}$ is incident to exactly one line ℓ_p fully contained in V . As $p \in V \setminus \mathcal{C}$ and $\sigma(\ell_i) = V$, for $i = 1, 2, 3$, it follows that ℓ_p intersects ℓ_1, ℓ_2 , and ℓ_3 .

If ℓ_1, ℓ_2 , and ℓ_3 are pairwise skew, p belongs to the regulus $R_{\ell_1, \ell_2, \ell_3}$ of all lines intersecting ℓ_1, ℓ_2 , and ℓ_3 . We have thus proved that $V \setminus \mathcal{C}$ is contained in $R_{\ell_1, \ell_2, \ell_3}$, and as $R_{\ell_1, \ell_2, \ell_3}$ is irreducible, it follows that $V = R_{\ell_1, \ell_2, \ell_3}$.

If ℓ_1, ℓ_2 , and ℓ_3 are concurrent but not coplanar then, arguing similarly, V is a cone with their common intersection point as an apex. Since a (non-planar) cone has no exceptional lines, as is easily checked, we may ignore this case.

Finally if any pair among ℓ_1, ℓ_2, ℓ_3 , say ℓ_1, ℓ_2 , are parallel then V must be the plane that they span, contrary to assumption. If ℓ_1 and ℓ_2 intersect at a point ξ , disjoint from ℓ_3 , then V is the union of the plane spanned by ℓ_1 and ℓ_2 and the plane spanned by ξ and ℓ_3 , again a contradiction.

Having exhausted all possible cases, the proof of (i) is complete.

(ii) By Theorem 2.3 and (i), all the lines that are fully contained in V , except for possibly two such lines, are parameterized by an irreducible algebraic curve Σ_0 in the Plücker space \mathbb{P}^5 . Let p be an exceptional point of V . The set Σ' of lines incident to p is an algebraic curve contained in the irreducible curve Σ_0 , implying that $\Sigma' = \Sigma_0$. This clearly implies that there is at most one exceptional point (and then it does not contain any exceptional line), and the proof of (ii) is complete too. \square

Remark. We refer the reader to Guth and Katz [14, Lemma 3.5, Corollary 3.6], for yet another (somewhat more compact) proof of this lemma.

Generic projections preserve non-planarity. In the analysis in Section 3, the goal is to project \mathbb{R}^d onto some generic 3-flat so that non-coplanar triples of lines do not project to coplanar triples. This is easily achieved by repeated applications of the following technical result, reducing the dimension one step at a time.

Lemma 2.5. *Let ℓ_1, ℓ_2, ℓ_3 be three non-coplanar lines in \mathbb{R}^d . Then, under a generic projection of \mathbb{R}^d onto some hyperplane H , the respective images $\ell_1^*, \ell_2^*, \ell_3^*$ of these lines are still non-coplanar.*

Proof. Assume without loss of generality that the (generic) hyperplane H onto which we project passes through the origin of \mathbb{R}^d , and let w denote the unit vector normal to H . The projection $h : \mathbb{R}^d \mapsto H$ is then given by $h(v) = v - (v \cdot w)w$.

Assume first that two of the three given lines, say ℓ_1, ℓ_2 , are skew (i.e., not coplanar). Let $\tilde{\ell}_1, \tilde{\ell}_2$ denote their projection onto H . If $\tilde{\ell}_1, \tilde{\ell}_2$ are coplanar they are either intersecting or parallel. If they are intersecting, then there are points $p_1 \in \ell_1, p_2 \in \ell_2$ that project to the same point, i.e., $p_1 - p_2$ has the same direction as w . Then w belongs to the set $\left\{ \frac{p_1 - p_2}{\|p_1 - p_2\|} \mid p_1 \in \ell_1, p_2 \in \ell_2 \right\}$. Since this is a two-dimensional set, it will be avoided for a generic choice of w , which is a generic point in \mathbb{S}^{d-1} , a set that is at least three-dimensional.

If $\tilde{\ell}_1, \tilde{\ell}_2$ are parallel, let v_1, v_2 denote the directions of ℓ_1, ℓ_2 . Since $v_1 - (v_1 \cdot w)w$ and $v_2 - (v_2 \cdot w)w$ are vectors in the directions of $\tilde{\ell}_1, \tilde{\ell}_2$, and are thus parallel, it follows that w must be a linear combination of v_1 and v_2 . Since $\|w\| = 1$, the resulting set of possible directions is only one-dimensional, and, again, it will be avoided with a generic choice of w .

We may therefore assume that every pair of lines among ℓ_1, ℓ_2, ℓ_3 are coplanar. Since these three lines are not all coplanar, the only two possibilities are that either they are all mutually parallel, or all concurrent.

Assume first that they are concurrent, say they all pass through the origin (even though the origin belongs to H , this still involves no loss of generality). Their projections are in

the directions $v_i - (v_i \cdot w)w$, for $i = 1, 2, 3$. If these projections are coplanar then there exist coefficients $\alpha_1, \alpha_2, \alpha_3$, not all zero, such that $\sum_i \alpha_i (v_i - (v_i \cdot w)w) = 0$. That is, putting $u := \sum_i \alpha_i v_i$, we have $u = (u \cdot w)w$, so u is parallel to w . In this case w belongs to the set $\left\{ \frac{\sum_i \alpha_i v_i}{\|\sum_i \alpha_i v_i\|} \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \text{ or } \mathbb{C} \right\}$. Again, being a two-dimensional set, it will be avoided by a generic choice of w .

In the remaining case, the lines ℓ_1, ℓ_2, ℓ_3 are mutually parallel, i.e., they all have the same direction v . Put, for $i = 1, 2, 3$, $\ell_i = \{p_i + tv\}_{t \in \mathbb{R}}$, and choose p_i so that $p_i \cdot v = 0$. The plane π_0 spanned by p_1, p_2, p_3 is projected to the plane π spanned by the points $p_i^* = p_i - (p_i \cdot w)w$, for $i = 1, 2, 3$ (since p_1, p_2, p_3 are not collinear, they will not project into collinear points in a generic projection), and the three lines project into a common plane if and only if their projections are fully contained in π , meaning that the projection $v^* = v - (v \cdot w)w$ is parallel to π , so it must be a linear combination of p_1^*, p_2^* , and p_3^* . A similar argument to those used above shows that a generic choice of w will avoid the resulting two-dimensional set of forbidden directions. This completes the proof. \square